A characterization of Hermitian varieties as codewords

Angela Aguglia
Dipartimento di Meccanica, Matematica e Management
Politecnico di Bari
Bari, Italy
angela.aguglia@poliba.it

Daniele Bartoli
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
Perugia, Italy
daniele.bartoli@unipg.it

Leo Storme
Department of Mathematics
Ghent University
Ghent, Belgium
leo.storme@ugent.be

Zsuzsa Weiner
MTA–ELTE Geometric and Algebraic Combinatorics Research Group
and Department of Computer Science
ELTE Eötvös Loránd University
Budapest, Hungary
weiner@cs.elte.hu

Submitted: Jan 1, 2012; Accepted: Jan 2, 2012; Published: XX
Mathematics Subject Classifications: 51E20, 94B05

Abstract

It is known that the Hermitian varieties are codewords in the code defined by the points and hyperplanes of the projective spaces $\mathrm{PG}(r,q^2)$. In finite geometry, also quasi-Hermitian varieties are defined. These are sets of points of $\mathrm{PG}(r,q^2)$ of the same size as a non-singular Hermitian variety of $\mathrm{PG}(r,q^2)$, having the same intersection sizes with the hyperplanes of $\mathrm{PG}(r,q^2)$. In the planar case, this reduces to the definition of a unital. A famous result of Blokhuis, Brouwer, and Wilbrink states that every unital in the code of the points and lines of $\mathrm{PG}(2,q^2)$ is a Hermitian curve. We prove a similar result for the quasi-Hermitian varieties in $\mathrm{PG}(3,q^2)$, $q=p^h$, as well as in $\mathrm{PG}(r,q^2)$, $q=p$ prime, or $q=p^2$, p prime, and $r \geqslant 4$.

Keywords: Hermitian variety; incidence vector; codes of projective spaces; quasi-Hermitian variety

1 Introduction

Consider the non-singular Hermitian varieties $\mathcal{H}(r, q^2)$ in $\mathcal{H}(r, q^2)$. A non-singular Hermitian variety $\mathcal{H}(r, q^2)$ in $\mathcal{H}(r, q^2)$ is the set of absolute points of a Hermitian polarity of $\mathcal{H}(r, q^2)$. Many properties of a non-singular Hermitian variety $\mathcal{H}(r, q^2)$ in $\mathcal{H}(r, q^2)$ are known. In particular, its size is $(q^{r+1}+(-1)^r)(q^r-(-1)^r)/(q^2-1)$, and its intersection numbers with the hyperplanes of $\mathcal{H}(r, q^2)$ are equal to $(q^r+(-1)^{r-1})(q^{r-1}-(-1)^{r-1})/(q^2-1)$, in case the hyperplane is a non-tangent hyperplane to $\mathcal{H}(r, q^2)$, and equal to $1+q^2(q^{r-1}+(-1)^r)(q^{r-2}-(-1)^r)/(q^2-1)$ in case the hyperplane is a tangent hyperplane to $\mathcal{H}(r, q^2)$; see [16].

Quasi-Hermitian varieties $\mathcal{V}$ in $\mathcal{H}(r, q^2)$ are generalizations of the non-singular Hermitian variety $\mathcal{H}(r, q^2)$ so that $\mathcal{V}$ and $\mathcal{H}(r, q^2)$ have the same size and the same intersection numbers with hyperplanes.

Obviously, a Hermitian variety $\mathcal{H}(r, q^2)$ can be viewed as a trivial quasi–Hermitian variety and we call $\mathcal{H}(r, q^2)$ the *classical quasi–Hermitian variety of* $\mathcal{H}(r, q^2)$. In the 2-dimensional case, $\mathcal{H}(r, q^2)$ is also known as the classical example of a *unital* of the projective plane $\mathcal{H}(2, q^2)$.

As far as we know, the only known non-classical quasi-Hermitian varieties of $\mathcal{H}(r, q^2)$ were constructed in [1, 2, 8, 9, 14, 15].

In [6], it is shown that a unital in $\mathcal{H}(2, q^2)$ is a Hermitian curve if and only if it is in the $\mathbb{F}_p$-code spanned by the lines of $\mathcal{H}(2, q^2)$, with $q = p^h$, p prime and $h \in \mathbb{N}$.

In this article, we prove the following result.

Theorem 1.1. *A quasi-Hermitian variety $\mathcal{V}$ of $\mathcal{H}(r, q^2)$, with $r = 3$ and $q = p^h \neq 4$, p prime, or $r \geqslant 4$, $q = p \geqslant 5$, or $r \geqslant 4$, $q = p^2$, $p \neq 2$ prime, is classical if and only if it is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\mathcal{H}(r, q^2)$.*

Furthermore we consider *singular quasi-Hermitian varieties*, that is point sets having the same number of points as a singular Hermitian variety $\mathcal{S}$ and for which the intersection numbers with respect to hyperplanes are also the intersection numbers of $\mathcal{S}$ with respect to hyperplanes. We show that Theorem 1.1 also holds in the case in which $\mathcal{V}$ is assumed to be a singular quasi-Hermitian variety of $\mathcal{H}(r, q^2)$.

2 Preliminaries

A subset $\mathcal{K}$ of $\mathcal{H}(r, q^2)$ is a k_{n,r,q^2} if n is a fixed integer, with $1 \leqslant n \leqslant q^2$, such that:

(i) $|\mathcal{K}| = k$;

(ii) $|\ell \cap \mathcal{K}| = 1$, n, or $q^2 + 1$ for each line ℓ;

(iii) $|\ell \cap \mathcal{K}| = n$ for some line ℓ.

A point P of $\mathcal{K}$ is *singular* if every line through P is either a unisecant or a line of $\mathcal{K}$. The set $\mathcal{K}$ is called *singular* or *non-singular* according as it has singular points or not.

Furthermore, a subset $\mathcal{K}$ of $\mathcal{H}(r, q^2)$ is called *regular* if

(a) $\mathcal{K}$ is a k_{n,r,q^2};

(b) $3 \leqslant n \leqslant q^2 - 1$;

(c) no planar section of $\mathcal{K}$ is the complement of a set of type $(0, q^2 + 1 - n)$.

Theorem 2.1. *[10, Theorem 19.5.13] Let $\mathcal{K}$ be a $k_{n,3,q^2}$ in $\mathcal{H}(3, q^2)$, where q is any prime power and $n \neq \frac{1}{2}q^2 + 1$. Suppose furthermore that every point in $\mathcal{K}$ lies on at least one n-secant. Then $n = q + 1$ and $\mathcal{K}$ is a non-singular Hermitian surface.*

Theorem 2.2. *[12, Theorem 23.5.19] If $\mathcal{K}$ is a regular, non-singular k_{n,r,q^2}, with $r \geqslant 4$ and $q > 2$, then $\mathcal{K}$ is a non-singular Hermitian variety.*

Theorem 2.3. *[12, Th. 23.5.1] If $\mathcal{K}$ is a singular $k_{n,3,q^2}$ in $\mathcal{H}(3, q^2)$, with $3 \leqslant n \leqslant q^2 - 1$, $q > 2$, then the following holds: $\mathcal{K}$ is n planes through a line or a cone with vertex a point and base $\mathcal{K}'$ a plane section of type*

I. *a unital;*

II. *a subplane $\mathcal{H}(2, q)$;*

III. *a set of type $(0, n - 1)$ plus an external line;*

IV. *the complement of a set of type $(0, q^2 + 1 - n)$.*

Theorem 2.4. *[12, Lemma 23.5.2 and Th. 25.5.3] If $\mathcal{K}$ is a singular $k_{n,r,q}$ with $r \geqslant 4$, then the singular points of $\mathcal{K}$ form a subspace Π_d of dimension d and one of the following possibilities holds:*

1. *$d = r - 1$ and $\mathcal{K}$ is a hyperplane;*

2. *$d = r - 2$ and $\mathcal{K}$ consists of $n > 1$ hyperplanes through Π_d;*

3. *$d \leqslant r - 3$ and $\mathcal{K}$ is equal to a cone $\Pi_d\mathcal{K}'$, with π_d as vertex and with $\mathcal{K}$ as base, where $\mathcal{K}'$ is a non singular $k_{n,r-d-1,q}$.*

A *multiset* in $\mathcal{H}(r, q)$ is a set in which multiple instances of the elements are allowed.

