Sharp lower bounds for the spectral radius of uniform hypergraphs concerning degrees*

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Abstract

Let $\mathcal{A}(H)$ and $\mathcal{Q}(H)$ be the adjacency tensor and signless Laplacian tensor of an r-uniform hypergraph H. Denote by $\rho(H)$ and $\rho(\mathcal{Q}(H))$ the spectral radii of $\mathcal{A}(H)$ and $\mathcal{Q}(H)$, respectively. In this paper, we present a lower bound on $\rho(H)$ in terms of vertex degrees and we characterize the extremal hypergraphs attaining the bound, which solves a problem posed by Nikiforov [Analytic methods for uniform hypergraphs, Linear Algebra Appl. 457 (2014) 455–535]. Also, we prove a lower bound on $\rho(\mathcal{Q}(H))$ concerning degrees and give a characterization of the extremal hypergraphs attaining the bound.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph with n vertices, and $A(G)$ be the adjacency matrix of G. Let $\rho(G)$ be the spectral radius of G, and d_i be the degree of vertex i of G, $i = 1, 2, \ldots, n$. In 1957, Collatz and Sinogowitz [4] showed that the spectral radius $\rho(G)$ of a graph G is greater than or equal to the average degree $\overline{d}$, and the equality holds if and only if G is regular. In 1988, Hofmeister [8] strengthened Collatz and Sinogowitz's result by proving a lower bound on $\rho(G)$ in terms of degrees of vertices of G as follows:

$$\rho(G) \geqslant \left(\frac{1}{n}\sum_{i=1}^{n} d_i^2\right)^{\frac{1}{2}}. \tag{1}$$

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Furthermore, if G is connected, then equality holds if and only if G is either a regular graph or a semiregular bipartite graph (see details in [8] and [24]). The inequality (1) has many important applications in spectral graph theory (see [6, 12, 13]).

In recent years, research on spectra of hypergraphs via tensors has drawn extensive interest, accompanied by a rapid development in tensor spectral theory. A *hypergraph* $H = (V, E)$ consists of a (finite) set V and a collection E of non-empty subsets of V (see [1]). The elements of V are called *vertices* and the elements of E are called *hyperedges*, or simply *edges* of the hypergraph. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph H explicitly by $V(H)$ and $E(H)$, respectively. An r-*uniform* hypergraph is a hypergraph in which every edge has size r. Throughout this paper, we denote by $V(H) = [n] := \{1, 2, \ldots, n\}$ the vertex set of a hypergraph H. For a vertex $i \in V(H)$, the *degree* of i, denoted by $d_H(i)$ or simply by d_i, is the number of edges containing i. If each vertex of H has the same degree, we say that the hypergraph H is *regular*. The minimum and maximum degrees among the vertices of H are denoted by $\delta(H)$ and $\Delta(H)$, respectively. For different i, $j \in V(H)$, i and j are said to be *adjacent*, written $i \sim j$, if there is an edge of H containing both i and j. A *walk* of hypergraph H is defined to be an alternating sequence of vertices and edges $i_1 e_1 i_2 e_2 \cdots i_\ell e_\ell i_{\ell+1}$ satisfying that $\{i_j, i_{j+1}\} \subseteq e_j \in E(H)$ for $1 \leqslant j \leqslant \ell$. A walk is called a *path* if all vertices and edges in the walk are distinct. A hypergraph H is called *connected* if for any vertices i, j, there is a walk connecting i and j. For positive integers r and n, a real *tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_r})$ of order r and dimension n refers to a multidimensional array (also called *hypermatrix*) with entries $a_{i_1 i_2 \cdots i_r}$ such that $a_{i_1 i_2 \cdots i_r} \in \mathbb{R}$ for all i_1, i_2, $\ldots$, $i_r \in [n]$. We say that tensor $\mathcal{A}$ is *symmetric* if its entries $a_{i_1 i_2 \cdots i_r}$ are invariant under any permutation of its indices.

Recently, Nikiforov [14] presented some analytic methods for studying uniform hypergraphs, and posed the following question (see [14, Question 11.5]):

Question 1 ([14])**.** Suppose that H is an r-uniform hypergraph on n vertices ($r \geqslant 3$). Let d_i be the degree of vertex i, $i \in [n]$, and $\rho(H)$ be the spectral radius of H. Is it always true

$$\rho(H) \geqslant \left(\frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} ?$$

In this paper, we focus on the above question, and give a solution to Question 1. Our main results can be stated as follows.

Theorem 2. *Suppose that H is an r-uniform hypergraph on n vertices ($r \geqslant 3$). Let d_i be the degree of vertex i of H, and $\rho(H)$ be the spectral radius of H. Then*

$$\rho(H) \geqslant \left(\frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} .$$

If H is connected, then the equality holds if and only if H is regular.

