The structure of delta-matroids with width one twists

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Abstract

The width of a delta-matroid is the difference in size between a maximal and minimal feasible set. We give a Rough Structure Theorem for delta-matroids that admit a twist of width one. We apply this theorem to give an excluded-minor characterisation of delta-matroids that admit a twist of width at most one.

Keywords: delta-matroid; excluded minor; matroid; partial dual; twist; width

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1 Introduction, results and notation

Delta-matroids are a generalisation of matroids and were introduced by A. Bouchet in [2]. Delta-matroid theory can be thought of as generalising topological graph theory in the same way that matroid theory can be thought of as generalising graph theory (see, e.g., [7]). In delta-matroid theory, feasible sets fulfill the same role as bases do in matroid theory, but feasible sets are in general not all of the same size. Notions of deletion, contraction and minors exist for delta-matroids, so there is the possibility of characterising minor-closed families of delta-matroids by their excluded minors. One of the most fundamental operations in delta-matroid theory is the twist, and a basic integer invariant of a delta-matroid is its width. We show that delta-matroids with twists of width at most k form a minor-closed family, of which we give an excluded-minor characterization in the case $k = 1$. We also examine how the structure of a delta-matroid determines the width of the delta-matroids that are in its equivalence class under twists.

Formally, a *delta-matroid* $D = (E, \mathcal{F})$ consists of a finite set E and a non-empty set $\mathcal{F}$ of subsets of E that satisfies the *Symmetric Exchange Axiom*: for all $X, Y \in \mathcal{F}$, if there is an element $u \in X \triangle Y$, then there is an element $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$. Here $X \triangle Y$ denotes the symmetric difference of sets X and Y. Note that it may be the case that $u = v$ in the Symmetric Exchange Axiom. Elements of $\mathcal{F}$ are called *feasible sets* and E is the *ground set*. We often use $\mathcal{F}(D)$ and $E(D)$ to denote the set of feasible sets and the ground set, respectively, of D. A *matroid* is a delta-matroid whose feasible sets are all of the same size. In this case the feasible sets are called *bases*. This definition of a matroid is a straightforward reformulation of the standard one in terms of bases.

In general a delta-matroid has feasible sets of different sizes. The *width* of a delta-matroid, defined by Bouchet in [4] and denoted by $w(D)$, is the difference between the sizes of its largest and smallest feasible sets: $w(D) := \max_{F \in \mathcal{F}} |F| - \min_{F \in \mathcal{F}} |F|$.

Twists, introduced by Bouchet in [2], are one of the fundamental operations of delta-matroid theory. Given a delta-matroid $D = (E, \mathcal{F})$ and some subset $A \subseteq E$, the *twist* of D with respect to A, denoted by $D * A$, is the delta-matroid given by $(E, \{A \triangle F : F \in \mathcal{F}\})$. (At times we write $D * e$ for $D * \{e\}$.) Note that the "empty twist" is $D * \emptyset = D$. The *dual* of D, written D^*, is equal to $D * E$. Moreover, in general, the twist can be thought of as a "partial dual" operation on delta-matroids.

Forming the twist of a delta-matroid usually changes the sizes of its feasible sets and its width. Here we are interested in the problem of recognising when a delta-matroid has a twist of small width. Our results are a Rough Structure Theorem for delta-matroids that have a twist of width one, and an excluded-minor characterisation of delta-matroids that have a twist of width at most one. While in this paper we are interested in the smallest width among all twists of a delta-matroid, the largest width among all twists has been investigated in, e.g., [10].

To state the Rough Structure Theorem we need the following definitions. Let $D = (E, \mathcal{F})$ be a delta-matroid and let $\mathcal{F}_{\min}$ be the set of feasible sets of minimum size. Observe that $D_{\min} := (E, \mathcal{F}_{\min})$ is a matroid. To see this suppose that F_1 and F_2 are feasible sets of minimum size, and e is an element of $F_1 - F_2$. Then $e \in F_1 \triangle F_2$ and there exists

$f \in F_1 \triangle F_2$ such that $F_1 \triangle \{e, f\} \in \mathcal{F}$. Because F_1 and F_2 are feasible sets of minimum size, $f \in F_2 - F_1$. Thus the feasible sets of minimum size obey the axioms defining the bases of a matroid. For a matroid M with ground set E, a subset A of E is said to be a *separator* of M if A is a union of components of M. Note that both $\emptyset$ and E are always separators. In terms of the matroid rank function, where the rank $r(X)$ of a set $X \subseteq E$ is defined to be the size of the largest intersection of X with a basis of M, the set A is a separator if and only if $r(A) + r(E - A) = r(M)$. Throughout the paper we use $\overline{A}$ for the complement $E - A$ of A, and $D|X$ denotes the restriction of D to $X \subseteq E$ (see the beginning of Section 2 for its definition).