Result 2.5. *[17, Remark 2.4 and Lemma 2.5] Let $\mathcal{M}$ be a multiset in $\mathcal{H}(2, q)$, $17 < q$, $q = p^h$, where p is prime. Assume that the number of lines intersecting $\mathcal{M}$ in not $k \pmod p$ points is δ. Then, the number s of non $k \pmod p$ secants through any point of $\mathcal{M}$ satisfies $qs - s(s - 1) \leqslant \delta$. In particular, if $\delta < \frac{3}{16}(q + 1)^2$, then the number of non $k \pmod p$ secants through any point is at most $\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^3}$ or at least $q + 1 - (\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^3})$.*

Property 2.6 ([17]). *Let $\mathcal{M}$ be a multiset in $\mathcal{H}(2, q)$, $q = p^h$, where p is prime. Assume that there are δ lines that intersect $\mathcal{M}$ in not $k \pmod p$ points. If through a point there are more than $q/2$ lines intersecting $\mathcal{M}$ in not $k \pmod p$ points, then there exists a value r such that the intersection multiplicity of at least $2\frac{\delta}{q+1} + 5$ of these lines with $\mathcal{M}$ is r.*

Result 2.7 ([17])**.** *Let $\mathcal{M}$ be a multiset in $\mathcal{H}(2,q)$, $17 < q$, $q = p^h$, where p is prime. Assume that the number of lines intersecting $\mathcal{M}$ in not $k \pmod p$ points is δ, where $\delta < (\lfloor\sqrt{q}\rfloor + 1)(q + 1 - \lfloor\sqrt{q}\rfloor)$. Assume furthermore that Property 2.6 holds. Then there exists a multiset $\mathcal{M}'$ with the property that it intersects every line in $k \pmod p$ points and the number of different points in $(\mathcal{M} \cup \mathcal{M}') \setminus (\mathcal{M} \cap \mathcal{M}')$ is exactly $\lceil \frac{\delta}{q+1} \rceil$.*

Result 2.8 ([17])**.** *Let B be a proper point set in $\mathcal{H}(2,q)$, $17 < q$. Suppose that B is a codeword of the lines of $\mathcal{H}(2,q)$. Assume also that $|B| < (\lfloor\sqrt{q}\rfloor + 1)(q+1-\lfloor\sqrt{q}\rfloor)$. Then B is a linear combination of at most $\lceil \frac{|B|}{q+1} \rceil$ lines.* □

3 Proof of Theorem 1.1

Let V be the vector space of dimension $q^{2r} + q^{2(r-1)} + \cdots + q^2 + 1$ over the prime field $\mathbb{F}_p$, where the coordinate positions for the vectors in V correspond to the points of $\mathcal{H}(r,q^2)$ in some fixed order. If S is a subset of points in $\mathcal{H}(r,q^2)$, then let v^S denote the vector in V with coordinate 1 in the positions corresponding to the points in S and with coordinate 0 in all other positions; that is v^S is the *characteristic vector* of S. Let $\mathcal{C}_p$ denote the subspace of V spanned by the characteristic vectors of all the hyperplanes in $\mathcal{H}(r,q^2)$. This code C_p is called *the linear code of* $\mathcal{H}(r,q^2)$.

From [13, Theorem 1], we know that the characteristic vector $v^{\mathcal{V}}$ of a Hermitian variety $\mathcal{V} \in \mathcal{H}(r,q^2)$ is in $\mathcal{C}_p$. So from now on, we will assume that $\mathcal{V}$ is a quasi-Hermitian variety in $\mathcal{H}(r,q^2)$ and $v^{\mathcal{V}} \in \mathcal{C}_p$. In the remainder of this section, we will show that $\mathcal{V}$ is a classical Hermitian variety for specific values of q.

The next lemmas hold for $r \geqslant 3$ and for any $q = p^h$, p prime, $h \geqslant 1$.

Lemma 3.1. *Every line of $\mathcal{H}(r,q^2)$, $q = p^h$, p prime, $h \geqslant 1$, meets $\mathcal{V}$ in $1 \pmod p$ points.*

Proof. We may express

$$v^{\mathcal{V}} = v^{H_1} + \cdots + v^{H_t},$$

where $H_1, \ldots, H_t$ are (not necessarily distinct) hyperplanes of $\mathcal{H}(r,q^2)$. Denote by $\cdot$ the usual dot product. We get $v^{\mathcal{V}} \cdot v^{\mathcal{V}} = |\mathcal{V}| \equiv 1 \pmod p$. On the other hand,

$$v^{\mathcal{V}} \cdot v^{\mathcal{V}} = v^{\mathcal{V}} \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod p,$$

since every hyperplane of $\mathcal{H}(r,q^2)$ meets $\mathcal{V}$ in $1 \pmod p$ points. Hence, we have $t \equiv 1 \pmod p$. Finally, for a line ℓ of $\mathcal{H}(r,q^2)$,

$$v^{\ell} \cdot v^{\mathcal{V}} = v^{\ell} \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod p,$$

as every line of $\mathcal{H}(r,q^2)$ meets a hyperplane in 1 or $q^2 + 1$ points. That is, $|\ell \cap \mathcal{V}| \equiv 1 \pmod p$ and in particular no lines of $\mathcal{H}(r,q^2)$ are external to $\mathcal{V}$. □

Remark 3.2. The preceding proof also shows that $\mathcal{V}$ is a linear combination of $1 \pmod p$ (not necessarily distinct) hyperplanes, all having coefficient one.

Lemma 3.3. *For every hyperplane H of $\mathcal{H}(r, q^2)$, $q = p^h$, p prime, $h \geqslant 1$, the intersection $H \cap \mathcal{V}$ is in the code of points and hyperplanes of H itself.*

Proof. Let Σ denote the set of all hyperplanes of $\mathcal{H}(r, q^2)$. By assumption,

$$v^{\mathcal{V}} = \sum_{H_i \in \Sigma} \lambda_i v^{H_i}. \tag{1}$$

For every $H \in \Sigma$, let π denote a hyperplane of H; then $\pi = H_{j_1} \cap \cdots \cap H_{j_{q^2+1}}$, where $H_{j_1}, \ldots, H_{j_{q^2+1}}$ are the hyperplanes of $\mathcal{H}(r, q^2)$ through π. We assume $H = H_{j_{q^2+1}}$. For every hyperplane π of H, we set

$$\lambda_\pi = \sum_{k=1,\ldots,q^2+1} \lambda_{j_k},$$

where λ_{j_k} is the coefficient in (1) of $v^{H_{j_k}}$ and H_{j_k} is one of the q^2+1 hyperplanes through π.

Now, consider

$$T = \sum_{\pi \in \Sigma'} \lambda_\pi v^\pi, \tag{2}$$

where Σ' is the set of all hyperplanes in H. We are going to show that

$$T = v^{\mathcal{V} \cap H}.$$

In fact, it is clear that at the positions belonging to the points outside of H we see zeros. At a position belonging to a point in H, we see the original coefficients of $v^{\mathcal{V}}$ plus $(|\Sigma'| - 1)\lambda_{j_{q^2+1}}$. Note that this last term is $0 \pmod p$, hence $T = v^{\mathcal{V} \cap H}$. □

Corollary 3.4. *For every subspace S of $\mathcal{H}(r, q^2)$, $q = p^h$, p prime, $h \geqslant 1$, the intersection $S \cap \mathcal{V}$ is in the code of points and hyperplanes of S itself.* □

Remark 3.5. Lemma 3.3 and Corollary 3.4 are valid for $\mathcal{V}$ any set of points in $\mathcal{H}(r, q^2)$ whose incidence vector belongs to the code of points and hyperplanes of $\mathcal{H}(r, q^2)$. In particular, it follows that for every plane π the intersection $\pi \cap \mathcal{V}$ is a codeword of the points and lines of π, $\pi \cap \mathcal{V}$ has size $1 \pmod p$ and so it is a linear combination of $1 \pmod p$ not necessarily distinct lines.

Lemma 3.6. *Let ℓ be a line of $\mathcal{H}(r, q^2)$. Then there exists at least one plane through ℓ meeting $\mathcal{V}$ in δ points, with $\delta \leqslant q^3 + q^2 + q + 1$.*

Proof. By way of contradiction, assume that all planes through ℓ meet $\mathcal{V}$ in more than $q^3 + q^2 + q + 1$ points. Set $x = |\ell \cap \mathcal{V}|$. We get

$$\frac{(q^{r+1} + (-1)^r)(q^r - (-1)^r)}{q^2 - 1} > m(q^3 + q^2 + q + 1 - x) + x, \tag{3}$$

where $m = q^{2(r-2)} + q^{2(r-3)} + \cdots + q^2 + 1$ is the number of planes in $\mathcal{H}(r, q^2)$ through ℓ. From (3), we obtain $x > q^2 + 1$, a contradiction. □

Lemma 3.7. *For each line ℓ of $\mathcal{H}(r, q^2)$, $q > 4$ and $q = p^h$, p odd prime, $h \geqslant 1$, either $|\ell \cap \mathcal{V}| \leqslant q + 1$ or $|\ell \cap \mathcal{V}| \geqslant q^2 - q + 1$.*

Proof. Let ℓ be a line of $\mathcal{H}(r, q^2)$ and let π be a plane through ℓ such that $|\pi \cap \mathcal{V}| \leqslant q^3 + q^2 + q + 1$; Lemma 3.6 shows that such a plane exists. Set $B = \pi \cap \mathcal{V}$. By Corollary 3.4, B is a codeword of the code of the lines of π, so we can write it as a linear combination of some lines of π, that is $\sum_i \lambda_i v^{e_i}$, where v^{e_i} are the characteristic vectors of the lines e_i in π.