Theorem 3. *Let H be a connected r-uniform hypergraph on n vertices ($r \geqslant 3$). Suppose that d_i, $i \in [n]$, is the degree of vertex i, and $\rho(\mathcal{Q}(H))$ is the spectral radius of the signless Laplacian tensor $\mathcal{Q}(H)$. Then*

$$\rho(\mathcal{Q}(H)) \geqslant 2 \left(\frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}},$$

with equality if and only if H is regular.

2 Preliminaries

In this section we review some basic notations and necessary conclusions. Denote the set of nonnegative vectors (positive vectors) of dimension n by $\mathbb{R}_+^n$ ($\mathbb{R}_{++}^n$). Let $||x||_r = (\sum_{i=1}^n |x_i|^r)^{1/r}$ denote the r-norm of a vector $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$. The unit tensor of order r and dimension n is the tensor $\mathcal{I}_n = (\delta_{i_1 i_2 \cdots i_r})$, whose entry is 1 if $i_1 = i_2 = \cdots = i_r$ and 0 otherwise.

The following general product of tensors was defined by Shao [20], which is a generalization of the matrix case.

Definition 4 ([20])**.** Let $\mathcal{A}$ (and $\mathcal{B}$) be an order $r \geqslant 2$ (and order $k \geqslant 1$), dimension n tensor. Define the product $\mathcal{AB}$ to be the following tensor $\mathcal{C}$ of order $(r-1)(k-1)+1$ and dimension n

$$c_{i\alpha_1 \cdots \alpha_{r-1}} = \sum_{i_2, \dots, i_r = 1}^{n} a_{i i_2 \cdots i_r} b_{i_2 \alpha_1} \cdots b_{i_r \alpha_{r-1}} \quad (i \in [n], \alpha_1, \dots, \alpha_{r-1} \in [n]^{k-1}).$$

From the above definition, let $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ be a column vector of dimension n. Then $\mathcal{A}x$ is a vector in $\mathbb{C}^n$, whose i-th component is as the following

$$(\mathcal{A}x)_i = \sum_{i_2, \dots, i_r = 1}^{n} a_{i i_2 \cdots i_r} x_{i_2} \cdots x_{i_r}, \quad i \in [n]$$

and

$$x^{\mathrm{T}}(\mathcal{A}x) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

In 2005, Lim [10] and Qi [17] independently introduced the concepts of tensor eigenvalues and the spectra of tensors. Let $\mathcal{A}$ be an order r and dimension n tensor, $x = (x_1, x_2 \dots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ be a column vector of dimension n. If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x = \lambda x^{[r-1]},$$

then λ is called an *eigenvalue* of $\mathcal{A}$, x is called an *eigenvector* of $\mathcal{A}$ corresponding to the eigenvalue λ, where $x^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \dots, x_n^{r-1})^{\mathrm{T}}$. The *spectral radius* $\rho(\mathcal{A})$ of $\mathcal{A}$ is the maximum modulus of the eigenvalues of $\mathcal{A}$. It was proved that λ is an eigenvalue of $\mathcal{A}$ if and only if it is a root of the characteristic polynomial of $\mathcal{A}$ (see details in [21]).

In 2012, Cooper and Dutle [5] defined the adjacency tensors for r-uniform hypergraphs.

Definition 5 ([5, 19])**.** Let $H = (V(H), E(H))$ be an r-uniform hypergraph on n vertices. The adjacency tensor of H is defined as the order r and dimension n tensor $\mathcal{A}(H) = (a_{i_1 i_2 \cdots i_r})$, whose $(i_1 i_2 \cdots i_r)$-entry is

$$a_{i_1 i_2 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{i_1, i_2, \ldots, i_r\} \in E(H), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(H)$ be an order r and dimension n diagonal tensor with its diagonal element $d_{ii\cdots i}$ being d_i, the degree of vertex i, for all $i \in [n]$. Then $\mathcal{L}(H) = \mathcal{D}(H) - \mathcal{A}(H)$ is the Laplacian tensor of H, and $\mathcal{Q}(H) = \mathcal{D}(H) + \mathcal{A}(H)$ is the signless Laplacian tensor of H.

For an r-uniform hypergraph H, denote the spectral radius of $\mathcal{A}(H)$ by $\rho(H)$. It should be announced that spectral radius defined in [14] differ from this paper, while for an r-uniform hypergraph H the spectral radius defined in [14] equals to $(r-1)!\rho(H)$. This is not essential and does not effect the result.