We now state the first of our two main results: a Rough Structure Theorem for delta-matroids admitting a twist of width one.

Theorem 1. *Let $D = (E, \mathcal{F})$ be a delta-matroid. Then D has a twist of width one if and only if there is some $A \subseteq E$ such that*

1. *A is a separator of $D_{\min}$,*
2. *$D|A$ is a matroid, and*
3. *$D|\overline{A}$ is of width one.*

We actually prove a result that is stronger than Theorem 1. This stronger result appears below as Theorem 7 and the present theorem follows immediately from it.

As an application of Theorem 1, we find an excluded-minor characterisation of the class of delta-matroids that have a twist of width one as our second main result, Theorem 3. This class of delta-matroids is shown to be minor closed in Proposition 9, and its set of excluded minors comprises the delta-matroids in the following definition together with their twists.

Definition 2. Let D_1 denote the delta-matroid on the elements a, b with feasible sets

$$\mathcal{F}(D_1) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Let D_2 and D_3 denote the delta-matroids on the elements a, b, c with feasible sets given by

$$\mathcal{F}(D_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\},$$
$$\mathcal{F}(D_3) = \{\emptyset, \{a, b\}, \{b, c\}, \{a, c\}\}.$$

Throughout this paper D_1, D_2 and D_3 refer exclusively to these delta-matroids. Let $\mathcal{D}_{[3]}$ be the set of all twists of these delta-matroids. Note that $D_i \in \mathcal{D}_{[3]}$ for all $i \in \{1, 2, 3\}$ via the empty twist.

Theorem 3. *A delta-matroid has a twist of width at most one if and only if it has no minor isomorphic to a member of $\mathcal{D}_{[3]}$.*

The proof of this theorem appears at the end of Section 3.

A delta-matroid is *even* if the difference in size between any two feasible sets is even. We note that the excluded minors of twists of matroids (i.e., twists of width zero delta-matroids) have been shown, to be $(\{a\}, \{\emptyset, \{a\}\})$, D_3, and $D_3 * \{a\}$ by A. Duchamp [9, Corollary 4.3]. This result can be recovered from Theorem 3 by noting that a delta-matroid has a twist of width zero if and only if it is both even and has a twist of width at most one, and that a delta-matroid is even if and only if it has no minor isomorphic to $(\{a\}, \{\emptyset, \{a\}\})$. The latter result is easily recovered from Lemma 5.4 of [4].

2 The proof of the Rough Structure Theorem

We recall some standard matroid and delta-matroid terminology. Given a delta-matroid $D = (E, \mathcal{F})$ and element $e \in E$, if e is in every feasible set of D then we say that e is a *coloop* of D. If e is in no feasible set of D, then we say that e is a *loop* of D. If $e \in E$ is not a coloop, then D *delete* e, denoted by $D \setminus e$, is the delta-matroid $(E - e, \{F : F \in \mathcal{F} \text{ and } F \subseteq E - e\})$. If $e \in E$ is not a loop, then D *contract* e, denoted by D/e, is the delta-matroid $(E - e, \{F - e : F \in \mathcal{F} \text{ and } e \in F\})$. If $e \in E$ is a loop or coloop, then we define $D/e = D \setminus e$. Useful identities that we use frequently are $D/e = (D*e) \setminus e$ and $D \setminus e = (D*e)/e$. Note that deleting or contracting an element from a delta-matroid corresponds to taking a subset of its feasible sets and removing an element from either none of them or all of them, and consequently cannot increase the width. If D' is a delta-matroid obtained from D by a sequence of deletions and contractions, then D' is independent of the order of the deletions and contractions used in its construction, so we can define $D \setminus X/Y$ for disjoint subsets X and Y of E, as the result of deleting each element in X and contracting each element in Y in some order. A *minor* of D is any delta-matroid that is obtained from D by a (possibly empty) sequence of deletions and contractions. The *restriction* of D to a subset A of E, written $D|A$, is equal to $D \setminus \overline{A}$. A delta-matroid is *normal* exactly when the empty set is among its feasible sets. Note that if D is a normal delta-matroid then F is feasible in $D|A$ if and only if $F \subseteq A$ and $F \in \mathcal{F}(D)$.

The *connectivity function* λ_M of a matroid M on ground set E with rank function r is defined on all subsets A of E by $\lambda_M(A) = r(A) + r(\overline{A}) - r(E)$. Recall that A is said to be a *separator* of M if A is a (possibly empty) union of components of M. This happens if and only if $\lambda_M(A) = 0$. Moreover, A is a separator if and only if $\overline{A}$ is a separator.

