# A characterization of Hermitian varieties as codewords


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**Abstract**

It is known that the Hermitian varieties are codewords in the code defined by the points and hyperplanes of the projective spaces $\mathrm{PG}(r, q^2)$. In finite geometry, also quasi-Hermitian varieties are defined. These are sets of points of $\mathrm{PG}(r, q^2)$ of the same size as a non-singular Hermitian variety of $\mathrm{PG}(r, q^2)$, having the same intersection sizes with the hyperplanes of $\mathrm{PG}(r, q^2)$. In the planar case, this reduces to the definition of a unital. A famous result of Blokhuis, Brouwer, and Wilbrink states that every unital in the code of the points and lines of $\mathrm{PG}(2, q^2)$ is a Hermitian curve. We prove a similar result for the quasi-Hermitian varieties in $\mathrm{PG}(3, q^2)$, $q = p^h$, as well as in $\mathrm{PG}(r, q^2)$, $q = p$ prime, or $q = p^2$, $p$ prime, and $r \geqslant 4$.



**Keywords:** Hermitian variety; incidence vector; codes of projective spaces; quasi-Hermitian variety

## 1 Introduction

Consider the non-singular Hermitian varieties $\mathcal{H}(r,q^2)$ in $\mathcal{H}(r,q^2)$. A non-singular Hermitian variety $\mathcal{H}(r,q^2)$ in $\mathcal{H}(r,q^2)$ is the set of absolute points of a Hermitian polarity of $\mathcal{H}(r,q^2)$. Many properties of a non-singular Hermitian variety $\mathcal{H}(r,q^2)$ in $\mathcal{H}(r,q^2)$ are known. In particular, its size is $(q^{r+1}+(-1)^r)(q^r-(-1)^r)/(q^2-1)$, and its intersection numbers with the hyperplanes of $\mathcal{H}(r,q^2)$ are equal to $(q^r+(-1)^{r-1})(q^{r-1}-(-1)^{r-1})/(q^2-1)$, in case the hyperplane is a non-tangent hyperplane to $\mathcal{H}(r,q^2)$, and equal to $1+q^2(q^{r-1}+(-1)^r)(q^{r-2}-(-1)^r)/(q^2-1)$ in case the hyperplane is a tangent hyperplane to $\mathcal{H}(r,q^2)$; see [16].

Quasi-Hermitian varieties $\mathcal{V}$ in $\mathcal{H}(r,q^2)$ are generalizations of the non-singular Hermitian variety $\mathcal{H}(r,q^2)$ so that $\mathcal{V}$ and $\mathcal{H}(r,q^2)$ have the same size and the same intersection numbers with hyperplanes.

Obviously, a Hermitian variety $\mathcal{H}(r,q^2)$ can be viewed as a trivial quasi–Hermitian variety and we call $\mathcal{H}(r,q^2)$ the *classical quasi–Hermitian variety of* $\mathcal{H}(r,q^2)$. In the 2-dimensional case, $\mathcal{H}(r,q^2)$ is also known as the classical example of a *unital* of the projective plane $\mathcal{H}(2,q^2)$.

As far as we know, the only known non-classical quasi-Hermitian varieties of $\mathcal{H}(r,q^2)$ were constructed in [1, 2, 8, 9, 14, 15].

In [6], it is shown that a unital in $\mathcal{H}(2,q^2)$ is a Hermitian curve if and only if it is in the $\mathbb{F}_p$-code spanned by the lines of $\mathcal{H}(2,q^2)$, with $q=p^h$, $p$ prime and $h\in\mathbb{N}$.

In this article, we prove the following result.

**Theorem 1.1.** *A quasi-Hermitian variety $\mathcal{V}$ of $\mathcal{H}(r,q^2)$, with $r=3$ and $q=p^h\neq 4$, $p$ prime, or $r\geqslant 4$, $q=p\geqslant 5$, or $r\geqslant 4$, $q=p^2$, $p\neq 2$ prime, is classical if and only if it is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\mathcal{H}(r,q^2)$.*

Furthermore we consider *singular quasi-Hermitian varieties*, that is point sets having the same number of points as a singular Hermitian variety $\mathcal{S}$ and for which the intersection numbers with respect to hyperplanes are also the intersection numbers of $\mathcal{S}$ with respect to hyperplanes. We show that Theorem 1.1 also holds in the case in which $\mathcal{V}$ is assumed to be a singular quasi-Hermitian variety of $\mathcal{H}(r,q^2)$.

## 2 Preliminaries

A subset $\mathcal{K}$ of $\mathcal{H}(r,q^2)$ is a $k_{n,r,q^2}$ if $n$ is a fixed integer, with $1\leqslant n\leqslant q^2$, such that:

(i) $|\mathcal{K}|=k$;

(ii) $|\ell\cap\mathcal{K}|=1$, $n$, or $q^2+1$ for each line $\ell$;

(iii) $|\ell\cap\mathcal{K}|=n$ for some line $\ell$.

A point $P$ of $\mathcal{K}$ is *singular* if every line through $P$ is either a unisecant or a line of $\mathcal{K}$. The set $\mathcal{K}$ is called *singular* or *non-singular* according as it has singular points or not.

Furthermore, a subset $\mathcal{K}$ of $\mathcal{H}(r,q^2)$ is called *regular* if

(a) $\mathcal{K}$ is a $k_{n,r,q^2}$;

(b) $3 \leqslant n \leqslant q^2 - 1$;

(c) no planar section of $\mathcal{K}$ is the complement of a set of type $(0, q^2 + 1 - n)$.

**Theorem 2.1.** *[10, Theorem 19.5.13] Let $\mathcal{K}$ be a $k_{n,3,q^2}$ in $\mathcal{H}(3, q^2)$, where $q$ is any prime power and $n \neq \frac{1}{2}q^2 + 1$. Suppose furthermore that every point in $\mathcal{K}$ lies on at least one $n$-secant. Then $n = q + 1$ and $\mathcal{K}$ is a non-singular Hermitian surface.*

**Theorem 2.2.** *[12, Theorem 23.5.19] If $\mathcal{K}$ is a regular, non-singular $k_{n,r,q^2}$, with $r \geqslant 4$ and $q > 2$, then $\mathcal{K}$ is a non-singular Hermitian variety.*

**Theorem 2.3.** *[12, Th. 23.5.1] If $\mathcal{K}$ is a singular $k_{n,3,q^2}$ in $\mathcal{H}(3, q^2)$, with $3 \leqslant n \leqslant q^2 - 1$, $q > 2$, then the following holds: $\mathcal{K}$ is $n$ planes through a line or a cone with vertex a point and base $\mathcal{K}'$ a plane section of type*

I. *a unital;*

II. *a subplane $\mathcal{H}(2, q)$;*

III. *a set of type $(0, n-1)$ plus an external line;*

IV. *the complement of a set of type $(0, q^2 + 1 - n)$.*

**Theorem 2.4.** *[12, Lemma 23.5.2 and Th. 25.5.3] If $\mathcal{K}$ is a singular $k_{n,r,q}$ with $r \geqslant 4$, then the singular points of $\mathcal{K}$ form a subspace $\Pi_d$ of dimension $d$ and one of the following possibilities holds:*

1. *$d = r - 1$ and $\mathcal{K}$ is a hyperplane;*
2. *$d = r - 2$ and $\mathcal{K}$ consists of $n > 1$ hyperplanes through $\Pi_d$;*
3. *$d \leqslant r - 3$ and $\mathcal{K}$ is equal to a cone $\Pi_d \mathcal{K}'$, with $\pi_d$ as vertex and with $\mathcal{K}$ as base, where $\mathcal{K}'$ is a non singular $k_{n,r-d-1,q}$.*

A *multiset* in $\mathcal{H}(r, q)$ is a set in which multiple instances of the elements are allowed.