In [7], the weak irreducibility of nonnegative tensors was defined. It was proved that an r-uniform hypergraph H is connected if and only if its adjacency tensor $\mathcal{A}(H)$ is weakly irreducible (see [16]). Clearly, this shows that if H is connected, then $\mathcal{A}(H)$, $\mathcal{L}(H)$ and $\mathcal{Q}(H)$ are all weakly irreducible. The following result for nonnegative tensors is a part of Perron–Frobenius theorem.

Theorem 6 ([2, 7])**.** *Let $\mathcal{A}$ be a nonnegative tensor of order r and dimension n. Then we have the following statements.*

(1) *$\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector corresponding to it.*

(2) *If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with the unique eigenvector $x \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.*

Theorem 7 ([18])**.** *Let $\mathcal{A}$ be a nonnegative symmetric tensor of order r and dimension n. Then we have*

$$\rho(\mathcal{A}) = \max\left\{x^{\mathrm{T}}(\mathcal{A}x) \mid x \in \mathbb{R}^n_+, ||x||_r = 1\right\}.$$

Furthermore, $x \in \mathbb{R}^n_+$ with $||x||_r = 1$ is an optimal solution of the above optimization problem if and only if it is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\rho(\mathcal{A})$.

The following concept of *direct products* (also called *Kronecker product*) of tensors was defined in [20], which is a generalization of the direct products of matrices.

Definition 8 ([20])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two order r tensors with dimension n and m, respectively. Define the direct product $\mathcal{A} \otimes \mathcal{B}$ to be the following tensor of order r and dimension mn (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

$$(\mathcal{A} \otimes \mathcal{B})_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = a_{i_1 i_2 \cdots i_r} b_{j_1 j_2 \cdots j_r}.$$

In particular, if $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$ are two column vectors with dimension n and m, respectively. Then

$$x \otimes y = (x_1y_1, \ldots, x_1y_m, x_2y_1, \ldots, x_2y_m, \ldots, x_ny_1, \ldots, x_ny_m)^{\mathrm{T}}.$$

The following basic results can be found in [20].

Proposition 9 ([20])**.** *The following conclusions hold.*

(1) $(\mathcal{A}_1 + \mathcal{A}_2) \otimes \mathcal{B} = \mathcal{A}_1 \otimes \mathcal{B} + \mathcal{A}_2 \otimes \mathcal{B}$.

(2) $(\lambda\mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\lambda\mathcal{B}) = \lambda(\mathcal{A} \otimes \mathcal{B})$, $\lambda \in \mathbb{C}$.

(3) $(\mathcal{A} \otimes \mathcal{B})(\mathcal{C} \otimes \mathcal{D}) = (\mathcal{A}\mathcal{C}) \otimes (\mathcal{B}\mathcal{D})$.

Theorem 10 ([20])**.** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two order r tensors with dimension n and m, respectively. Let λ be an eigenvalue of $\mathcal{A}$ with corresponding eigenvector u, and μ be an eigenvalue of $\mathcal{B}$ with corresponding eigenvector v. Then $\lambda\mu$ is an eigenvalue of $\mathcal{A} \otimes \mathcal{B}$ with corresponding eigenvector $u \otimes v$.*

3 Proof of Theorem 2

In this section, we shall give a proof of Theorem 2.

Proof of Theorem 2. Let H be an r-uniform hypergraph with spectral radius $\rho(H)$ and vertex set $V(H) = [n]$, and denote by d_i the degree of vertex i of H, $i = 1, 2, \ldots, n$.

We now define an r-uniform hypergraph $\widetilde{H}$ as follows. The hypergraph $\widetilde{H}$ has vertex set $V(H) \times [r]$, and $\{(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)\} \in E(\widetilde{H})$ if and only if $\{i_1, i_2, \ldots, i_r\} \in E(H)$ and $j_1, j_2, \ldots, j_r$ are pairwise distinct. Let $\mathcal{A}(H) = (a_{i_1i_2\cdots i_r})$ be the adjacency tensor of H. We define an order r and dimension r tensor $\mathcal{B} = (b_{j_1j_2\cdots j_r})$ as follows:

$$b_{j_1j_2\cdots j_r} = \begin{cases} 1, & \text{if } j_1, j_2, \ldots, j_r \text{ are pairwise distinct,} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We denote the adjacency tensor of $\widetilde{H}$ by $\mathcal{A}(\widetilde{H})$, in which the set of subscripts is taken as $[n] \times [r]$ in the lexicographic order.

Claim 1. If H is connected, then $\widetilde{H}$ is connected.

Proof of Claim 1. It suffices to show that for any (i, j), $(s, t) \in V(\widetilde{H})$, there exists a walk connecting them. We distinguish the following two cases.
Case 1. $i \neq s$, $j \neq t$.