We will use Bouchet's analogue of the rank function for delta-matroids from [3]. For a delta-matroid $D = (E, \mathcal{F})$, it is denoted by ρ_D or simply ρ when D is clear from the context. Its value on a subset A of E is given by

$$\rho(A) := |E| - \min\{|A \bigtriangleup F| : F \in \mathcal{F}\}.$$

The following results give expressions for the width of a twist of a delta-matroid.

Lemma 4. *Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$. Then the width, $w(D * A)$, of the twist of D by A is given by $w(D * A) = \rho(\overline{A}) - |E| + \rho(A)$.*

Proof. A largest feasible set in $D*A$ has size $\max\{|F\triangle A| : F \in \mathcal{F}(D)\}$. Take $F' \in \mathcal{F}$ such that $|F'\triangle A|$ is maximal. Then $|F'\triangle \overline{A}|$ is minimal. As $\rho(\overline{A}) = |E| - \min\{|F\triangle \overline{A}| : F \in \mathcal{F}\}$, we see that $\rho(\overline{A}) = |E| - |F' \triangle \overline{A}| = |F' \triangle A|$. Hence a largest feasible set in $D * A$ has size equal to $\rho(\overline{A})$.

Next, the size of a smallest feasible set in $D * A$ is $|E|$ minus the size of a largest feasible set in $(D * A)^* = D * \overline{A}$. By an application of the above, it follows that the size of a smallest feasible set in $D * A$ is $|E| - \rho(A)$. Hence $w(D * A) = \rho(\overline{A}) - |E| + \rho(A)$. □

Theorem 5. *Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$. Then the width, $w(D * A)$, of the twist of D by A is given by*

$$w(D * A) = w(D|A) + w(D|\overline{A}) + 2\,\lambda_{D_{\min}}(A).$$

Proof. We let r and n be the rank and nullity functions, respectively, of $D_{\min}$. From [7], we know that $w(D|A) = \rho(A) - r(A) - n(E) + n(A)$. As $n(A) = |A| - r(A)$ and $n(E) = |E| - r(E)$,

$$\begin{aligned}
w(D|A) &+ w(D|\overline{A}) \\
&= \rho(A) - r(A) - |E| + r(E) + |A| - r(A) \\
&\quad + \rho(\overline{A}) - r(\overline{A}) - |E| + r(E) + |\overline{A}| - r(\overline{A}) \\
&= \rho(\overline{A}) - |E| + \rho(A) - 2(r(A) + r(\overline{A}) - r(E)) \\
&= w(D * A) - 2(\lambda_{D_{\min}}(A)),
\end{aligned}$$

where we have applied the previous lemma to obtain the last equality. □

The following two theorems are immediate consequences of Theorem 5. The Rough Structure Theorem, Theorem 1, follows immediately from the second of them.

Theorem 6 (Chun et al [8])**.** *Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$. Then $D*A$ is a matroid if and only if A is a separator of $D_{\min}$, and both $D|A$ and $D|\overline{A}$ are matroids.*

Theorem 7. *Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$. Then $D * A$ has width one if and only if A is a separator of $D_{\min}$, and one of $D|A$ and $D|\overline{A}$ is a matroid and the other has width one.*

For convenience, we write down the following straightforward corollary. It provides the form of the Rough Structure Theorem that we use to find excluded minors in the next section.

Corollary 8. *Let $D = (E, \mathcal{F})$ be a normal delta-matroid. Then the following hold.*

1. *D has a twist of width zero if and only if there exists $A \subseteq E$ such that $D|A$ and $D|\overline{A}$ are both matroids.*

2. *D has a twist of width one if and only if there exists $A \subseteq E$ such that $D|A$ is a matroid and $D|\overline{A}$ is of width one.*

Proof. This is a straightforward consequence of the fact that if $\emptyset$ is feasible in D, then $D_{\min}$ is the matroid on $E(D)$ where each element is a loop, thus every set $A \subseteq E$ is a separator of $D_{\min}$. □

3 The proof of the excluded-minor characterisation

We begin this section by verifying that the class of delta-matroids in question is indeed minor-closed.

Proposition 9. *For each $k \in \mathbb{N}_0$, the set of delta-matroids with a twist of width at most k is minor-closed.*

Proof. Let $D = (E, \mathcal{F})$ and suppose $w(D * A) \leqslant k$ for some $A \subseteq E$. If E is empty then D has width zero and has no minors other than itself, so assume not and let $e \in E$. If $e \notin A$ then $(D \setminus e) * A = (D * A) \setminus e$, and $(D/e) * A = ((D * e) \setminus e) * A = ((D * e) * A) \setminus e = ((D * A) * e) \setminus e = (D * A)/e$. Similarly, if $e \in A$ then $e \notin A - e$, so using and extending the previous argument, $(D/e) * (A - e) = (D * (A - e))/e = ((D * A) * e)/e = (D * A) \setminus e$, and $(D \setminus e) * (A - e) = (D * (A - e)) \setminus e = ((D * A) * e) \setminus e = (D * A)/e$. In each case we see that D/e and $D \setminus e$ have a twist equal to either $(D * A)/e$ or $(D * A) \setminus e$. Since deletion and contraction never increase width it follows that D/e and $D \setminus e$ have twists of width at most $w(D * A) \leqslant k$. The result follows. □