Let B^* be the multiset consisting of the lines e_i, with multiplicity λ_i, in the dual plane of π. The weight of the codeword B is at most $q^3 + q^2 + q + 1$, hence in the dual plane this is the number of lines intersecting B^* in not $0 \pmod p$ points. Actually, as B is a proper set, we know that each non $0 \pmod p$ secant of B^* must be a $1 \pmod p$ secant. Using Result 2.5, with $\delta = q^3 + q^2 + q + 1$, in $\mathcal{H}(2, q^2)$, the number of non $0 \pmod p$ secants through any point is at most

$$\frac{\delta}{q^2+1} + 2\frac{\delta^2}{(q^2+1)^3} = q + 1 + 2\frac{(q+1)^2}{q^2+1} < q + 4$$

or at least

$$q^2 + 1 - \left(\frac{\delta}{q^2+1} + 2\frac{\delta^2}{(q^2+1)^3}\right) > q^2 - q - 3.$$

In the original plane π, this means that each line intersects B in either at most $q + 3$ or in at least $q^2 - q - 2$ points. Since such lines must be $1 \pmod p$ secants and $p > 2$, then each line intersects B in either at most $q + 1$ or in at least $q^2 - q + 1$ points. $\square$

Proposition 3.8. *Assume that π is a plane of $\mathcal{H}(r, q^2)$, $q > 4$, and $q = p^h$, p odd prime, $h \geqslant 1$, such that $|\pi \cap \mathcal{V}| \leqslant q^3 + 2q^2$. Furthermore, suppose also that there exists a line ℓ meeting $\pi \cap \mathcal{V}$ in at least $q^2 - q + 1$ points, when $q^3 + 1 \leqslant |\pi \cap \mathcal{V}|$. Then $\pi \cap \mathcal{V}$ is a linear combination of at most $q + 1$ lines, each with weight 1.*

Proof. Let B be the point set $\pi \cap \mathcal{V}$. By Corollary 3.4, B is the corresponding point set of a codeword c of lines of π, that is $c = \sum_i \lambda_i v^{e_i}$, where lines of π are denoted by e_i. Let C^* be the multiset in the dual plane containing the dual of each line e_i with multiplicity λ_i. Clearly the number of lines intersecting C^* in not $0 \pmod p$ points is $w(c) = |B|$. Note also, that every line that is not a $0 \pmod p$ secant is a $1 \pmod p$ secant, as B is a proper point set.

Our very first aim is to show that c is a linear combination of at most $q + 3$ different lines. When $|B| < q^3 + 1$, then, by Result 2.8, it is a linear combination of at most q different lines.

Next assume that $|B| \geqslant q^3 + 1$. From the assumption of the proposition, we know that there exists a line ℓ meeting $\pi \cap \mathcal{V}$ in at least $q^2 - q + 1$ points and from Lemma 3.7, we also know that each line intersects B in either at most $q + 1$ or in at least $q^2 - q + 1$ points. Hence, if we add the line ℓ to c with multiplicity -1, we reduce the weight by at least $q^2 - q + 1 - q$ and by at most $q^2 + 1$. If $w(c - v^\ell) < q^3 + 1$, then from the above we know that $c - v^\ell$ is a linear combination of $\lceil \frac{w(c - v^\ell)}{q^2+1} \rceil$ lines. Hence, c is a linear combination of at most $q+1$ lines. If $w(c - v^\ell) \geqslant q^3+1$, then $w(c) \geqslant q^3+q^2-2q-2$ (see above) and so it follows that through any point of B, there passes at least one line intersecting B in at least q^2-q+1 points. This means that we easily find three lines ℓ_1, ℓ_2, and ℓ_3 intersecting B in at least q^2-q+1 points. Since $w(c) \leqslant q^3+2q^2$, we get that $w(c - v^{\ell_1} - v^{\ell_2} - v^{\ell_3}) \leqslant q^3+2q^2-3\cdot(q^2-2q-2) < q^3+1$. Hence, similarly as before, we get that c is a linear combination of at most $q+3$ lines.

Next we show that each line in the linear combination (that constructs c) has weight 1. Take a line ℓ which is in the linear combination with coefficient $\lambda \neq 0$. Then there are at least $q^2+1-(q+2)$ positions, such that the corresponding point is in ℓ and the value at that position is λ. As B is a proper set, this yields that $\lambda = 1$. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of c must be 1 (mod p); hence it can be at most $q+1$. □

Proposition 3.9. *Assume that π is a plane of $\mathcal{H}(r, q^2)$, $q > 4$, and $q = p^h$, p odd prime, $h \geqslant 1$, such that $|\pi \cap \mathcal{V}| \leqslant q^3+2q^2$. Furthermore, suppose that every line meets $\pi \cap \mathcal{V}$ in at most $q+1$ points. Then $\pi \cap \mathcal{V}$ is a classical unital.*

Proof. Again let $B = \pi \cap \mathcal{V}$ and first assume that $|B| < q^3+1$. Proposition 3.8 shows that B is a linear combination of at most $q+1$ lines, each with weight 1. But this yields that these lines intersect B in at least q^2+1-q points. So this case cannot occur.

Hence, $q^3+1 \leqslant |B| \leqslant q^3+2q^2$. We are going to prove that there exists at least a tangent line to B in π. Let t_i be the number of lines meeting B in i points. Set $x = |B|$. Then double counting arguments give the following equations for the integers t_i.

$$\begin{cases} \sum_{i=1}^{q+1} t_i = q^4+q^2+1 \\ \sum_{i=1}^{q+1} it_i = x(q^2+1) \\ \sum_{i=1}^{q+1} i(i-1)t_i = x(x-1). \end{cases} \tag{4}$$

Consider $f(x) = \sum_{i=1}^{q+1}(i-2)(q+1-i)t_i$. From (4), we get

$$f(x) = -x^2 + x[(q^2+1)(q+2)+1] - 2(q+1)(q^4+q^2+1).$$

Since $f(q^3/2) > 0$, whereas $f(q^3+1) < 0$ and $f(q^3+2q^2) < 0$, it follows that if $q^3+1 \leqslant x \leqslant q^3+2q^2$, then $f(x) < 0$ and thus t_1 must be different from zero. Consider now the quantity $\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i$. We have that

$$\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i = f(x) + \sum_{i=1}^{q+1}(q+1-i)t_i = f(x) + (q+1)\sum_{i=1}^{q+1} t_i - \sum_{i=1}^{q+1} it_i$$

$$= f(x)+(q+1)(q^4+q^2+1)-x(q^2+1) = -x^2+x[(q^2+1)(q+1)+1]-(q+1)(q^4+q^2+1).$$

Since $\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i \geqslant 0$, we have that $x \leqslant \frac{(q^2+1)(q+1)+1+(q^3-q^2-q)}{2} = q^3+1$. Therefore, $x = q^3+1$ and $\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i = 0$.

Since $(i-1)(q+1-i) > 0$, for $2 \leqslant i \leqslant q$, we obtain $t_2 = t_3 = \cdots = t_q = 0$, that is, B is a set of q^3+1 points such that each line is a 1-secant or a $(q+1)$-secant of B. Namely, B is a unital and precisely a classical unital since B is a codeword of π [6]. □

The above two propositions and Lemma 3.7 imply the following corollary.