Since H is connected, there exists a path $i = i_1e_1i_2 \cdots i_\ell e_\ell i_{\ell+1} = s$. Since $r \geqslant 3$, there exist j' such that $j' \neq j$, $j' \neq t$. From the definition of $\widetilde{H}$, if ℓ is odd, we have

$$\begin{cases} (i_h, j) \sim (i_{h+1}, j'), & h = 1, 3, \ldots, \ell - 2, \\ (i_k, j') \sim (i_{k+1}, j), & k = 2, 4, \ldots, \ell - 1, \\ (i_\ell, j) \sim (s, t). \end{cases}$$

If ℓ is even, we obtain

$$\begin{cases} (i_h, j) \sim (i_{h+1}, j'), & h = 1, 3, \dots, \ell - 1, \\ (i_k, j') \sim (i_{k+1}, j), & k = 2, 4, \dots, \ell - 2, \\ (i_\ell, j') \sim (s, t). \end{cases}$$

Hence there exists a walk connecting (i, j) and (s, t).

Case 2. $i = s$, $j \neq t$.

Since $r \geqslant 3$, there exist i' and j' such that $i' \neq i$, $j' \neq j$, $j' \neq t$. According to Case 1 we know that there is a path connecting (i, j) and (i', j'). Noting that $i' \neq s$ and $j' \neq t$, there is a path connecting (i', j') and (s, t) by Case 1. So there exists a walk connecting (i, j) and (s, t), as desired. The proof of the claim is completed.

Claim 2. $\mathcal{A}(\widetilde{H}) = \mathcal{A}(H) \otimes \mathcal{B}$.

Proof of Claim 2. From the definition of $\widetilde{H}$, it follows that

$$(\mathcal{A}(\widetilde{H}))_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{i_1, i_2, \dots, i_r\} \in E(H), b_{j_1 j_2 \cdots j_r} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

According to Definition 8, $\mathcal{A}(H) \otimes \mathcal{B}$ is an order r and dimension rn tensor, whose entries are given by

$$(\mathcal{A}(H) \otimes \mathcal{B})_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = a_{i_1 i_2 \cdots i_r} b_{j_1 j_2 \cdots j_r}.$$

If $\{i_1, i_2, \dots, i_r\} \in E(H)$ and $b_{j_1 j_2 \cdots j_r} = 1$, then

$$(\mathcal{A}(H) \otimes \mathcal{B})_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = \frac{1}{(r-1)!},$$

and 0 otherwise. Hence $\mathcal{A}(\widetilde{H}) = \mathcal{A}(H) \otimes \mathcal{B}$, as desired.

Claim 3. $\rho(\mathcal{B}) = (r - 1)!$.

Proof of Claim 3. Let $e = (1, 1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^r$. It follows from Theorem 7 that

$$\rho(\mathcal{B}) \geqslant \frac{e^{\mathrm{T}}(\mathcal{B}e)}{||e||_r^r} = \frac{r!}{r} = (r - 1)!.$$

On the other hand, let $z = (z_1, z_2, \dots, z_r) \in \mathbb{R}_+^r$ be a nonnegative eigenvector corresponding to $\rho(\mathcal{B})$ with $||z||_r = 1$. By AM–GM inequality, we have

$$\rho(\mathcal{B}) = z^{\mathrm{T}}(\mathcal{B}z) = r! z_1 z_2 \cdots z_r \leqslant r! \left(\frac{z_1^r + z_2^r + \cdots + z_r^r}{r} \right) = (r - 1)!,$$

with equality holds if and only if

$$z_1 = z_2 = \cdots = z_r = \frac{1}{\sqrt[r]{r}}.$$

Therefore, $\rho(\mathcal{B}) = (r-1)!$. The proof of the claim is completed.

Claim 4. $\rho(\widetilde{H}) = (r-1)!\rho(H)$.

Proof of Claim 4. By Claim 2, $\mathcal{A}(\widetilde{H}) = \mathcal{A}(H) \otimes \mathcal{B}$. We consider the following two cases depending on whether or not H is connected.
Case 1. H is connected.

Since H is connected, we have $\mathcal{A}(H)$ is weakly irreducible. From Theorem 6, let u be the positive eigenvector corresponding to the eigenvalue $\rho(H)$. Then by Theorem 10, $\rho(H)\rho(\mathcal{B})$ is an eigenvalue of $\mathcal{A}(H) \otimes \mathcal{B}$ with a positive eigenvector $u \otimes e$. By Claim 1, $\widetilde{H}$ is connected. It follows from Theorem 6 and Claim 3 that $(r-1)!\rho(H)$ must be the spectral radius of $\mathcal{A}(H) \otimes \mathcal{B}$, i.e., $\rho(\widetilde{H}) = (r-1)!\rho(H)$.
Case 2. H is disconnected.