Lemma 10. *Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$. Then*

$$\{H : H \text{ is a minor of } D * A\} = \{J * (A \cap E(J)) : J \text{ is a minor of } D\}.$$

Proof. In the proof of Proposition 9 it was shown that if $e \notin A$ then $(D * A)/e = (D/e) * A$ and $(D * A) \setminus e = (D \setminus e) * A$, whereas if $e \in A$ then $(D * A) \setminus e = (D/e) * (A - e)$ and $(D * A)/e = (D \setminus e) * (A - e)$. The result follows immediately from this. □

Before stating the next lemma we define the delta-matroids D_4 and D_5 on elements a, b, c with feasible sets given by

$$\mathcal{F}(D_4) = \{\emptyset, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\},$$
$$\mathcal{F}(D_5) = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, c\}\}.$$

Both are twists of D_2.

Lemma 11. *Let D be a normal delta-matroid. Then D has a twist of width at most 1, or contains a minor isomorphic to one of $D_1, \ldots, D_5$.*

Proof. For any normal delta-matroid D, set

$$L := \{x \in E(D) : \{x\} \in \mathcal{F}(D)\} \quad \text{and} \quad \overline{L} = E(D) - L.$$

(Technically we should record the fact that L depends upon D in the notation, however we avoid doing this for notational simplicity. This should cause no confusion.) Note that L may be empty. Construct a (simple) graph G_D as follows. Take one vertex v_x for each element $x \in \overline{L}$, and add one other vertex v_L. The edges of G_D arise from certain two-element feasible sets of D. Add an edge $v_x v_y$ to G_D for each pair $x, y \in \overline{L}$ with $\{x, y\} \in \mathcal{F}(D)$; add an edge $v_x v_L$ to G_D if $\{x, z\} \in \mathcal{F}(D)$ for some $z \in L$.

We consider two cases: when G_D is bipartite, and when it is not. We will show that if G_D is bipartite then D must have a twist of width at most one or a minor isomorphic to D_1 or D_2; if G_D is not bipartite then it must have a minor isomorphic to D_1, D_3, D_4, or D_5.

Case 1. Suppose that G_D is bipartite. Fix a 2-colouring of G_D. Let A be the set of elements in $E(D)$ that correspond to the vertices in the colour class containing v_L, except v_L itself, together with the elements in L, and let $\overline{A} \subseteq E(D)$ be the set of elements corresponding to the vertices in the colour class not containing v_L.

We start by showing

$$D|\overline{A} \cong U_{0,|\overline{A}|}, \tag{1}$$

where $U_{0,|\overline{A}|}$ denotes the uniform matroid with rank zero and $|\overline{A}|$ elements.

To see why (1) holds, note that $\mathcal{F}(D|\overline{A}) = \{F : F \subseteq \overline{A} \text{ and } F \in \mathcal{F}(D)\}$. Since the elements in $\overline{A}$ correspond to vertices in $\overline{L}$, no feasible sets of $D|\overline{A}$ have size one. Furthermore, $\mathcal{F}(D|\overline{A})$ cannot contain any sets of size two since, by the construction of G_D, whenever $\{x, y\} \in \mathcal{F}(D)$ the corresponding vertices v_x and v_y are in different colour classes. Since $\emptyset \in \mathcal{F}(D|\overline{A})$, the Symmetric Exchange Axiom ensures that there are no other feasible sets. (If $F \in \mathcal{F}(D|\overline{A})$ with $F \neq \emptyset$, take $x \in \emptyset \triangle F$. Then by the Symmetric Exchange Axiom $\emptyset \triangle \{x, y\}$ must be in $\mathcal{F}(D|\overline{A})$ for some y, but there are no feasible sets of size one or two.) This completes the justification of (1).

Next we examine the feasible sets in $D|A$. Trivially $\emptyset \in \mathcal{F}(D|A)$. The set of feasible sets of $D|A$ of size one is $\{F \in \mathcal{F}(D|A) : |F| = 1\} = \{F \in \mathcal{F}(D) : |F| = 1\} = \{\{x\} : x \in L\}$.

If $\mathcal{F}(D|A)$ contains a set $\{x, y\}$ of size two then $x, y \in L$ as otherwise there would be an edge $v_x v_y$ in G_D whose ends are in the same colour class. It follows in this case that $D|\{x, y\}$ is a minor of D isomorphic to D_1.

