Equidistributions of Mahonian statistics over pattern avoiding permutations

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Submitted: Jul 6, 2017; Accepted: Dec 24, 2017; Published: Jan 12, 2018
Mathematics Subject Classifications: 05A05, 05A15, 05A19

Abstract

A Mahonian d-function is a Mahonian statistic that can be expressed as a linear combination of vincular pattern functions of length at most d. Babson and Steingrímsson classified all Mahonian 3-functions up to trivial bijections and identified many of them with well-known Mahonian statistics in the literature. We prove a host of Mahonian 3-function equidistributions over permutations in $\mathcal{S}_n$ avoiding a single classical pattern in $\mathcal{S}_3$. Tools used include block decomposition, Dyck paths and generating functions.

Keywords: Mahonian statistic; Equidistribution; st-Wilf equivalence, Pattern avoidance; Dyck path statistic; Polyomino

Contents

1 Introduction

A *combinatorial statistic* on a set S is a map stat : $S \to \mathbb{N}$. The *distribution* of stat over S is given by the coefficients of the generating function $\sum_{\sigma \in S} q^{\mathrm{stat}(\sigma)}$. Let $\mathcal{S}_n$ be the set of permutations $\sigma = a_1 a_2 \cdots a_n$ of the letters $[n] = \{1, 2, \ldots, n\}$ and let $\sigma(k)$ denote the entry a_k. Let $\mathcal{S} = \bigcup_{n \geqslant 0} \mathcal{S}_n$. The *inversion set* of $\sigma \in \mathcal{S}_n$ is defined by $\mathrm{Inv}(\sigma) = \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$. A particularly well-studied statistic on $\mathcal{S}_n$ is $\mathrm{inv} : \mathcal{S}_n \to \mathbb{N}$, given by $\mathrm{inv}(\sigma) = |\operatorname{Inv}(\sigma)|$. An elegant formula for the distribution of the inversion statistic was found in 1839 by Rodrigues [27]

$$\sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!,$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$ and $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. The *descent set* of σ is defined by $\mathrm{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. In 1915 MacMahon [25] showed that inv has the same distribution as another statistic, now called the *major index* (due to MacMahon's profession as a major in the british army) [17], given by $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i$. We also write $\mathrm{imaj}(\sigma) = \mathrm{maj}(\sigma^{-1})$. In honor of MacMahon any permutation statistic with the same distribution as maj is called *Mahonian*. Mahonian statistics are well-studied in the literature. Since MacMahon's initial work, many new Mahonian statistics have been identified. Babson and Steingrímsson [1] showed that almost all (at the time) known Mahonian statistics can be expressed as linear combinations of statistics counting occurrences of vincular patterns. They made several further conjectures regarding new vincular pattern-based Mahonian statistics. These have since been proved and reproved at various levels of refinement by a number of authors (see e.g., [4, 7, 18, 33]). Two sequences of integers $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots b_n$ are said to be *order isomorphic* provided $a_i < a_j$ if and only if $b_i < b_j$ for all $1 \leqslant i < j \leqslant n$. A *vincular pattern* (also known as *generalized pattern*) of length m is a pair (π, X) where π is a permutation in $\mathcal{S}_m$ and $X \subseteq \{0, 1, \ldots, m\}$ is a set of adjacencies. Adjacencies are indicated by underlining the adjacent entries in π (see Example 1). If $0 \in X$ (respectively, $m \in X$), then we denote this by adding a square bracket at the beginning (respectively, end) of the pattern π. If $X = \emptyset$, then (π, X) coincides with the definition of a *classical pattern*. A permutation $\sigma = a_1 a_2 \cdots a_n \in \mathcal{S}_n$ *contains the vincular pattern* (π, X) if there is an m-tuple $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ such that the following criteria are satisfied

- $a_{i_1} a_{i_2} \cdots a_{i_m}$ is order-isomorphic to π,
- $i_{j+1} = i_j + 1$ for each $j \in X \setminus \{0, m\}$ and
- $i_1 = 1$ if $0 \in X$ and $i_m = n$ if $m \in X$.

We also say that $a_{i_1} a_{i_2} \cdots a_{i_m}$ is an *occurrence* of π in σ. We say that σ *avoids* π if σ contains no occurrences of π. We denote the set of permutations in $\mathcal{S}_n$ avoiding the pattern π by $\mathcal{S}_n(\pi)$. Moreover if Π is a set of patterns, then we set $\mathcal{S}_n(\Pi) = \bigcap_{\pi \in \Pi} \mathcal{S}_n(\pi)$.

In this paper we shall also need an additional generalization of vincular patterns, allowing us to restrict occurrences to particular value requirements. Let $\upsilon = (\upsilon_1, \ldots, \upsilon_m)$

where $\upsilon_i \in \mathbb{N} \sqcup \{\text{-}\}$. Define a *value-restricted vincular pattern* $(\pi, X)\big|_\upsilon$ to be a triple (π, X, υ) where (π, X) is a vincular pattern. We say that $a_{i_1} a_{i_2} \cdots a_{i_m}$ is an *occurrence* of $(\pi, X)\big|_\upsilon$ in σ if it is an occurrence of the vincular pattern (π, X) and $a_{i_j} = \upsilon_j$ whenever $\upsilon_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Note in particular that $(\pi, X)\big|_{(\text{-},\ldots,\text{-})} = (\pi, X)$. Every value-restricted vincular pattern $(\pi, X)\big|_\upsilon$ gives rise to a permutation statistic $(\pi, X)\big|_\upsilon : \mathcal{S}_n \to \mathbb{N}$ called a *pattern function* counting the number of occurrences of $(\pi, X)\big|_\upsilon$ in a given permutation $\sigma \in \mathcal{S}_n$ (see Example 1). The *length* of $(\pi, X)\big|_\upsilon : \mathcal{S}_n \to \mathbb{N}$ is defined as the length of the underlying vincular pattern (π, X).

Example 1. Let $\sigma = 246153$.

Pattern π	X	Occurrences in σ
231	$\emptyset$	$241, 261, 461, 463, 453$
$[231$	$\{0\}$	$241, 261$
$\underline{23}1$	$\{1\}$	$241, 461, 463$
$2\underline{31}$	$\{2\}$	$261, 461, 453$
$\underline{231}$	$\{1,2\}$	461
$2\underline{31}]$	$\{2,3\}$	453
$\underline{23}1\big\|_{(\text{-},6,\text{-})}$	$\{1\}$	$461, 463$

We also have $(231)\sigma = 5$, $[231)\sigma = 2$, $(\underline{23}1)\sigma = 3$, $(2\underline{31})\sigma = 3$, $(\underline{231})\sigma = 1$, $(2\underline{31}]\sigma = 1$ and $(\underline{23}1)\big|_{(\text{-},6,\text{-})}\sigma = 2$. On the other hand, the permutation $\sigma = 215346$ avoids the pattern $\pi = 231$ (and hence all the patterns in the table above).

In this paper we mainly study equidistributions of the form

$$\sum_{\sigma \in \mathcal{S}_n(\Pi_1)} q^{\mathrm{stat}_1(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\Pi_2)} q^{\mathrm{stat}_2(\sigma)} \tag{1}$$

where Π_1, Π_2 are sets of patterns and $\mathrm{stat}_1, \mathrm{stat}_2$ are permutation statistics. We will almost exclusively focus on the case where Π_i consists of a single classical pattern of length three and stat_i is a Mahonian statistic. The equidistributions we prove are summarized in §5, Table 2. Although Mahonian statistics are equidistributed over $\mathcal{S}_n$, they need not be equidistributed over pattern avoiding sets of permutations. For instance maj and inv are not equidistributed over $\mathcal{S}_n(\pi)$ for any classical pattern $\pi \in \mathcal{S}_3$. Neither do the existing bijections in the literature for proving equidistribution over $\mathcal{S}_n$ necessarily restrict to bijections over $\mathcal{S}_n(\pi)$ (cf. [1, 4, 7, 18, 33]). Therefore whenever such an equidistribution is present, we must usually seek a new bijection which simultaneously preserves statistic and pattern avoidance. Another motivation for studying equidistributions over permutations avoiding a classical pattern of length three is that $|\mathcal{S}_n(\pi)| = C_n$ for all $\pi \in \mathcal{S}_3$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number (see [22]). Therefore equidistributions of this kind induce equidistributions between statistics on other Catalan objects (and vice versa) whenever we have bijections where the statistics translate in an appropriate fashion. We prove several results in this vein where an exchange between statistics on $\mathcal{S}_n(\pi)$, Dyck paths and polyominoes takes place. In general, studying the generating function (1) provides a rich source of interesting q-analogues to well-known sequences enumerated by pattern avoidance and raises new questions about the coefficients of such polynomials.

Equidistributions such as (1) has been studied in the past. For instance, Burstein and Elizalde proved the following result involving the Mahonian *Denert statistic*

$$\mathrm{den}(\sigma) = \mathrm{inv}(\mathrm{Exc}(\sigma)) + \mathrm{inv}(\mathrm{NExc}(\sigma)) + \sum_{\substack{i\in[n]\\ \sigma(i)>i}} i,$$

where $\mathrm{Exc}(\sigma) = (\sigma(i))_{\sigma(i)>i}$ and $\mathrm{NExc}(\sigma) = (\sigma(i))_{\sigma(i)\leqslant i}$.

Theorem 2 (Burstein-Elizalde [5]). *For any $n \geqslant 1$,*

$$\sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(321)} q^{\mathrm{den}(\sigma)}.$$

Two sets of patterns Π_1 and Π_2 are said to be *Wilf-equivalent* if $|\mathcal{S}_n(\Pi_1)| = |\mathcal{S}_n(\Pi_2)|$ for all $n \geqslant 0$. Sagan and Savage [28] coined a q-analogue of this concept. Two sets of patterns Π_1 and Π_2 are said to be *st-Wilf equivalent* with respect to the statistic $\mathrm{st} : \mathcal{S} \to \mathbb{N}$ if (1) holds with $\mathrm{stat}_1 = \mathrm{st} = \mathrm{stat}_2$ for any $n \geqslant 0$. Let $[\Pi]_{\mathrm{st}}$ denote the st-Wilf class of the set Π. This concept have been studied at several places in the literature. An overview of the st-Wilf classification of single and multiple classical patterns of length three can be found in the table below.

st	Reference
maj, inv	Dokos-Dwyer-Johnson-Sagan-Selsor [14]
charge	Killpatrick [20]
fp, exc, des	Elizalde [15, 16]
peak, valley	Baxter [2]
peak, valley, head, last, lir, rir, lrmin, rank, comp, ldr	Claesson-Kitaev [11]

In particular it was shown in [14] that $I_n(132;q) = I_n(213;q) = C_n(q)$ and $I_n(231;q) = I_n(312;q) = \tilde{C}_n(q)$ where

$$I_n(\pi;q) = \sum_{\sigma\in\mathcal{S}_n(\pi)} q^{\mathrm{inv}(\sigma)},$$

$$C_n(q) = \sum_{k=0}^{n-1} q^{(k+1)(n-k)} C_k(q) C_{n-k-1}(q), \;\; C_0(q) = 1,$$

$$\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^{k} \tilde{C}_k(q) \tilde{C}_{n-k-1}(q), \;\; \tilde{C}_0(q) = 1.$$

The polynomial $C_n(q)$ is known as the *Carlitz-Riordan q-analogue of the Catalan numbers* and have been studied by numerous authors (though no explicit formula is known). Similar recursions for maj have been studied in [8, 14].

To decompose pattern avoiding permutations we will require some notation. Given permutations $\tau \in \mathcal{S}_k$ and $\sigma_1, \sigma_2 \ldots, \sigma_k \in \mathcal{S}$, the *inflation* of τ by $\sigma_1, \sigma_2 \ldots, \sigma_k$ is the permutation $\tau[\sigma_1, \sigma_2, \ldots, \sigma_k]$ obtained by replacing each entry $\tau(i)$ by a block of length $|\sigma_i|$ order isomorphic to σ_i for $i = 1, \ldots, k$ such that the blocks are externally order-isomorphic to τ.

Example 3. $231[21, 1, 213] = 546213$.

Let $\sigma \in \mathcal{S}_n$. Recall that the *descent set* of σ is given by $\operatorname{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. The set of *descent bottoms* (resp. *descent tops*) of σ is given by $\operatorname{DB}(\sigma) = \{\sigma(i+1) : i \in \operatorname{Des}(\sigma)\}$ (resp. $\operatorname{DT}(\sigma) = \{\sigma(i) : i \in \operatorname{Des}(\sigma)\}$). Likewise the *ascent set* of σ is given by $\operatorname{Asc}(\sigma) = \{i : \sigma(i) < \sigma(i+1)\}$ and we define the set of *ascent bottoms* (resp. *ascent tops*) of σ to be $\operatorname{AB}(\sigma) = \{\sigma(i) : i \in \operatorname{Asc}(\sigma)\}$ (resp. $\operatorname{AT}(\sigma) = \{\sigma(i+1) : i \in \operatorname{Asc}(\sigma)\}$). An entry $\sigma(j)$ is called a *left-to-right maxima* if $\sigma(j) > \sigma(i)$ for all $i < j$. Let $\operatorname{LRMax}(\sigma)$ denote the set of left-to-right maxima in σ and let $\operatorname{lrmax}(\sigma) = |\operatorname{LRMax}(\sigma)|$. Similarly an entry $\sigma(j)$ is called a *left-to-right minima* if $\sigma(j) < \sigma(i)$ for all $i < j$. Let $\operatorname{LRMin}(\sigma)$ denote the set of left-to-right minima in σ and let $\operatorname{lrmin}(\sigma) = |\operatorname{LRMin}(\sigma)|$. We call $\sigma(i)$ a *pinnacle* if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and $\sigma(i)$ a *trough* if $\sigma(i-1) > \sigma(i) < \sigma(i+1)$.