Let $\varepsilon > 0$ and $\mathcal{A}_\varepsilon = \mathcal{A}(H) + \varepsilon \mathcal{J}_1$, $\mathcal{B}_\varepsilon = \mathcal{B} + \varepsilon \mathcal{J}_2$, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are order r tensors with all entries 1 with dimension n and r, respectively. Then $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are both positive tensors, and therefore are weakly irreducible. Using the similar arguments as Case 1, we have

$$\rho(\mathcal{A}_\varepsilon \otimes \mathcal{B}_\varepsilon) = \rho(\mathcal{A}_\varepsilon)\rho(\mathcal{B}_\varepsilon).$$

Notice that the maximal absolute value of the roots of a complex polynomial is a continuous function on the coefficients of the polynomial. Take the limit $\varepsilon \to 0$ on both sides of the above equation, we obtain the desired result. The proof of the claim is completed.

It is clear that $\widetilde{H}$ is an r-partite hypergraph with partition

$$V(\widetilde{H}) = \bigcup_{i=1}^{r} \left(V(H) \times \{i\}\right).$$

We define a vector $x \in \mathbb{R}^{rn}$ as follows:

$$x_{(i,j)} = \begin{cases} \frac{a_i}{\sqrt[r]{rn}}, & \text{if } i = 1, 2, \ldots, n, j = 1, \\ \frac{1}{\sqrt[r]{rn}}, & \text{otherwise}, \end{cases} \tag{3}$$

where $a_1, a_2, \ldots, a_n \geqslant 0$ and $a_1^r + a_2^r + \cdots + a_n^r = n$. It is obvious that $d_{\widetilde{H}}((i,j)) = (r-1)!d_i$ for any $i \in [n]$, $j \in [r]$. By Theorem 7, we deduce that

$$\begin{aligned} \rho(\widetilde{H}) \geqslant x^{\mathrm{T}}(\mathcal{A}(\widetilde{H})x) &= r \sum_{\{(i_1,j_1),\ldots,(i_r,j_r)\} \in E(\widetilde{H})} x_{(i_1,j_1)} x_{(i_2,j_2)} \cdots x_{(i_r,j_r)} \\ &= r\left[\sum_{i=1}^{n} \frac{a_i}{\sqrt[r]{rn}} \cdot \left(\frac{1}{\sqrt[r]{rn}}\right)^{r-1} \cdot (r-1)!d_i\right] \\ &= \frac{(r-1)!}{n} \sum_{i=1}^{n} a_i d_i. \end{aligned} \tag{4}$$

It follows from $a_1^r + a_2^r + \cdots + a_n^r = n$ and Hölder inequality that

$$\begin{aligned}\sum_{i=1}^{n} a_i d_i &\leqslant \left(\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}} \cdot \left(\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \\ &= \sqrt[r]{n}\left(\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}\end{aligned}$$

with equality if and only if

$$a_i = \frac{\sqrt[r]{n} d_i^{\frac{1}{r-1}}}{\sqrt[r]{\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}}}, \ i = 1, 2, \ldots, n. \tag{5}$$

Now we set a_i as (5). In the light of (4) and (5) we have

$$\rho(H) = \frac{\rho(\widetilde{H})}{(r-1)!} \geqslant \frac{1}{n}\sum_{i=1}^{n} a_i d_i = \left(\frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}. \tag{6}$$

Now we give a characterization of extremal hypergraphs achieving the equality in (6). Suppose first the equality holds in (6). Then the vector $x \in \mathbb{R}^{rn}$ defined by (3) is an eigenvector corresponding to $\rho(\widetilde{H})$ by Theorem 7. Note that H is connected, by Theorem 6, we let $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n_{++}$ be a positive eigenvector corresponding to $\rho(H)$. From Theorem 10 and Claim 4, it follows that $u \otimes e$ is a positive eigenvector to $\rho(\widetilde{H})$. By Claim 1 and Theorem 6, $\widetilde{H}$ is connected, and we see that x and $u \otimes e$ are linearly dependent. Notice that

$$x = \frac{1}{\sqrt[r]{rn}}(\underbrace{a_1, 1, \ldots, 1}_{r}, \underbrace{a_2, 1, \ldots, 1}_{r}, \ldots, \underbrace{a_n, 1, \ldots, 1}_{r})^{\mathrm{T}} \in \mathbb{R}^{rn}_{++}$$

and

$$u \otimes e = (\underbrace{u_1, u_1, \ldots, u_1}_{r}, \underbrace{u_2, u_2, \ldots, u_2}_{r}, \ldots, \underbrace{u_n, u_n, \ldots, u_n}_{r})^{\mathrm{T}} \in \mathbb{R}^{rn}_{++}.$$

Consequently, $u_1 = u_2 = \cdots = u_n$, which implies that H is regular.