**Result 2.5.** *[17, Remark 2.4 and Lemma 2.5] Let $\mathcal{M}$ be a multiset in $\mathcal{H}(2, q)$, $17 < q$, $q = p^h$, where $p$ is prime. Assume that the number of lines intersecting $\mathcal{M}$ in not $k \pmod p$ points is $\delta$. Then, the number $s$ of non $k \pmod p$ secants through any point of $\mathcal{M}$ satisfies $qs - s(s-1) \leqslant \delta$. In particular, if $\delta < \frac{3}{16}(q+1)^2$, then the number of non $k \pmod p$ secants through any point is at most $\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^3}$ or at least $q + 1 - (\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^3})$.*

**Property 2.6** ([17])**.** *Let $\mathcal{M}$ be a multiset in $\mathcal{H}(2, q)$, $q = p^h$, where $p$ is prime. Assume that there are $\delta$ lines that intersect $\mathcal{M}$ in not $k \pmod p$ points. If through a point there are more than $q/2$ lines intersecting $\mathcal{M}$ in not $k \pmod p$ points, then there exists a value $r$ such that the intersection multiplicity of at least $2\frac{\delta}{q+1} + 5$ of these lines with $\mathcal{M}$ is $r$.*

**Result 2.7** ([17]). *Let $\mathcal{M}$ be a multiset in $\mathcal{H}(2,q)$, $17 < q$, $q = p^h$, where $p$ is prime. Assume that the number of lines intersecting $\mathcal{M}$ in not $k \pmod p$ points is $\delta$, where $\delta < (\lfloor\sqrt{q}\rfloor + 1)(q + 1 - \lfloor\sqrt{q}\rfloor)$. Assume furthermore that Property 2.6 holds. Then there exists a multiset $\mathcal{M}'$ with the property that it intersects every line in $k \pmod p$ points and the number of different points in $(\mathcal{M} \cup \mathcal{M}') \setminus (\mathcal{M} \cap \mathcal{M}')$ is exactly $\lceil \frac{\delta}{q+1} \rceil$.*

**Result 2.8** ([17]). *Let $B$ be a proper point set in $\mathcal{H}(2,q)$, $17 < q$. Suppose that $B$ is a codeword of the lines of $\mathcal{H}(2,q)$. Assume also that $|B| < (\lfloor\sqrt{q}\rfloor + 1)(q + 1 - \lfloor\sqrt{q}\rfloor)$. Then $B$ is a linear combination of at most $\lceil \frac{|B|}{q+1} \rceil$ lines.* □

## 3 Proof of Theorem 1.1

Let $V$ be the vector space of dimension $q^{2r} + q^{2(r-1)} + \cdots + q^2 + 1$ over the prime field $\mathbb{F}_p$, where the coordinate positions for the vectors in $V$ correspond to the points of $\mathcal{H}(r,q^2)$ in some fixed order. If $S$ is a subset of points in $\mathcal{H}(r,q^2)$, then let $v^S$ denote the vector in $V$ with coordinate 1 in the positions corresponding to the points in $S$ and with coordinate 0 in all other positions; that is $v^S$ is the *characteristic vector* of $S$. Let $\mathcal{C}_p$ denote the subspace of $V$ spanned by the characteristic vectors of all the hyperplanes in $\mathcal{H}(r,q^2)$. This code $C_p$ is called *the linear code of* $\mathcal{H}(r,q^2)$.

From [13, Theorem 1], we know that the characteristic vector $v^{\mathcal{V}}$ of a Hermitian variety $\mathcal{V} \in \mathcal{H}(r,q^2)$ is in $\mathcal{C}_p$. So from now on, we will assume that $\mathcal{V}$ is a quasi-Hermitian variety in $\mathcal{H}(r,q^2)$ and $v^{\mathcal{V}} \in \mathcal{C}_p$. In the remainder of this section, we will show that $\mathcal{V}$ is a classical Hermitian variety for specific values of $q$.

The next lemmas hold for $r \geqslant 3$ and for any $q = p^h$, $p$ prime, $h \geqslant 1$.

**Lemma 3.1.** *Every line of $\mathcal{H}(r,q^2)$, $q = p^h$, $p$ prime, $h \geqslant 1$, meets $\mathcal{V}$ in $1 \pmod p$ points.*

*Proof.* We may express
$$v^{\mathcal{V}} = v^{H_1} + \cdots + v^{H_t},$$
where $H_1, \ldots, H_t$ are (not necessarily distinct) hyperplanes of $\mathcal{H}(r,q^2)$. Denote by $\cdot$ the usual dot product. We get $v^{\mathcal{V}} \cdot v^{\mathcal{V}} = |\mathcal{V}| \equiv 1 \pmod p$. On the other hand,
$$v^{\mathcal{V}} \cdot v^{\mathcal{V}} = v^{\mathcal{V}} \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod p,$$
since every hyperplane of $\mathcal{H}(r,q^2)$ meets $\mathcal{V}$ in $1 \pmod p$ points. Hence, we have $t \equiv 1 \pmod p$. Finally, for a line $\ell$ of $\mathcal{H}(r,q^2)$,
$$v^{\ell} \cdot v^{\mathcal{V}} = v^{\ell} \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod p,$$
as every line of $\mathcal{H}(r,q^2)$ meets a hyperplane in 1 or $q^2 + 1$ points. That is, $|\ell \cap \mathcal{V}| \equiv 1 \pmod p$ and in particular no lines of $\mathcal{H}(r,q^2)$ are external to $\mathcal{V}$. □

**Remark 3.2.** The preceding proof also shows that $\mathcal{V}$ is a linear combination of $1 \pmod p$ (not necessarily distinct) hyperplanes, all having coefficient one.

**Lemma 3.3.** *For every hyperplane $H$ of $\mathcal{H}(r, q^2)$, $q = p^h$, $p$ prime, $h \geqslant 1$, the intersection $H \cap \mathcal{V}$ is in the code of points and hyperplanes of $H$ itself.*

*Proof.* Let $\Sigma$ denote the set of all hyperplanes of $\mathcal{H}(r, q^2)$. By assumption,

$$v^{\mathcal{V}} = \sum_{H_i \in \Sigma} \lambda_i v^{H_i}. \tag{1}$$

For every $H \in \Sigma$, let $\pi$ denote a hyperplane of $H$; then $\pi = H_{j_1} \cap \cdots \cap H_{j_{q^2+1}}$, where $H_{j_1}, \ldots, H_{j_{q^2+1}}$ are the hyperplanes of $\mathcal{H}(r, q^2)$ through $\pi$. We assume $H = H_{j_{q^2+1}}$. For every hyperplane $\pi$ of $H$, we set

$$\lambda_\pi = \sum_{k=1,\ldots,q^2+1} \lambda_{j_k},$$

where $\lambda_{j_k}$ is the coefficient in (1) of $v^{H_{j_k}}$ and $H_{j_k}$ is one of the $q^2 + 1$ hyperplanes through $\pi$.

Now, consider

$$T = \sum_{\pi \in \Sigma'} \lambda_\pi v^\pi, \tag{2}$$

where $\Sigma'$ is the set of all hyperplanes in $H$. We are going to show that

$$T = v^{\mathcal{V} \cap H}.$$

In fact, it is clear that at the positions belonging to the points outside of $H$ we see zeros. At a position belonging to a point in $H$, we see the original coefficients of $v^{\mathcal{V}}$ plus $(|\Sigma'| - 1)\lambda_{j_{q^2+1}}$. Note that this last term is $0 \pmod p$, hence $T = v^{\mathcal{V} \cap H}$. $\square$

**Corollary 3.4.** *For every subspace $S$ of $\mathcal{H}(r, q^2)$, $q = p^h$, $p$ prime, $h \geqslant 1$, the intersection $S \cap \mathcal{V}$ is in the code of points and hyperplanes of $S$ itself.* $\square$

**Remark 3.5.** Lemma 3.3 and Corollary 3.4 are valid for $\mathcal{V}$ any set of points in $\mathcal{H}(r, q^2)$ whose incidence vector belongs to the code of points and hyperplanes of $\mathcal{H}(r, q^2)$. In particular, it follows that for every plane $\pi$ the intersection $\pi \cap \mathcal{V}$ is a codeword of the points and lines of $\pi$, $\pi \cap \mathcal{V}$ has size $1 \pmod p$ and so it is a linear combination of $1 \pmod p$ not necessarily distinct lines.

**Lemma 3.6.** *Let $\ell$ be a line of $\mathcal{H}(r, q^2)$. Then there exists at least one plane through $\ell$ meeting $\mathcal{V}$ in $\delta$ points, with $\delta \leqslant q^3 + q^2 + q + 1$.*

*Proof.* By way of contradiction, assume that all planes through $\ell$ meet $\mathcal{V}$ in more than $q^3 + q^2 + q + 1$ points. Set $x = |\ell \cap \mathcal{V}|$. We get

$$\frac{(q^{r+1} + (-1)^r)(q^r - (-1)^r)}{q^2 - 1} > m(q^3 + q^2 + q + 1 - x) + x, \tag{3}$$

where $m = q^{2(r-2)} + q^{2(r-3)} + \cdots + q^2 + 1$ is the number of planes in $\mathcal{H}(r, q^2)$ through $\ell$. From (3), we obtain $x > q^2 + 1$, a contradiction. $\square$

**Lemma 3.7.** *For each line $\ell$ of $\mathcal{H}(r,q^2)$, $q>4$ and $q=p^h$, $p$ odd prime, $h\geqslant 1$, either $|\ell\cap\mathcal{V}|\leqslant q+1$ or $|\ell\cap\mathcal{V}|\geqslant q^2-q+1$.*

*Proof.* Let $\ell$ be a line of $\mathcal{H}(r,q^2)$ and let $\pi$ be a plane through $\ell$ such that $|\pi\cap\mathcal{V}|\leqslant q^3+q^2+q+1$; Lemma 3.6 shows that such a plane exists. Set $B=\pi\cap\mathcal{V}$. By Corollary 3.4, $B$ is a codeword of the code of the lines of $\pi$, so we can write it as a linear combination of some lines of $\pi$, that is $\sum_i \lambda_i v^{e_i}$, where $v^{e_i}$ are the characteristic vectors of the lines $e_i$ in $\pi$.