Example 4. Let $\sigma = 271985346$. Then $\operatorname{Des}(\sigma) = \{2, 4, 5, 6\}$, $\operatorname{DB}(\sigma) = \{1, 3, 5, 8\}$, $\operatorname{DT}(\sigma) = \{5, 7, 8, 9\}$, $\operatorname{Asc}(\sigma) = \{1, 3, 7, 8\}$, $\operatorname{AB}(\sigma) = \{1, 2, 3, 4\}$, $\operatorname{AT}(\sigma) = \{4, 6, 7, 9\}$, $\operatorname{LRMax}(\sigma) = \{2, 7, 9\}$, $\operatorname{LRMin}(\sigma) = \{2, 1\}$. The pinnacles of σ are given by $\{7, 9\}$ and the troughs of σ by $\{1, 3\}$.

If $\sigma = a_1 a_2 \cdots a_{n-1} a_n$, then the *reverse* of σ is given by $\sigma^r = a_n a_{n-1} \cdots a_2 a_1$ and the *complement* of σ by $\sigma^c = (n-a_1+1)(n-a_2+1) \cdots (n-a_{n-1}+1)(n-a_n+1)$. The *inverse* of σ (in the group theoretical sense) is denoted by σ^{-1}. The operations complement, reverse and inverse are often referred to as *trivial bijections* and together they generate a group isomorphic to the Dihedral group D_4 of order 8 acting on $\mathcal{S}_n$. If π is a classical pattern and $g \in D_4$, then it is not difficult to see that $\sigma \in \mathcal{S}_n(\pi)$ if and only if $\sigma^g \in \mathcal{S}_n(\pi^g)$. However if π is a non-classical pattern, then avoidance is not necessarily closed under inverse in any similar way. E.g. $\sigma = 6274251$ avoids the vincular pattern $\pi = \underline{12}3$, but $\sigma^{-1} = 7254613$ avoids no vincular pattern (π, X) of length three with $X = \{1\}$ or $X = \{2\}$. Therefore taking the inverse should not be viewed as a 'trivial bijection' in the same sense as complement and reverse when it comes to vincular patterns.

In Table 1 we list the vincular pattern definitions of the Mahonian statistics that we shall consider from [1]. The references in Table 1 indicate where the Mahonian nature of the statistics was first proved. Some of these statistics where originally defined in a slightly different form. See [1] for their translation into vincular pattern functions.

For example, Foata and Zeilberger introduced the Mahonian statistic mak in [18] where it was essentially defined as

$$\operatorname{mak}(\sigma) = \sum_{\alpha \in \operatorname{DB}(\sigma)} \alpha + (\underline{31}2)\sigma. \tag{2}$$

Name	Vincular pattern definition	Reference
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon [25]
inv	$(\underline{23}1) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	MacMahon [25]
mak	$(1\underline{32}) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Foata-Zeilberger [19]
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng [13]
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng [13]
bast	$(\underline{13}2) + (\underline{21}3) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[1]
bast′	$(\underline{13}2) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[1]
bast″	$(1\underline{32}) + (3\underline{12}) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson[1]
foze	$(\underline{21}3) + (3\underline{21}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger [18]
foze′	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger [18]
foze″	$(\underline{23}1) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger [18]
sist	$(\underline{13}2) + (\underline{13}2) + (2\underline{13}) + (\underline{21})$	Simion-Stanton [28]
sist′	$(\underline{13}2) + (\underline{13}2) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [28]
sist″	$(\underline{13}2) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [28]

Table 1: Mahonian 3-functions.

It is easy to see that

$$\sum_{\alpha \in \mathrm{DB}(\sigma)} \alpha = ((1\underline{32}) + (\underline{32}1) + (\underline{21}))\sigma.$$

The statistic mad introduced by Clarke-Steingrímsson-Zeng in [13] is defined similarly by replacing the sum of descent bottoms by the sum of descent differences, i.e., the sum of the differences between the two letters of a descent.

According to [1], Table 1 is the complete list of *Mahonian 3-functions* (up to trivial bijections), i.e., Mahonian statistics that can be written as a sum of vincular pattern functions of length at most three. Since some of these statistics have received no conventional name in the literature, we will take the liberty of naming them according to the initials of the authors who first proved their Mahonian nature.

2 Equidistributions via direct bijection

The equidistributions proved in this section are shown by directly exhibiting a bijection. The bijections are based on standard decompositions of pattern avoiding permutations, or rely on specifying data by which pattern avoiding permutations are uniquely determined. In many cases we are able to find a more refined equidistribution. We begin by proving that maj and mak are related via the inverse map over certain pattern avoiding sets of permutations. This may seem unexpected given that vincular patterns do not behave as straightforwardly under the inverse map as they do under complement and reverse.

Proposition 5. *Let $\sigma \in \mathcal{S}_n(\pi)$ where $\pi \in \{132, 213, 231, 312\}$. Then*

$$\mathrm{mak}(\sigma) = \mathrm{imaj}(\sigma).$$

Moreover for any $n \geqslant 1$,

$$\sum_{\sigma \in \mathcal{S}_n(\pi)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\pi^{-1})} q^{\mathrm{mak}(\sigma)} t^{\mathrm{des}(\sigma)}.$$

Proof. Let $\sigma \in \mathcal{S}_n(231)$. If $\mathrm{Des}(\sigma) = \{i_1, \ldots, i_k\}$, then by [32, Lemma 3.1] we have that

$$\mathrm{Des}(\sigma^{-1}) = \{\sigma(i_1) - 1, \ldots, \sigma(i_k) - 1\}.$$

In particular $\mathrm{des}(\sigma) = \mathrm{des}(\sigma^{-1})$. Note that

$$\sigma(i_j) = \sigma(i_j + 1) + (\underline{31}2)\big|_{(\sigma(i_j), \sigma(i_j+1), \text{-})}\sigma + 1, \tag{3}$$

for $j = 1, \ldots, k$. Indeed if $\sigma(i_j + 1) < \alpha < \sigma(i_j)$, then α must appear to the right of the descent i_j in σ, otherwise $\alpha\sigma(i_j)\sigma(i_j + 1)$ is an occurrence of 231 (which is forbidden). Therefore $\sigma(i_j)\sigma(i_j+1)\alpha$ is an occurrence of $(\underline{31}2)\big|_{(\sigma(i_j), \sigma(i_j+1), \text{-})}$ in σ for every α such that $\sigma(i_j + 1) < \alpha < \sigma(i_j)$. Thus (3) follows.

Hence by (3) and (2) we have

$$\begin{aligned} \mathrm{imaj}(\sigma) &= \sum_{j=1}^{k} (\sigma(i_j) - 1) \\ &= \sum_{j=1}^{k} \Big(\sigma(i_j + 1) + (\underline{31}2)\big|_{(\sigma(i_j), \sigma(i_j+1), \text{-})}\Big) \\ &= \sum_{\alpha \in \mathrm{DB}(\sigma)} \alpha + (\underline{31}2)\sigma \\ &= \mathrm{mak}(\sigma). \end{aligned}$$

The statement is proved similarly for remaining choices of π and those analogous arguments are omitted. □

Remark 6. By Proposition 5 and [32, Corollary 4.1] it follows that

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{maj}(\sigma) + \mathrm{mak}(\sigma)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \tag{4}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$. The right hand side of (4) is known as *MacMahon's q-analogue of the Catalan numbers* [26].

The following lemma regarding the structure of $\mathcal{S}_n(321)$ is part of the folklore of pattern avoidance (see e.g., [22]).

Lemma 7. *We have $\sigma \in \mathcal{S}_n(321)$ if and only if the elements of $[n] \setminus \mathrm{LRMax}(\sigma)$ form an increasing subsequence of σ.*

Theorem 8. *For any $n \geqslant 1$,*

$$\sum_{\sigma\in\mathcal{S}_n(321)} q^{\mathrm{maj}(\sigma)}\mathbf{x}^{\mathrm{DB}(\sigma)}\mathbf{y}^{\mathrm{DT}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(321)} q^{\mathrm{mak}(\sigma)}\mathbf{x}^{\mathrm{DB}(\sigma)}\mathbf{y}^{\mathrm{DT}(\sigma)},$$
$$\sum_{\sigma\in\mathcal{S}_n(123)} q^{\mathrm{maj}(\sigma)}\mathbf{x}^{\mathrm{AB}(\sigma)}\mathbf{y}^{\mathrm{AT}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(123)} q^{\mathrm{mak}(\sigma)}\mathbf{x}^{\mathrm{AB}(\sigma)}\mathbf{y}^{\mathrm{AT}(\sigma)}.$$

Proof. Let $\sigma \in \mathcal{S}_n(321)$. By Lemma 7 we may decompose σ as

$$\sigma = u_1v_1u_2v_2\cdots u_tv_t,$$

where $u_1,\ldots,u_t$ are non-empty factors of left-to-right maxima in σ and $v_1,\ldots,v_t$ are non-empty factors (except possibly v_t) such that $v_1v_2\cdots v_t$ is an increasing subword. Assume first that $v_t \neq \emptyset$. Let $M_i = \max(u_i)$ and $m_i = \min(v_i)$ for $i = 1,\ldots,t$. Clearly $\mathrm{DB}(\sigma) = \{m_i : 1 \leqslant i \leqslant t\}$ and $\mathrm{DT}(\sigma) = \{M_i : 1 \leqslant i \leqslant t\}$. Let $\bar{u}_i = u_i \setminus M_i$ and $\bar{v}_i = v_i \setminus m_i$ for $i = 1,\ldots,t$. Write $\bar{u} = \bar{u}_1\cdots\bar{u}_t$ and $\bar{v} = \bar{v}_1\cdots\bar{v}_t$.

We now define an involution

$$\phi : \mathcal{S}_n(321) \to \mathcal{S}_n(321) \tag{5}$$

such that $\mathrm{maj}(\phi(\sigma)) = \mathrm{mak}(\sigma)$, preserving all pairs of descent top and descent bottoms. For convenience, set $M_0 = -\infty$ and $M_{t+1} = \infty$. Let u'_k denote the unique increasing word of the letters in the set

$$\{\alpha \in \bar{v} : M_{k-1} < \alpha < M_k\},$$

with M_k adjoined at the end and let v'_k denote the unique increasing word of the letters in the set

$$\{\beta \in \bar{u} : m_k < \beta < M_{k+1}\},$$

with m_k adjoined at the beginning for $k = 1,\ldots,t$. Define

$$\phi(\sigma) = \begin{cases} u'_1v'_1\cdots u'_tv'_t & \text{if } v_t \neq \emptyset \\ \phi(u_1v_1\cdots u_{t-1}v_{t-1})u_t & \text{if } v_t = \emptyset \end{cases}.$$

Thus ϕ effectively swaps $\bar{u} = \mathrm{LRMax}(\sigma) \setminus \mathrm{DT}(\sigma)$ with $\bar{v} = [n] \setminus (\mathrm{LRMax}(\sigma) \cup \mathrm{DB}(\sigma))$ (when $v_t \neq \emptyset$) and $\mathrm{DB}(\phi(\sigma)) = \mathrm{DB}(\sigma)$, $\mathrm{DT}(\phi(\sigma)) = \mathrm{DT}(\sigma)$. Hence ϕ is an involution.

We have

$$
\begin{aligned}
(2\underline{31})\sigma &= \sum_{\beta\in\bar{u}} (2\underline{31})\big|_{(\beta,\text{-},\text{-})}\sigma \\
&= \sum_{\beta\in\bar{u}} (\max\{k : m_k < \beta\} - \min\{k : M_k > \beta\} + 1) \\
&= \sum_{\beta\in\bar{u}} (\max\{k : m_k < \phi(\beta)\} - \min\{k : M_k > \phi(\beta)\} + 1) \\
&= \sum_{\beta\in\bar{u}} (\underline{31}2)\big|_{(\text{-},\text{-},\phi(\beta))}\phi(\sigma) \\
&= (\underline{31}2)\phi(\sigma),
\end{aligned}
$$

since under the involution ϕ, each $\beta \in \mathrm{LRMax}(\sigma) \setminus \mathrm{DT}(\sigma)$ precisely passes the number of descent bottoms that are less than it to its right. Therefore β is involved in the same number of $2\underline{31}$ occurrences in σ as $\phi(\beta)$ is involved in $\underline{31}2$ occurrences in $\phi(\sigma)$. Hence

$$
\begin{aligned}
\mathrm{mak}(\phi(\sigma)) &= ((1\underline{32}) + (\underline{32}1) + (\underline{21}))\phi(\sigma) + (\underline{31}2)\phi(\sigma) \\
&= \sum_{\alpha\in\mathrm{DB}(\phi(\sigma))} \alpha + (\underline{31}2)\phi(\sigma) \\
&= \sum_{\alpha\in\mathrm{DB}(\sigma)} \alpha + (2\underline{31})\sigma \\
&= \mathrm{maj}(\sigma).
\end{aligned}
$$

The statement is proved analogously over $\mathcal{S}(123)$. □

Example 9. Let ϕ be the involution (5) in Theorem 8 and let $\sigma = 561237948 \in \mathcal{S}_9(321)$. Then

$$561237948 \overset{\phi}{\mapsto} 236189457,$$

where the black letters indicate the fixed pairs of descent tops and descent bottoms, red letters denote non-descent top left-to-right maxima and blue letters denote non-descent bottom non-left-to-right maxima. The involution swaps the role of red and blue letters while keeping consecutive pairs of black letters together in the same relative order.

Proposition 10. *We have*

$$
\begin{aligned}
[123]_{\mathrm{mak}} &= \{123\}, \\
[321]_{\mathrm{mak}} &= \{321\}, \\
[132]_{\mathrm{mak}} &= \{132,\ 312\} = [312]_{\mathrm{mak}}, \\
[213]_{\mathrm{mak}} &= \{213,\ 231\} = [231]_{\mathrm{mak}}.
\end{aligned}
$$

Proof. As shown in [14, Theorem 2.6] the map $\phi : \mathcal{S}_n(132) \to \mathcal{S}_n(231)$ recursively defined by

$$\phi(231[\sigma_1, 1, \sigma_2]) = 132[\phi(\sigma_1), 1, \phi(\sigma_2)],$$

is a descent preserving bijection implying that $[132]_{\text{maj}} = [231]_{\text{maj}}$. Thus by Proposition 5 we have

$$\sum_{\sigma\in\mathcal{S}_n(132)} q^{\text{mak}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(132)} q^{\text{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(231)} q^{\text{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(312)} q^{\text{mak}(\sigma)}.$$

Hence $[132]_{\text{mak}} = [312]_{\text{mak}}$. The remaining mak-Wilf equivalence is proved similarly invoking Proposition 5. The inequivalences between the four classes is easily verified by hand or with computer. ☐

Remark 11. The charge statistic is also a Mahonian statistic related to maj via trivial bijections by $\text{maj}(\sigma) = \text{charge}(((\sigma^r)^c)^{-1})$ (see [20]). It is worth noting that the mak-Wilf classes in Proposition 10 coincide with the charge-Wilf classes identified in [20].