Corollary 3.10. *Assume that π is a plane of $\mathcal{H}(r,q^2), q > 4$ and $q = p^h$, p odd prime, $h \geqslant 1$, such that $|\pi \cap \mathcal{V}| \leqslant q^3 + 2q^2$. Then $\pi \cap \mathcal{V}$ is a linear combination of at most $q+1$ lines, each with weight* 1*, or it is a classical unital.* □

Corollary 3.11. *Suppose that π is a plane of $\mathcal{H}(r,q^2), q > 4$ and $q = p^h$, p odd prime, $h \geqslant 1$, containing exactly q^3+1 points of $\mathcal{V}$. Then $\pi \cap \mathcal{V}$ is a classical unital.*

Proof. Let B be the point set $\pi \cap \mathcal{V}$. We know that B is the support of a codeword of lines of π. By Proposition 3.8, if there is a line intersecting B in at least q^2-q+1 points, then B is a linear combination of at most $q+1$ lines, each with multiplicity 1. First of all note that a codeword that is a linear combination of $q+1$ lines has weight at least $(q^2+1)(q+1) - 2\binom{q+1}{2}$, that is exactly q^3+1. In fact, in a linear combination of $q+1$ lines the minimum number of points is obtained if there is a hole at the intersection of any two lines. There are $\binom{q+1}{2}$ intersections and each intersection is counted twice, therefore we have to subtract $2\binom{q+1}{2}$. To achieve this, we need that the intersection points of any two lines from such a linear combination are all different and the sum of the coefficients of any two lines is zero; which is clearly not the case (as all the coefficients are 1). From Remark 3.5, in this case B would be a linear combination of at most $q+1-p$ lines and so its weight would be less than q^3+1, a contradiction. Hence, there is no line intersecting B in at least q^2-q+1 points, so Proposition 3.9 finishes the proof. □

3.1 Case $r = 3$

In $\mathcal{H}(3,q^2)$, each plane intersects $\mathcal{V}$ in either q^3+1 or q^3+q^2+1 points since these are the intersection numbers of a quasi-Hermitian variety with a plane of $\mathcal{H}(3,q^2)$.

3.1.1 $q = p$

Let $\mathcal{V}$ be a quasi-Hermitian variety of $\mathcal{H}(3,p^2)$, p prime.

Lemma 3.12. *Every plane π of $\mathcal{H}(3,p^2)$ sharing p^3+1 points with $\mathcal{V}$ intersects $\mathcal{V}$ in a unital of π.*

Proof. Set $U = \pi \cap \mathcal{V}$. Let P be a point in U. Assume that every line ℓ in π through the point P meets U in at least $p+1$ points. We get $|\pi \cap \mathcal{V}| = p^3+1 \geqslant (p^2+1)p+1 = p^3+p+1$, which is impossible.

Thus, P lies on at least one tangent line to U and this implies that U is a minimal blocking set in π of size p^3+1. From a result obtained by Bruen and Thas, see [7], it follows that U is a unital of π and hence every line in π meets U in either 1 or $p+1$ points. □

Lemma 3.13. *Let π be a plane in $\mathcal{H}(3,p^2)$ such that $|\pi \cap \mathcal{V}| = p^3+p^2+1$, then every line in π meets $\pi \cap \mathcal{V}$ in either 1 or $p+1$ or p^2+1 points.*

Proof. Set $C = \pi \cap \mathcal{V}$ and Let m be a line in π such that $|m \cap C| = s$ with $s \neq 1$ and $s \neq p+1$. Thus, from Lemma 3.12, every plane through m has to meet $\mathcal{V}$ in p^3+p^2+1 points and thus

$$|\mathcal{V}| = (p^2+1)(p^3+p^2+1-s)+s,$$

which gives $s = p^2+1$. □

From Lemmas 3.12 and 3.13, it follows that every line in $\mathcal{H}(3,p^2)$ meets $\mathcal{V}$ in either 1 or, $p+1$ or, p^2+1 points.

3.1.2 $q = p^h$, $q \geqslant 5$ odd

Let $\mathcal{V}$ be a quasi-Hermitian variety of $\mathcal{H}(3,q^2)$, $q \geqslant 5$ odd.

Lemma 3.14. *Let π be a plane in $\mathcal{H}(3,q^2)$ such that $|\pi \cap \mathcal{V}| = q^3+q^2+1$, then every line in π meets $\pi \cap \mathcal{V}$ in either 1, $q+1$ or q^2+1 points.*

Proof. Set $C = \pi \cap \mathcal{V}$ and let m be a line in π such that $|m \cap C| = s$, with $s \neq 1$ and $s \neq q+1$. Thus, from Corollary 3.11, every plane through m has to meet $\mathcal{V}$ in q^3+q^2+1 points and thus

$$|\mathcal{V}| = (q^2+1)(q^3+q^2+1-s)+s,$$

which gives $s = q^2+1$. □

From Corollary 3.11 and Lemma 3.14, it follows that every line in $\mathcal{H}(3,q^2)$ meets $\mathcal{V}$ in either 1, $q+1$, or q^2+1 points.

3.1.3 $q = 2^h$, $h > 2$

Let $\mathcal{V}$ be a quasi-Hermitian variety of $\mathcal{H}(3,2^{2h})$, $h > 2$.

Lemma 3.15. *For each line ℓ of $\mathcal{H}(3,2^{2h})$, $h > 2$, either $|\ell \cap \mathcal{V}| \leqslant q+1$ or $|\ell \cap \mathcal{V}| \geqslant q^2-q-1$.*

Proof. Let ℓ be a line of $\mathcal{H}(3,2^{2h})$. Since ℓ is at least a tangent to $\mathcal{V}$, there exists a plane through ℓ meeting $\mathcal{V}$ in q^3+q^2+1 points. Let π be a plane through ℓ such that $|\pi \cap \mathcal{V}| = q^3+q^2+1$. Set $B = \pi \cap \mathcal{V}$. As before, by Corollary 3.4, B is a codeword of the code of the lines of π, so we can write it as a linear combination of some lines of π, that is $\sum_i \lambda_i v^{e_i}$, where v^{e_i} are the characteristic vectors of the lines e_i in π.

Let B^* be the multiset consisting of the lines e_i, with multiplicity λ_i, in the dual plane of π. The weight of the codeword B is q^3+q^2+1, hence in the dual plane this is the number of lines intersecting B^* in not $0 \pmod p$ points. Actually, as B is a proper set, we know that each non $0 \pmod p$ secant of B^* must be a $1 \pmod p$ secant. Using Result 2.5, with $\delta = q^3+q^2+1$ in $\mathcal{H}(2,2^{2h})$, the number s of non $0 \pmod p$ secants through any point of B satisfies the inequality $s^2-(q^2+1)s-(q^3+q^2+1) \geqslant 0$. Since the determinant, $(q^2+1)^2+4(q^3+q^2+1) > ((q^2+1)-2(q+3))^2$, we get $s < q+3$ or $s > q^2-q-2$

In the original plane π, this means that each line intersects B in either at most $q+2$ or in at least q^2-q-1 points. Since such lines must be $1 \pmod p$ secants and $p=2$, then each line intersects B in either at most $q+1$ or in at least q^2-q-1 points. □

Let α be a plane meeting $\mathcal{V}$ in a point set B' of size q^3+q^2+1 points. We want to prove that α contains some s-secant, with s at least q^2-q-1. Assume on the contrary that each line in α meets $\mathcal{V}$ in at most $q+1$ points. Let P be a point of B' and consider the q^2+1 lines through P. We get $q^3+q^2+1 \leqslant (q^2+1)q+1$, a contradiction. Therefore, there exists a line ℓ in α meeting B' in at least q^2-q-1 points. We are going to show that B' is a linear combination of exactly $q+1$ lines each with weight 1.

Again, by Corollary 3.4, B' is the corresponding point set of a codeword c' of lines of π, that is $c' = \sum_i \lambda_i v^{e_i}$, where lines of π are denoted by e_i. Let C'^* be the multiset in the dual plane containing the dual of each line e_i with multiplicity λ_i. As before, the number of lines intersecting C'^* in not $0 \pmod p$ points is $w(c') = |B'| = q^3+q^2+1$ and every line that is not a $0 \pmod p$ secant is a $1 \pmod p$ secant, as B' is a proper point set.

Hence, if we add the line ℓ to c with multiplicity 1, we reduce the weight by at least $q^2-q-1-q-2 = q^2-2q-3$ and at most by q^2+1. Now, through any point of B', there passes at least one line intersecting B' in at least q^2-q-1 points. Thus, we easily find two lines ℓ_1 and ℓ_2 intersecting B in at least q^2-q-1 points. We get that $w(c'-v^{\ell_1}-v^{\ell_2}) < q^3+1$. Hence, similarly as before, we get that $c'-v^{\ell_1}-v^{\ell_2}$ is a linear combination of $\lceil \frac{w(c'-v^{\ell_1}-v^{\ell_2})}{q^2+1} \rceil$ lines. Hence, c' is a linear combination of at most $q+2$ lines. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of c' must be $1 \pmod p$; hence, as $p=2$ it can be at most $q+1$. For $p=2$, a codeword that is a linear combination of at most $q+1$ lines each with weight 1, has weight at most q^3+q^2+1 and this is achieved when the $q+1$ lines are concurrent. This implies that each line in α is either a 1 or $q+1$ or q^2+1-secant to $\mathcal{V}$. Now consider a line m' that is an s-secant to $\mathcal{V}$ with s different from $1, q+1$, and also different from q^2+1. Each plane through m' has to meet $\mathcal{V}$ in q^3+1 points. From $|V| = q^5+q^3+q^2+1 = (q^2+1)(q^3+1-s)+s$ we get $s=0$, a contradiction.