Let $B^*$ be the multiset consisting of the lines $e_i$, with multiplicity $\lambda_i$, in the dual plane of $\pi$. The weight of the codeword $B$ is at most $q^3+q^2+q+1$, hence in the dual plane this is the number of lines intersecting $B^*$ in not $0 \pmod p$ points. Actually, as $B$ is a proper set, we know that each non $0 \pmod p$ secant of $B^*$ must be a $1 \pmod p$ secant. Using Result 2.5, with $\delta=q^3+q^2+q+1$, in $\mathcal{H}(2,q^2)$, the number of non $0 \pmod p$ secants through any point is at most

$$\frac{\delta}{q^2+1}+2\frac{\delta^2}{(q^2+1)^3}=q+1+2\frac{(q+1)^2}{q^2+1}<q+4$$

or at least

$$q^2+1-\left(\frac{\delta}{q^2+1}+2\frac{\delta^2}{(q^2+1)^3}\right)>q^2-q-3.$$

In the original plane $\pi$, this means that each line intersects $B$ in either at most $q+3$ or in at least $q^2-q-2$ points. Since such lines must be $1 \pmod p$ secants and $p>2$, then each line intersects $B$ in either at most $q+1$ or in at least $q^2-q+1$ points. $\square$

**Proposition 3.8.** *Assume that $\pi$ is a plane of $\mathcal{H}(r,q^2)$, $q>4$, and $q=p^h$, $p$ odd prime, $h\geqslant 1$, such that $|\pi\cap\mathcal{V}|\leqslant q^3+2q^2$. Furthermore, suppose also that there exists a line $\ell$ meeting $\pi\cap\mathcal{V}$ in at least $q^2-q+1$ points, when $q^3+1\leqslant|\pi\cap\mathcal{V}|$. Then $\pi\cap\mathcal{V}$ is a linear combination of at most $q+1$ lines, each with weight $1$.*

*Proof.* Let $B$ be the point set $\pi\cap\mathcal{V}$. By Corollary 3.4, $B$ is the corresponding point set of a codeword $c$ of lines of $\pi$, that is $c=\sum_i\lambda_i v^{e_i}$, where lines of $\pi$ are denoted by $e_i$. Let $C^*$ be the multiset in the dual plane containing the dual of each line $e_i$ with multiplicity $\lambda_i$. Clearly the number of lines intersecting $C^*$ in not $0 \pmod p$ points is $w(c)=|B|$. Note also, that every line that is not a $0 \pmod p$ secant is a $1 \pmod p$ secant, as $B$ is a proper point set.

Our very first aim is to show that $c$ is a linear combination of at most $q+3$ different lines. When $|B|<q^3+1$, then, by Result 2.8, it is a linear combination of at most $q$ different lines.

Next assume that $|B|\geqslant q^3+1$. From the assumption of the proposition, we know that there exists a line $\ell$ meeting $\pi\cap\mathcal{V}$ in at least $q^2-q+1$ points and from Lemma 3.7, we also know that each line intersects $B$ in either at most $q+1$ or in at least $q^2-q+1$ points. Hence, if we add the line $\ell$ to $c$ with multiplicity $-1$, we reduce the weight by at least $q^2-q+1-q$ and by at most $q^2+1$. If $w(c-v^\ell)<q^3+1$, then from the above we know that $c-v^\ell$ is a linear combination of $\lceil\frac{w(c-v^\ell)}{q^2+1}\rceil$ lines. Hence, $c$ is a linear

combination of at most $q+1$ lines. If $w(c-v^{\ell}) \geqslant q^3+1$, then $w(c) \geqslant q^3+q^2-2q-2$ (see above) and so it follows that through any point of $B$, there passes at least one line intersecting $B$ in at least $q^2-q+1$ points. This means that we easily find three lines $\ell_1$, $\ell_2$, and $\ell_3$ intersecting $B$ in at least $q^2-q+1$ points. Since $w(c) \leqslant q^3+2q^2$, we get that $w(c-v^{\ell_1}-v^{\ell_2}-v^{\ell_3}) \leqslant q^3+2q^2-3\cdot(q^2-2q-2) < q^3+1$. Hence, similarly as before, we get that $c$ is a linear combination of at most $q+3$ lines.

Next we show that each line in the linear combination (that constructs $c$) has weight 1. Take a line $\ell$ which is in the linear combination with coefficient $\lambda \neq 0$. Then there are at least $q^2+1-(q+2)$ positions, such that the corresponding point is in $\ell$ and the value at that position is $\lambda$. As $B$ is a proper set, this yields that $\lambda = 1$. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of $c$ must be 1 (mod $p$); hence it can be at most $q+1$. $\square$

**Proposition 3.9.** *Assume that $\pi$ is a plane of $\mathcal{H}(r,q^2)$, $q>4$, and $q=p^h$, $p$ odd prime, $h \geqslant 1$, such that $|\pi \cap \mathcal{V}| \leqslant q^3+2q^2$. Furthermore, suppose that every line meets $\pi \cap \mathcal{V}$ in at most $q+1$ points. Then $\pi \cap \mathcal{V}$ is a classical unital.*

*Proof.* Again let $B = \pi \cap \mathcal{V}$ and first assume that $|B| < q^3+1$. Proposition 3.8 shows that $B$ is a linear combination of at most $q+1$ lines, each with weight 1. But this yields that these lines intersect $B$ in at least $q^2+1-q$ points. So this case cannot occur.

Hence, $q^3+1 \leqslant |B| \leqslant q^3+2q^2$. We are going to prove that there exists at least a tangent line to $B$ in $\pi$. Let $t_i$ be the number of lines meeting $B$ in $i$ points. Set $x = |B|$. Then double counting arguments give the following equations for the integers $t_i$.

$$\begin{cases} \sum_{i=1}^{q+1} t_i = q^4+q^2+1 \\ \sum_{i=1}^{q+1} it_i = x(q^2+1) \\ \sum_{i=1}^{q+1} i(i-1)t_i = x(x-1). \end{cases} \tag{4}$$

Consider $f(x) = \sum_{i=1}^{q+1}(i-2)(q+1-i)t_i$. From (4), we get

$$f(x) = -x^2 + x[(q^2+1)(q+2)+1] - 2(q+1)(q^4+q^2+1).$$

Since $f(q^3/2) > 0$, whereas $f(q^3+1) < 0$ and $f(q^3+2q^2) < 0$, it follows that if $q^3+1 \leqslant x \leqslant q^3+2q^2$, then $f(x) < 0$ and thus $t_1$ must be different from zero. Consider now the quantity $\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i$. We have that

$$\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i = f(x) + \sum_{i=1}^{q+1}(q+1-i)t_i = f(x) + (q+1)\sum_{i=1}^{q+1} t_i - \sum_{i=1}^{q+1} it_i$$

$$= f(x)+(q+1)(q^4+q^2+1)-x(q^2+1) = -x^2+x[(q^2+1)(q+1)+1]-(q+1)(q^4+q^2+1).$$

Since $\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i \geqslant 0$, we have that $x \leqslant \frac{(q^2+1)(q+1)+1+(q^3-q^2-q)}{2} = q^3+1$. Therefore, $x = q^3+1$ and $\sum_{i=1}^{q+1}(i-1)(q+1-i)t_i = 0$.

Since $(i-1)(q+1-i) > 0$, for $2 \leqslant i \leqslant q$, we obtain $t_2 = t_3 = \cdots = t_q = 0$, that is, $B$ is a set of $q^3+1$ points such that each line is a 1-secant or a $(q+1)$-secant of $B$. Namely, $B$ is a unital and precisely a classical unital since $B$ is a codeword of $\pi$ [6]. $\square$

The above two propositions and Lemma 3.7 imply the following corollary.