Now assume that $\mathcal{F}(D|A)$ does not contain a set of size two. If $\mathcal{F}(D|A)$ has no sets of size one then, arguing via the Symmetric Exchange Axiom as in the justification of (1), we have $D|A \cong U_{0,|A|}$. Taken together with (1), this implies that A satisfies the conditions of the first part of Corollary 8, so D has a twist of width zero.

Suppose that $\mathcal{F}(D|A)$ does contain a set of size one. If it contains no sets of size greater than one then $D|A$ is of width one, and by combining this with (1), it follows from Corollary 8 that D has a twist of width one ($D * A$ and $D * \overline{A}$ are such twists). On the other hand, if $\mathcal{F}(D|A)$ does contain a set of size greater than one, then, as it does not contain a set of size two, the Symmetric Exchange Axiom guarantees there is a set in $\mathcal{F}(D|A)$ of size exactly three. (If not, let F be a minimum sized feasible set with $|F| > 1$. Then $|F| > 3$, and $F \setminus \{x, y\}$ is feasible and of size at least two for some $x, y \in \emptyset \triangle F$ contradicting the minimality of $|F|$.) Let $\{x, y, z\} \in \mathcal{F}(D|A)$. Then after possibly relabelling its elements, the collection of feasible sets of $D|\{x, y, z\}$ is one of

$$\{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y, z\}\}, \quad \{\emptyset, \{x\}, \{y\}, \{x, y, z\}\}, \quad \{\emptyset, \{x\}, \{x, y, z\}\}.$$

Only the first of the three cases is possible as the Symmetric Exchange Axiom fails for the other two showing that neither is the collection of feasible sets of a delta-matroid. In fact, the second and third set systems are isomorphic to T_2^* and T_1^*, respectively, which Bonin, Chun, and Noble [1] showed to be among the excluded minors for the class of delta-matroids. Hence, restricting D to $\{x, y, z\}$ results in a minor isomorphic to D_2.

Thus we have shown that if G_D is bipartite then D has a twist of width at most one or contains a minor isomorphic to D_1 or D_2. This completes the proof of Case 1.

Case 2. Suppose that G_D is non-bipartite. We will show that D contains a minor isomorphic to one of D_1, D_3, D_4 or D_5 by induction on the length of a shortest odd cycle in G_D.

For the base of the induction suppose that G_D has an odd cycle C of length three. There are two sub-cases, when v_L is not in C and when it is. Note that the former sub-case includes the situation where $L = \emptyset$.

Sub-case 2.1. Suppose that v_L is not in C. Let $x, y, z \in E(D)$ be the elements corresponding to the three vertices of C. We have $x, y, z \in \overline{L}$, so $\{x\}, \{y\}, \{z\} \notin \mathcal{F}(D)$. From the three edges of C we have $\{x, y\}, \{y, z\}, \{z, x\} \in \mathcal{F}(D)$. It follows that $D|\{x, y, z\}$ is isomorphic to either D_3 or D_4 giving the required minor.

Sub-case 2.2. Suppose that v_L is in C. Let v_x, v_y, v_L be the vertices in C. The edges of C give that $\{x, y\} \in \mathcal{F}(D)$, and since $x, y \in \overline{L}$ we have $\{x\}, \{y\} \notin \mathcal{F}(D)$. We also know that there are elements $\alpha, \beta \in L$ such that $\{\alpha\}, \{\beta\}, \{x, \alpha\}, \{y, \beta\} \in \mathcal{F}(D)$, where possibly $\alpha = \beta$.

If $\alpha = \beta$ then $D|\{x, y, \alpha\}$ must have feasible sets

$$\{\emptyset, \{\alpha\}, \{x, \alpha\}, \{y, \alpha\}, \{x, y\}\} \quad \text{or} \quad \{\emptyset, \{\alpha\}, \{x, \alpha\}, \{y, \alpha\}, \{x, y\}, \{\alpha, x, y\}\}. \qquad (2)$$

The first case gives a minor of D isomorphic to D_5; in the second case, $(D|\{x, y, \alpha\})/\alpha$ is a minor of D isomorphic to D_1.

If $\alpha \neq \beta$ then the feasible sets of $D|\{x, y, \alpha, \beta\}$ of size zero or one are exactly $\emptyset$, $\{\alpha\}$, and $\{\beta\}$. From G_D, the feasible sets of size two include $\{x, \alpha\}, \{y, \beta\}, \{x, y\}$. If $\{y, \alpha\}$ is also feasible then $D|\{x, y, \alpha\}$ is isomorphic to one of the delta-matroids arising from (2), so D has a minor isomorphic to D_1 or D_5. The case when $\{x, \beta\}$ is feasible is similar. If $\{\alpha, \beta\}$ is feasible then $D|\{\alpha, \beta\}$ is isomorphic to D_1.

