Bipartite Ramsey numbers for bipartite graphs of small bandwidth

Lili Shen
College of Mathematics and Computer Science
Fuzhou University
Fuzhou, PR China
sllbeilin112@163.com

Qizhong Lin[*,†] Qinghai Liu [‡]
Center for Discrete Mathematics
Fuzhou University
Fuzhou, PR China
{linqizhong, qliu}@fzu.edu.cn

Submitted: Sep 17, 2017; Accepted: Apr 4, 2018; Published: Apr 27, 2018

Abstract

A graph $H = (W, E_H)$ is said to have *bandwidth* at most b if there exists a labeling of W as $w_1, w_2, \ldots, w_n$ such that $|i - j| \leqslant b$ for every edge $w_i w_j \in E_H$, and a *bipartite balanced* (β, Δ)*-graph* H is a bipartite graph with bandwidth at most $\beta|W|$ and maximum degree at most Δ, and furthermore it has a proper 2-coloring $\chi : W \to [2]$ such that $||\chi^{-1}(1)| - |\chi^{-1}(2)|| \leqslant \beta|\chi^{-1}(2)|$. We prove that for any fixed $0 < \gamma < 1$ and integer $\Delta \geqslant 1$, there exist a constant $\beta = \beta(\gamma, \Delta) > 0$ and a natural number n_0 such that for every balanced (β, Δ)-graph H on $n \geqslant n_0$ vertices the bipartite Ramsey number $br(H, H)$ is at most $(1 + \gamma)n$. In particular, $br(C_{2n}, C_{2n}) = (2 + o(1))n$.

Mathematics Subject Classifications: 05D10

1 Introduction

For graphs G, H_1 and H_2, denote $G \to (H_1, H_2)$ by that any red/blue edge coloring of G containing either a red copy of H_1 or a blue copy of H_2. In Ramsey Theory, a well-known problem is to determine the Ramsey number $r(H_1, H_2)$, i.e. the minimum integer N such that $K_N \to (H_1, H_2)$. We refer the readers to the book by Graham, Rothschild and Spencer [9] for an overview and a survey by Conlon, Fox and Sudakov [7] for many recent developments.

A natural generalization of the above problem is to determine the bipartite Ramsey number $br(H_1, H_2)$, the minimum integer N such that $K_{N,N} \to (H_1, H_2)$, where H_1 and

[*]Supported by NSFC(11671088) and NSFFP(2016J01017).
[†]Corresponding author.
[‡]Supported by NSFC(11301086).

H_2 are bipartite graphs. Since an edge coloring of K_{2N} induces an edge coloring of $K_{N,N}$, we have

$$r(H_1, H_2) \leqslant 2br(H_1, H_2).$$

The bipartite Ramsey numbers involving complete bipartite graphs have attracted most attention. Beineke and Schwenk [1] conjectured that $br(K_{t,n}, K_{t,n}) = 2^t(n-1)+1$ for any $n \geqslant t \geqslant 1$, which is trivially true if $t = 1$. In particular, the authors in [1] proved that $br(K_{2,n}, K_{2,n}) \leqslant 4n-3$, and $br(K_{2,n}, K_{2,n}) = 4n-3$ if there is a Hadamard matrix of order $2(n-1)$ for odd n, and $br(K_{3,n}, K_{3,n}) \geqslant 8n-7$ if there is a Hadamard matrix of order $4(n-1)$. Irving [12] proved $br(K_{3,n}, K_{3,n}) \leqslant 8n-7$, which together with the lower bound obtained in [1] gave $br(K_{3,n}, K_{3,n}) = 8n-7$ for infinitely many values of n. However, for the diagonal case, Irving disproved the general conjecture by showing that $br(K_{n,n}, K_{n,n}) < 2^{n-1}(n-1)$ for $n \geqslant 21$. Thomason [23] proved that $br(K_{t,n}, K_{t,n}) \leqslant 2^t(n-1)+1$ for any $n \geqslant t \geqslant 1$. For fixed $t \geqslant 2$, Li, Tang and Zang [14] proved that $br(K_{t,n}, K_{t,n}) = (1+o(1))2^t n$ as $n \to \infty$. It was shown that

$$(1+o(1))\frac{\sqrt{2}}{e}n2^{n/2} \leqslant br(K_{n,n}, K_{n,n}) \leqslant (1+o(1))2^{n+1}\log n,$$