**Corollary 3.10.** *Assume that $\pi$ is a plane of $\mathcal{H}(r, q^2), q > 4$ and $q = p^h$, $p$ odd prime, $h \geqslant 1$, such that $|\pi \cap \mathcal{V}| \leqslant q^3 + 2q^2$. Then $\pi \cap \mathcal{V}$ is a linear combination of at most $q+1$ lines, each with weight 1, or it is a classical unital.* $\square$

**Corollary 3.11.** *Suppose that $\pi$ is a plane of $\mathcal{H}(r, q^2), q > 4$ and $q = p^h$, $p$ odd prime, $h \geqslant 1$, containing exactly $q^3+1$ points of $\mathcal{V}$. Then $\pi \cap \mathcal{V}$ is a classical unital.*

*Proof.* Let $B$ be the point set $\pi \cap \mathcal{V}$. We know that $B$ is the support of a codeword of lines of $\pi$. By Proposition 3.8, if there is a line intersecting $B$ in at least $q^2-q+1$ points, then $B$ is a linear combination of at most $q+1$ lines, each with multiplicity 1. First of all note that a codeword that is a linear combination of $q+1$ lines has weight at least $(q^2+1)(q+1) - 2\binom{q+1}{2}$, that is exactly $q^3+1$. In fact, in a linear combination of $q+1$ lines the minimum number of points is obtained if there is a hole at the intersection of any two lines. There are $\binom{q+1}{2}$ intersections and each intersection is counted twice, therefore we have to subtract $2\binom{q+1}{2}$. To achieve this, we need that the intersection points of any two lines from such a linear combination are all different and the sum of the coefficients of any two lines is zero; which is clearly not the case (as all the coefficients are 1). From Remark 3.5, in this case $B$ would be a linear combination of at most $q+1-p$ lines and so its weight would be less than $q^3+1$, a contradiction. Hence, there is no line intersecting $B$ in at least $q^2-q+1$ points, so Proposition 3.9 finishes the proof. $\square$

### 3.1 Case $r = 3$

In $\mathcal{H}(3, q^2)$, each plane intersects $\mathcal{V}$ in either $q^3+1$ or $q^3+q^2+1$ points since these are the intersection numbers of a quasi-Hermitian variety with a plane of $\mathcal{H}(3, q^2)$.

#### 3.1.1 $q = p$

Let $\mathcal{V}$ be a quasi-Hermitian variety of $\mathcal{H}(3, p^2)$, $p$ prime.

**Lemma 3.12.** *Every plane $\pi$ of $\mathcal{H}(3, p^2)$ sharing $p^3+1$ points with $\mathcal{V}$ intersects $\mathcal{V}$ in a unital of $\pi$.*

*Proof.* Set $U = \pi \cap \mathcal{V}$. Let $P$ be a point in $U$. Assume that every line $\ell$ in $\pi$ through the point $P$ meets $U$ in at least $p+1$ points. We get $|\pi \cap \mathcal{V}| = p^3+1 \geqslant (p^2+1)p+1 = p^3+p+1$, which is impossible.

Thus, $P$ lies on at least one tangent line to $U$ and this implies that $U$ is a minimal blocking set in $\pi$ of size $p^3+1$. From a result obtained by Bruen and Thas, see [7], it follows that $U$ is a unital of $\pi$ and hence every line in $\pi$ meets $U$ in either 1 or $p+1$ points. $\square$

**Lemma 3.13.** *Let $\pi$ be a plane in $\mathcal{H}(3,p^2)$ such that $|\pi \cap \mathcal{V}| = p^3 + p^2 + 1$, then every line in $\pi$ meets $\pi \cap \mathcal{V}$ in either $1$ or $p+1$ or $p^2+1$ points.*

*Proof.* Set $C = \pi \cap \mathcal{V}$ and Let $m$ be a line in $\pi$ such that $|m \cap C| = s$ with $s \neq 1$ and $s \neq p+1$. Thus, from Lemma 3.12, every plane through $m$ has to meet $\mathcal{V}$ in $p^3 + p^2 + 1$ points and thus

$$|\mathcal{V}| = (p^2+1)(p^3+p^2+1-s)+s,$$

which gives $s = p^2 + 1$. □

From Lemmas 3.12 and 3.13, it follows that every line in $\mathcal{H}(3,p^2)$ meets $\mathcal{V}$ in either 1 or, $p+1$ or, $p^2+1$ points.

### 3.1.2 $q = p^h$, $q \geqslant 5$ odd

Let $\mathcal{V}$ be a quasi-Hermitian variety of $\mathcal{H}(3,q^2)$, $q \geqslant 5$ odd.

**Lemma 3.14.** *Let $\pi$ be a plane in $\mathcal{H}(3,q^2)$ such that $|\pi \cap \mathcal{V}| = q^3 + q^2 + 1$, then every line in $\pi$ meets $\pi \cap \mathcal{V}$ in either $1$, $q+1$ or $q^2+1$ points.*

*Proof.* Set $C = \pi \cap \mathcal{V}$ and let $m$ be a line in $\pi$ such that $|m \cap C| = s$, with $s \neq 1$ and $s \neq q+1$. Thus, from Corollary 3.11, every plane through $m$ has to meet $\mathcal{V}$ in $q^3+q^2+1$ points and thus

$$|\mathcal{V}| = (q^2+1)(q^3+q^2+1-s)+s,$$

which gives $s = q^2 + 1$. □

From Corollary 3.11 and Lemma 3.14, it follows that every line in $\mathcal{H}(3,q^2)$ meets $\mathcal{V}$ in either 1, $q+1$, or $q^2+1$ points.

### 3.1.3 $q = 2^h$, $h > 2$

Let $\mathcal{V}$ be a quasi-Hermitian variety of $\mathcal{H}(3,2^{2h})$, $h > 2$.

**Lemma 3.15.** *For each line $\ell$ of $\mathcal{H}(3,2^{2h})$, $h > 2$, either $|\ell \cap \mathcal{V}| \leqslant q+1$ or $|\ell \cap \mathcal{V}| \geqslant q^2 - q - 1$.*

*Proof.* Let $\ell$ be a line of $\mathcal{H}(3,2^{2h})$. Since $\ell$ is at least a tangent to $\mathcal{V}$, there exists a plane through $\ell$ meeting $\mathcal{V}$ in $q^3+q^2+1$ points. Let $\pi$ be a plane through $\ell$ such that $|\pi \cap \mathcal{V}| = q^3 + q^2 + 1$. Set $B = \pi \cap \mathcal{V}$. As before, by Corollary 3.4, $B$ is a codeword of the code of the lines of $\pi$, so we can write it as a linear combination of some lines of $\pi$, that is $\sum_i \lambda_i v^{e_i}$, where $v^{e_i}$ are the characteristic vectors of the lines $e_i$ in $\pi$.

Let $B^*$ be the multiset consisting of the lines $e_i$, with multiplicity $\lambda_i$, in the dual plane of $\pi$. The weight of the codeword $B$ is $q^3+q^2+1$, hence in the dual plane this is the number of lines intersecting $B^*$ in not $0 \pmod p$ points. Actually, as $B$ is a proper set, we know that each non $0 \pmod p$ secant of $B^*$ must be a $1 \pmod p$ secant. Using Result 2.5, with $\delta = q^3+q^2+1$ in $\mathcal{H}(2,2^{2h})$, the number $s$ of non $0 \pmod p$ secants through any point of $B$ satisfies the inequality $s^2-(q^2+1)s-(q^3+q^2+1) \geqslant 0$. Since the determinant, $(q^2+1)^2+4(q^3+q^2+1) > ((q^2+1)-2(q+3))^2$, we get $s < q+3$ or $s > q^2-q-2$

In the original plane $\pi$, this means that each line intersects $B$ in either at most $q+2$ or in at least $q^2-q-1$ points. Since such lines must be $1 \pmod p$ secants and $p=2$, then each line intersects $B$ in either at most $q+1$ or in at least $q^2-q-1$ points. $\square$

Let $\alpha$ be a plane meeting $\mathcal{V}$ in a point set $B'$ of size $q^3+q^2+1$ points. We want to prove that $\alpha$ contains some $s$-secant, with $s$ at least $q^2-q-1$. Assume on the contrary that each line in $\alpha$ meets $\mathcal{V}$ in at most $q+1$ points. Let $P$ be a point of $B'$ and consider the $q^2+1$ lines through $P$. We get $q^3+q^2+1 \leqslant (q^2+1)q+1$, a contradiction. Therefore, there exists a line $\ell$ in $\alpha$ meeting $B'$ in at least $q^2-q-1$ points. We are going to show that $B'$ is a linear combination of exactly $q+1$ lines each with weight 1.

Again, by Corollary 3.4, $B'$ is the corresponding point set of a codeword $c'$ of lines of $\pi$, that is $c' = \sum_i \lambda_i v^{e_i}$, where lines of $\pi$ are denoted by $e_i$. Let $C'^*$ be the multiset in the dual plane containing the dual of each line $e_i$ with multiplicity $\lambda_i$. As before, the number of lines intersecting $C'^*$ in not $0 \pmod p$ points is $w(c') = |B'| = q^3+q^2+1$ and every line that is not a $0 \pmod p$ secant is a $1 \pmod p$ secant, as $B'$ is a proper point set.