Conversely, if H is a connected regular hypergraph, it is easy to see that the equality (6) holds. $\square$

We give the Power Mean inequality which will be used in the following remark: Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. If $0 < p < q$, then

$$\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{1/p} \leqslant \left(\frac{x_1^q + x_2^q + \cdots + x_n^q}{n}\right)^{1/q}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

Remark 11. It is known that the spectral radius $\rho(H)$ of H is greater than or equal to the average degree [5], i.e.,

$$\rho(H) \geqslant \frac{\sum_{i=1}^{n} d_i}{n}.$$

It follows from Power Mean inequality that

$$\left(\frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \geqslant \frac{\sum_{i=1}^{n} d_i}{n}.$$

Therefore Theorem 2 has a better estimation for spectral radius of H.

4 Proof of Theorem 3

In this section, we shall give a proof of Theorem 3.

Proof of Theorem 3. Let H be a connected r-uniform hypergraph with vertex set $V(H) = [n]$. Let $\widetilde{H}$ be the r-uniform hypergraph as defined in Theorem 2. Suppose that $\mathcal{B}$ is the order r and dimension r tensor given by (2), and $\mathcal{I}_r$ is the unit tensor of order r and dimension r. We have the following claims.

Claim 5. $\mathcal{Q}(\widetilde{H}) = (r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r) + \mathcal{A}(H) \otimes \mathcal{B}$.

Proof of Claim 5. Recall that $d_{\widetilde{H}}((i,j)) = (r-1)!d_i$ for any $i \in [n]$, $j \in [r]$, we have

$$\mathcal{D}(\widetilde{H})_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = \begin{cases}(r-1)!d_i, & \text{if } i_1 = \cdots = i_r = i, j_1 = \cdots = j_r = j,\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, by Definition 8 we obtain that

$$[(r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r)]_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = (r-1)!\mathcal{D}(H)_{i_1 i_2\cdots i_r}(\mathcal{I}_r)_{j_1 j_2\cdots j_r}.$$

If $i_1 = i_2 = \cdots = i_r = i$, $i \in [n]$ and $j_1 = j_2 = \cdots = j_r = j$, then

$$[(r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r)]_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = (r-1)!d_i$$

and 0 otherwise. It follows that

$$\mathcal{D}(\widetilde{H}) = (r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r).$$

Therefore, we have

$$\mathcal{Q}(\widetilde{H}) = \mathcal{D}(\widetilde{H}) + \mathcal{A}(\widetilde{H}) = (r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r) + \mathcal{A}(H) \otimes \mathcal{B}.$$

The proof of the claim is completed.

Claim 6. $\rho(\mathcal{Q}(\widetilde{H})) = (r-1)!\rho(\mathcal{Q}(H))$.

Proof of Claim 6. Since H is connected, $\mathcal{Q}(H)$ is weakly irreducible. From Theorem 6, we let u be the positive eigenvector to $\rho(\mathcal{Q}(H))$. Let $e = (1, 1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^r_{++}$. By Proposition 9, we deduce that

$$\begin{aligned}
\mathcal{Q}(\widetilde{H})(u \otimes e) &= [(r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r) + \mathcal{A}(H) \otimes \mathcal{B}](u \otimes e) \\
&= [(r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r)](u \otimes e) + [\mathcal{A}(H) \otimes \mathcal{B}](u \otimes e) \\
&= (r-1)!(\mathcal{D}(H)u) \otimes (\mathcal{I}_r e) + (\mathcal{A}(H)u) \otimes (\mathcal{B}e) \\
&= (r-1)!(\mathcal{D}(H)u) \otimes e + (r-1)![(\mathcal{A}(H)u) \otimes e] \\
&= (r-1)![\mathcal{D}(H)u + \mathcal{A}(H)u] \otimes e \\
&= (r-1)!(\mathcal{Q}(H)u) \otimes e.
\end{aligned}$$

It follows from $\mathcal{Q}(H)u = \rho(\mathcal{Q}(H))u^{[r-1]}$ that

$$\mathcal{Q}(\widetilde{H})(u \otimes e) = (r-1)!\rho(\mathcal{Q}(H))(u \otimes e)^{[r-1]},$$

which yields that $u \otimes e$ is a positive eigenvector of $\mathcal{Q}(\widetilde{H})$ corresponding to $(r-1)!\rho(\mathcal{Q}(H))$. Note that H is connected, then $\widetilde{H}$ is connected by Claim 1. Therefore $\mathcal{Q}(\widetilde{H})$ is weakly irreducible. By Theorem 6, $(r-1)!\rho(\mathcal{Q}(H))$ is the spectral radius of signless Laplacian tensor $\mathcal{Q}(\widetilde{H})$, as claimed.