Thus each line of $\mathcal{H}(3, 2^{2h})$ meets $\mathcal{V}$ in either 1 or $q+1$ or q^2+1 points.

Proof of Theorem 1.1 (case $r=3$): From all previous lemmas of this section, it follows that every line in $\mathcal{H}(3,q^2)$, with $q=p^h \neq 4$ and p any prime, meets $\mathcal{V}$ in either 1, $q+1$, or q^2+1 points. Now, suppose on the contrary that there exists a singular point P on $\mathcal{V}$; this means that all lines through P are either tangents or (q^2+1)-secants to $\mathcal{V}$. Take a plane π which does not contain P. Then $|\mathcal{V}| = q^2|\pi \cap \mathcal{V}|+1$ and since the two possible sizes of the planar sections are q^3+1 or q^3+q^2+1, we get a contradiction. Thus, every point in $\mathcal{V}$ lies on at least one $(q+1)$-secant and, from Theorem 2.1, we obtain that $\mathcal{V}$ is a Hermitian surface. □

3.2 Case $r \geqslant 4$ and $q = p \geqslant 5$

We first prove the following result.

Lemma 3.16. *If π is a plane of $\mathcal{H}(r,p^2)$, which is not contained in $\mathcal{V}$, then either*

$$|\pi \cap \mathcal{V}| = p^2 + 1 \text{ or } |\pi \cap \mathcal{V}| \geqslant p^3 + 1.$$

Proof. Let π be a plane of $\mathcal{H}(r,p^2)$ and set $B = \pi \cap \mathcal{V}$. By Remark 3.5, B is a linear combination of 1 (mod p) not necessarily distinct lines.

If $|B| < p^3 + 1$, then by Result 2.8, B is a linear combination of at most p distinct lines. This and the previous observation yield that when $|B| < p^3+1$, then it is the scalar multiple of one line; hence $|B| = p^2 + 1$. □

Proposition 3.17. *Let π be a plane of $\mathcal{H}(r,p^2)$, such that $|\pi\cap\mathcal{V}| \leqslant p^3+p^2+p+1$. Then $B = \pi \cap \mathcal{V}$ is either a classical unital or a linear combination of $p+1$ concurrent lines or just one line, each with weight* 1.

Proof. From Corollary 3.10, we have that B is either a linear combination of at most $p+1$ lines or a classical unital. In the first case, since B intersects every line in 1 (mod p) points and B is a proper point set, the only possibilities are that B is a linear combination of $p + 1$ concurrent lines or just one line, each with weight 1. □

Proof of Theorem 1.1 (case $r \geqslant 4$, $q = p$): Consider a line ℓ of $\mathcal{H}(r,p^2)$ which is not contained in $\mathcal{V}$. By Lemma 3.6, there is a plane π through ℓ such that $|\pi \cap \mathcal{V}| \leqslant q^3 + q^2 + q + 1$. From Proposition 3.17, we have that ℓ is either a unisecant or a $(p + 1)$-secant of $\mathcal{V}$ and we also have that $\mathcal{V}$ has no plane section of size $(p+1)(p^2+1)$. Finally, it is easy to see like in the previous case $r = 3$, that $\mathcal{V}$ has no singular points, thus $\mathcal{V}$ turns out to be a Hermitian variety of $\mathcal{H}(r,p^2)$ (Theorem 2.2).

3.3 Case $r \geqslant 4$ and $q = p^2$, p odd

Assume now that $\mathcal{V}$ is a quasi-Hermitian variety of $\mathcal{H}(r,p^4)$, with $r \geqslant 4$.

Lemma 3.7 states that every line contains at most $p^2 + 1$ points of $\mathcal{V}$ or at least $p^4 - p^2 + 1$ points of $\mathcal{V}$.

Lemma 3.18. *If ℓ is a line of $\mathcal{H}(r,p^4)$, such that $|\ell \cap \mathcal{V}| \geqslant p^4 - p^2 + 1$, then $|\ell \cap \mathcal{V}| \geqslant p^4 - p + 1$.*

Proof. Set $|\ell \cap \mathcal{V}| = p^4 - x + 1$, where $x \leqslant p^2$. It suffices to prove that $x < p + 2$. Let π be a plane through ℓ and $B = \pi \cap \mathcal{V}$. Choose π such that $|B| = |\pi \cap \mathcal{V}| \leqslant p^6 + p^4 + p^2 + 1$ (Lemma 3.6). Then, by Proposition 3.8, B is a linear combination of at most p^2+1 lines, each with weight 1. Let c be the codeword corresponding to B. We observe that ℓ must be one of the lines of c, otherwise $|B \cap \ell| \leqslant p^2 + 1$, which is impossible. Thus if P is a point in $\ell \setminus B$, then through P there pass at least $p-1$ other lines of c. If $x \geqslant p+2$, then the number of lines necessary to define the codeword c would be at least $(p+2)(p-1)+1$, a contradiction. □

Lemma 3.19. *For each plane π of $\mathcal{H}(r,p^4)$, either $|\pi\cap\mathcal{V}|\leqslant p^6+2p^4-p^2-p+1$ or $|\pi\cap\mathcal{V}|\geqslant p^8-p^5+p^4-p+1$.*

Proof. Let $B=\pi\cap\mathcal{V}$, $x=|B|$, and let t_i be the number of lines in π meeting B in i points. Then, in this case, Equations (4) read

$$\begin{cases}\sum_{i=1}^{p^4+1} t_i = p^8+p^4+1\\[2ex] \sum_{i=1}^{p^4+1} it_i = x(p^4+1)\\[2ex] \sum_{i=1}^{p^4+1} i(i-1)t_i = x(x-1).\end{cases} \tag{5}$$

Set $f(x)=\sum_{i=1}^{p^4+1}(p^2+1-i)(i-(p^4-p+1))t_i$. From (5) we obtain

$$f(x)=-x^2+[(p^4+1)(p^4+p^2-p+1)+1]x-(p^8+p^4+1)(p^2+1)(p^4-p+1).$$

Because of Lemma 3.18, we get $f(x)\leqslant 0$, while $f(p^6+2p^4-p^2+1)>0$, $f(p^8-p^5+p^4-p)>0$. This finishes the proof of the lemma. □

Lemma 3.20. *If π is a plane of $\mathcal{H}(r,p^4)$, such that $|\pi\cap\mathcal{V}|\geqslant p^8-p^5+p^4-p+1$, then either π is entirely contained in $\mathcal{V}$ or $\pi\cap\mathcal{V}$ consists of $p^8-p^5+p^4+1$ points and it only contains i-secants, with $i\in\{1,p^4-p+1,p^4+1\}$.*

Proof. Set $S=\pi\setminus\mathcal{V}$. Suppose that there exists some point $P\in S$. We have the following two possibilities: either each line of the pencil with center at P is a (p^4-p+1)-secant or only one line through P is an i-secant, with $1\leqslant i\leqslant p^2+1$, whereas the other p^4 lines through P are (p^4-p+1)-secants. In the former case, when there are no i-secants, $1\leqslant i\leqslant p^2+1$, each line ℓ in π either is disjoint from S or it meets S in p points since ℓ is a (p^4-p+1)-secant. This implies that S is a maximal arc and this is impossible for $p\neq 2$ [4, 5].

In the latter case, we observe that the size of $\pi\cap\mathcal{V}$ must be $p^8-p^5+p^4+i$, where $1\leqslant i\leqslant p^2+1$. Next, we denote by t_s the number of s-secants in π, where $s\in\{i,p^4-p+1,p^4+1\}$. We have that

$$\begin{cases}\sum_s t_s = p^8+p^4+1\\[2ex] \sum_s st_s=(p^4+1)(p^8-p^5+p^4+i)\\[2ex] \sum_s s(s-1)t_s=(p^8-p^5+p^4+i)(p^8-p^5+p^4+i-1).\end{cases} \tag{6}$$

From (6) we get

$$t_i=\frac{p(p^4-p-i+1)(p^5-i+1)}{p(p^4-p-i+1)(p^4-i+1)}=\frac{p^5-i+1}{p^4-i+1}, \tag{7}$$

and we can see that the only possibility for t_i to be an integer is $ip-p-i+1=0$, that is $i=1$. For $i=1$, we get $|B|=p^8-p^5+p^4+1$. □

Lemma 3.21. *If π is a plane of $\mathcal{H}(r, p^4)$, not contained in $\mathcal{V}$ and which does not contain any $(p^4 - p + 1)$-secant, then $\pi \cap \mathcal{V}$ is either a classical unital or the union of i concurrent lines, with $1 \leqslant i \leqslant p^2 + 1$.*

Proof. Because of Lemmas 3.18, 3.19 and 3.20, the plane π meets $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points. Furthermore, each line of π which is not contained in $\mathcal{V}$ is an i-secant, with $1 \leqslant i \leqslant p^2 + 1$ (Lemma 3.18 and the sentence preceding Lemma 3.18). Set $B = \pi \cap \mathcal{V}$. If in π there are no $(p^4 + 1)$-secants to B, then $|B| \leqslant p^6 + p^2 + 1$ and by Proposition 3.9 it follows that B is a classical unital.