Remark 12. It can be checked that maj, inv and mak are the only statistics in Table 1 with non-singleton st-Wilf classes for single classical patterns of length three.

The bijection (5) in Theorem 8 induces an interesting equidistribution on shortened polyominoes. A *shortened polyomino* is a pair (P, Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying

i. P and Q begin at the same vertex and end at the same vertex.

ii. P stays weakly above Q and the two paths can share E-steps but not N-steps.

Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$. For $(P, Q) \in \mathcal{H}_n$, let $\text{Proj}_P^Q(i)$ denote the step $j \in [n]$ of P that is the *projection of the i^{th} step of Q on P*. Let

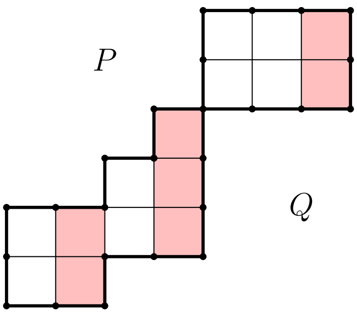
$$\text{Valley}(Q) = \{i : Q_i Q_{i+1} = EN\}$$

denote the set of indices of the *valleys* in Q and let $\text{nval}(Q) = |\,\text{Valley}(Q)|$. Moreover for each $i \in [n]$ define

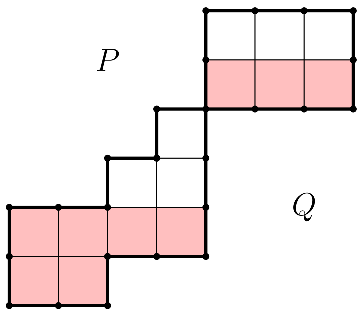
$$\text{area}_{(P,Q)}(i) = \#\text{squares between the } i^{\text{th}} \text{ step of } Q \text{ and the } j^{\text{th}} \text{ step of } P,$$

where $j = \text{Proj}_P^Q(i)$. Consider the statistics *valley-column area* and *valley-row area* of (P, Q) given by

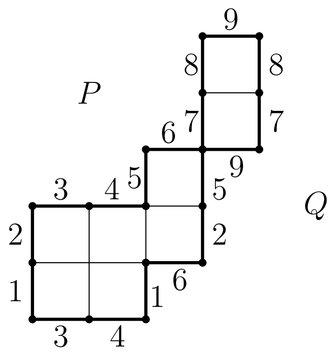
$$\text{vcarea}(P, Q) = \sum_{i\in\text{Valley}(Q)} \text{area}_{(P,Q)}(i),$$
$$\text{vrarea}(P, Q) = \sum_{i\in\text{Valley}(Q)} \text{area}_{(P,Q)}(i + 1).$$



(a) $\mathrm{vcarea}(P,Q) = 2+3+2 = 7$



(b) $\mathrm{vrarea}(P,Q) = 2+4+3 = 9$



The bijection Υ. Here $\Upsilon(P,Q) = 341625978 \in \mathcal{S}_9(321)$.

Theorem 13. *For any $n \geqslant 1$,*

$$\sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vcarea}(P,Q)} t^{\mathrm{nval}(Q)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vrarea}(P,Q)} t^{\mathrm{nval}(Q)}.$$

Proof. We begin by recalling a bijection $\Upsilon : \mathcal{H}_n \to \mathcal{S}_n(321)$ due to Cheng-Eu-Fu [9]. Given $(P,Q) \in \mathcal{H}_n$, set $\mathrm{Label}_P(i) = i$ and $\mathrm{Label}_Q(i) = \mathrm{Label}_P(\mathrm{Proj}_P^Q(i))$. Then

$$\Upsilon(P,Q) = \mathrm{Label}_Q(1) \cdots \mathrm{Label}_Q(n) \in \mathcal{S}_n(321)$$

is a bijection.

Let $(P,Q) \in \mathcal{H}_n$ and $i \in \mathrm{Valley}(Q)$. The definition of Υ immediately gives

$$\mathrm{Valley}(P,Q) = \mathrm{Des}(\Upsilon(P,Q)).$$

In particular $\mathrm{Label}_Q(i+1) < \mathrm{Label}_Q(i)$. Let $s = \mathrm{Proj}_P^Q(i+1)$ and $t = \mathrm{Proj}_P^Q(i)$. Then $s < t$ and

$$\begin{aligned}
\mathrm{area}_{(P,Q)}(i) &= |\{j : P_j = N,\ s \leqslant j \leqslant t\}| \\
&= |\{j : \mathrm{Label}_Q(i+1) \leqslant \mathrm{Label}_Q(j) < \mathrm{Label}_Q(i),\ j > i\}| \\
&= 1 + (\underline{31}2)\big|_{(\mathrm{Label}_Q(i),\, \mathrm{Label}_Q(i+1),\, \text{-})} \Upsilon(P,Q).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{area}_{(P,Q)}(i+1) &= |\{j : P_j = E,\ s \leqslant j \leqslant t\}| \\
&= |\{j : \mathrm{Label}_Q(i+1) < \mathrm{Label}_Q(j) \leqslant \mathrm{Label}_Q(i),\ j \leqslant i\}| \\
&= 1 + (2\underline{31})\big|_{(\text{-},\, \mathrm{Label}_Q(i),\, \mathrm{Label}_Q(i+1))} \Upsilon(P,Q).
\end{aligned}$$

Let $\phi : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ be the bijection (5) from Theorem 8. Recall that $(\underline{31}2)\phi(\sigma) = (2\underline{31})\sigma$ and $\mathrm{des}(\phi(\sigma)) = \mathrm{des}(\sigma)$ for all $\sigma \in \mathcal{S}_n(321)$. Let $\Phi : \mathcal{H}_n \to \mathcal{H}_n$ be the bijection

$$\Phi = \Upsilon^{-1} \circ \phi \circ \Upsilon,$$

and set $(P', Q') = \Phi(P, Q)$. Then

$$\begin{aligned}
\mathrm{vcarea}(\Phi(P,Q)) &= \sum_{i \in \mathrm{Valley}(Q')} \mathrm{area}_{(P',Q')}(i) \\
&= \sum_{i \in \mathrm{Valley}(Q')} \Big(1 + (\underline{31}2)\big|_{(\mathrm{Label}_{Q'}(i),\, \mathrm{Label}_{Q'}(i+1),\, \text{-})} \Upsilon(P',Q')\Big) \\
&= \sum_{i \in \mathrm{Des}(\phi(\Upsilon(P,Q)))} \Big(1 + (\underline{31}2)\big|_{(\phi(\mathrm{Label}_Q(i)),\, \phi(\mathrm{Label}_Q(i+1)),\, \text{-})} \phi(\Upsilon(P,Q))\Big) \\
&= (\mathrm{des} + (\underline{31}2))\phi(\Upsilon(P,Q)) \\
&= (\mathrm{des} + (2\underline{31}))\Upsilon(P,Q) \\
&= \sum_{i \in \mathrm{Valley}(Q)} \Big(1 + (2\underline{31})\big|_{(\text{-},\, \mathrm{Label}_Q(i),\, \mathrm{Label}_Q(i+1))} \Upsilon(P,Q)\Big) \\
&= \sum_{i \in \mathrm{Valley}(Q)} \mathrm{area}_{(P,Q)}(i+1) \\
&= \mathrm{vrarea}(P,Q).
\end{aligned}$$

Since $\mathrm{Valley}(P,Q) = \mathrm{Des}(\Upsilon(P,Q))$ and $\mathrm{des}(\phi(\sigma)) = \mathrm{des}(\sigma)$ it follows that $\mathrm{nval}(Q') = \mathrm{nval}(Q)$. This concludes the proof. □

Below we provide a brief account for a well-known lemma due to Simion and Schmidt which will be used to justify the bijection in the next theorem.

Lemma 14 (Simion-Schmidt [29]). *A permutation $\sigma \in \mathcal{S}(132)$ is uniquely determined by the values and positions of its left-to-right minima.*

Proof. It is clear that the left-to-right minima are positioned in decreasing order relative to each other. Now fill in the remaining numbers from left to right, for each empty position i choosing the smallest remaining entry that is larger than the closest left-to-right minima m in position before i. If the remaining numbers are not entered in this unique way and y is placed before x where $y > x$, then myx is an occurrence of the pattern 132. □

Theorem 15. *For any $n \geqslant 1$,*

$$\sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{LRMin}(\sigma)} = \sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{foze}(\sigma)} \mathbf{x}^{\mathrm{LRMin}(\sigma)}$$

Proof. Let $\sigma \in \mathcal{S}_n(132)$. It is not difficult to see that $\mathrm{LRMin}(\sigma) = \mathrm{DB}(\sigma) \cup \{\sigma(1)\}$. Indeed if $\sigma(i) \in \mathrm{DB}(\sigma)$ and $\sigma(j) < \sigma(i)$ for some $j < i$, then $\sigma(j)\sigma(i-1)\sigma(i)$ is an occurrence of 132. Hence by Lemma 14 we have that σ is uniquely determined equivalently by its first letter, $\mathrm{Des}(\sigma)$ and $\mathrm{DB}(\sigma)$. We define a map $\phi : \mathcal{S}_n(132) \to \mathcal{S}_n(132)$ by requiring

$$\begin{aligned} \phi(\sigma)(1) &= \sigma(1), \\ \mathrm{DB}(\phi(\sigma)) &= \mathrm{DB}(\sigma), \\ \mathrm{Des}(\phi(\sigma)) &= \{n - \sigma(i) + 1 : i \in \mathrm{Des}(\sigma)\}. \end{aligned}$$

We claim that a permutation $\phi(\sigma) \in \mathcal{S}_n(132)$ with the above requirements exists. If the claim holds, then the image of σ is uniquely determined by the data above and therefore ϕ is well-defined. It also immediately follows that ϕ is a bijection.

Let $i_1 < \cdots < i_m$ be the descents of σ. Suppose

$$n - \sigma(i_{j_1}) + 1 < \cdots < n - \sigma(i_{j_m}) + 1.$$

To show that ϕ is well-defined we show that the insertion procedure from Lemma 14 is always valid. Given a descent bottom (i.e. left-to-right minima) $\sigma(i_k + 1)$ in position $n - \sigma(i_{j_k}) + 2$ we must show that there exists enough remaining numbers greater than $\sigma(i_k + 1)$ to fill in the gap to the next descent bottom $\sigma(i_{k+1} + 1)$. Within the filling procedure, next after the descent bottom $\sigma(i_k + 1)$, there exists

$$n - \sigma(i_k + 1) - (n - \sigma(i_{j_k}) + 1) = \sigma(i_{j_k}) - \sigma(i_k + 1) - 1$$

numbers remaining that are greater than $\sigma(i_k + 1)$. There are

$$(n - \sigma(i_{j_{k+1}}) + 2) - (n - \sigma(i_{j_k}) + 2) - 1 = \sigma(i_{j_k}) - \sigma(i_{j_{k+1}}) - 1$$

positions to fill in the gap between the descent bottoms $\sigma(i_k + 1)$ and $\sigma(i_{k+1} + 1)$. By minimality

$$\sigma(i_{j_k}) - \sigma(i_{j_{k+1}}) \leqslant \sigma(i_{j_k}) - \sigma(i_k) \leqslant \sigma(i_{j_k}) - \sigma(i_k + 1),$$

so there are enough numbers remaining to fill in the gap. Hence ϕ is well-defined. Finally,

$$\begin{aligned} \mathrm{maj}(\phi(\sigma)) &= \sum_{i \in \mathrm{Des}(\phi(\sigma))} i \\ &= \sum_{i \in \mathrm{Des}(\sigma)} (n - \sigma(i) + 1) \\ &= \sum_{\alpha \in \mathrm{DT}(\sigma)} (n - \alpha) + \mathrm{des}(\sigma) \\ &= ((\underline{21}3) + (3\underline{21}))\sigma + (\underline{21})\sigma \\ &= \mathrm{foze}(\sigma). \end{aligned}$$

Since also $\phi(\mathrm{LRMin}(\sigma)) = \mathrm{LRMin}(\sigma)$, the theorem follows. $\square$

Below we provide an additional list of information uniquely determining permutations in $\mathcal{S}_n(231)$.

Lemma 16. *A permutation $\sigma \in \mathcal{S}_n(231)$ is uniquely determined by any of the following data:*

(i) The values and positions of right-to-left minima.

(ii) The last letter, ascents and ascent bottoms.

(iii) The pairs $P(\sigma) = \{(p,t) : p$ pinnacle and t its following trough$\}$.

(iv) The pairs $Q(\sigma) = \{(\alpha,\beta) : \alpha$ descent top and β its following descent bottom$\}$.

(v) The pairs $R(\sigma) = \Big\{\Big(\alpha,\ (\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma\Big) : \alpha$ descent top$\Big\}$.

Proof.

(i) Suppose the values and positions of right-to-left minima are fixed in σ. Then $\sigma^r \in \mathcal{S}_n(132)$ and the values and positions of the left-to-right minima in σ^r are fixed. By Lemma 14 this information uniquely determines σ^r. Hence σ is uniquely determined.

(ii) Follows directly from (i) since the positions and values of right-to-left minima are given by the positions and values of the ascents and ascent bottoms together with the last letter.