Let $x \in \mathbb{R}^{rn}$ be the column vector defined by (3). By Theorem 7, we have

$$\begin{aligned}
\rho(\mathcal{Q}(\widetilde{H})) &\geqslant x^{\mathrm{T}}(\mathcal{Q}(\widetilde{H})x) = x^{\mathrm{T}}(\mathcal{D}(\widetilde{H})x) + x^{\mathrm{T}}(\mathcal{A}(\widetilde{H})x) \\
&= \sum_{j=1}^{r}\sum_{i=1}^{n} \mathcal{D}(\widetilde{H})_{(i,j)(i,j)\cdots(i,j)} x^r_{(i,j)} + x^{\mathrm{T}}(\mathcal{A}(\widetilde{H})x) \\
&= \frac{(r-1)!}{rn}\sum_{i=1}^{n}(a_i^r + r - 1)d_i + x^{\mathrm{T}}(\mathcal{A}(\widetilde{H})x).
\end{aligned} \tag{7}$$

Furthermore, by AM–GM inequality, we have

$$a_i^r + r - 1 = a_i^r + \underbrace{1 + 1 + \cdots + 1}_{r-1} \geqslant r a_i, \ i \in [n]. \tag{8}$$

So it follows from (4), (7) and (8) that

$$\begin{aligned}
\rho(\mathcal{Q}(\widetilde{H})) &\geqslant \frac{(r-1)!}{rn}\sum_{i=1}^{n}(a_i^r + r - 1)d_i + \frac{(r-1)!}{n}\sum_{i=1}^{n} a_i d_i \\
&\geqslant \frac{2(r-1)!}{n}\sum_{i=1}^{n} a_i d_i.
\end{aligned}$$

By (5), we obtain

$$\rho(\mathcal{Q}(H)) = \frac{\rho(\mathcal{Q}(\widetilde{H}))}{(r-1)!} \geqslant \frac{2}{n}\sum_{i=1}^{n} a_i d_i = 2\left(\frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}. \tag{9}$$

Suppose that the equality holds in (9). Then the vector $x \in \mathbb{R}^{rn}$ defined by (3) is an eigenvector corresponding to $\rho(\mathcal{Q}(\widetilde{H}))$ by Theorem 7 and $a_i = 1$, $i \in [n]$ by (8). Recall that $\mathcal{Q}(\widetilde{H})$ is weakly irreducible. By Claim 6, $u \otimes e$ is a positive eigenvector to $\rho(\mathcal{Q}(\widetilde{H}))$. We see that x and $u \otimes e$ are linearly dependent by Theorem 6. Therefore, $d_1 = d_2 = \cdots = d_n$, which implies that H is regular.

Conversely, if H is a regular connected hypergraph, it is straightforward to verify that the equality (9) holds. □

Remark 12. Recently, Nikiforov introduces the concept of odd-colorable hypergraphs [15], which is a generalization of bipartite graphs. Let $r \geqslant 2$ and r be even. An r-uniform hypergraph H with $V(H) = [n]$ is called odd-colorable if there exists a map $\varphi : [n] \to [r]$ such that for any edge $\{i_1, i_2, \dots, i_r\}$ of H, we have

$$\varphi(i_1) + \varphi(i_2) + \cdots + \varphi(i_r) \equiv \frac{r}{2} \pmod{r}.$$

It was proved that if H is a connected r-uniform hypergraph, then $\rho(\mathcal{L}(H)) = \rho(\mathcal{Q}(H))$ if and only if r is even and H is odd-colorable [23]. Therefore we have

$$\rho(\mathcal{L}(H)) \geqslant 2\left(\frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}$$

for a connected odd-colorable hypergraph H, which generalizes a result in [24, Corollary 10].

5 Concluding remarks

In this paper, we generalize a lower bound of Hofmeister to an r-uniform hypergraph H as follows:

$$\rho(H) \geqslant \left(\frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}},$$

and characterize the extremal hypergraphs attaining the bound. In [14], Nikiforov showed that the exponent $\frac{r}{r-1}$ is the best possible by proving the following statement: If $r \geqslant 2$ and $\varepsilon > 0$, there is an r-uniform hypergraph H on n vertices such that

$$\rho(H) < \left(\frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}+\varepsilon}\right)^{\frac{1}{r/(r-1)+\varepsilon}}.$$