where the upper bound by Conlon [6] improved that by Irving [12], while the lower bound by Hattingh and Henning [11] is similar to Spencer's lower bound [20] for ordinary Ramsey numbers by using Lovász Local Lemma, and here and henceforth logarithmic function has natural base 2. For fixed $t \geqslant 2$, Caro and Rousseau [5] proved that

$$c_1\left(\frac{n}{\log n}\right)^{(t+1)/2} < br(K_{t,t}, K_{n,n}) < c_2\left(\frac{n}{\log n}\right)^t,$$

where $c_i = c_i(t) > 0$ are constants. Recently, Lin and Li [16] proved that the order of magnitude of $br(K_{t,n}, K_{n,n})$ is $n^{t+1}/(\log n)^t$, but the orders of magnitude of $br(K_{n,n}, K_{n,n})$ and $br(K_{t,t}, K_{n,n})$, that are more interesting, seems to be very hard to obtain. For more bipartite Ramsey numbers involving small complete bipartite graphs, see e.g. [1, 4, 12, 11].

For the bipartite Ramsey numbers involving non-complete bipartite graphs, there are not too many references. Let P_n be the path of order n, and let C_n be the cycle of order n. A celebrated result by Faudree and Schelp [8] determined the "Ramsey bipartite number pair" $B(P_n, P_m)$ for all $m, n \geqslant 1$, which implied the corresponding bipartite Ramsey number $br(P_n, P_m)$. However, there are not too many references on the bipartite Ramsey numbers $br(C_{2n}, C_{2m})$. In [24, 25], Zhang, Sun and Wu obtained that $br(C_{2n}, C_4) = n+1$, and $br(C_{2n}, C_6) = n+2$ for $n \geqslant 4$.

In this paper, we mainly consider the bipartite balanced (β, Δ)-graphs.

Theorem 1. *For each fixed $0 < \gamma < 1$ and every natural number $\Delta \geqslant 1$, there exist a constant $\beta = \beta(\gamma, \Delta) > 0$ and a natural number n_0 such that for every bipartite balanced (β, Δ)-graph H on $n \geqslant n_0$ vertices we have*

$$br(H, H) \leqslant (1+\gamma)n.$$

From Theorem 1, the following corollary follows easily and the lower bound construction will be presented below.

Corollary 2. *Let C_{2n} be the cycle of length $2n$. Then*

$$br(C_{2n}, C_{2n}) = (2 + o(1))n.$$

Proof. Indeed, note that C_{2n} defined on $[2n] = \{1, 2, \ldots, 2n\}$ is a balanced $(1/n, 2)$-graph if we label the vertices and denote the cycle as

$$1, 3, 5, \ldots, 2n - 1, 2n, 2n - 2, \ldots, 4, 2, 1,$$

so Theorem 1 implies that $br(C_{2n}, C_{2n}) \leqslant (2+o(1))n$ for sufficiently large n. On the other hand, consider the bipartite graph F with bipartition $U = \{u_1, u_2, \ldots, u_{2n-2}\}$ and $V = \{v_1, v_2, \ldots, v_{2n-2}\}$, in which $u_i v_j$ is an edge in F if and only if $u_i \in U$ and $1 \leqslant j \leqslant n - 1$. Clearly, F does not contain C_{2n} as a subgraph, and its complement $\overline{F}$ (which is restricted in $K_{2n-2,2n-2}$) also contains no copy of C_{2n}. This gives that $br(C_{2n}, C_{2n}) \geqslant 2n - 1$. Therefore, we have the above corollary as desired. $\square$

2 Preliminary results

Let A be a set of positive integers and $A_n = A \cap [n]$. In the 1930s, Erdős and Turán conjectured that if $\overline{\lim\limits_{n\to\infty}} \frac{|A_n|}{n} > 0$, then A contains arbitrarily long arithmetic progressions. The conjecture for the arithmetic progression of length 3 was proved by Roth [18, 19]. The full conjecture was proved by Szemerédi [21] with a deep and complicated combinatorial argument. In the proof, Szemerédi used a result, which is now called the bipartite regularity lemma, and then he proved the general Regularity Lemma, see [22]. The lemma has become a powerful tool in extremal graph theory. For many applications, we refer the readers to the survey of Komlós and Simonovits [13] and related references.