(iii) Consider the pinnacle-trough decomposition

$$\sigma = a_1 p_1 d_1 t_1 \cdots a_{m-1} p_{m-1} d_{m-1} t_{m-1} a_m p_k d_m$$

where p_i and t_i are pinnacles resp. troughs and a_i and d_i are (possibly empty) increasing resp. decreasing words for $i = 1, \dots, m$.

We claim that the pairs in P are relatively positioned in increasing order of the valleys. Indeed let $(p,t),(p',t') \in P(\sigma)$. Without loss assume $t < t'$. Suppose (for a contradiction) that (p',t') is ordered before (p,t) in σ. Note that $t' < p$, otherwise $t'\alpha p$ is an occurrence of 231, where α is the ascent top following t'. This in turn implies that $t'pt$ is an occurrence of 231 giving a contradiction. Therefore (p,t) is ordered before (p',t') proving the claim.

Next we claim that the decreasing words d_j are uniquely determined. Going from right to left, let d_j be the unique decreasing word of all remaining letters (in value) between p_j and t_j for $j = m, \dots, 1$. If we do not insert the letters this way and $t_j < \sigma_i < p_j$, where σ_i is positioned before p_j (and hence t_j) then $\sigma_i p_j t_j$ is an occurrence of 231 which is forbidden.

Finally we show that the increasing words a_j are uniquely determined. Suppose a_j contains a letter α such that $\alpha > t_j$. Since $\alpha < p_j$ it follows that $\alpha p_j t_j$ is an occurrence of 231. Therefore all letters of a_i are smaller than t_j. Hence a_j is given by the unique increasing word of all letters α such that $t_{j-1} < \alpha < t_j$ for $j = 1, \dots, m$. Hence σ is uniquely determined.

(iv) Partition the letters in $\mathrm{DB}(\sigma) \cup \mathrm{DT}(\sigma)$ into maximal consecutive decreasing subwords $d_1, \ldots, d_m$ based on the pairs in $Q(\sigma)$. The top element of each decreasing subword d_i must be a pinnacle and the bottom element trough. This information uniquely determines σ as per part (iii).

(v) Note that $\alpha \in \mathrm{DT}(\sigma)$ is the largest letter in an occurrence of $\underline{13}2$ in σ if and only if α is a pinnacle. Therefore the pinnacles are the descent tops α with $(\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma > 0$. Given a pinnacle p and the closest trough t to its right, any letter σ_i such that $t < \sigma_i < p$ must be in position after v, otherwise $\sigma_i pt$ is an occurrence of 231. Hence $(\underline{13}2)\big|_{(\text{-},p,\text{-})}\sigma$ precisely represents the difference between p and t. In other words $t = p - (\underline{13}2)\big|_{(\text{-},p,\text{-})}\sigma$. Hence σ is uniquely determined by part (iii). □

Theorem 17. *For $n \geqslant 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mak}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{foze}(\sigma)} t^{\mathrm{des}(\sigma)}.$$

Proof. Let $\sigma \in \mathcal{S}_n(231)$. Note that for $\alpha \in \mathrm{DT}(\sigma)$ we have $(\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma \leqslant \alpha - 2$ since there are at most $\alpha - 2$ numbers between α and its immediately preceding ascent bottom (if present). Thus the function

$$\begin{aligned} f_\sigma : \mathrm{DT}(\sigma) &\to [n] \\ \alpha &\mapsto (n - \alpha + 2) + (\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma \end{aligned}$$

is well-defined.

We claim that f_σ is injective by induction on n. Consider the inflation form $\sigma = 132[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in \mathcal{S}_k(231)$ and $\sigma_2 \in \mathcal{S}_{n-k-1}(231)$. Let $\mathrm{DT}_{\leqslant k}(\sigma) = \{\alpha \in \mathrm{DT}(\sigma) : \alpha \leqslant k\}$ and $\mathrm{DT}_{>k}(\sigma) = \{\alpha \in \mathrm{DT}(\sigma) : \alpha > k\}$. By induction $f_{\sigma_1} : \mathrm{DT}(\sigma_1) \to [k]$ is injective and $f_\sigma(\alpha) = n - k + f_{\sigma_1}(\alpha)$ for every $\alpha \in \mathrm{DT}_{\leqslant}(\sigma)$. Hence $f_\sigma\big|_{\mathrm{DT}_{\leqslant k}(\sigma)}$ is injective. By induction $f_{\sigma_2} : \mathrm{DT}(\sigma_2) \to [n-k-1]$ is injective and $f_\sigma(\alpha) = 1 + f_{\sigma_2}(\alpha - k)$ for every $\alpha \in \mathrm{DT}_{>k}(\sigma)$. Hence $f_\sigma\big|_{\mathrm{DT}_{>k}(\sigma)}$ is injective. Finally note that $f_\sigma(n) = 2 + |\sigma_2|$ if $\sigma_1 \neq \emptyset$ and $f_\sigma(n) = 2$ if $\sigma_1 = \emptyset$. Therefore for all $\alpha \in \mathrm{DT}_{\leqslant k}(\sigma)$ and $\beta \in \mathrm{DT}_{>k}(\sigma)$ we have

$$f_\sigma(\alpha) \geqslant (n - k + 2) > f_\sigma(n) > n - k \geqslant f_\sigma(\beta),$$

if $\sigma_1 \neq \emptyset$ and

$$f_\sigma(\alpha) \geqslant (n - k + 2) > f_\sigma(\beta) > 2 = f_\sigma(n),$$

if $\sigma_1 = \emptyset$. Hence f_σ is injective on all of $\mathrm{DT}(\sigma)$.

Define a map $\phi : \mathcal{S}_n(231) \to \mathcal{S}_n(231)$ by setting the pairs of descent tops and descent bottoms in $\phi(\sigma)$ to $Q(\phi(\sigma)) = \{(f_\sigma(\alpha), n - \alpha + 1) : \alpha \in \mathrm{DT}(\sigma)\}$. By Lemma 16 (iv) this data uniquely determines $\phi(\sigma)$. Note that the pairs are well-defined since f_σ is injective and $f_\sigma(\alpha) > n - \alpha + 1$ for all $\alpha \in \mathrm{DT}(\sigma)$.

We claim that ϕ is a bijection. By Lemma 16 (iv) we may uniquely associate σ with a set of pairs $R(\sigma) = \left\{\left(\alpha,\, (\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma\right) : \alpha \in \mathrm{DT}(\sigma)\right\}$. It suffices to show that ϕ is injective. Let $\pi_1, \pi_2 \in \mathcal{S}_n(231)$ such that $\pi_1 \neq \pi_2$. If $\mathrm{DT}(\pi_1) \neq \mathrm{DT}(\pi_2)$, then $\mathrm{DB}(\phi(\pi_1)) \neq \mathrm{DB}(\phi(\pi_2))$, so $\phi(\pi_1) \neq \phi(\pi_2)$. Assume therefore $\mathrm{DT}(\pi_1) = \mathrm{DT}(\pi_2)$. Since $\pi_1 \neq \pi_2$ we have by uniqueness that $R(\pi_1) \neq R(\pi_2)$. Therefore there exists $\alpha \in \mathrm{DT}(\pi_1) = \mathrm{DT}(\pi_2)$ such that $f_{\pi_1}(\alpha) \neq f_{\pi_2}(\alpha)$. Thus $Q(\phi(\pi_1)) \neq Q(\phi(\pi_2))$ which again implies that $\phi(\pi_1) \neq \phi(\pi_2)$. Hence ϕ is injective and therefore a bijection.

It remains to show that $\mathrm{mak}(\phi(\sigma)) = \mathrm{foze}(\sigma)$. Note that

$$((1\underline{32}) + (\underline{32}1) + (\underline{21}))\sigma = \sum_{\beta \in \mathrm{DB}(\sigma)} \beta.$$

Since there are no occurrences of $2\underline{31}$ in σ by assumption, the letters between each pair of descent top and descent bottom occur to the right of the pair. Therefore the number of occurrences of $\underline{31}2$ in σ is given precisely by

$$\sum_{(\alpha,\beta) \in Q(\sigma)} (\alpha - \beta - 1).$$

Hence

$$\mathrm{mak}(\sigma) = \sum_{\alpha \in \mathrm{DT}(\sigma)} (\alpha - 1).$$

On the other hand note that

$$((\underline{21}3) + (3\underline{21}) + (\underline{21}))\sigma = \sum_{\alpha \in \mathrm{DT}(\sigma)} (n - \alpha + 1).$$

Thus

$$\begin{aligned} \mathrm{foze}(\sigma) &= \sum_{\alpha \in \mathrm{DT}(\sigma)} (n - \alpha + 1) + (\underline{13}2)\sigma \\ &= \sum_{\alpha \in \mathrm{DT}(\sigma)} \Big(n - \alpha + 1 + (\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma \Big) \\ &= \sum_{\alpha \in \mathrm{DT}(\sigma)} (f_\sigma(\alpha) - 1). \end{aligned}$$

Hence

$$\mathrm{mak}(\phi(\sigma)) = \sum_{\alpha' \in \mathrm{DT}(\phi(\sigma))} (\alpha' - 1) = \sum_{\alpha \in \mathrm{DT}(\sigma)} (f(\alpha) - 1) = \mathrm{foze}(\sigma).$$

Finally since $\mathrm{des}(\phi(\sigma)) = \mathrm{des}(\sigma)$, the theorem follows. □

Remark 18. By combining Theorem 17 with Proposition 5 we may deduce further equidistributions between maj and foze, see Table 2 in §5 for a summary.

3 Equidistributions via Dyck paths

A *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never go below the x-axis. For convenience we denote the up-steps by U and the down-steps by D enabling us to encode a Dyck path as a *Dyck word* (we will refer to the two notions interchangeably). Let $\mathcal{D}_n$ denote the set of all Dyck paths of length $2n$ and set $\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n$. For $P \in \mathcal{D}_n$, let $|P| = 2n$ denote the length of P. There are many statistics associated with Dyck paths in the literature. Here we will consider several Dyck path statistics that are intimately related with the inv statistic on pattern avoiding permutations.

Let $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$. A *double rise* in P is a subword UU and a *double fall* in P a subword DD. Let $\mathrm{dr}(P)$ (resp. $\mathrm{df}(P)$) denote the number of double rises (resp. double falls) in P. A *peak* in P is an up-step followed by a down-step, in other words, a subword of the form UD. Let $\mathrm{Peak}(P) = \{p : s_p s_{p+1} = UD\}$ denote the set of indices of the peaks in P and $\mathrm{npea}(P) = |\,\mathrm{Peak}(P)|$. For $p \in \mathrm{Peak}(P)$ define the *position* of p, $\mathrm{pos}_P(p)$, resp. the *height* of p, $\mathrm{ht}_P(p)$, to be the x resp. y-coordinate of its highest point. A *valley* in P is a down step followed by an up step, in other words, a subword of the form DU. Let $\mathrm{Valley}(P) = \{v : s_v s_{v+1} = DU\}$ denote the set of indices of the valleys in P and $\mathrm{nval}(P) = |\,\mathrm{Valley}(P)|$. For $v \in \mathrm{Valley}(P)$ define the *position* of v, $\mathrm{pos}_P(v)$, resp. the *height* of v, $\mathrm{ht}_P(v)$, to be the x resp. y-coordinate of its lowest point. For each $v \in \mathrm{Valley}(P)$, there is a corresponding *tunnel* which is the subword $s_i \cdots s_v$ of P where i is the step after the first intersection of P with the line $y = \mathrm{ht}_P(v)$ to the left of step v (see Figure 2). The length, $v - i$, of a tunnel is always an even number. Let $\mathrm{Tunnel}(P) = \{(i,j) : s_i \cdots s_j \text{ tunnel in } P\}$ denote the set of pairs of beginning and end indices of the tunnels in P. Cheng et.al. [8] define the statistics *sumpeaks* and *sumtunnels* given respectively by

$$\mathrm{spea}(P) = \sum_{p\in\mathrm{Peak}(P)} (\mathrm{ht}_P(p) - 1),$$
$$\mathrm{stun}(P) = \sum_{(i,j)\in\mathrm{Tunnel}(P)} (j - i)/2.$$

Let $\mathrm{Up}(P) = \{i : s_i = U\}$ denote the indices of the set of U-steps in P and $\mathrm{Down}(P) = \{i : s_i = D\}$ the set of indices of the D-steps in P. Given $i \in [2n]$ define the *height* of the step i in P, $\mathrm{ht}_P(i)$, to be the y-coordinate of its lowest point. Define the statistics *sumups* and *sumdowns* by

$$\mathrm{sups}(P) = \sum_{i\in\mathrm{Up}(P)} \lceil \mathrm{ht}_P(i)/2 \rceil$$
$$\mathrm{sdow}(P) = \sum_{i\in\mathrm{Down}(P)} \lfloor \mathrm{ht}_P(i)/2 \rfloor$$

Define the *area* of P, denoted $\mathrm{area}(P)$, to be the number of complete $\sqrt{2} \times \sqrt{2}$ tiles that fit between P and the x-axis (cf [21]).

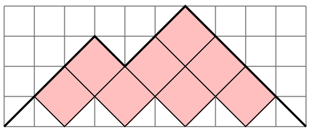


Figure 1: $\mathrm{area}(P) = 8$.