As mentioned in Section 1, for a graph G, $\rho(G) \geqslant \overline{d}$ with equality if and only if G is regular. A point of view inspired by this fact is to consider the difference between the spectral radius and the average degree as a measure of irregularity. In some sense, the more regular graph is the closer the gap between the spectral radius and the average degree. Relatedly, Hong [9] posed an open problem on minimizing the difference between the spectral radius and the average degree of a connected graph of given order and size (see Problem 3 in [9]). Let us note that Cioabă and Gregory [3] and Nikiforov [13] presented some lower bounds of $\rho(G) - \overline{d}$ in terms of the degrees and therefore extended and improved the inequality $\rho(G) \geqslant \overline{d}$. It is interesting to generalize these results to hypergraphs. Predictably, some of them can not be generalized straightforwardly from the graph results as the hypergraph spectral theory are quite intricate. Recently, Si and Yuan [22] prove that

$$\rho(H) - \overline{d} \geqslant \frac{r}{2n}\left[\sqrt[r]{2}(\Delta^{\frac{r}{r-1}} + \delta^{\frac{r}{r-1}})^{\frac{r-1}{r}} - (\Delta + \delta)\right]$$

for an r-uniform hypergraph H, which generalizes and improves a result in [3, Corollary 3]. Moreover, Liu, Kang and Shan [11] investigate the irregularity of uniform hypergraphs, and present some bounds of $\rho(H) - \overline{d}$ with respect to the measures of irregularity of uniform hypergraphs, which extend some results of Nikiforov [12] to uniform hypergraphs.

References

[1] A. Bretto. *Hypergraph Theory: An Introduction*. Springer, 2013.

[2] K. C. Chang, K. Pearson, T. Zhang. Perron–Frobenius theorem for nonnegative tensors. *Commun. Math. Sci.*, 6:507–520, 2008.

[3] S. M. Cioabă, D. A. Gregory. Large matchings from eigenvalues. *Linear Algebra Appl.*, 422:308–317, 2007.

[4] L. Collatz, U. Sinogowitz. Spektren endlicher Grafen. *Abh. Math. Sem. Univ. Hamburg*, 21:63–77, 1957.

[5] J. Cooper, A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra Appl.*, 436:3268–3299, 2012.

[6] M. Fiedler, V. Nikiforov. Spectral radius and Hamiltonicity of graphs. *Linear Algebra Appl.*, 432:2170–2173, 2010.

[7] S. Friedland, A. Gaubert, L. Han. Perron–Frobenius theorems for nonnegative multilinear forms and extensions. *Linear Algebra Appl.*, 438:738–749, 2013.

[8] M. Hofmeister. Spectral radius and degree sequence. *Math. Nachr.*, 139:37–44, 1988.

[9] Y. Hong. Bounds of eigenvalues of graphs. *Discrete Math.*, 123:65–74, 1993.

[10] L. H. Lim. Singular values and eigenvalues of tensors: a variational approach. In *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing* (CAMSAP05), 1:129–132, 2005.

[11] L. Liu, L. Kang, E. Shan. On the irregularity of uniform hypergraphs. *European J. Combin.*, 71:22–32, 2008.

[12] V. Nikiforov. Eigenvalues and degree deviation in graphs. *Linear Algebra Appl.*, 414:347–360, 2006.

[13] V. Nikiforov. Bounds on graph eigenvalues II. *Linear Algebra Appl.*, 427:183–189, 2007.

[14] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra Appl.*, 457:455–535, 2014.

[15] V. Nikiforov. Hypergraphs and hypermatrices with symmetric spectrum. *Linear Algebra Appl.*, 519:1–18, 2017.

[16] K. Pearson, T. Zhang. On spectral hypergraph theory of the adjacency tensor. *Graphs Combin.*, 30:1233–1248, 2014.

[17] L. Qi. Eigenvalues of a real supersymmetric tensor. *J. Symbolic Comput.*, 40:1302–1324, 2005.

[18] L. Qi. Symmetric nonnegative tensors and copositive tensors. *Linear Algebra Appl.*, 439:228–238, 2013.

[19] L. Qi. H^+-eigenvalue of Laplacian and signless Laplacian tensors. *Commun. Math. Sci.*, 12:1045–1064, 2014.

[20] J. Shao. A general product of tensors with applications. *Linear Algebra Appl.*, 439:2350–2366, 2013.

[21] J. Shao, H. Shan, B. Wu. Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs. *Linear Multilinear Algebra*, 63(12):2359–2372, 2015.

[22] X. Si, X. Yuan. On the spectral radii and principal eigenvectors of uniform hypergraphs. *Discrete Math., Alg. and Appl.*, 9(4): 2017.

[23] X. Yuan, L. Qi, J. Shao, C. Ouyang. Some properties and applications of odd-colorable r-hypergraphs. *Discrete Appl. Math.*, 236:446–452, 2018.

[24] A. Yu, M. Lu, F. Tian. On the spectral radius of graphs. *Linear Algebra Appl.*, 387:41–49, 2004.