The case that remains is when the feasible sets of $D|\{x, y, \alpha, \beta\}$ of size at most two are exactly

$$\emptyset, \{\alpha\}, \{\beta\}, \{x, \alpha\}, \{y, \beta\}, \{x, y\}.$$

By applying the Symmetric Exchange Axiom to each of the triples

$$(\{\alpha\}, \{x, y\}, y),\ (\{\beta\}, \{x, y\}, x),\ (\{\beta\}, \{x, \alpha\}, x) \text{ and } (\{\alpha\}, \{y, \beta\}, y),$$

where the three components of the triple play the roles of F_1, F_2 and u in the Symmetric Exchange Axiom, one can show that each of the three element sets,

$$\{\alpha, x, y\}, \{\beta, x, y\}, \{\alpha, \beta, x\}, \{\alpha, \beta, y\},$$

is feasible in $D|\{x, y, \alpha, \beta\}$. Finally, $\{\alpha, \beta, x, y\}$ may or may not be feasible.

If $\{\alpha, \beta, x, y\}$ is feasible then $(D|\{x, y, \alpha, \beta\})/\{x, y\}$ is isomorphic to D_1; if $\{\alpha, \beta, x, y\}$ is not feasible then $(D|\{x, y, \alpha, \beta\})/\{\alpha\}$ is isomorphic to D_5.

This completes the base of the induction.

For the inductive hypothesis, we assume that, for some odd $n > 3$, if D' is a normal delta-matroid and $G_{D'}$ has an odd cycle of length less than n, then D' has a minor isomorphic to D_1, D_3, D_4, or D_5.

Suppose that a shortest odd cycle C of G_D has length n. Again there are two sub-cases: when v_L is not in C and when it is.

Sub-case 2.3. Suppose that v_L is not in C. Let $C = v_{x_1}v_{x_2}\dots v_{x_n}v_{x_1}$. Since each $x_i \in \overline{L}$ and C is the shortest odd cycle in G_D,

$$\emptyset, \{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_n, x_1\} \tag{3}$$

is a complete list of the feasible sets of size at most two in $D|\{x_1, \dots, x_n\}$.

Next, we show

$$\{x_i, x_j, x_k\} \notin \mathcal{F}(D|\{x_1, \dots, x_n\}), \quad \text{for any distinct } 1 \leqslant i, j, k \leqslant n. \tag{4}$$

To see why (4) holds, first note that, since $n > 3$, every set of three distinct vertices in the cycle includes a non-adjacent pair. If $\{x_i, x_j, x_k\}$ were feasible in $D|\{x_1, \dots, x_n\}$, then, without loss of generality, $\{x_j, x_k\} \notin \mathcal{F}(D|\{x_1, \dots, x_n\})$. As $x_i \in \{x_i, x_j, x_k\} \triangle \emptyset$, an application of the Symmetric Exchange Axiom would imply that $\{x_i, x_j, x_k\} \triangle \{x_i, z\}$ is feasible for some $z \in \{x_i, x_j, x_k\}$. Thus $\{x_j, x_k\}, \{x_j\}$, or $\{x_k\}$ would be feasible, a contradiction to (3). Thus (4) holds.

Next we show that, taking indices modulo n,

$$\{x_i, x_{i+1}, x_j, x_{j+1}\} \in \mathcal{F}(D|\{x_1, \dots, x_n\}), \tag{5}$$

for any i and j such that $1 \leqslant i, j \leqslant n$ and $i, i+1, j, j+1$ are pairwise distinct. By (3), $\{x_i, x_{i+1}\}$ and $\{x_j, x_{j+1}\}$ are feasible. As x_j is in their symmetric difference, by the Symmetric Exchange Axiom, $\{x_i, x_{i+1}\} \triangle \{x_j, y\}$ is feasible for some $y \in \{x_i, x_{i+1}, x_j, x_{j+1}\}$. Thus $\{x_i, x_{i+1}, x_j\}, \{x_i, x_j\}, \{x_{i+1}, x_j\}$ or $\{x_i, x_{i+1}, x_j, x_{j+1}\}$ is feasible. First suppose that neither x_{i+1} and x_j nor x_{j+1} and x_i are adjacent in C. By (3) and (4), $\{x_i, x_{i+1}, x_j, x_{j+1}\}$ is feasible. Alternatively, if x_{i+1} and x_j are adjacent then the Symmetric Exchange Axiom implies that $\{x_i, x_{i+1}\} \triangle \{x_{i+3}, z\}$ is feasible for some $z \in \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$. Again, (3) and (4) imply that $\{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$ must be feasible. The other case is identical. This completes the justification of (5).