Burstein and Elizalde [5] define a statistic which they call the 'mass' of P. We will define two versions of it, one pertaining to the U-steps and one to the D-steps. For each $i \in \mathrm{Up}(P)$ define the *mass* of i, $\mathrm{mass}_P(i)$, as follows. If $s_{i+1} = D$, then $\mathrm{mass}_P(i) = 0$. If $s_{i+1} = U$, then P has a subword of the form $s_i U P_1 D P_2 D$ where P_1, P_2 are Dyck paths and we define $\mathrm{mass}_P(i) = |P_2|/2$. In other words, the mass is half the number of steps between the matching D-steps of two consecutive U-steps. The part of the Dyck path P contributing to the mass of each of the first three U-steps is highlighted with matching colours in Figure 2. Define

$$\mathrm{mass}_{\mathrm{U}}(P) = \sum_{i \in \mathrm{Up}(P)} \mathrm{mass}_P(i).$$

The statistic $\mathrm{mass}_{\mathrm{U}}$ coincides with the mass statistic defined by Burstein and Elizalde [5]. Analogously if $i \in \mathrm{Down}(P)$, define $\mathrm{mass}_P(i) = 0$ if $s_{i-1} = U$. If $s_{i-1} = D$, then P has a subword of the form $UP_1UP_2Ds_i$ where P_1, P_2 are Dyck paths and we define $\mathrm{mass}_{\mathrm{D}}(s) = |P_1|/2$. In other words, the mass is half the number of steps between the matching U-steps of two consecutive D-steps. Define

$$\mathrm{mass}_{\mathrm{D}}(P) = \sum_{i \in \mathrm{Down}(P)} \mathrm{mass}_P(i).$$

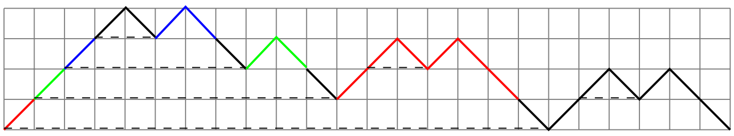


Figure 2: The tunnel lengths of a Dyck path (indicated with dashes) and the mass associated with the first three up-steps is highlighted with matching colours.

Next we give a description of the various Dyck path bijections that will be referenced. The *standard bijection* $\Delta : \mathcal{S}_n(231) \to \mathcal{D}_n$ can be defined recursively by

$$\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2),$$

where $\sigma = 213[1, \sigma_1, \sigma_2]$. We will also (with abuse of notation) define the *standard bijection* $\Delta : \mathcal{S}_n(312) \to \mathcal{D}_n$ recursively by

$$\Delta(\sigma) = \Delta(\sigma_1)U\Delta(\sigma_2)D,$$

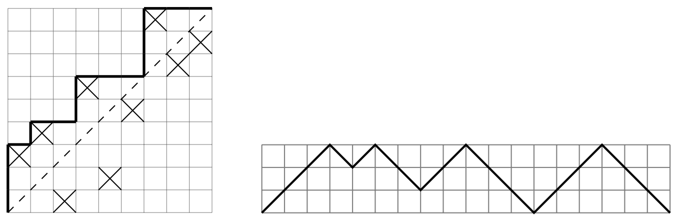


Figure 3: The Dyck path $\Gamma(\sigma)$ corresponding to $\sigma = 341625978$.

where $\sigma = 132[\sigma_1, \sigma_2, 1]$. There is also a non-recursive description of Δ due to Krattenthaler, see [23].

We now define another well-known map $\Gamma : \mathcal{S}_n(321) \to \mathcal{D}_n$ due to Krattenthaler [23] which also appears in a slightly different form in the work of Elizalde [15]. Let $\sigma \in \mathcal{S}_n(321)$ and consider an $n \times n$ array with crosses in positions (i, π_i) for $1 \leqslant i \leqslant n$, where the first coordinate is the column number, increasing from left to right, and the second coordinate is the row number, increasing from bottom to top. Consider the path with north and east steps from the lower-left corner to the upper-right corner of the array, whose right turns occur at the crosses (i, σ_i) with $\sigma_i \geqslant i$. Define $\Gamma(\sigma)$ to be the Dyck path obtained from this path by reading a U-step for every north step and a D-step for every east step of the path. The bijection is illustrated in Figure 3.

Theorem 19 (Krattenthaler [23], Elizalde [15])**.**
For each $n \geqslant 1$ the map $\Gamma : \mathcal{S}_n(321) \to \mathcal{D}_n$ is a bijection.

Theorem 20 (Cheng-Elizalde-Kasraoui-Sagan [8])**.**
We have $\mathrm{inv}(\sigma) = \mathrm{spea}(\Gamma(\sigma))$ *and* $\mathrm{lrmax}(\sigma) = \mathrm{npea}(\Gamma(\sigma))$ *for all* $\sigma \in \mathcal{S}_n(321)$.

Next we define a Dyck path bijection $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ due to Cheng et.al. [8] that is weight preserving between the statistics spea and stun.

First we define a bijection $\delta : \bigsqcup_{k=0}^{n-1} \mathcal{D}_k \times \mathcal{D}_{n-k-1} \to \mathcal{D}_n$ as follows. Given two Dyck paths

$$Q = U^{a_1} D^{b_1} U^{a_2} D^{b_2} \cdots U^{a_s} D^{b_s} \in \mathcal{D}_k \text{ and } R = U^{c_1} D^{d_1} U^{c_2} D^{d_2} \cdots U^{c_t} D^{d_t} \in \mathcal{D}_{n-k-1}$$

where all exponents are positive, define $\delta(Q, R)$ by

$$\delta(Q, R) = U^{a_1+1} D^{b_1+1} U^{a_2} D^{b_2} \cdots U^{a_s} D^{b_s},$$

if $R = \emptyset$ and define

$$\delta(Q, R) = U^{a_1+1} D U^{a_2} D^{b_1} U^{a_3} D^{b_2} \cdots U^{a_s} D^{b_{s-1}} U^{c_1} D^{b_s+d_1} U^{c_2} D^{d_2} \cdots U^{c_t} D^{d_t},$$

if $R \neq \emptyset$. If $Q = \emptyset$ the same definition works with the convention that $a_1 = b_1 = 0$.

Let $P \in \mathcal{D}_n$ and $(Q, R) = \delta^{-1}(P)$. Define $\Psi(\emptyset) = \emptyset$ and for $n \geqslant 1$

$$\Psi(P) = \begin{cases} UD\Psi(Q) & \text{if } R = \emptyset \\ U\Psi(R)D & \text{if } Q = \emptyset, \\ U\Psi(Q)D\Psi(R) & \text{otherwise.} \end{cases}$$

Theorem 21 (Cheng-Elizalde-Kasraoui-Sagan [8]). *The map* $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ *is a bijection such that* $\mathrm{spea}(P) = \mathrm{stun}(\Psi(P))$ *and* $\mathrm{npea}(P) = n - \mathrm{nval}(\Psi(P))$ *for all* $P \in \mathcal{D}_n$. *In particular*

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} t^{\mathrm{npea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{stun}(P)} t^{n-\mathrm{nval}(P)}$$

for all $P \in \mathcal{D}_n$.

We will now interpret mad over both $\mathcal{S}_n(231)$ and $\mathcal{S}_n(312)$ in terms of Dyck path statistics under Δ. The following theorem is a straightforward modification of Theorem 3.11 in [5].

Theorem 22. *For all* $\sigma \in \mathcal{S}_n(231)$, $\pi \in \mathcal{S}_n(312)$ *and* $P \in \mathcal{D}_n$ *we have*

(i) $\mathrm{mad}(\sigma) = \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma)) + \mathrm{dr}(\Delta(\sigma))$,

(ii) $\mathrm{mad}(\pi) = 2\,\mathrm{mass}_{\mathrm{D}}(\Delta(\pi)) + \mathrm{df}(\Delta(\pi))$,

(iii) a bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ *such that* $\mathrm{sups}(P) = \mathrm{mass}_{\mathrm{U}}(\Theta(P)) + \mathrm{dr}(\Theta(P))$.

Proof.

(i) Let $\sigma \in \mathcal{S}_n(231)$ and decompose $\sigma = 213[1, \sigma_1, \sigma_2]$. If we assume $\sigma_1 \neq \emptyset$, then we may further decompose σ_1 and write $\sigma = 42135[1, 1, \sigma_3, \sigma_4, \sigma_2]$. In particular $(\underline{31}2)\big|_{(\sigma(1),\sigma(2),\text{-})}\sigma = |\sigma_4|$. Since

$$\Delta(\sigma) = UU\Delta(\sigma_3)D\Delta(\sigma_4)D\Delta(\sigma_2),$$

we have by induction that

$$\begin{aligned}\mathrm{mass}_{\mathrm{U}}(\Delta(\sigma)) &= \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma_3)) + \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma_4)) + \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma_2)) + |\Delta(\sigma_4)|/2 \\ &= (\underline{31}2)\sigma_3 + (\underline{31}2)\sigma_4 + (\underline{31}2)\sigma_2 + |\sigma_4| \\ &= (\underline{31}2)\sigma.\end{aligned}$$

and

$$\begin{aligned}\mathrm{dr}(\Delta(\sigma)) &= \mathrm{dr}(\Delta(\sigma_1)) + \mathrm{dr}(\Delta(\sigma_2)) + 1 \\ &= \mathrm{des}(\sigma_1) + \mathrm{des}(\sigma_2) + 1 \\ &= \mathrm{des}(\sigma).\end{aligned}$$

Hence $\mathrm{mass}_{\mathrm{U}}(\Delta(\sigma)) + \mathrm{dr}(\Delta(\sigma)) = \mathrm{mad}(\sigma)$.

(ii) Let $\pi \in \mathcal{S}_n(312)$ and decompose $\pi = 132[\pi_1, \pi_2, 1]$. Assuming $\pi_2 \neq \emptyset$ we may write $\pi = 13542[\pi_1, \pi_3, \pi_4, 1, 1]$. In particular $(2\underline{31})\big|_{(\text{-},\pi(n-1),\pi(n))}\pi = |\pi_3|$. Since

$$\Delta(\pi) = \Delta(\pi_1)U\Delta(\pi_3)U\Delta(\pi_4)DD,$$

it follows by an induction similar to part (i) that $\mathrm{mass}_{\mathrm{D}}(\Delta(\pi)) = (2\underline{31})\pi$ and $\mathrm{df}(\Delta(\pi)) = \mathrm{des}(\pi)$. Hence $2\,\mathrm{mass}_{\mathrm{D}}(\Delta(\pi)) + \mathrm{df}(\Delta(\pi)) = \mathrm{mad}(\pi)$.

(iii) Construct a recursive bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ as follows. Let $P \in \mathcal{D}_n$. If $P = P_1 \cdots P_r$ where P_i is a Dyck path returning to the x-axis for the first time at its endpoint, then define $\Theta(P) = \Theta(P_1) \cdots \Theta(P_r)$. Assume therefore $r = 1$ and write

$$P = UUQ_1DUQ_2D \cdots UQ_sDD,$$

provided $P \neq UD$, where $Q_1, \ldots, Q_s$ are Dyck paths. Define

$$\Theta(P) = \begin{cases} \emptyset & \text{if } P = \emptyset, \\ UD & \text{if } P = UD, \\ U^{s+1}D\Theta(Q_1)D\Theta(Q_2)D \cdots \Theta(Q_s)D & \text{otherwise.} \end{cases}$$

The map Θ is clearly a bijection. Note that

$$\operatorname{sups}(P) = \sum_{i=1}^{s} \operatorname{sups}(Q_i) + \frac{1}{2}\sum_{i=1}^{s} |Q_i| + s,$$

$$\operatorname{mass}_{\mathrm{U}}(\Theta(P)) + \operatorname{dr}(\Theta(P)) = \sum_{i=1}^{s}(\operatorname{mass}_{\mathrm{U}}(\Theta(Q_i)) + \operatorname{dr}(\Theta(Q_i)) + \frac{1}{2}\sum_{i=1}^{s} |\Theta(Q_i)| + s.$$

Hence by induction it follows that $\operatorname{mass}_{\mathrm{U}}(\Theta(P)) + \operatorname{dr}(\Theta(P)) = \operatorname{sups}(P)$. $\square$

Theorem 23. *There exists a bijection* $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ *such that* $\operatorname{stun}(P) = \operatorname{mass}_{\mathrm{U}}(\Phi(P)) + \operatorname{dr}(\Phi(P))$. *In particular, for any* $n \geqslant 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\operatorname{mass}_{\mathrm{U}}(P) + \operatorname{dr}(P)}.$$

Proof. Let $P \in \mathcal{D}_n$ and consider the decomposition

$$P = UP_1D \cdots UP_{m-1}DUP_mD,$$

where $P_1, \ldots, P_{m-1}, P_m$ are (possibly empty) Dyck paths. Define $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ recursively by

$$\Phi(P) = \begin{cases} \emptyset, & \text{if } P = \emptyset \\ UD\Phi(P_1), & \text{if } m = 1 \\ UUU^{m-2}D^{m-2}D\Phi(P_1) \cdots \Phi(P_{m-1})D\Phi(P_m), & \text{if } m > 1 \end{cases}$$

It is not difficult to verify by induction that Φ is a bijection from the recursion. It remains to show that $\operatorname{stun}(P) = \operatorname{mass}_{\mathrm{U}}(\Phi(P)) + \operatorname{dr}(\Phi(P))$. We argue by induction on n. The statement holds for $P = \emptyset$. If $m = 1$, then by induction

$$\begin{aligned} \operatorname{stun}(P) &= \operatorname{stun}(P_1) \\ &= \operatorname{mass}_{\mathrm{U}}(\Phi(P_1)) + \operatorname{dr}(\Phi(P_1)) \\ &= \operatorname{mass}_{\mathrm{U}}(UD\Phi(P_1)) + \operatorname{dr}(UD\Phi(P_1)) \\ &= \operatorname{mass}_{\mathrm{U}}(\Phi(P)) + \operatorname{dr}(\Phi(P)). \end{aligned}$$

Suppose $m > 1$. Note that

$$\mathrm{mass}_{\mathrm{U}}(UUP_0DP_1 \cdots P_{m-1}DP_m) = \sum_{i=0}^{m} \mathrm{mass}_{\mathrm{U}}(P_i) + \sum_{i=1}^{m-1} |P_i|/2$$

and that $\mathrm{mass}_{\mathrm{U}}(U^kD^k) = 0$ for all $k \geqslant 0$. Hence by induction

$$\begin{aligned}
\mathrm{stun}(P) &= \mathrm{stun}(P_m) + \sum_{i=1}^{m-1}(\mathrm{stun}(P_i) + (|P_i| + 2)/2) \\
&= \mathrm{mass}_{\mathrm{U}}(\Phi(P_m)) + \mathrm{dr}(\Phi(P_m)) + \sum_{i=1}^{m-1} [(\mathrm{mass}_{\mathrm{U}}(\Phi(P_i)) + \mathrm{dr}(\Phi(P_i)) + (|P_i| + 2)/2)] \\
&= \left(\mathrm{mass}_{\mathrm{U}}(U^{m-2}D^{m-2}) + \sum_{i=1}^{m} \mathrm{mass}_{\mathrm{U}}(\Phi(P_i)) + \sum_{i=1}^{m-1} |\Phi(P_i)|/2\right) \\
&\quad + \left((m-1) + \sum_{i=1}^{m} \mathrm{dr}(\Phi(P_i))\right) \\
&= \mathrm{mass}_{\mathrm{U}}(\Phi(P)) + \mathrm{dr}(\Phi(P)),
\end{aligned}$$

as required. □

Corollary 24. *For any $n \geqslant 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)}.$$

Proof. By Theorem 19, Theorem 20, Theorem 21, Theorem 22 (i) and Theorem 23 we have the following diagram of weight preserving bijections

$$\begin{tikzcd}
(\mathcal{S}_n(321), \mathrm{inv}) \arrow[r, "\Gamma"] \arrow[d, "\phi"] & (\mathcal{D}_n, \mathrm{spea}) \arrow[r, "\Psi"] & (\mathcal{D}_n, \mathrm{stun}) \arrow[d, "\Phi"] \\
(\mathcal{S}_n(231), \mathrm{mad}) \arrow[rr, "\Delta"] & & (\mathcal{D}_n, \mathrm{mass}_{\mathrm{U}} + \mathrm{dr})
\end{tikzcd}$$

Thus

$$\phi = \Delta^{-1} \circ \Phi \circ \Psi \circ \Gamma$$

is our sought bijection with $\mathrm{inv}(\sigma) = \mathrm{mad}(\phi(\sigma))$ for all $\sigma \in \mathcal{S}_n(321)$. □

The following corollary answers a question of Burstein and Elizalde in [5].