2.1 The regularity method

Let $G = (V, E)$ be a graph on n vertices. The density of G is given by $d_G = |E|/\binom{n}{2}$. For disjoint vertex sets $A, B \subseteq V$, let $E_G(A, B)$ denote the number of edges of G with one endpoint in A and the other in B, and the density of (A, B) is

$$d_G(A, B) = \frac{E_G(A, B)}{|A||B|}.$$

For $\epsilon > 0, d \leqslant 1$, a pair (A, B) is ϵ-regular if for all $X \subseteq A$, $Y \subseteq B$ with $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $|d_G(X, Y) - d_G(A, B)| < \epsilon$. Then, (A, B) is called (ϵ, d)-regular if it is ϵ-regular and $d_G(A, B) \geqslant d$. Moreover, (A, B) is called (ϵ, d)-super-regular if it is ϵ-regular and $deg_G(u) > d|B|$ for all $u \in A$ and $deg_G(v) > d|A|$ for all $v \in B$, where $deg_G(u)$ is the degree of a vertex u in G.

The following property says that subgraphs of a regular pair are regular.

Claim 3. *If (A, B) is (ϵ, d)-regular, and let $X \subseteq A$ and $Y \subseteq B$ with $|X| \geqslant \alpha|A|$ and $|Y| \geqslant \alpha|B|$ for some $\alpha > \epsilon$. Then (X, Y) is ϵ'-regular such that $|d(A, B) - d(X, Y)| < \epsilon$, where $\epsilon' = \max\{\frac{\epsilon}{\alpha}, 2\epsilon\}$.*

Let us have another property that any regular pair has a large subgraph which is super-regular, and we include a proof for completeness.

Claim 4. *For $0 < \epsilon < 1/2$ and $d \leqslant 1$, if (A, B) is (ϵ, d)-regular with $|A| = |B| = m$ then there exist $A_1 \subseteq A$ and $B_1 \subseteq B$ with $|A_1| = |B_1| = (1-\epsilon)m$ such that (A_1, B_1) is $(2\epsilon, d - 2\epsilon)$-super-regular.*

Proof. Let $X \subseteq A$ consists of vertices with at most $(d-\epsilon)|B|$ neighbors in B. Since $e(X, B) \leqslant |X|(d-\epsilon)|B|$ we have $|d(X, B) - d| \geqslant \epsilon$. According to the definition of ϵ-regular, we know that $|X| < \epsilon m$. Similarly, let $Y \subseteq B$ consists of vertices with at most $(d-\epsilon)|A|$ neighbors in A, we know that $|Y| < \epsilon m$.

Take $A_1 \subseteq A \setminus X$ and $B_1 \subseteq B \setminus Y$ with $|A_1| = |B_1| = (1-\epsilon)m$. Clearly, each vertex of A_1 has at least $(d-\epsilon)m - \epsilon m = (d-2\epsilon)m$ neighbors in B_1, and similarly each vertex of B_1 has at least $(d-2\epsilon)m$ neighbors in A_1. On the other hand, for any subset $S \subseteq A_1$ and $T \subseteq B_1$. If $|S| \geqslant 2\epsilon|A_1|$ and $|T| \geqslant 2\epsilon|B_1|$, then clearly $|S| \geqslant \epsilon m$ and $|T| \geqslant \epsilon m$. From the fact that (A, B) is (ϵ, d)-regular, we have

$$|d(S, T) - d(A_1, B_1)| \leqslant |d(S, T) - d(A, B)| + |d(A_1, B_1) - d(A, B)| < 2\epsilon.$$

This completes the proof. $\square$

In this paper, we shall use the following bipartite form of the regularity lemma. In [3], Böttcher, Heinig and Taraz give a proof (sketch), one can also find a detailed proof in Lin and Li [15, Lemma 5]. For a bipartite graph G with bipartition $(V^{(1)}, V^{(2)})$, a partition $\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}$ for each $V^{(s)}$ $(s = 1, 2)$ is said to be (ϵ, d)-regular on $R = R([k], [k]; E_R)$ if $ij \in E_R$ implies that $(V_i^{(1)}, V_j^{(2)})$ is (ϵ, d)-regular in G for $i, j \in [k]$. If such a partition exists, we also say that R is an (ϵ, d)-reduced graph of G.