Hence, if we add the line $\ell$ to $c$ with multiplicity 1, we reduce the weight by at least $q^2-q-1-q-2 = q^2-2q-3$ and at most by $q^2+1$. Now, through any point of $B'$, there passes at least one line intersecting $B'$ in at least $q^2-q-1$ points. Thus, we easily find two lines $\ell_1$ and $\ell_2$ intersecting $B$ in at least $q^2-q-1$ points. We get that $w(c'-v^{\ell_1}-v^{\ell_2}) < q^3+1$. Hence, similarly as before, we get that $c'-v^{\ell_1}-v^{\ell_2}$ is a linear combination of $\lceil \frac{w(c'-v^{\ell_1}-v^{\ell_2})}{q^2+1} \rceil$ lines. Hence, $c'$ is a linear combination of at most $q+2$ lines. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of $c'$ must be $1 \pmod p$; hence, as $p=2$ it can be at most $q+1$. For $p=2$, a codeword that is a linear combination of at most $q+1$ lines each with weight 1, has weight at most $q^3+q^2+1$ and this is achieved when the $q+1$ lines are concurrent. This implies that each line in $\alpha$ is either a 1 or $q+1$ or $q^2+1$-secant to $\mathcal{V}$. Now consider a line $m'$ that is an $s$-secant to $\mathcal{V}$ with $s$ different from $1, q+1$, and also different from $q^2+1$. Each plane through $m'$ has to meet $\mathcal{V}$ in $q^3+1$ points. From $|V| = q^5+q^3+q^2+1 = (q^2+1)(q^3+1-s)+s$ we get $s=0$, a contradiction.

Thus each line of $\mathcal{H}(3, 2^{2h})$ meets $\mathcal{V}$ in either 1 or $q+1$ or $q^2+1$ points.

**Proof of Theorem 1.1 (case $r=3$):** From all previous lemmas of this section, it follows that every line in $\mathcal{H}(3, q^2)$, with $q = p^h \neq 4$ and $p$ any prime, meets $\mathcal{V}$ in either 1, $q+1$, or $q^2+1$ points. Now, suppose on the contrary that there exists a singular point $P$ on $\mathcal{V}$; this means that all lines through $P$ are either tangents or $(q^2+1)$-secants to $\mathcal{V}$. Take a plane $\pi$ which does not contain $P$. Then $|\mathcal{V}| = q^2|\pi \cap \mathcal{V}|+1$ and since the two possible sizes of the planar sections are $q^3+1$ or $q^3+q^2+1$, we get a contradiction. Thus, every point in $\mathcal{V}$ lies on at least one $(q+1)$-secant and, from Theorem 2.1, we obtain that $\mathcal{V}$ is a Hermitian surface. $\square$

### 3.2 Case $r \geqslant 4$ and $q = p \geqslant 5$

We first prove the following result.

**Lemma 3.16.** *If $\pi$ is a plane of $\mathcal{H}(r,p^2)$, which is not contained in $\mathcal{V}$, then either*

$$|\pi \cap \mathcal{V}| = p^2 + 1 \text{ or } |\pi \cap \mathcal{V}| \geqslant p^3 + 1.$$

*Proof.* Let $\pi$ be a plane of $\mathcal{H}(r,p^2)$ and set $B = \pi \cap \mathcal{V}$. By Remark 3.5, $B$ is a linear combination of $1 \pmod p$ not necessarily distinct lines.

If $|B| < p^3 + 1$, then by Result 2.8, $B$ is a linear combination of at most $p$ distinct lines. This and the previous observation yield that when $|B| < p^3 + 1$, then it is the scalar multiple of one line; hence $|B| = p^2 + 1$. □

**Proposition 3.17.** *Let $\pi$ be a plane of $\mathcal{H}(r,p^2)$, such that $|\pi \cap \mathcal{V}| \leqslant p^3 + p^2 + p + 1$. Then $B = \pi \cap \mathcal{V}$ is either a classical unital or a linear combination of $p+1$ concurrent lines or just one line, each with weight 1.*

*Proof.* From Corollary 3.10, we have that $B$ is either a linear combination of at most $p+1$ lines or a classical unital. In the first case, since $B$ intersects every line in $1 \pmod p$ points and $B$ is a proper point set, the only possibilities are that $B$ is a linear combination of $p+1$ concurrent lines or just one line, each with weight 1. □

**Proof of Theorem 1.1 (case $r \geqslant 4$, $q = p$):** Consider a line $\ell$ of $\mathcal{H}(r,p^2)$ which is not contained in $\mathcal{V}$. By Lemma 3.6, there is a plane $\pi$ through $\ell$ such that $|\pi \cap \mathcal{V}| \leqslant q^3 + q^2 + q + 1$. From Proposition 3.17, we have that $\ell$ is either a unisecant or a $(p+1)$-secant of $\mathcal{V}$ and we also have that $\mathcal{V}$ has no plane section of size $(p+1)(p^2+1)$. Finally, it is easy to see like in the previous case $r = 3$, that $\mathcal{V}$ has no singular points, thus $\mathcal{V}$ turns out to be a Hermitian variety of $\mathcal{H}(r,p^2)$ (Theorem 2.2).

### 3.3 Case $r \geqslant 4$ and $q = p^2$, $p$ odd

Assume now that $\mathcal{V}$ is a quasi-Hermitian variety of $\mathcal{H}(r,p^4)$, with $r \geqslant 4$.

Lemma 3.7 states that every line contains at most $p^2 + 1$ points of $\mathcal{V}$ or at least $p^4 - p^2 + 1$ points of $\mathcal{V}$.

**Lemma 3.18.** *If $\ell$ is a line of $\mathcal{H}(r,p^4)$, such that $|\ell \cap \mathcal{V}| \geqslant p^4 - p^2 + 1$, then $|\ell \cap \mathcal{V}| \geqslant p^4 - p + 1$.*

*Proof.* Set $|\ell \cap \mathcal{V}| = p^4 - x + 1$, where $x \leqslant p^2$. It suffices to prove that $x < p+2$. Let $\pi$ be a plane through $\ell$ and $B = \pi \cap \mathcal{V}$. Choose $\pi$ such that $|B| = |\pi \cap \mathcal{V}| \leqslant p^6 + p^4 + p^2 + 1$ (Lemma 3.6). Then, by Proposition 3.8, $B$ is a linear combination of at most $p^2 + 1$ lines, each with weight 1. Let $c$ be the codeword corresponding to $B$. We observe that $\ell$ must be one of the lines of $c$, otherwise $|B \cap \ell| \leqslant p^2 + 1$, which is impossible. Thus if $P$ is a point in $\ell \setminus B$, then through $P$ there pass at least $p-1$ other lines of $c$. If $x \geqslant p+2$, then the number of lines necessary to define the codeword $c$ would be at least $(p+2)(p-1)+1$, a contradiction. □

**Lemma 3.19.** *For each plane $\pi$ of $\mathcal{H}(r,p^4)$, either $|\pi\cap\mathcal{V}|\leqslant p^6+2p^4-p^2-p+1$ or $|\pi\cap\mathcal{V}|\geqslant p^8-p^5+p^4-p+1$.*

*Proof.* Let $B=\pi\cap\mathcal{V}$, $x=|B|$, and let $t_i$ be the number of lines in $\pi$ meeting $B$ in $i$ points. Then, in this case, Equations (4) read

$$\begin{cases}\sum_{i=1}^{p^4+1} t_i = p^8+p^4+1\\ \sum_{i=1}^{p^4+1} it_i = x(p^4+1)\\ \sum_{i=1}^{p^4+1} i(i-1)t_i = x(x-1).\end{cases}\tag{5}$$

Set $f(x)=\sum_{i=1}^{p^4+1}(p^2+1-i)(i-(p^4-p+1))t_i$. From (5) we obtain

$$f(x)=-x^2+[(p^4+1)(p^4+p^2-p+1)+1]x-(p^8+p^4+1)(p^2+1)(p^4-p+1).$$

Because of Lemma 3.18, we get $f(x)\leqslant 0$, while $f(p^6+2p^4-p^2+1)>0$, $f(p^8-p^5+p^4-p)>0$. This finishes the proof of the lemma. □

**Lemma 3.20.** *If $\pi$ is a plane of $\mathcal{H}(r,p^4)$, such that $|\pi\cap\mathcal{V}|\geqslant p^8-p^5+p^4-p+1$, then either $\pi$ is entirely contained in $\mathcal{V}$ or $\pi\cap\mathcal{V}$ consists of $p^8-p^5+p^4+1$ points and it only contains $i$-secants, with $i\in\{1,p^4-p+1,p^4+1\}$.*

*Proof.* Set $S=\pi\setminus\mathcal{V}$. Suppose that there exists some point $P\in S$. We have the following two possibilities: either each line of the pencil with center at $P$ is a $(p^4-p+1)$-secant or only one line through $P$ is an $i$-secant, with $1\leqslant i\leqslant p^2+1$, whereas the other $p^4$ lines through $P$ are $(p^4-p+1)$-secants. In the former case, when there are no $i$-secants, $1\leqslant i\leqslant p^2+1$, each line $\ell$ in $\pi$ either is disjoint from $S$ or it meets $S$ in $p$ points since $\ell$ is a $(p^4-p+1)$-secant. This implies that $S$ is a maximal arc and this is impossible for $p\neq 2$ [4, 5].