Corollary 25. *There exists a bijection $\Lambda : \mathcal{D}_n \to \mathcal{D}_n$ such that* $\mathrm{spea}(P) = \mathrm{sups}(\Lambda(P))$. *In particular for any $n \geqslant 1$,*

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{sups}(P)}.$$

Proof. By Theorem 21, Theorem 22 (iii) and Theorem 23 we have the following diagram of weight preserving bijections

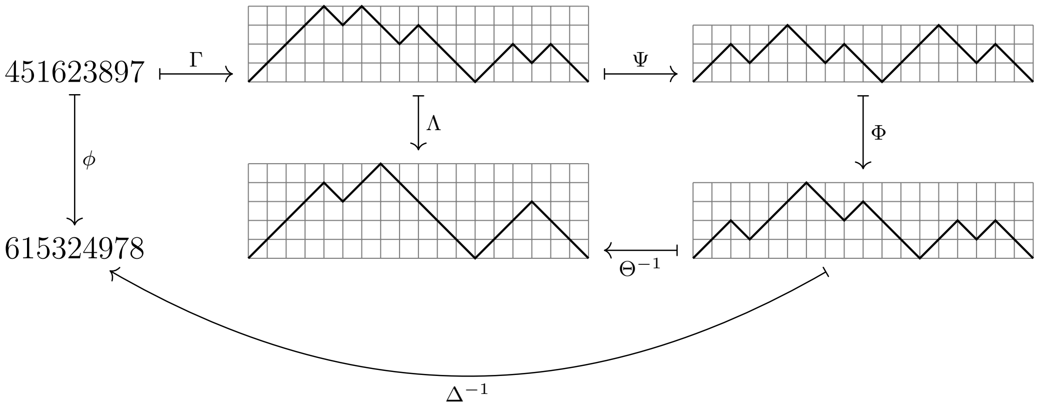
$$\begin{array}{ccc} (\mathcal{D}_n, \mathrm{spea}) & \xrightarrow{\quad\Psi\quad} & (\mathcal{D}_n, \mathrm{stun}) \\ \Big\downarrow \Lambda & & \Big\downarrow \Phi \\ (\mathcal{D}_n, \mathrm{sups}) & \xrightarrow{\quad\Theta\quad} & (\mathcal{D}_n, \mathrm{mass}_{\mathrm{U}} + \mathrm{dr}) \end{array}$$

Hence

$$\Lambda = \Theta^{-1} \circ \Phi \circ \Psi$$

is the required bijection. $\square$

Example 26. The below diagram shows an example of the intermediate images under the bijections ϕ and Λ from Corollary 24 and Corollary 25.



For each Dyck path $P \in \mathcal{D}_n$, Kim et.al. [21] construct two bijections $\mathrm{DTS}(P, \cdot)$ and $\mathrm{DTR}(P, \cdot)$ from the set of linear extensions of the chord poset of P to the set of cover-inclusive Dyck tilings with lower path P (see [21] for terminology). In the special case where $P = (UD)^n$ and the set of linear extensions is restricted to $\mathcal{S}_n(312)$, it follows from [21, Theorem 2.3] that $\mathrm{DTS}(P, \cdot)$ and $\mathrm{DTR}(P, \cdot)$ induce bijections $\theta_{\mathrm{DTS}} : \mathcal{S}_n(312) \to \mathcal{D}_n$ and $\theta_{\mathrm{DTR}} : \mathcal{S}_n(312) \to \mathcal{D}_n$. We remark that the restriction is over $\mathcal{S}_n(231)$ in [21] due to difference in notation. By [21, Theorem 2.4] and [21, Theorem 6.1] it moreover follows that

$$\mathrm{inv}(\sigma) = \mathrm{area}(\theta_{\mathrm{DTS}}(\sigma)), \tag{6}$$

$$\mathrm{mad}(\sigma) = \mathrm{area}(\theta_{\mathrm{DTR}}(\sigma)) \tag{7}$$

for all $\sigma \in \mathcal{S}_n(312)$. Therefore we get a bijection $\theta : \mathcal{S}_n(312) \to \mathcal{S}_n(312)$ given by

$$\theta = \theta_{\mathrm{DTS}}^{-1} \circ \theta_{\mathrm{DTR}},$$

satisfying $\mathrm{mad}(\theta(\sigma)) = \mathrm{inv}(\sigma)$. Hence we obtain the following theorem.

Theorem 27 (Kim-Mésáros-Panova-Wilson [21]).
For any $n \geqslant 1$,

$$\sum_{\sigma \in \mathcal{S}_n(312)} q^{\operatorname{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(312)} q^{\operatorname{inv}(\sigma)}.$$

Corollary 28. *For any* $n \geqslant 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{area}(P)} = \sum_{P \in \mathcal{D}_n} q^{2\operatorname{mass}_{\mathrm{D}}(P) + \operatorname{df}(P)}.$$

Proof. Combine Theorem 22 (ii) with (7). □

Below we find an interpretation of Theorem 2 in terms of Dyck path statistics. Part of the answer is given by a bijection $\Omega : \mathcal{S}_n(231) \to \mathcal{D}_n$ due to Stump [32] which we now define. Let $\sigma \in \mathcal{S}_n(231)$. Suppose $\operatorname{Des}(\sigma) = \{i_1, \ldots, i_k\}$ and $\operatorname{iDes} = \{i'_j \in \operatorname{Des}(\sigma^{-1})\}$ such that $i_1 < \cdots < i_k$ and $i'_1 < \cdots < i'_k$ (recall that $\operatorname{des}(\sigma) = \operatorname{des}(\sigma^{-1})$ via e.g. the argument in Proposition 5). For notational purposes set $i_{k+1} = n = i'_{k+1}$. Define a Dyck path $\Omega(\sigma)$ by starting with i'_1 U-steps, followed by i_1 D-steps, followed by $i'_2 - i'_1$ U-steps, followed by $i_2 - i_1$ D-steps, followed by $i'_3 - i'_2$ U-steps, and so on, ending with $i_{k+1} - i_k$ D-steps. Define the statistic

$$\beta(P) = \sum_{v \in \operatorname{Valley}(P)} |\{j \leqslant \operatorname{pos}_P(v) : s_j = D\}|,$$

for each Dyck path $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$.

Theorem 29 (Stump [32]). *The map* $\Omega : \mathcal{S}_n(231) \to \mathcal{D}_n$ *is a well-defined bijection such that* $\operatorname{maj}(\sigma) = \beta(\Omega(\sigma))$ *for all* $\sigma \in \mathcal{S}_n(231)$.

Proposition 30. *For all* $\sigma \in \mathcal{S}_n(231)$ *and* $\pi \in \mathcal{S}_n(321)$ *we have*

$$\operatorname{maj}(\sigma) = \sum_{v \in \operatorname{Valley}(\Omega(\sigma))} \frac{\operatorname{pos}_{\Omega(\sigma)}(v) - \operatorname{ht}_{\Omega(\sigma)}(v)}{2},$$

$$\operatorname{den}(\pi) = \operatorname{npea}(\Gamma(\pi)) + \sum_{p \in \operatorname{Peak}(\Gamma(\pi))} \frac{\operatorname{pos}_{\Gamma(\pi)}(p) - \operatorname{ht}_{\Gamma(\pi)}(p)}{2}.$$

Proof. As in [5, Theorem 2.5], observe that

$$\operatorname{den}(\pi) = \sum_{\substack{i \in [n] \\ \pi(i) > i}} i,$$

for all $\pi \in \mathcal{S}_n(321)$. In the definition of Krattenthaler's bijection Γ, each $i \in [n]$ such that $\pi(i) > i$ corresponds to a column i in the array containing a box above the main diagonal. In other words it corresponds to the number of east steps in the lattice path that occur to the left of the box. In the Dyck path $\Gamma(\pi) = s_1 \cdots s_{2n}$ this is reflected in the statistic

$$|\{j \leqslant \operatorname{pos}_{\Gamma(\pi)}(p) : s_j = D\}| + 1,$$

associated with each $p \in \mathrm{Peak}(\Gamma(\pi))$. We have the following two obvious relations

$$
\begin{aligned}
|\{j \leqslant \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = U\}| - |\{j \leqslant \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| &= \mathrm{ht}_{\Gamma(\pi)}(p), \\
|\{j \leqslant \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = U\}| + |\{j \leqslant \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| &= \mathrm{pos}_{\Gamma(\pi)}(p),
\end{aligned}
$$

for each $p \in \mathrm{Peak}(\Gamma(\pi))$. Hence

$$
\begin{aligned}
\mathrm{den}(\pi) &= \sum_{p \in \mathrm{Peak}(\Gamma(\pi))} (|\{j \leqslant \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| + 1) \\
&= \mathrm{npea}(\Gamma(\pi)) + \sum_{p \in \mathrm{Peak}(\Gamma(\pi))} \frac{\mathrm{pos}_{\Gamma(\pi)}(p) - \mathrm{ht}_{\Gamma(\pi)}(p)}{2}.
\end{aligned}
$$

The first statement in the proposition follows from Theorem 29 and a similar observation to above. □

Remark 31. By Theorem 2, the Dyck path statistics in Proposition 30 are equidistributed over $\mathcal{D}_n$.

4 Equidistributions via generating functions

In this section we use generating functions to derive the equidistributions (albeit non-bijectively) between Mahonian statistics over $\mathcal{S}_n(\pi)$. We also provide a recursion for a more general statistic involving arbitrary linear combinations of vincular pattern functions of length three. This recursion generalizes for instance the recursions in [14].

Theorem 32. *We have*

$$
\sum_{\sigma \in \mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{sist}(\sigma)} z^{|\sigma|} = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{qz}{1 - \cfrac{q^2 z}{1 - \cfrac{q^2 z}{\ddots}}}}}} \tag{8}
$$

$$
\sum_{\sigma \in \mathcal{S}(312)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma \in \mathcal{S}(213)} q^{\mathrm{sist}(\sigma)} z^{|\sigma|} = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{q^2 z}{1 - \cfrac{q^3 z}{1 - \cfrac{q^4 z}{\ddots}}}}}}. \tag{9}
$$

Proof. Note that over $\mathcal{S}(231)$ we have mad $= (\underline{31}2) + (\underline{21})$. The reverse of sist (i.e. the statistic obtained by reversing all vincular patterns) is given by rsist $= (\underline{31}2)+(\underline{12})$. Hence (8) is equivalent to proving

$$\sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{rsist}(\sigma)} z^{|\sigma|}.$$

Let $\sigma \in \mathcal{S}(231)$ and decompose $\sigma = 213[1, \sigma_1, \sigma_2]$. Then we obtain the recursion

$$\begin{aligned}\mathrm{rsist}(\sigma) &= [12)\sigma_1 + \delta_{\sigma_2\neq\emptyset} + \mathrm{rsist}(\sigma_1) + \mathrm{rsist}(\sigma_2),\\ [12)\sigma &= |\sigma_2|,\end{aligned}$$

where δ denotes the Kronecker delta. Let

$$F(q,t,z) = \sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{rsist}(\sigma)} t^{[12)\sigma} z^{|\sigma|}.$$

Then

$$\begin{aligned}F(q,t,z) = 1 + z &\left(\sum_{\sigma_1\in\mathcal{S}(231)} q^{\mathrm{rsist}(\sigma_1)} q^{[12)\sigma_1} z^{|\sigma_1|}\right)\\ &+ qz\left(\sum_{\sigma_1\in\mathcal{S}(231)} q^{\mathrm{rsist}(\sigma_1)} q^{[12)\sigma_1} z^{|\sigma_1|}\right)\left(\sum_{\sigma_2\in\mathcal{S}(231)} q^{\mathrm{rsist}(\sigma_2)} (zt)^{|\sigma_2|} - 1\right)\\ &= 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,zt) - 1).\end{aligned}$$

Substituting $t = 1$ and $t = q$ we obtain the equation system

$$\begin{cases} F(q,1,z) = 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,z) - 1)\\ F(q,q,z) = 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,qz) - 1)\end{cases}$$

Eliminating $F(q,q,z)$ and solving for $F(q,1,z)$ we obtain

$$F(q,1,z) = \cfrac{1}{1 - \cfrac{z}{1 - qzF(q,1,qz)}},$$

which gives the continued fraction in the theorem. Similarly letting

$$G(q,z,t) = \sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} t^{[12)} z^{|\sigma|},$$

then we obtain the recursive relation

$$G(q,t,z) = 1 + zG(q,1,zt) + qzG(q,1,zt)(G(q,q,z) - 1).$$

Substituting $t = 1$ and $t = q$ as before and solving for $G(q, 1, z)$ we obtain the same continued fraction expansion as above, proving the desired equidistribution.