Combining (3)–(5) gives that all of $\emptyset, \{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-2}, x_1\}$, but none of $\{x_1\}, \dots, \{x_{n-2}\}$, are feasible in $(D|\{x_1, \dots, x_n\})/\{x_{n-1}, x_n\}$. Consequently the graph $G_{(D|\{x_1,\dots,x_n\})/\{x_{n-1},x_n\}}$ has a shorter odd cycle than G_D. By the inductive hypothesis, $(D|\{x_1, \dots, x_n\})/\{x_{n-1}, x_n\}$ and hence D has a minor isomorphic to one of D_1, D_3, D_4 or D_5.

Sub-case 2.4. Suppose that v_L is in C. Let $C = v_L v_{x_2} v_{x_3} \dots v_{x_n} v_L$. The edges of the cycle give that, for each $2 \leqslant i \leqslant n-1$, $\{x_i, x_{i+1}\} \in \mathcal{F}(D)$. Also, for $2 \leqslant i \leqslant n$, since

$x_i \in \overline{L}$ we have $\{x_i\} \notin \mathcal{F}(D)$. We also know that there are elements $\alpha, \beta \in L$ such that $\{\alpha\}, \{\beta\}, \{\alpha, x_2\}, \{\beta, x_n\} \in \mathcal{F}(D)$ where possibly $\alpha = \beta$. In the case where $\alpha \neq \beta$, if $\{\alpha, \beta\} \in \mathcal{F}(D)$, then $D|\{\alpha, \beta\}$ is isomorphic to D_1, therefore we assume $\{\alpha, \beta\} \notin \mathcal{F}(D)$. The following analysis covers both the case where $\alpha = \beta$ and the case where $\alpha \neq \beta$.

Using that C is a shortest odd cycle, the feasible sets of $D|\{\alpha, \beta, x_2, \dots, x_n\}$ of size at most two are exactly

$$\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \dots, \{x_{n-1}, x_n\}, \{\beta, x_n\}. \tag{6}$$

An argument similar to the justification of (4) gives that

$$\{x_i, x_j, x_k\} \notin \mathcal{F}(D|\{\alpha, \beta, x_2, \dots, x_n\}), \quad \text{for any distinct } 2 \leqslant i, j, k \leqslant n. \tag{7}$$

However

$$\{\alpha, x_{n-1}, x_n\}, \{\beta, x_{n-1}, x_n\} \in \mathcal{F}(D|\{\alpha, \beta, x_2, \dots, x_n\}). \tag{8}$$

To see this note that $x_{n-1} \in \{\alpha\} \bigtriangleup \{x_{n-1}, x_n\}$, so the Symmetric Exchange Axiom gives that one of $\{\alpha, x_{n-1}\}$, $\{x_{n-1}\}$, or $\{\alpha, x_{n-1}, x_n\}$ is feasible, and we know from (6) that the feasible set must be the third option. That $\{\beta, x_{n-1}, x_n\}$ is feasible follows from a similar argument.

We next show that for each $2 \leqslant i < n-2$,

$$\{\alpha, x_2, x_{n-1}, x_n\}, \{x_i, x_{i+1}, x_{n-1}, x_n\}, \{\beta, x_{n-2}, x_{n-1}, x_n\} \in \mathcal{F}(D|\{\alpha, \beta, x_2, \dots, x_n\}). \tag{9}$$

For this, first consider $x_2 \in \{x_{n-1}, x_n\} \bigtriangleup \{\alpha, x_2\}$. The Symmetric Exchange Axiom implies that $\{x_{n-1}, x_n\} \bigtriangleup \{x_2, z\}$ is feasible for some $z \in \{\alpha, x_2, x_{n-1}, x_n\}$. By (6) and (7), $z = \alpha$, thus $\{\alpha, x_2, x_{n-1}, x_n\}$ is feasible. Next, to show that $\{x_i, x_{i+1}, x_{n-1}, x_n\}$ is feasible, we take $x_i \in \{x_{n-1}, x_n\} \bigtriangleup \{x_i, x_{i+1}\}$ and apply the Symmetric Exchange Axiom as above to see that $\{x_{n-1}, x_n\} \bigtriangleup \{x_i, z\}$ is feasible, where z must equal x_{i+1}. Lastly, to show that $\{\beta, x_{n-2}, x_{n-1}, x_n\}$ is feasible, we first show that $\{\beta, x_{n-2}, x_n\} \notin \mathcal{F}(D|\{\alpha, \beta, x_2, \dots, x_n\})$. If $\{\beta, x_{n-2}, x_n\}$ were feasible, then since $x_{n-2} \in \emptyset \bigtriangleup \{\beta, x_{n-2}, x_n\}$, the Symmetric Exchange Axiom would give $\{x_{n-2}\}$, $\{\beta, x_{n-2}\}$ or $\{x_{n-2}, x_n\}$ as feasible, a contradiction. Now showing that $\{\beta, x_{n-2}, x_{n-1}, x_n\}$ is feasible comes from taking $x_{n-2} \in \{\beta, x_n\} \bigtriangleup \{x_{n-2}, x_{n-1}\}$. The Symmetric Exchange Axiom gives that $\{\beta, x_n\} \bigtriangleup \{x_{n-2}, z\}$ is feasible for some $z \in \{\beta, x_{n-2}, x_{n-1}, x_n\}$, of which $z = x_{n-1}$ is the only possibility.