In the latter case, we observe that the size of $\pi\cap\mathcal{V}$ must be $p^8-p^5+p^4+i$, where $1\leqslant i\leqslant p^2+1$. Next, we denote by $t_s$ the number of $s$-secants in $\pi$, where $s\in\{i,p^4-p+1,p^4+1\}$. We have that

$$\begin{cases}\sum_s t_s = p^8+p^4+1\\ \sum_s st_s = (p^4+1)(p^8-p^5+p^4+i)\\ \sum_s s(s-1)t_s = (p^8-p^5+p^4+i)(p^8-p^5+p^4+i-1).\end{cases}\tag{6}$$

From (6) we get

$$t_i=\frac{p(p^4-p-i+1)(p^5-i+1)}{p(p^4-p-i+1)(p^4-i+1)}=\frac{p^5-i+1}{p^4-i+1},\tag{7}$$

and we can see that the only possibility for $t_i$ to be an integer is $ip-p-i+1=0$, that is $i=1$. For $i=1$, we get $|B|=p^8-p^5+p^4+1$. □

**Lemma 3.21.** *If $\pi$ is a plane of $\mathcal{H}(r, p^4)$, not contained in $\mathcal{V}$ and which does not contain any $(p^4 - p + 1)$-secant, then $\pi \cap \mathcal{V}$ is either a classical unital or the union of $i$ concurrent lines, with $1 \leqslant i \leqslant p^2 + 1$.*

*Proof.* Because of Lemmas 3.18, 3.19 and 3.20, the plane $\pi$ meets $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points. Furthermore, each line of $\pi$ which is not contained in $\mathcal{V}$ is an $i$-secant, with $1 \leqslant i \leqslant p^2 + 1$ (Lemma 3.18 and the sentence preceding Lemma 3.18). Set $B = \pi \cap \mathcal{V}$. If in $\pi$ there are no $(p^4 + 1)$-secants to $B$, then $|B| \leqslant p^6 + p^2 + 1$ and by Proposition 3.9 it follows that $B$ is a classical unital.

If there is a $(p^4 + 1)$-secant to $B$ in $\pi$, then arguing as in the proof of Proposition 3.8, we get that $B$ is still a linear combination of $m$ lines, with $m \leqslant p^2 + 1$. Each of these $m$ lines is a $(p^4 + 1)$-secant to $\mathcal{V}$. In fact if one of these lines, say $v$, was an $s$-secant, with $1 \leqslant s \leqslant p^2 + 1$, then through each point $P \in v \setminus B$, there would pass at least $p$ lines of the codeword corresponding to $B$ and hence $B$ would be a linear combination of at least $(p^4 + 1 - s)(p - 1) + 1 > p^2 + 1$ lines, which is impossible.

We are going to prove that these $m$ lines, say $\ell_1, \ldots, \ell_m$, are concurrent. Assume on the contrary that they are not. We can assume that through a point $P \in \ell_n$, there pass at least $p + 1$ lines of our codeword but there is a line $\ell_j$ which does not pass through $P$. Thus through at least $p + 1$ points on $\ell_j$, there are at least $p + 1$ lines of our codeword and thus we find at least $(p + 1)p + 1 > m$ lines of $B$, a contradiction. $\square$

**Lemma 3.22.** *A plane $\pi$ of $\mathcal{H}(r, p^4)$ meeting $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points and containing a $(p^4 - p + 1)$-secant to $\mathcal{V}$ has at most $(p^2 + 1)(p^4 - p + 1)$ points of $\mathcal{V}$.*

*Proof.* Let $\ell$ be a line of $\pi$ which is a $(p^4 - p + 1)$-secant to $\mathcal{V}$. In this case, $\pi \cap \mathcal{V}$ is a linear combination of at most $p^2 + 1$ lines, each with weight 1 (Proposition 3.8). A line not in the codeword can contain at most $p^2 + 1$ points. In particular, since $\ell$ contains more than $p^4 - p^2 + 1$ points of $\mathcal{V}$, $\ell$ is a line of the codeword and hence through each of the missing points of $\ell$ there are at least $p$ lines of the codeword corresponding to $B$. On these $p$ lines we can see at most $p^4 - p + 1$ points of $\mathcal{V}$.

So let $\ell_1, \ell_2, \ldots, \ell_p$ be $p$ lines of the codeword through a point of $\ell \setminus \mathcal{V}$. Each of these lines contains at most $p^4 - p + 1$ points of $\mathcal{V}$. Thus these $p$ lines contain together at most $p(p^4 - p + 1)$ points of $\mathcal{V}$. Now take any other line of the codeword, say $e$. If $e$ goes through the common point of the lines $\ell_i$, then there is already one point missing from $e$, so adding $e$ to our set, we can add at most $p^4 - p + 1$ points. If $e$ does not go through the common point, then it intersects $\ell_i$ in $p$ different points. These points either do not belong to the set $\pi \cap \mathcal{V}$ or they belong to the set $\pi \cap \mathcal{V}$, but we have already counted them when we counted the points of $\ell_i$, so again $e$ can add at most $p^4 - p + 1$ points to the set $\pi \cap \mathcal{V}$. Thus adding the lines of the codewords one by one to $\ell_i$ and counting the number of points, each time we add only at most $p^4 - p + 1$ points to the set $\pi \cap \mathcal{V}$. Hence, the plane $\pi$ contains at most $(p^2 + 1)(p^4 - p + 1)$ points of $\mathcal{V}$. $\square$

**Lemma 3.23.** *Let $\pi$ be a plane of $\mathcal{H}(r, p^4)$, containing an $i$-secant, $1 < i < p^2 + 1$, to $\mathcal{V}$. Then $\pi \cap \mathcal{V}$ is either the union of $i$ concurrent lines or it is a linear combination of $p^2 + 1$ lines (each with weight 1) so that they form a subplane of order $p$, minus $p$ concurrent lines.*

*Proof.* By Lemmas 3.19, 3.20 and 3.21, $\pi$ meets $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points and must contain a $(p^4 - p + 1)$-secant to $\mathcal{V}$ or $\pi \cap \mathcal{V}$ is the union of $i$ concurrent lines. Hence, from now on, we assume that $\pi$ contains a $(p^4 - p + 1)$-secant. By Result 2.8, such a plane is a linear combination of at most $p^2 + 1$ lines. As before each line from the linear combination has weight 1. Note that the above two statements imply that a line of the linear combination will be either a $(p^4 + 1)$-secant or a $(p^4 - p + 1)$-secant.

As we have at most $p^2 + 1$ lines in the combination, a $(p^4 - p + 1)$-secant must be one of these lines. This also means that through each of the $p$ missing points of this line, there must pass at least $p - 1$ other lines from the linear combination. Hence, we already get $(p - 1)p + 1$ lines.

In the case in which the linear combination contains exactly $p^2 - p + 1$ lines, then from each of these lines there are exactly $p$ points missing and through each missing point there are exactly $p$ lines from the linear combination. Hence, the missing points and these lines form a projective plane of order $p - 1$, a contradiction as $p > 3$.

Therefore, as the number of the lines of the linear combination must be $1 \pmod p$ and at most $p^2 + 1$, we can assume that the linear combination contains $p^2 + 1$ lines. We are going to prove that through each point of the plane there pass either 0, 1, $p$ or $p + 1$ lines from the linear combination. From earlier arguments, we know that the number of lines through one point $P$ is 0 or 1 (mod $p$). Assume to the contrary that through $P$ there pass at least $p + 2$ of such lines. These $p^2 + 1$ lines forming the linear combination are not concurrent, so there is a line $\ell$ not through $P$. Through each of the intersection points of $\ell$ and a line through $P$, there pass at least $p - 1$ more other lines of the linear combination, so in total we get at least $(p - 1)(p + 2) + 1$ lines forming the linear combination, a contradiction.

Since there are $p^2 + 1$ lines forming the linear combination and through each point of the plane there pass either 0, 1, $p$ or $p + 1$ of these lines, we obtain that on a $(p^4 - p + 1)$-secant there is exactly one point, say $P$, through which there pass exactly $p + 1$ lines from the linear combination and $p$ points, not in the quasi Hermitian variety, through each of which there pass exactly $p$ lines.