The second statement in the theorem is proved similarly. Over $\mathcal{S}(312)$ we have mad $=$ $(2\underline{31}) + (2\underline{31}) + (\underline{21})$. Let $\sigma \in \mathcal{S}(312)$ and decompose $\sigma = 132[\sigma_1, \sigma_2, 1]$. Then we obtain the recursion

$$\begin{aligned} \mathrm{mad}(\sigma) &= 2 \cdot (12]\sigma_2 + \delta_{\sigma_2 \neq \emptyset} + \mathrm{mad}(\sigma_1) + \mathrm{mad}(\sigma_2), \\ (12]\sigma &= |\sigma_1|. \end{aligned}$$

Letting $F(q, t, z) = \sum_{\sigma \in \mathcal{S}(312)} q^{\mathrm{mad}(\sigma)} t^{(12]\sigma} z^{|\sigma|}$ we thus obtain

$$F(q, t, z) = 1 + zF(q, 1, zt) + qzF(q, 1, zt)(F(q, q^2, z) - 1).$$

Putting $t = 1$ and $t = q^2$, eliminating $F(q, q^2, z)$ from the resulting equation system and solving for $F(q, 1, z)$ we obtain the continued fraction expansion in the theorem.

A similar argument for rsist over $\mathcal{S}(312)$ gives a matching continued fraction expansion. We leave the details to the reader. $\square$

Remark 33. In [8, Corollary 8.6] it was proved that the continued fraction expansion of the generating function of inv over $\mathcal{S}(321)$ matches that of (8). This gives an alternative proof of Corollary 24.

Remark 34. For mad, the continued fractions (8) and (9) may also be deduced from the following more refined continued fraction in [12, Theorem 22] by specializing $(x, y, p, q) =$ $(1, q, 0, q) = 1$ resp. $(x, y, p, q) = (1, p, p^2, 0)$ and using the fact that $\sigma \in \mathcal{S}(2\underline{31})$ if and only if $\sigma \in \mathcal{S}(231)$ (see [10, Lemma 2]),

$$\sum_{\sigma \in \mathcal{S}} x^{\delta_{\sigma \neq \emptyset} + (\underline{12})\sigma} y^{(\underline{21})\sigma} p^{(2\underline{31})\sigma} q^{(\underline{31}2)\sigma} z^{|\sigma|} = \cfrac{1}{1 - \cfrac{x[1]_{p,q} z}{1 - \cfrac{y[1]_{p,q} z}{1 - \cfrac{x[2]_{p,q} z}{1 - \cfrac{y[2]_{p,q} z}{1 - \cfrac{x[3]_{p,q} z}{\ddots}}}}}}$$

where $[n]_{p,q} = q^{n-1} + pq^{n-2} + \cdots + p^{n-2}q + p^{n-1}$ and δ denotes the Kronecker delta.

Using almost identical arguments to Theorem 32 we may moreover prove the following equidistributions.

Theorem 35. *For any $n \geqslant 1$*

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} &= \sum_{\sigma \in \mathcal{S}_n(132)} q^{\mathrm{sist}'(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{sist}''(\sigma)}, \\ \sum_{\sigma \in \mathcal{S}_n(312)} q^{\mathrm{mad}(\sigma)} &= \sum_{\sigma \in \mathcal{S}_n(132)} q^{\mathrm{foze}'(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{sist}'(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(132)} q^{\mathrm{sist}''(\sigma)}. \end{aligned}$$

By combining Theorem 32 and Theorem 35 with Theorem 27 and Corollary 24 we may deduce further equidistributions between inv and the statistics $\mathrm{foze}', \mathrm{sist}, \mathrm{sist}'$ and sist'', see Table 2 in §5 for a summary.

For each $k \geqslant 1$, let $\iota_{k-1} = (12\cdots k)$ denote the statistic that counts the number of increasing subsequences of length k in a permutation. Define ι_{-1} by $\iota_{-1}(\sigma) = 1$ for all $\sigma \in \mathcal{S}$ (i.e. we declare all permutations to have exactly one subsequence of length 0). We will now find a statistic expressed as a linear combination of ι_k's which is equidistributed with the continued fraction (8). We will derive this statistic using the Catalan continued fraction framework of Brändén-Claesson-Steingrímsson[3]. Let

$$\mathcal{A} = \{A : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} : A_{nk} = 0 \text{ for all but finitely many } k \text{ for each } n\}$$

be the ring of infinite matrices with a finite number of non-zero entries in each row. Note in particular that the matrices in $\mathcal{A}$ are indexed starting from 0. With each $A \in \mathcal{A}$ associate a family of statistics $\{\langle \boldsymbol{\iota}, A_k \rangle\}_{k\geqslant 0}$ where $\boldsymbol{\iota} = (\iota_0, \iota_1, \dots)$, A_k is the k^{th} column of A, and

$$\langle \boldsymbol{\iota}, A_k \rangle = \sum_{i=0}^{\infty} A_{ik}\iota_i.$$

Let $\mathbf{q} = (q_0, q_1, \dots)$, where $q_0, q_1, \dots$ are indeterminates. For each $A \in \mathcal{A}$ define

$$F_A(\mathbf{q}) = \sum_{\sigma\in\mathcal{S}(132)} \prod_{k\geqslant 0} q_k^{\langle \boldsymbol{\iota}, A_k\rangle(\sigma)},$$

$$C_A(\mathbf{q}) = \cfrac{1}{1 - \cfrac{\prod q_k^{A_{0k}}}{1 - \cfrac{\prod q_k^{A_{1k}}}{1 - \cfrac{\prod q_k^{A_{2k}}}{1 - \cfrac{\prod q_k^{A_{3k}}}{1 - \cfrac{\prod q_k^{A_{4k}}}{\ddots}}}}}}$$

Theorem 36 (Brändén-Claesson-Steingrímsson[3])**.** *Let $A \in \mathcal{A}$ and $B = \left(\binom{i}{j}\right)_{i,j\geqslant 0}$. Then*

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}),$$

and conversely

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

In particular, all continued fractions $C_A(\mathbf{q})$ are generating functions of statistics on $\mathcal{S}(132)$ expressed as (possibly infinite) linear combinations of ι_k's.

Define the permutation statistic

$$\mathrm{inc} = \iota_1 + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \iota_k.$$

Note that inc is not a Mahonian statistic.

Proposition 37. *We have*

$$\sum_{\sigma\in\mathcal{S}(132)} q^{\mathrm{inc}(\sigma)} z^{|\sigma|} = \cfrac{1}{1-\cfrac{z}{1-\cfrac{qz}{1-\cfrac{qz}{1-\cfrac{q^2z}{1-\cfrac{q^2z}{\ddots}}}}}} \tag{10}$$

Proof. Comparing (10) with the definition of $C_A(\mathbf{q})$ we get

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $B^{-1} = \left((-1)^{i-j}\binom{i}{j}\right)_{i,j\geqslant 0}$. In $B^{-1}A$ we see that columns $2,3,\dots$ are zero columns and that column 1 is equal to $(1,0,0,\dots)^T$ since $\sum_{k\geqslant 0}(-1)^{n-k}\binom{n}{k} = \delta_{n0}$ where δ_{ij} denotes the Kronecker delta. The entries $(B^{-1}A)_{k0}$ in column 0 are given by

$$(B^{-1}A)_{n0} = \sum_{i\geqslant 0}\lfloor (i+1)/2\rfloor (-1)^{k-i}\binom{k}{i} = \begin{cases} 0, & \text{if } k=0 \\ 1, & \text{if } k=1 \\ (-1)^{k-1}2^{k-2}, & \text{if } k>1. \end{cases}$$

Hence the proposition follows from Theorem 36. □

Remark 38. Applying the same argument to the continued fraction (9) it is easy to see that Theorem 36 gives equidistribution with

$$\sum_{\sigma\in\mathcal{S}(132)} q^{\iota_1(\sigma)} z^{|\sigma|} = \sum_{\sigma\in\mathcal{S}(312)} q^{\mathrm{inv}(\sigma)} z^{|\sigma|}.$$

Corollary 39. *For any $n\geqslant 1$,*

$$\sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{inc}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)}$$

Proof. Follows by combining Corollary 24, Theorem 32 and Proposition 37. □

Proposition 40. *Let $\Delta : \mathcal{S}(132) \to \mathcal{D}$ denote the standard bijection defined by $\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2)$ where $\sigma = 231[\sigma_1, 1, \sigma_2] \in \mathcal{S}(132)$. Then*

$$\mathrm{inc}(\sigma) = \mathrm{sdow}(\Delta(\sigma))$$

for all $\sigma \in \mathcal{S}(132)$.

Proof. In [23] (see also [3]) Krattenthaler shows that

$$\iota_k(\sigma) = \sum_{i \in \mathrm{Down}(\Delta(\sigma))} \binom{\mathrm{ht}_{\Delta(\sigma)}(i) - 1}{k},$$

for all $\sigma \in \mathcal{S}(132)$. Hence

$$\begin{aligned}\mathrm{inc}(\sigma) &= \sum_{i \in \mathrm{Down}(\Delta(\sigma))} \left(\binom{\mathrm{ht}_{\Delta(\sigma)}(i) - 1}{1} + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \binom{\mathrm{ht}_{\Delta(\sigma)}(i) - 1}{k} \right) \\ &= \sum_{i \in \mathrm{Down}(\Delta(\sigma))} \lfloor \mathrm{ht}_{\Delta(\sigma)}(i)/2 \rfloor \\ &= \mathrm{sdow}(\Delta(\sigma)),\end{aligned}$$

for all $\sigma \in \mathcal{S}(132)$. □

Since the Mahonian statistics in Table 1 are linear combinations of vincular patterns of length at most three, it is natural to consider the following more general statistic.

Definition 41. Let $\mathcal{P} = \{a\underline{bc} : abc \in \mathcal{S}_3\} \cup \{\underline{ab}c : abc \in \mathcal{S}_3\} \cup \{\underline{21}\}$ and $\boldsymbol{\alpha} = (\alpha_\rho) \in \mathbb{N}^{\mathcal{P}}$. Define the statistic $\mathrm{stat}_{\boldsymbol{\alpha}} : \mathcal{S} \to \mathbb{N}$ by

$$\mathrm{stat}_{\boldsymbol{\alpha}}(\sigma) = \sum_{\rho \in \mathcal{P}} \alpha_\rho (\rho)\sigma,$$

for all $\sigma \in \mathcal{S}$.

Let *head* and *last* be the statistics defined by $\mathrm{head}(\sigma) = \sigma(1)$ and $\mathrm{last}(\sigma) = \sigma(n)$ for all $\sigma \in \mathcal{S}_n$. We associate to $\mathrm{stat}_{\boldsymbol{\alpha}}$ the following generating function for each set Π of patterns

$$F_n(\Pi, \boldsymbol{\alpha}; q, t, u, v) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\mathrm{stat}_{\boldsymbol{\alpha}}(\sigma)} t^{\mathrm{des}(\sigma)} u^{\mathrm{head}(\sigma)} v^{\mathrm{last}(\sigma)}.$$

Theorem 42. *We have*

$$\begin{aligned}&F_n(312, \boldsymbol{\alpha}; q, t, u, v) \\ &= q^{C(0)} uv F_{n-1}\left(312, \boldsymbol{\alpha}; q, q^{A_2(0)}t, q^{B_2}, v\right) + q^{C(n-1)} tuv F_{n-1}\left(312, \boldsymbol{\alpha}; q, q^{A_1(n-1)}t, u, q^{B_1}\right) \\ &\quad + \sum_{k=1}^{n-2} q^{C(k)} tuv^k F_k\left(312, \boldsymbol{\alpha}; q, q^{A_1(k)}t, u, q^{B_1}\right) F_{n-k-1}\left(312, \boldsymbol{\alpha}; q, q^{A_2(k)}t, q^{B_2}, v\right),\end{aligned}$$

where

$$
\begin{aligned}
A_1(k) &= \alpha_{3\underline{21}} - \alpha_{\underline{23}1} + (n-k-1)\left(\alpha_{\underline{21}3} - \alpha_{\underline{12}3}\right),\\
A_2(k) &= (k+1)\left(\alpha_{1\underline{32}} - \alpha_{1\underline{23}}\right),\\
B_1 &= \alpha_{2\underline{31}} - \alpha_{3\underline{21}},\\
B_2 &= \alpha_{\underline{13}2} - \alpha_{\underline{12}3},\\
C(k) &= (k\alpha_{\underline{12}3} - \alpha_{\underline{21}3})(n-k-1) - \delta_{k<n-1}\alpha_{\underline{13}2} + \delta_{k>0}(n-k-1)\alpha_{\underline{21}3}\\
&\quad + \delta_{k>0}(k-1)\alpha_{\underline{23}1} + \delta_{k<n-1}(k+1)(n-k-2)\alpha_{1\underline{23}}\\
&\quad + \delta_{k<n-1}k\alpha_{2\underline{13}} - \delta_{k>0}\alpha_{2\underline{31}} + k\alpha_{3\underline{21}} + \delta_{k>0}\alpha_{\underline{21}},
\end{aligned}
$$

and δ denotes the Kronecker delta.