If there is a $(p^4 + 1)$-secant to B in π, then arguing as in the proof of Proposition 3.8, we get that B is still a linear combination of m lines, with $m \leqslant p^2 + 1$. Each of these m lines is a $(p^4 + 1)$-secant to $\mathcal{V}$. In fact if one of these lines, say v, was an s-secant, with $1 \leqslant s \leqslant p^2 + 1$, then through each point $P \in v \setminus B$, there would pass at least p lines of the codeword corresponding to B and hence B would be a linear combination of at least $(p^4 + 1 - s)(p - 1) + 1 > p^2 + 1$ lines, which is impossible.

We are going to prove that these m lines, say $\ell_1, \ldots, \ell_m$, are concurrent. Assume on the contrary that they are not. We can assume that through a point $P \in \ell_n$, there pass at least $p + 1$ lines of our codeword but there is a line ℓ_j which does not pass through P. Thus through at least $p + 1$ points on ℓ_j, there are at least $p + 1$ lines of our codeword and thus we find at least $(p + 1)p + 1 > m$ lines of B, a contradiction. □

Lemma 3.22. *A plane π of $\mathcal{H}(r, p^4)$ meeting $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points and containing a $(p^4 - p + 1)$-secant to $\mathcal{V}$ has at most $(p^2 + 1)(p^4 - p + 1)$ points of $\mathcal{V}$.*

Proof. Let ℓ be a line of π which is a $(p^4 - p + 1)$-secant to $\mathcal{V}$. In this case, $\pi \cap \mathcal{V}$ is a linear combination of at most $p^2 + 1$ lines, each with weight 1 (Proposition 3.8). A line not in the codeword can contain at most $p^2 + 1$ points. In particular, since ℓ contains more than $p^4 - p^2 + 1$ points of $\mathcal{V}$, ℓ is a line of the codeword and hence through each of the missing points of ℓ there are at least p lines of the codeword corresponding to B. On these p lines we can see at most $p^4 - p + 1$ points of $\mathcal{V}$.

So let $\ell_1, \ell_2, \ldots, \ell_p$ be p lines of the codeword through a point of $\ell \setminus \mathcal{V}$. Each of these lines contains at most $p^4 - p + 1$ points of $\mathcal{V}$. Thus these p lines contain together at most $p(p^4 - p + 1)$ points of $\mathcal{V}$. Now take any other line of the codeword, say e. If e goes through the common point of the lines ℓ_i, then there is already one point missing from e, so adding e to our set, we can add at most $p^4 - p + 1$ points. If e does not go through the common point, then it intersects ℓ_i in p different points. These points either do not belong to the set $\pi \cap \mathcal{V}$ or they belong to the set $\pi \cap \mathcal{V}$, but we have already counted them when we counted the points of ℓ_i, so again e can add at most $p^4 - p + 1$ points to the set $\pi \cap \mathcal{V}$. Thus adding the lines of the codewords one by one to ℓ_i and counting the number of points, each time we add only at most $p^4 - p + 1$ points to the set $\pi \cap \mathcal{V}$. Hence, the plane π contains at most $(p^2 + 1)(p^4 - p + 1)$ points of $\mathcal{V}$. □

Lemma 3.23. *Let π be a plane of $\mathcal{H}(r, p^4)$, containing an i-secant, $1 < i < p^2 + 1$, to $\mathcal{V}$. Then $\pi \cap \mathcal{V}$ is either the union of i concurrent lines or it is a linear combination of $p^2 + 1$ lines (each with weight 1) so that they form a subplane of order p, minus p concurrent lines.*

Proof. By Lemmas 3.19, 3.20 and 3.21, π meets $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points and must contain a $(p^4 - p + 1)$-secant to $\mathcal{V}$ or $\pi \cap \mathcal{V}$ is the union of i concurrent lines. Hence, from now on, we assume that π contains a $(p^4 - p + 1)$-secant. By Result 2.8, such a plane is a linear combination of at most $p^2 + 1$ lines. As before each line from the linear combination has weight 1. Note that the above two statements imply that a line of the linear combination will be either a $(p^4 + 1)$-secant or a $(p^4 - p + 1)$-secant.

As we have at most $p^2 + 1$ lines in the combination, a $(p^4 - p + 1)$-secant must be one of these lines. This also means that through each of the p missing points of this line, there must pass at least $p - 1$ other lines from the linear combination. Hence, we already get $(p - 1)p + 1$ lines.

In the case in which the linear combination contains exactly $p^2 - p + 1$ lines, then from each of these lines there are exactly p points missing and through each missing point there are exactly p lines from the linear combination. Hence, the missing points and these lines form a projective plane of order $p - 1$, a contradiction as $p > 3$.

Therefore, as the number of the lines of the linear combination must be $1 \pmod p$ and at most $p^2 + 1$, we can assume that the linear combination contains $p^2 + 1$ lines. We are going to prove that through each point of the plane there pass either 0, 1, p or $p + 1$ lines from the linear combination. From earlier arguments, we know that the number of lines through one point P is 0 or 1 (mod p). Assume to the contrary that through P there pass at least $p + 2$ of such lines. These $p^2 + 1$ lines forming the linear combination are not concurrent, so there is a line ℓ not through P. Through each of the intersection points of ℓ and a line through P, there pass at least $p - 1$ more other lines of the linear combination, so in total we get at least $(p - 1)(p + 2) + 1$ lines forming the linear combination, a contradiction.

Since there are $p^2 + 1$ lines forming the linear combination and through each point of the plane there pass either 0, 1, p or $p + 1$ of these lines, we obtain that on a $(p^4 - p + 1)$-secant there is exactly one point, say P, through which there pass exactly $p + 1$ lines from the linear combination and p points, not in the quasi Hermitian variety, through each of which there pass exactly p lines.

If all the $p^2 + 1$ lines forming the linear combination, were $(p^4 - p + 1)$-secants then the number of points through which there pass exactly p lines would be $(p^2 + 1)p/p$. On the other hand, through P there pass $p + 1$ $(p^4 - p + 1)$-secants, hence we already get $(p + 1)p$ such points, a contradiction. Thus, there exists a line m of the linear combination that is a $(p^4 + 1)$-secant. From the above arguments, on this line there are exactly p points through each of which there pass exactly $p + 1$ lines, whereas through the rest of the points of the line m there pass no other lines of the linear combination.

Assume that there is a line $m' \neq m$ of the linear combination that is also a $(p^4 + 1)$-secant. Then there is a point Q on m' but not on m through which there pass $p + 1$ lines. This would mean that there are at least $p + 1$ points on m, through which there pass more than 2 lines of the linear combination, a contradiction.

Hence, there is exactly one line m of the linear combination that is a $(p^4 + 1)$-secant and all the other lines of the linear combination are $(p^4 - p + 1)$-secants. It is easy to check that the points through which there are more than 2 lines plus the $(p^4 - p + 1)$-secants form a dual affine plane. Hence our lemma follows. □

Lemma 3.24. *There are no i-secants to $\mathcal{V}$, with $1 < i < p^2+1$.*

Proof. By Lemma 3.23, if a plane π contains an i-secant, $1 < i < p^2+1$, then $\pi \cap \mathcal{V}$ is a linear combination of either i concurrent lines or lines of an embedded subplane of order p minus p concurrent lines. In the latter case, if $i > 1$ then an i-secant is at least a (p^2-p+1)-secant. In fact, since the p^2+1 lines of the linear combination do not intersect outside of $\mathcal{H}(2,p)$, each of the p^2+1 lines of the linear combination is at least a (p^4-p+1)-secant, whereas the p concurrent lines are 1-secants. Also, all the other lines of $\mathcal{H}(2,p^4)$ intersect $\mathcal{H}(2,p)$ in either 1 or zero points. If they intersect $\mathcal{H}(2,p)$ in zero points they are (p^2+1)-secants to $\mathcal{V}$. If they intersect $\mathcal{H}(2,p)$ in a unique point P, then they are at least (p^2-p+1)-secants since P lies on p or $p+1$ lines of the linear combination and each line intersects $\mathcal{V}$ in $1 \pmod p$ points.