If all the $p^2 + 1$ lines forming the linear combination, were $(p^4 - p + 1)$-secants then the number of points through which there pass exactly $p$ lines would be $(p^2 + 1)p/p$. On the other hand, through $P$ there pass $p + 1$ $(p^4 - p + 1)$-secants, hence we already get $(p + 1)p$ such points, a contradiction. Thus, there exists a line $m$ of the linear combination that is a $(p^4 + 1)$-secant. From the above arguments, on this line there are exactly $p$ points through each of which there pass exactly $p + 1$ lines, whereas through the rest of the points of the line $m$ there pass no other lines of the linear combination.

Assume that there is a line $m' \neq m$ of the linear combination that is also a $(p^4 + 1)$-secant. Then there is a point $Q$ on $m'$ but not on $m$ through which there pass $p + 1$ lines. This would mean that there are at least $p + 1$ points on $m$, through which there pass more than 2 lines of the linear combination, a contradiction.

Hence, there is exactly one line $m$ of the linear combination that is a $(p^4 + 1)$-secant and all the other lines of the linear combination are $(p^4 - p + 1)$-secants. It is easy to check that the points through which there are more than 2 lines plus the $(p^4 - p + 1)$-secants form a dual affine plane. Hence our lemma follows. □

**Lemma 3.24.** *There are no $i$-secants to $\mathcal{V}$, with $1 < i < p^2 + 1$.*

*Proof.* By Lemma 3.23, if a plane $\pi$ contains an $i$-secant, $1 < i < p^2 + 1$, then $\pi \cap \mathcal{V}$ is a linear combination of either $i$ concurrent lines or lines of an embedded subplane of order $p$ minus $p$ concurrent lines. In the latter case, if $i > 1$ then an $i$-secant is at least a $(p^2 - p + 1)$-secant. In fact, since the $p^2 + 1$ lines of the linear combination do not intersect outside of $\mathcal{H}(2, p)$, each of the $p^2 + 1$ lines of the linear combination is at least a $(p^4 - p + 1)$-secant, whereas the $p$ concurrent lines are 1-secants. Also, all the other lines of $\mathcal{H}(2, p^4)$ intersect $\mathcal{H}(2, p)$ in either 1 or zero points. If they intersect $\mathcal{H}(2, p)$ in zero points they are $(p^2 + 1)$-secants to $\mathcal{V}$. If they intersect $\mathcal{H}(2, p)$ in a unique point $P$, then they are at least $(p^2 - p + 1)$-secants since $P$ lies on $p$ or $p + 1$ lines of the linear combination and each line intersects $\mathcal{V}$ in $1 \pmod p$ points.

Hence, if there is an $i$-secant with $1 < i < p^2 - p + 1$, say $\ell$, we get that for each plane $\alpha$ through $\ell$, $\alpha \cap \mathcal{V}$ is a linear combination of $i$ concurrent lines. Therefore

$$|\mathcal{V}| = m(ip^4 + 1 - i) + i, \tag{8}$$

where $m = p^{4(r-2)} + p^{4(r-3)} + \ldots + p^4 + 1$ is the number of planes in $\mathcal{H}(r, p^4)$ through $\ell$.

Setting $r = 2\sigma + \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$ according to $r$ is even or odd, we can write

$$|\mathcal{V}| = 1 + p^4 + \cdots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon+1)} + \cdots + p^{4(r-1)})p^2$$

Hence, (8) becomes

$$\begin{array}{l} 1 + p^4 + \cdots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon+1)} + \cdots + p^{4(r-1)})p^2 \\ -(p^{4(r-2)} + p^{4(r-3)} + \cdots + p^4 + 1) = ip^{4(r-1)}. \end{array} \tag{9}$$

Since $\sigma \geqslant 2$, we see that $p^{4(r-1)}$ does not divide the left hand side of (9), a contradiction.

Thus, there can only be 1-, $(p^2 - p + 1)$-, $(p^2 + 1)$-, $(p^4 - p + 1)$- or $(p^4 + 1)$-secants to $\mathcal{V}$. Now, suppose that $\ell$ is a $(p^2 - p + 1)$-secant to $\mathcal{V}$. Again by Lemma 3.23, each plane through $\ell$ either has $x = (p^2 - p + 1)p^4 + 1$ or $y = p^2(p^4 - p) + p^4 + 1$ points of $\mathcal{V}$. Next, denote by $t_j$ the number of $j$-secant planes through $\ell$ to $\mathcal{V}$. We get

$$\left\{ \begin{array}{l} t_x + t_y = m \\ t_x(x - p^2 + p - 1) + t_y(y - p^2 + p - 1) + p^2 - p + 1 = |\mathcal{V}|. \end{array} \right. \tag{10}$$

Recover the value of $t_y$ from the first equation and substitute it in the second. We obtain

$$(m - t_y)(p^6 - p^5 + p^4 - p^2 + p) + t_y(p^6 + p^4 - p^3 - p^2 + p) + p^2 - p + 1 = |\mathcal{V}|,$$

that is,

$$p^3(p^2 - 1)t_y = |\mathcal{V}| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 + p - 1.$$

The case $t_y = 0$ must be excluded, since by direct computations $|\mathcal{V}| \neq m(p^6 - p^5 + p^4 - p^2 + p) + p^2 - p + 1$. It is easy to check that $|\mathcal{V}| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 + p - 1$ is not divisible by $p + 1$ and hence, $t_y$ turns out not to be an integer, which is impossible. □

**Lemma 3.25.** *No plane meeting $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points contains a $(p^4 - p + 1)$-secant.*

*Proof.* Let $\pi$ be a plane of $\mathcal{H}(r, p^4)$ such that $|\pi \cap \mathcal{V}| \leqslant p^6 + 2p^4 - p^2 - p + 1$. It can contain only 1-, $(p^2 + 1)$-, $(p^4 - p + 1)$-, $(p^4 + 1)$-secants (Lemma 3.18 and Lemma 3.24). If $\pi \cap \mathcal{V}$ contains a $(p^4 - p + 1)$-secant, we know from Proposition 3.8 that it is a linear combination of at most $p^2 + 1$ lines, each with weight 1. Suppose that $e$ is a $(p^4 - p + 1)$-secant to $\pi \cap \mathcal{V}$. Let $P$ and $Q$ be two missing points of $e$. We know that there must be at least $p - 1$ other lines of the codeword through $P$ and $Q$. Let $f$ and $g$ be two such lines through $Q$. The number of lines of the codeword through $P$ is at most $2p$ and since the lines $f$ and $g$ are lines of the codeword through $Q$, they are $(p^4 - p + 1)$-secants. There exist at most $(p - 1)^2$ lines through $P$ distinct from $e$ intersecting $f$ or $g$ in a point not in $\mathcal{V}$. Therefore we can find a line, say $m$, of the plane through $P$, that intersects $f$ and $g$ in a point of $\mathcal{V}$ and that is not a line of the codeword. Then $|m \cap \mathcal{V}| \geqslant 1 + p$ since $|m \cap \mathcal{V}| \equiv 1 \pmod{p}$. Recall that $m$ is not a line of the codeword and the total number of lines of the codeword is at most $p^2 + 1$. Also, $m$ meets in $P$ at least $p$ lines of the codeword and therefore it can intersect all the remaining lines in at most $p^2 - p + 1$ pairwise distinct points. This means that the line $m$ satisfies $p + 1 \leqslant |m \cap \mathcal{V}| \leqslant p^2 - p + 1$, and this contradicts Lemma 3.24. □

**Lemma 3.26.** *There are no $(p^4 - p + 1)$-secants to $\mathcal{V}$.*

*Proof.* If there was a $(p^4 - p + 1)$-secant to $\mathcal{V}$, say $\ell$, then, by Lemma 3.25, all the planes through $\ell$ would contain at least $p^8 - p^5 + p^4 + 1$ points of $\mathcal{V}$, and thus

$$|\mathcal{V}| \geqslant (p^{4(r-2)} + p^{4(r-3)} + \cdots + p^4 + 1)(p^8 - p^5 + p) + p^4 - p + 1, \tag{11}$$

a contradiction. □

**Proof of Theorem 1.1 (case $r \geqslant 4$ and $q = p^2$):** Consider a line $\ell$ which is not contained in $\mathcal{V}$. From the preceding lemmas we have that $\ell$ is either a 1-secant or a $(p^2 + 1)$-secant of $\mathcal{V}$. Furthermore, $\mathcal{V}$ has no plane section of size $(p^2 + 1)(p^4 + 1)$ because of Lemma 3.21. Finally, as in the case $r = 3$, it is easy to see that $\mathcal{V}$ has no singular points, thus, by Theorem 2.2, $\mathcal{V}$ turns out to be a Hermitian variety of $\mathcal{H}(r, p^4)$.

## 4 Singular quasi-Hermitian varieties

In this section, we consider sets having the same behavior with respect to hyperplanes as singular Hermitian varieties.