From (6)–(9) it follows that all of $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha, x_2\}$, $\{x_2, x_3\}$, …, $\{\beta, x_{n-2}\}$, but none of $\{x_2\}$, …, $\{x_{n-2}\}$, are feasible in $D' = (D|\{\alpha, x_2, \dots, x_n, \beta\})/\{x_{n-1}, x_n\}$. Hence the graph $G_{D'}$ has a shorter odd cycle than G_D. The inductive hypothesis gives that D' and hence D has a minor isomorphic to one of D_1, D_3, D_4 or D_5. This completes the proof of the sub-case, and the lemma. □

We now apply Lemma 11 to prove our excluded-minor characterisation of the family of delta-matroids admitting a twist of width at most one.

Proof of Theorem 3. All twists of the delta-matroids D_1, D_2, D_3 are of width at least two. Since the set of delta-matroids with a twist of width at most one is minor-closed it follows that no minor of a delta-matroid with a twist of width at most one is isomorphic to a member of $\mathcal{D}_{[3]}$. This proves one direction of the theorem.

Conversely suppose that every twist of a delta-matroid $D = (E, \mathcal{F})$ is of width at least two. Let $A \in \mathcal{F}$. Then $D * A$ is a normal delta-matroid and in which every twist is of width at least two. By Lemma 11, $D * A$ has a minor isomorphic to one of $D_1, \dots, D_5$. Using the fact that D_4 and D_5 are both twists of D_2, it follows from Lemma 10 that D has a minor isomorphic to a member of $\mathcal{D}_{[3]}$. □

Remark 12. In the introduction, we mentioned the very close connection between delta-matroids and graphs in surfaces. Viewed as ribbon graphs, such graphs give rise to the minor-closed class of *ribbon graphic delta-matroids* in much the same way as (abstract) graphs give rise to the class of graphic matroids. For a full explanation see [7]. The width of a delta-matroid can be viewed as the analogue of the genus (or more precisely the Euler genus) of an embedded graph, while twisting is the analogue of S. Chmutov's partial duality of [6]. Thus characterising delta-matroids with a twist of width one is the analogue of characterising embedded graphs having a partial dual embedded in the real projective plane. The topological graph theoretical analogues of Theorems 1, 3 and 5 can be found in [11, 12]. In fact the ribbon graph results provide versions of Theorems 1, 3, and 5 that hold for the class of ribbon graphic delta-matroids.

For a ribbon graphic version of Theorem 3, if D is ribbon graphic and does not admit a twist of width at most one, then $D = D(G)$ for some ribbon graph G that does not have a partial dual of Euler genus at most one. By Theorem 1.1 of [12], it follows that one of the ribbon graphs X_1, X_2, or X_3 from Figure 1 of that paper must be a minor. By the compatibility of ribbon graph and delta-matroid deletion and contraction described in [7], one of $D(X_1) = D_1$, $D(X_2) = D_3$, or $D(X_3) = D_3 * a$ is a minor of D. The converse follows since the class is minor-closed. It is not difficult to see that neither D_2 nor any of its twists is ribbon-graphic, so they do not appear as excluded minors for this class, but several definitions from [7] are required to make the argument concise. Briefly, any ribbon-graphic delta-matroid with the same feasible sets of size at most two as D_2 must arise from a ribbon graph comprising a single vertex and three pairwise interlaced non-orientable loops. In any such ribbon-graphic delta-matroid there is no feasible set of size three. Thus D_2 is not ribbon graphic. As any twist of a ribbon-graphic delta-matroid is ribbon-graphic [4, 7], it follows that no twist of D_2 is ribbon-graphic. (An alternative way to see that D_2 is not ribbon graphic is to note that ribbon graphic delta-matroids are representable over $GF(2)$ and that D_2 is one of the excluded minors for that class [5].)

Similar reasoning shows that Theorem 3.4 of [11] gives Theorem 5 for ribbon graphic delta-matroids in the special case where $\lambda_{D_{\min}}(A) = 0$; and Theorem 4.3(2) of [11] results in Theorem 1 for ribbon graphic delta-matroids. Deducing these results uses the fact that A defines a biseparation of G if and only if it is a separator of $D(G)$ [7].

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