Hence, if there is an i-secant with $1 < i < p^2-p+1$, say ℓ, we get that for each plane α through ℓ, $\alpha \cap \mathcal{V}$ is a linear combination of i concurrent lines. Therefore

$$|\mathcal{V}| = m(ip^4+1-i)+i, \tag{8}$$

where $m = p^{4(r-2)} + p^{4(r-3)} + \ldots + p^4 + 1$ is the number of planes in $\mathcal{H}(r,p^4)$ through ℓ.

Setting $r = 2\sigma + \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$ according to r is even or odd, we can write

$$|\mathcal{V}| = 1 + p^4 + \ldots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon+1)} + \ldots + p^{4(r-1)})p^2$$

Hence, (8) becomes

$$\begin{array}{l} 1 + p^4 + \ldots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon+1)} + \ldots + p^{4(r-1)})p^2 \\ -(p^{4(r-2)} + p^{4(r-3)} + \ldots + p^4 + 1) = ip^{4(r-1)}. \end{array} \tag{9}$$

Since $\sigma \geqslant 2$, we see that $p^{4(r-1)}$ does not divide the left hand side of (9), a contradiction.

Thus, there can only be 1-, (p^2-p+1)-, (p^2+1)-, (p^4-p+1)- or (p^4+1)-secants to $\mathcal{V}$. Now, suppose that ℓ is a (p^2-p+1)-secant to $\mathcal{V}$. Again by Lemma 3.23, each plane through ℓ either has $x = (p^2-p+1)p^4+1$ or $y = p^2(p^4-p)+p^4+1$ points of $\mathcal{V}$. Next, denote by t_j the number of j-secant planes through ℓ to $\mathcal{V}$. We get

$$\left\{ \begin{array}{l} t_x + t_y = m \\ t_x(x-p^2+p-1) + t_y(y-p^2+p-1) + p^2-p+1 = |\mathcal{V}|. \end{array} \right. \tag{10}$$

Recover the value of t_y from the first equation and substitute it in the second. We obtain

$$(m-t_y)(p^6-p^5+p^4-p^2+p) + t_y(p^6+p^4-p^3-p^2+p) + p^2-p+1 = |\mathcal{V}|,$$

that is,

$$p^3(p^2-1)t_y = |\mathcal{V}| - m(p^6-p^5+p^4-p^2+p) - p^2+p-1.$$

The case $t_y = 0$ must be excluded, since by direct computations $|\mathcal{V}| \neq m(p^6-p^5+p^4-p^2+p)+p^2-p+1$. It is easy to check that $|\mathcal{V}| - m(p^6-p^5+p^4-p^2+p)-p^2+p-1$ is not divisible by $p+1$ and hence, t_y turns out not to be an integer, which is impossible. □

Lemma 3.25. *No plane meeting $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points contains a $(p^4 - p + 1)$-secant.*

Proof. Let π be a plane of $\mathcal{H}(r, p^4)$ such that $|\pi \cap \mathcal{V}| \leqslant p^6 + 2p^4 - p^2 - p + 1$. It can contain only 1-, $(p^2 + 1)$-, $(p^4 - p + 1)$-, $(p^4 + 1)$-secants (Lemma 3.18 and Lemma 3.24). If $\pi \cap \mathcal{V}$ contains a $(p^4 - p + 1)$-secant, we know from Proposition 3.8 that it is a linear combination of at most $p^2 + 1$ lines, each with weight 1. Suppose that e is a $(p^4 - p + 1)$-secant to $\pi \cap \mathcal{V}$. Let P and Q be two missing points of e. We know that there must be at least $p - 1$ other lines of the codeword through P and Q. Let f and g be two such lines through Q. The number of lines of the codeword through P is at most $2p$ and since the lines f and g are lines of the codeword through Q, they are $(p^4 - p + 1)$-secants. There exist at most $(p - 1)^2$ lines through P distinct from e intersecting f or g in a point not in $\mathcal{V}$. Therefore we can find a line, say m, of the plane through P, that intersects f and g in a point of $\mathcal{V}$ and that is not a line of the codeword. Then $|m \cap \mathcal{V}| \geqslant 1 + p$ since $|m \cap \mathcal{V}| \equiv 1 \pmod{p}$. Recall that m is not a line of the codeword and the total number of lines of the codeword is at most $p^2 + 1$. Also, m meets in P at least p lines of the codeword and therefore it can intersect all the remaining lines in at most $p^2 - p + 1$ pairwise distinct points. This means that the line m satisfies $p + 1 \leqslant |m \cap \mathcal{V}| \leqslant p^2 - p + 1$, and this contradicts Lemma 3.24. □

Lemma 3.26. *There are no $(p^4 - p + 1)$-secants to $\mathcal{V}$.*

Proof. If there was a $(p^4 - p + 1)$-secant to $\mathcal{V}$, say ℓ, then, by Lemma 3.25, all the planes through ℓ would contain at least $p^8 - p^5 + p^4 + 1$ points of $\mathcal{V}$, and thus

$$|\mathcal{V}| \geqslant (p^{4(r-2)} + p^{4(r-3)} + \cdots + p^4 + 1)(p^8 - p^5 + p) + p^4 - p + 1, \tag{11}$$

a contradiction. □

Proof of Theorem 1.1 (case $r \geqslant 4$ and $q = p^2$): Consider a line ℓ which is not contained in $\mathcal{V}$. From the preceding lemmas we have that ℓ is either a 1-secant or a $(p^2 + 1)$-secant of $\mathcal{V}$. Furthermore, $\mathcal{V}$ has no plane section of size $(p^2 + 1)(p^4 + 1)$ because of Lemma 3.21. Finally, as in the case $r = 3$, it is easy to see that $\mathcal{V}$ has no singular points, thus, by Theorem 2.2, $\mathcal{V}$ turns out to be a Hermitian variety of $\mathcal{H}(r, p^4)$.

4 Singular quasi-Hermitian varieties

In this section, we consider sets having the same behavior with respect to hyperplanes as singular Hermitian varieties.

Definition 4.1. *A* d-singular quasi-Hermitian variety *is a subset of points of $\mathcal{H}(r, q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension d.*

We prove the following result.

Theorem 4.2. *Let $\mathcal{S}$ be a d-singular quasi-Hermitian variety in $\mathcal{H}(r,q^2)$. Suppose that either*

- $r = 3$, $d = 0$, $q = p^h \neq 4$, $h \geqslant 1$, *p any prime, or*
- $r \geqslant 4$, $d \leqslant r - 3$, $q = p \geqslant 5$, *or*
- $r \geqslant 4$, $d \leqslant r - 3$, $q = p^2$, *p odd prime.*

Then $\mathcal{S}$ is a singular Hermitian variety with a singular space of dimension d if and only if its incidence vector is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\mathcal{H}(r,q^2)$.

Proof. Let $\mathcal{S}$ be a singular Hermitian variety of $\mathcal{H}(r,q^2)$. The characteristic vector $v^{\mathcal{S}}$ of $\mathcal{S}$ is in $\mathcal{C}_p$ since [13, Theorem 1] also holds for singular Hermitian varieties. Now assume that $\mathcal{S}$ is a d-singular quasi-Hermitian variety. As in the non-singular case, by Lemma 3.1, each line of $\mathcal{H}(r,q^2)$ intersects $\mathcal{S}$ in $1 \pmod p$ points.

4.1 Case $r = 3$

Suppose that $r = 3$ and therefore $d = 0$. Then $\mathcal{S}$ has $q^5 + q^2 + 1$ points. Let π be a plane of $\mathcal{H}(3,q^2)$. In this case, π meets $\mathcal{S}$ in either $q^2 + 1$, or $q^3 + 1$ or $q^3 + q^2 + 1$ points. In particular, a planar section of $\mathcal{S}$ with $q^2 + 1$ points is a line since it is a blocking set with respect to lines of a projective plane.

In the case in which q is a prime p, Lemma 3.12 and Lemma 3.13 are still valid in the singular case for $\mathcal{V} = \mathcal{S}$ and thus, the planar sections of $\mathcal{S}$ with p^3+1 or p^3+p^2+1 points have to be unitals or pencils of $p+1$ lines, respectively. Hence, each line of $\mathcal{H}(3,p^2)$ meets $\mathcal{S}$ in 1, $p+1$ or p^2+1 points.

When $q \geqslant 5$ is an odd prime power, Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10 are still valid in the singular case for $r = 3$.

Thus, if $|\pi \cap \mathcal{S}| = q^2+1$, then Proposition 3.8 implies that $\pi \cap \mathcal{S}$ is a line of π, whereas if $|\pi \cap \mathcal{S}| = q^3 + 1$, then Corollary 3.11 gives that $\pi \cap \mathcal{S}$ is a classical unital of π. Now suppose that $|\pi \cap \mathcal{S}| = q^3 + q^2 + 1$. Let ℓ be a line of π such that $|\ell \cap \mathcal{S}| = s$ with $s \neq 1, q+1, q^2+1$.