**Definition 4.1.** *A* $d$-singular quasi-Hermitian variety *is a subset of points of $\mathcal{H}(r, q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension $d$.*

We prove the following result.

**Theorem 4.2.** *Let $\mathcal{S}$ be a $d$-singular quasi-Hermitian variety in $\mathcal{H}(r, q^2)$. Suppose that either*

- $r = 3$, $d = 0$, $q = p^h \neq 4$, $h \geqslant 1$, $p$ *any prime, or*
- $r \geqslant 4$, $d \leqslant r - 3$, $q = p \geqslant 5$, *or*
- $r \geqslant 4$, $d \leqslant r - 3$, $q = p^2$, $p$ *odd prime.*

*Then $\mathcal{S}$ is a singular Hermitian variety with a singular space of dimension $d$ if and only if its incidence vector is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\mathcal{H}(r, q^2)$.*

*Proof.* Let $\mathcal{S}$ be a singular Hermitian variety of $\mathcal{H}(r, q^2)$. The characteristic vector $v^{\mathcal{S}}$ of $\mathcal{S}$ is in $\mathcal{C}_p$ since [13, Theorem 1] also holds for singular Hermitian varieties. Now assume that $\mathcal{S}$ is a $d$-singular quasi-Hermitian variety. As in the non-singular case, by Lemma 3.1, each line of $\mathcal{H}(r, q^2)$ intersects $\mathcal{S}$ in $1 \pmod p$ points.

### 4.1 Case $r = 3$

Suppose that $r = 3$ and therefore $d = 0$. Then $\mathcal{S}$ has $q^5 + q^2 + 1$ points. Let $\pi$ be a plane of $\mathcal{H}(3, q^2)$. In this case, $\pi$ meets $\mathcal{S}$ in either $q^2 + 1$, or $q^3 + 1$ or $q^3 + q^2 + 1$ points. In particular, a planar section of $\mathcal{S}$ with $q^2 + 1$ points is a line since it is a blocking set with respect to lines of a projective plane.

In the case in which $q$ is a prime $p$, Lemma 3.12 and Lemma 3.13 are still valid in the singular case for $\mathcal{V} = \mathcal{S}$ and thus, the planar sections of $\mathcal{S}$ with $p^3 + 1$ or $p^3 + p^2 + 1$ points have to be unitals or pencils of $p + 1$ lines, respectively. Hence, each line of $\mathcal{H}(3, p^2)$ meets $\mathcal{S}$ in 1, $p + 1$ or $p^2 + 1$ points.

When $q \geqslant 5$ is an odd prime power, Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10 are still valid in the singular case for $r = 3$.

Thus, if $|\pi \cap \mathcal{S}| = q^2 + 1$, then Proposition 3.8 implies that $\pi \cap \mathcal{S}$ is a line of $\pi$, whereas if $|\pi \cap \mathcal{S}| = q^3 + 1$, then Corollary 3.11 gives that $\pi \cap \mathcal{S}$ is a classical unital of $\pi$. Now suppose that $|\pi \cap \mathcal{S}| = q^3 + q^2 + 1$. Let $\ell$ be a line of $\pi$ such that $|\ell \cap \mathcal{S}| = s$ with $s \neq 1, q + 1, q^2 + 1$.

Each plane through $\ell$ must meet $\mathcal{S}$ in $q^3 + q^2 + 1$ points and this gives

$$(q^3 + q^2 + 1 - s)(q^2 + 1) + s = q^5 + q^2 + 1,$$

that is, $s = q^2 + q + 1$, which is impossible.

Thus in $\mathcal{H}(3, q^2)$, where $q$ is an odd prime power, each line intersects $\mathcal{S}$ in either 1, or $q + 1$ or $q^2 + 1$ points.

Next, assume $q = 2^h$, $h > 2$. Arguing as in the corresponding non-singular case, it turns out that a $(q^3 + q^2 + 1)$-plane meets $\mathcal{S}$ in a pencil of $q + 1$ lines. Now, assume that there is an $i$-secant line to $\mathcal{S}$, say $m$, with $2 < i < q$. Then, each plane through $m$ has to be a $(q^3 + 1)$-plane of $\mathcal{S}$. Counting the number of points of $\mathcal{S}$ by using all planes through $m$ we obtain $q^5 + q^2 + 1 = q^5 + q^3 + q^2 - iq^2 + 1$, hence $i = q$, a contradiction since every line contains $1 \pmod 2$ points of $\mathcal{V}$.

Therefore, each line of $\mathcal{H}(3, 2^{2h})$, $h \neq 2$ meets $\mathcal{S}$ in 1, $q+1$ or $q^2+1$ points. Finally, $\mathcal{S}$ is a $k_{q+1,3,q^2}$ for all $q \neq 4$. Also, $\mathcal{S}$ cannot be non-singular by assumption. When $q \neq 2$, Theorem 2.3 applies and $\mathcal{S}$ turns out to be a cone $\Pi_0\mathcal{S}'$ with $\mathcal{S}'$ of type I, II, III or IV as the possible intersection sizes with planes are $q^2+1, q^3+1, q^3+q^2+1$.

Possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $\mathcal{S} = \Pi_0\mathcal{H}$, where $\mathcal{H}$ is a non-singular Hermitian curve.

For $q = 2$, there is just one point set in $\mathcal{H}(3, 4)$ up to equivalence, meeting each line in 1, 3 or 5 points and each plane in 5, 9 or 13 points, that is the Hermitian cone, see [11, Theorem 19.6.8].

## 4.2 Case $r \geqslant 4$

Let $\ell$ be a line of $\mathcal{H}(r, q^2)$ containing $x < q^2+1$ points of $\mathcal{S}$. We are going to prove that there exists at least one plane through $\ell$ containing less than $q^3+q^2+q+1$ points of $\mathcal{S}$. If we suppose that all the planes through $\ell$ contain at least $q^3+q^2+q+1$ points of $\mathcal{S}$, then

$$q^{2(d+1)}\frac{(q^{r-d}+(-1)^{r-d-1})(q^{r-d-1}-(-1)^{r-d-1})}{q^2-1}+q^{2d}+q^{2(d-1)}+\cdots+q^2+1 \geqslant$$

$$m(q^3+q^2+q+1-x)+x,$$

where $m = q^{2(r-2)}+q^{2(r-3)}+\cdots+q^2+1$ is the number of planes through $\ell$ in $\mathcal{H}(r, q^2)$. We obtain $x > q^2+1$, a contradiction.

Therefore, there exists at least one plane through $\ell$ having less than $q^3+q^2+q+1$ points of $\mathcal{S}$ and hence Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10, are still valid in this singular case for any odd $q > 4$.

Next, we are going to prove that $\mathcal{S}$ is a $k_{q+1,r,q^2}$, with $q = p^h > 4$, $h = 1, 2$.

**Case $\boldsymbol{q = p \geqslant 5}$:** Let $\ell$ be a line of $\mathcal{H}(r, p^2)$. As we have seen, there is a plane $\pi$ through $\ell$ such that $|\pi \cap \mathcal{V}| < p^3+p^2+p+1$. Proposition 3.17 is still valid in this case and thus we have that $\ell$ is either a unisecant or a $(p+1)$-secant of $\mathcal{S}$. Furthermore, we also have that $\mathcal{S}$ has no plane section of size $(p+1)(p^2+1)$ and hence $\mathcal{S}$ is a regular $k_{p+1,r,p^2}$.

**Case $\boldsymbol{q = p^2}$, $\boldsymbol{p}$ odd:** We first observe that (8) and (11) hold true in the case in which $\mathcal{V}$ is assumed to be a singular quasi-Hermitian variety. This implies that all lemmas stated in the subparagraph 3.3 are valid in our case. Thus, we obtain that $\mathcal{S}$ is a $k_{p^2+1,r,p^4}$ and it is straightforward to check that $\mathcal{S}$ is also regular.

Finally, in both cases $q = p$ or $q = p^2$, we have that $\mathcal{S}$ is a singular $k_{q+1,r,q^2}$ because if $\mathcal{S}$ were a non-singular $k_{q+1,r,q^2}$, then, from Theorem 2.2, $\mathcal{S}$ would be a non-singular Hermitian variety and this is not possible by our assumptions.

Therefore, by Theorem 2.4, the only possibility is that $\mathcal{S}$ is a cone $\Pi_d\mathcal{S}'$, with $\mathcal{S}'$ a non-singular $k_{q+1,r-d-1,q^2}$. By Lemma 3.3, $\mathcal{S}'$ belongs to the code of points and hyperplanes of $\mathcal{H}(r-d-1, q^2)$. Since $r-d-1 \geqslant 2$, then, by [6] and Theorem 1.1, $\mathcal{S}'$ is a non-singular Hermitian variety and, therefore, $\mathcal{S}$ is a singular Hermitian variety with a vertex of dimension $d$. □

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