Proof. Let $\sigma \in \mathcal{S}_n(312)$ and consider the inflation form $\sigma = 213[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in \mathcal{S}_k(312)$ and $\sigma_2 \in \mathcal{S}_{n-k-1}(312)$. Then for each $\rho \in \mathcal{P}$ we get the recursive relations

$$(\rho)\sigma = (\rho)\sigma_1 + (\rho)\sigma_2 + m_\rho,$$

where

$$
\begin{aligned}
m_{\underline{12}3} &= [12)\sigma_2 + |\sigma_2|(\underline{12})\sigma_1, & m_{1\underline{23}} &= (|\sigma_1|+1)(\underline{12})\sigma_2,\\
m_{\underline{13}2} &= [21)\sigma_2, & m_{1\underline{32}} &= (|\sigma_1|+1)(\underline{21})\sigma_2,\\
m_{\underline{21}3} &= ((\underline{21}) + \delta_{\sigma_1\neq\emptyset})|\sigma_2|, & m_{2\underline{13}} &= |\sigma_1|\delta_{\sigma_2\neq\emptyset},\\
m_{\underline{23}1} &= (\underline{12})\sigma_1, & m_{2\underline{31}} &= (12]\sigma_1,\\
m_{\underline{32}1} &= (\underline{21})\sigma_1, & m_{3\underline{21}} &= (21]\sigma_1
\end{aligned}
$$

and $m_{\underline{21}} = \delta_{\sigma_1\neq\emptyset}$. It follows that $\mathrm{stat}_{\boldsymbol{\alpha}}$ satisfies the following recursion

$$\mathrm{stat}_{\boldsymbol{\alpha}}(\sigma) = \mathrm{stat}_{\boldsymbol{\alpha}}(\sigma_1) + \mathrm{stat}_{\boldsymbol{\alpha}}(\sigma_1) + \sum_{\rho\in\mathcal{P}} m_\rho.$$

We note that $|\sigma_1| = k$, $|\sigma_2| = n-k-1$, $(\underline{21})\sigma = \mathrm{des}(\sigma)$, $(\underline{12})\sigma = \delta_{\sigma\neq\emptyset}(|\sigma|-1) - \mathrm{des}(\sigma)$, $[21)\sigma = \mathrm{head}(\sigma) - \delta_{\sigma\neq\emptyset}$, $[12)\sigma = |\sigma| - \mathrm{head}(\sigma)$, $(12]\sigma = \mathrm{last}(\sigma) - \delta_{\sigma\neq\emptyset}$ and $(21]\sigma = |\sigma| - \mathrm{last}(\sigma)$ for all $\sigma \in \mathcal{S}_n(312)$. Converting these statements into generating functions proves the theorem. □

Remark 43. If $\alpha_{\underline{23}1} = \alpha_{\underline{31}2} = \alpha_{\underline{32}1} = \alpha_{\underline{21}} = 1$ and $\alpha_\rho = 0$ otherwise, then $\mathrm{stat}_{\boldsymbol{\alpha}} = \mathrm{inv}$ and $F(312, \boldsymbol{\alpha}; q, 1, 1, 1) = I_n(q) = \tilde{C}_n(q)$. Similarly if we choose $\boldsymbol{\alpha}$ such that $\mathrm{stat}_{\boldsymbol{\alpha}} = \mathrm{maj}$, then we recover the recursion in [14, Theorem 3.4] via the recursion for $F(312, \boldsymbol{\alpha}; q, t, 1, 1)$ in Theorem 42.

Recall the Simion-Schmidt bijection $\phi : \mathcal{S}_n(123) \to \mathcal{S}_n(132)$ which maps $\sigma \in \mathcal{S}_n(123)$ to the unique permutation in $\mathcal{S}_n(132)$ with the same left-to-right minima in the same positions as σ (cf Lemma 14). As explicitly noted by Claesson and Kitaev [11] this bijection clearly preserves the head statistic and hence $[123]_{\mathrm{head}} = [132]_{\mathrm{head}}$. Although head is not a Mahonian statistic we complete its st-Wilf classification below for all subsets of $\mathcal{S}_3$ of size at most three. Equivalences for subsets of larger size can easily be found using similar analysis on the inflation forms. These are less interesting and omitted for brevity. We note in particular that the single pattern distributions with respect to the head statistic are given by well-known refinements of the Catalan numbers.

Proposition 44. *We have*

$$\begin{aligned}
[123]_{\text{head}} &= \{123, 132\} = [132]_{\text{head}},\\
[321]_{\text{head}} &= \{321, 312\} = [312]_{\text{head}},\\
[231]_{\text{head}} &= \{213, 231\} = [213]_{\text{head}}\\
[123, 213]_{\text{head}} &= \{\{123, 213\}, \{132, 213\}, \{132, 231\}\}\\
[231, 321]_{\text{head}} &= \{\{231, 321\}, \{213, 312\}, \{231, 312\}\}\\
[213, 231, 321]_{\text{head}} &= \{\{213, 231, 321\}, \{213, 231, 312\}\}\\
[132, 213, 231]_{\text{head}} &= \{\{132, 213, 231\}, \{123, 213, 231\}\}\\
[132, 213, 321]_{\text{head}} &= \{\{132, 213, 321\}, \{132, 213, 312\}, \{132, 231, 321\},\\
&\qquad \{132, 231, 312\}, \{123, 213, 312\}\}.
\end{aligned}$$

Remaining subsets $\Pi \subseteq \mathcal{S}_3$ *of size at most three have singleton* head-*Wilf class. Moreover for any* $n \geqslant 1$

$$\begin{aligned}
\sum_{\sigma \in \mathcal{S}_n(123)} q^{\text{head}(\sigma)} &= \sum_{k=1}^{n} C_{n-1,k-1} q^k,\\
\sum_{\sigma \in \mathcal{S}_n(213)} q^{\text{head}(\sigma)} &= \sum_{k=1}^{n} C_{k-1} C_{n-k} q^k,\\
\sum_{\sigma \in \mathcal{S}_n(123,213)} q^{\text{head}(\sigma)} &= q + \sum_{k=2}^{n} 2^{k-2} q^k.
\end{aligned}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ *and* $C_{n,k} = \frac{n-k+1}{n+1}\binom{n+k}{n}$ (`A009766` *[31]*).

Proof. The map $\psi : \mathcal{S}_n(321) \to \mathcal{S}_n(312)$ given by $\psi(\sigma) = \phi(\sigma^c)^c$, where $\phi : \mathcal{S}_n(123) \to \mathcal{S}_n(132)$ is the Simion-Schmidt bijection, clearly satisfies $\text{head}(\psi(\sigma)) = \text{head}(\sigma)$. Hence $[321]_{\text{head}} = [312]_{\text{head}}$. Let $\sigma = a_1 a_2 \cdots a_n \in \mathcal{S}_n(132)$. According to the non-recursive description of the standard bijection $\Delta : \mathcal{S}_n(132) \to \mathcal{D}_n$ (due to Krattenthaler [23]), when a_i is read from left to right we adjoin as many U-steps as necessary to the path obtained thus far to reach height $h_j + 1$, followed by a D-step to height h_j. Here h_j is the number of letters in $a_{j+1} \cdots a_n$ which are larger than a_j. As such, the number of permutations $\sigma \in \mathcal{S}_n(132)$ with $\text{head}(\sigma) = k$ is given by the number of Dyck paths starting with exactly $n-k+1$ number of U-steps. These are equivalently enumerated by the number of lattice paths with steps $(1, 0)$ and $(0, 1)$ from $(1, n-k+1)$ to (n, n) staying weakly above the line $y = x$. By [24, Theorem 10.3.1] the number of such paths are given by

$$\binom{n+n-1-(n-k+1)}{n-(n-k+1)} - \binom{n+n-1-(n-k+1)}{n-1+1} = C_{n-1,k-1}.$$

The map $\varphi : \mathcal{S}_n(231) \to \mathcal{S}_n(213)$ recursively defined by

$$\varphi(213[1, \sigma_1, \sigma_2]) = 231[1, \varphi(\sigma_1), \varphi(\sigma_2)],$$

where $\sigma_1 \in \mathcal{S}_{k-1}(231)$ and $\sigma_2 \in \mathcal{S}_{n-k}(231)$, is clearly a head-preserving bijection. Hence $[231]_{\text{head}} = [213]_{\text{head}}$. Since $|\mathcal{S}_k(231)| = C_k$ it follows from the inflation form that there are $C_{k-1}C_{n-k}$ permutations $\sigma \in \mathcal{S}_n(231)$ with $\text{head}(\sigma) = k$.

If $\sigma \in \mathcal{S}_n(132, 231)$, then σ is either decomposed as $12[\sigma_1, 1]$ or as $21[1, \sigma_1]$ where $\sigma_1 \in \mathcal{S}_{n-1}(132, 231)$. Thus the letters $1, 2, \dots, n$ are in reverse order recursively placed at the beginning or at the end of the permutation. For σ to have $\text{head}(\sigma) = k$, the letters $k+1, \dots, n$ must be placed in increasing order at the end and k at the beginning. Remaining $k-1$ letters may be placed on either end giving two choices each (except for the last letter). Hence there exists 2^{k-2} permutations $\sigma \in \mathcal{S}_n(132, 231)$ with $\text{head}(\sigma) = k$ for $k > 1$.

Let $\iota_k = 12 \cdots k$ and $\delta_k = k \cdots 21$ for $k \geqslant 1$. If $\sigma \in \mathcal{S}_n(123, 213)$ and $\text{head}(\sigma) = k$, then $\sigma = 231[1, \delta_{n-k}, \sigma_1]$ for some $\sigma_1 \in \mathcal{S}_{k-1}(123, 213)$. It is easy to see that $|\mathcal{S}_k(123, 213)| = 2^{k-1}$ by induction. Hence $[132, 231]_{\text{head}} = [123, 213]_{\text{head}}$.

If $\sigma \in \mathcal{S}_n(132, 213)$, then $\sigma = 231[1, \iota_{n-k}, \sigma_1]$ where $\sigma_1 \in \mathcal{S}_{k-1}(132, 213)$. The map $\chi : \mathcal{S}_n(132, 213) \to \mathcal{S}_n(123, 213)$ recursively given by

$$\chi(231[1, \iota_{n-k}, \sigma_1]) = 231[1, \delta_{n-k}, \chi(\sigma_1)],$$

is clearly a head-preserving bijection. Hence $[132, 213]_{\text{head}} = [123, 213]_{\text{head}}$. Remaining equivalences and their distributions may be deduced from the fact that $\text{head}(\sigma^c) = n - \text{head}(\sigma) + 1$. The equivalences between the size three subsets can be proved similarly via bijections between their corresponding inflation forms (the inflation forms can be referenced in [14]). The details for these are left to the reader. $\square$

5 Summary and conjectures

In Table 2 we summarize the equidistributions proved in this paper (highlighted in black). In a given cell corresponding to stat_{row} and stat_{col}, a pair of patterns π_1, π_2 denotes the equidistribution

$$\sum_{\sigma \in \mathcal{S}_n(\pi_1)} q^{\text{stat}_{\text{row}}(\sigma)} \sum_{\sigma \in \mathcal{S}_n(\pi_2)} q^{\text{stat}_{\text{col}}(\sigma)}.$$

The equidistributions in Table 2 highlighted in blue were established in [14, 21]. The equidistributions between maj, bast′ and bast″ can be proved in a similar way to Proposition 5, since the inverse map is the right bijection in two of the cases and the rest can be deduced via the maj-Wilf equivalences from [14]. Remaining equidistributions were either proved directly or follow by combining equidistributions proved in this paper. For instance $\sum_{\sigma \in \mathcal{S}_n(213)} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(231)} q^{\text{foze}(\sigma)}$ is deduced by combining Proposition 5 and Theorem 17.

Conjecture 45. Table 2 is the complete table of Mahonian 3-function equidistributions over permutations avoiding a single classical pattern of length three.

We have verified all entries in Table 2 by computer for $n \leqslant 10$. Other than than the entries in Table 2 there are no additional equidistributions (over permutations avoiding a single classical pattern of length three) between the statistics in Table 1.

	maj	inv	mak	makl	mad	bast	bast′	bast″	foze	foze′	foze″	sist	sist′	sist″
maj	132, 231 213, 312		123, 123 132, 132 132, 312 213, 213 213, 231 231, 132 231, 312 312, 213 312, 231 321, 321	132, 231 213, 312 231, 231 312, 312 321, 321		132, 213 213, 231 231, 213 312, 231	132, 132 231, 132	213, 231 312, 231	132, 132 213, 231 231, 132 312, 231					
inv	•	132, 213 231, 312			231, 312 312, 312 321, 231					231, 132 312, 132 321, 213	231, 132 312, 132 321, 213	231, 213 312, 213 321, 132	231, 231 312, 231 321, 132	231, 132 312, 132 321, 231
mak	•	•	132, 312 213, 231	132, 231 213, 312 231, 312 312, 231 321, 321		132, 213 213, 231 231, 231 312, 213	132, 132 312, 132	213, 231 231, 231	132, 132 213, 231 231, 231 312, 132					
makl	•	•	•			132, 132 231, 213 312, 231	231, 132	312, 231	132, 213 231, 132 312, 231					
mad	•	•	•	•						231, 213 312, 132	231, 213 312, 132	231, 132 312, 213	132, 213 231, 132 312, 231	213, 213 231, 231 312, 132
bast	•	•	•	•	•		213, 132	231, 231	123, 123 213, 132 132, 213 231, 231 312, 312 321, 321					
bast′	•	•	•	•	•	•			132, 132					
bast″	•	•	•	•	•	•	•		231, 231					
foze	•	•	•	•	•	•	•	•						
foze′	•	•	•	•	•	•	•	•	•		132, 132 213, 213	132, 213 213, 132	132, 231 213, 132	132, 132 213, 231
foze″	•	•	•	•	•	•	•	•	•	•		213, 132 132, 213	213, 132 132, 231	132, 132 213, 231
sist	•	•	•	•	•	•	•	•	•	•	•		132, 132 213, 231 312, 312	132, 231 213, 132 231, 312
sist′	•	•	•	•	•	•	•	•	•	•	•	•		132, 231 231, 132
sist″	•	•	•	•	•	•	•	•	•	•	•	•	•	

Table 2: Previously established equidistributions in blue, equidistributions proved in this paper in black and conjectured equidistributions in red.

Note. The conjectured equidistributions in Table 2 between maj and bast (and consequently between mak and bast) were recently established by J. N. Chen [6].

Acknowledgements

The author is grateful to Petter Brändén, Samu Potka and Bruce Sagan for comments and discussions.

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