Each plane through ℓ must meet $\mathcal{S}$ in $q^3 + q^2 + 1$ points and this gives

$$(q^3 + q^2 + 1 - s)(q^2 + 1) + s = q^5 + q^2 + 1,$$

that is, $s = q^2 + q + 1$, which is impossible.

Thus in $\mathcal{H}(3,q^2)$, where q is an odd prime power, each line intersects $\mathcal{S}$ in either 1, or $q+1$ or q^2+1 points.

Next, assume $q = 2^h$, $h > 2$. Arguing as in the corresponding non-singular case, it turns out that a (q^3+q^2+1)-plane meets $\mathcal{S}$ in a pencil of $q+1$ lines. Now, assume that there is an i-secant line to $\mathcal{S}$, say m, with $2 < i < q$. Then, each plane through m has to be a (q^3+1)-plane of $\mathcal{S}$. Counting the number of points of $\mathcal{S}$ by using all planes through m we obtain $q^5 + q^2 + 1 = q^5 + q^3 + q^2 - iq^2 + 1$, hence $i = q$, a contradiction since every line contains $1 \pmod 2$ points of $\mathcal{V}$.

Therefore, each line of $\mathcal{H}(3, 2^{2h})$, $h \neq 2$ meets $\mathcal{S}$ in 1, $q+1$ or q^2+1 points. Finally, $\mathcal{S}$ is a $k_{q+1,3,q^2}$ for all $q \neq 4$. Also, $\mathcal{S}$ cannot be non-singular by assumption. When $q \neq 2$, Theorem 2.3 applies and $\mathcal{S}$ turns out to be a cone $\Pi_0\mathcal{S}'$ with $\mathcal{S}'$ of type I, II, III or IV as the possible intersection sizes with planes are q^2+1, q^3+1, q^3+q^2+1.

Possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $\mathcal{S} = \Pi_0\mathcal{H}$, where $\mathcal{H}$ is a non-singular Hermitian curve.

For $q = 2$, there is just one point set in $\mathcal{H}(3, 4)$ up to equivalence, meeting each line in 1, 3 or 5 points and each plane in 5, 9 or 13 points, that is the Hermitian cone, see [11, Theorem 19.6.8].

4.2 Case $r \geqslant 4$

Let ℓ be a line of $\mathcal{H}(r, q^2)$ containing $x < q^2+1$ points of $\mathcal{S}$. We are going to prove that there exists at least one plane through ℓ containing less than q^3+q^2+q+1 points of $\mathcal{S}$. If we suppose that all the planes through ℓ contain at least q^3+q^2+q+1 points of $\mathcal{S}$, then

$$q^{2(d+1)}\frac{(q^{r-d}+(-1)^{r-d-1})(q^{r-d-1}-(-1)^{r-d-1})}{q^2-1}+q^{2d}+q^{2(d-1)}+\cdots+q^2+1 \geqslant$$

$$m(q^3+q^2+q+1-x)+x,$$

where $m = q^{2(r-2)}+q^{2(r-3)}+\cdots+q^2+1$ is the number of planes through ℓ in $\mathcal{H}(r, q^2)$. We obtain $x > q^2+1$, a contradiction.

Therefore, there exists at least one plane through ℓ having less than q^3+q^2+q+1 points of $\mathcal{S}$ and hence Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10, are still valid in this singular case for any odd $q > 4$.

Next, we are going to prove that $\mathcal{S}$ is a k_{q+1,r,q^2}, with $q = p^h > 4$, $h = 1, 2$. **Case $\boldsymbol{q = p \geqslant 5}$:** Let ℓ be a line of $\mathcal{H}(r, p^2)$. As we have seen, there is a plane π through ℓ such that $|\pi \cap \mathcal{V}| < p^3+p^2+p+1$. Proposition 3.17 is still valid in this case and thus we have that ℓ is either a unisecant or a $(p+1)$-secant of $\mathcal{S}$. Furthermore, we also have that $\mathcal{S}$ has no plane section of size $(p+1)(p^2+1)$ and hence $\mathcal{S}$ is a regular k_{p+1,r,p^2}.
Case $\boldsymbol{q = p^2}$, $\boldsymbol{p}$ odd: We first observe that (8) and (11) hold true in the case in which $\mathcal{V}$ is assumed to be a singular quasi-Hermitian variety. This implies that all lemmas stated in the subparagraph 3.3 are valid in our case. Thus, we obtain that $\mathcal{S}$ is a k_{p^2+1,r,p^4} and it is straightforward to check that $\mathcal{S}$ is also regular.

Finally, in both cases $q = p$ or $q = p^2$, we have that $\mathcal{S}$ is a singular k_{q+1,r,q^2} because if $\mathcal{S}$ were a non-singular k_{q+1,r,q^2}, then, from Theorem 2.2, $\mathcal{S}$ would be a non-singular Hermitian variety and this is not possible by our assumptions.

Therefore, by Theorem 2.4, the only possibility is that $\mathcal{S}$ is a cone $\Pi_d\mathcal{S}'$, with $\mathcal{S}'$ a non-singular $k_{q+1,r-d-1,q^2}$. By Lemma 3.3, $\mathcal{S}'$ belongs to the code of points and hyperplanes of $\mathcal{H}(r-d-1, q^2)$. Since $r-d-1 \geqslant 2$, then, by [6] and Theorem 1.1, $\mathcal{S}'$ is a non-singular Hermitian variety and, therefore, $\mathcal{S}$ is a singular Hermitian variety with a vertex of dimension d. □

Acknowledgements

The research of the first two authors was partially supported by Ministry for Education, University and Research of Italy (MIUR) (Project PRIN 2012 "Geometrie di Galois e strutture di incidenza" - Prot. N. 2012XZE22K_005) and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM). The third and fourth author acknowledge the financial support of the Fund for Scientific Research - Flanders (FWO) and the Hungarian Academy of Sciences (HAS) project: Substructures of projective spaces (VS.073.16N). The fourth author was also partially supported by OTKA Grant K124950.

References

[1] A. Aguglia. Quasi-Hermitian varieties in $\mathrm{PG}(r, q^2)$, q even. *Contrib. Discrete Math.*, 8(1):31–37, 2013.

[2] A. Aguglia, A. Cossidente, and G. Korchmáros. On quasi-Hermitian varieties. *J. Combin. Des.*, 20(10):433–447, 2012.

[3] E. F. Assmus and J. D. Key. *Designs and their codes.* Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1992.

[4] S. Ball and A. Blokhuis. An easier proof of the maximal arcs conjecture. *Proc. Amer. Math. Soc.*, 126(11):3377–3380, 1998.

[5] S. Ball, A. Blokhuis, and F. Mazzocca. Maximal arcs in Desarguesian planes of odd order do not exist. *Combinatorica*, 17(1):31–41, 1997.

[6] A. Blokhuis, A. E. Brouwer, and H. Wilbrink. Hermitian unitals are code words. *Discrete Math.*, 97:63–68, 1991.

[7] A. A. Bruen and J. A. Thas. Blocking sets. *Geom. Dedicata*, 6:193–203, 1977.

[8] F. Buekenhout. Existence of unitals in finite translation planes of order q^2 with a kernel of order q. *Geom. Dedicata*, 5:189–194, 1976.

[9] S. De Winter and J. Schillewaert. A note on quasi-Hermitian varieties and singular quasi-quadrics. *Bull. Belg. Math. Soc. Simon Stevin*, 17(5):911–918, 2010.

[10] J. W. P. Hirschfeld. *Finite Projective Spaces of Three Dimensions.* Oxford Mathematical Monographs, Oxford University Press, Oxford, 1985.

[11] J. W. P. Hirschfeld. *Projective Geometries over Finite Fields, 2nd edn.* Oxford Mathematical Monographs, Oxford University Press, Oxford, 1998.

[12] J. W. P. Hirschfeld and J. A. Thas. *General Galois Geometries.* Oxford Mathematical Monographs, Oxford University Press, Oxford, 1991.

[13] J. D. Key. Hermitian varieties as Codewords. *Des. Codes Cryptogr.*, 1:255–259, 1991.

[14] R. Metz. On a class of unitals. *Geom. Dedicata*, 8:125–126, 1979.

[15] F. Pavese. Geometric constructions of two-character sets. *Discrete Math.*, 338(3):202–208, 2015.

[16] B. Segre. Forme e geometrie Hermitiane, con particolare riguardo al caso finito. *Ann. Mat. Pura Appl. (4)*, 70:1–201, 1965.

[17] T. Szőnyi and Zs. Weiner. Stability of k mod p multisets and small weight codewords of the code generated by the lines of $\mathrm{PG}(2, q)$. *Journal of Combinatorial Theory Ser. A*, to appear.