Lemma 5. *For any $\epsilon > 0$ and integer $k_0 \geqslant 1$, there exists $K_0 = K_0(\epsilon, k_0)$ such that every bipartite graph G with bipartition $(V^{(1)}, V^{(2)})$ satisfying $|V^{(1)}| = |V^{(2)}| \geqslant K_0$ has a partition $\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}$ for each $V^{(s)}$ for $s = 1, 2$, where k is the same for each part $V^{(s)}$ with $k_0 \leqslant k \leqslant K_0$, such that*

1. *$|V_1^{(s)}| = \cdots = |V_k^{(s)}|$ and $|V_0^{(s)}| \leqslant \epsilon|V^{(s)}|$ for $s = 1, 2$;*
2. *All but at most ϵk^2 pairs $(V_i^{(1)}, V_j^{(2)})$, $1 \leqslant i, j \leqslant k$, are ϵ-regular.*

It is known that the Blow-up Lemma guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently large super-regular pairs. In order to prove Theorem 1.1, we need to find a copy of H in a monochromatic graph G of any red/blue edge coloring of $K_{N,N}$. Similar to the Blow-up Lemma, the following Embedding Lemma allows us to embed H into G if H and G have "compatible" partitions.

Let $H = (W, E_H)$ be a graph, for a vertex $w \in W$, denote $N_H(w)$ by the neighborhood of w in H. For $S \subseteq W$, denote $N_H(S) = [\cup_{v \in S} N_H(v)] \setminus S$.

Definition 6. Let $H = (W, E_H)$ and $R = ([k], E_R)$ be graphs and let $R' = ([k], E_{R'})$ be a subgraph of R. We say that a vertex partition $W = (W_i)_{i\in[k]}$ of H is ϵ-compatible with a vertex partition $V = (V_i)_{i\in[k]}$ of a graph $G = (V, E_G)$ if the following holds.

For $i, j \in [k]$ with $i \neq j$, let $S_i \subseteq W_i$ that has some neighbours in W_j for $ij \in E_R \setminus E_{R'}$. Set $S = \cup S_i$ and $T_i = N_H(S) \cap (W_i \setminus S)$. Then

(1) $|W_i| \leqslant |V_i|$ for $i \in [k]$.

(2) $xy \in E_H$ for $x \in W_i, y \in W_j$ implies $ij \in E_R$.

(3) $|S_i| \leqslant \epsilon|V_i|$ for $i \in [k]$.

(4) $|T_i| \leqslant \epsilon \min\{|V_j| : ij \in E_{R'}\}$.

That is to say, from this definition, we particularly require that the edges of H run only between classes that correspond to a dense regular pair in G (condition (2)). The vertices that have neighbors not belong to the super-regular pairs are assumed to be few (condition (3)), and the neighbours of these vertices are assumed to be few too (condition (4)).

The following Embedding Lemma by Böttcher, Heinig and Taraz [2, 3] is crucial for us to prove our main result.

Lemma 7 (**Embedding Lemma**)**.** *For all $d > 0$ and integers $\Delta, r \geqslant 1$, there is a constant $\varepsilon = \varepsilon(d, \Delta, r) > 0$ such that the following holds. Let $G = (V, E)$ be an N-vertex graph that has a partition $(V_i)_{i\in[k]}$ with (ε, d)-reduced graph R on $[k]$, and $R' \subseteq R$ is (ε, d)-super-regular with connected components having at most r vertices each. Suppose $H = (W, E_H)$ is a graph of order $n \leqslant N$ with maximum degree $\Delta(H) \leqslant \Delta$ that has a partition $(W_i)_{i\in[k]}$ which is ε-compatible with $(V_i)_{i\in[k]}$. Then $H \subseteq G$.*

In the above lemma, if $R' = M$ is a matching of R, then $r = 2$, and so $\varepsilon = \varepsilon(d, \Delta)$ depends only on d and Δ.

2.2 Long path in bipartite reduced graph

In our proof, for any red/blue edge coloring of $K_{N,N}$, we will embed the bipartite graph H into one of the monochromatic subgraph G of $K_{N,N}$. From the regularity lemma, the reduced graph R corresponding to G has enough ε-regular pairs, and we need to find a long path of this bipartite graph R such that we can embed H into the subgraph induced by all edges corresponding to the regular pairs of this path. This can be achieved from the result by Gyárfás, Rousseau and Schelp[10, Theorem 1].

Lemma 8. *Let H be a bipartite graph with bipartition (U, V), where $|U| = k_1$ and $|V| = k_2$ $(k_1 \leqslant k_2)$. If $c \leqslant k_1/2$ and H contains no path P_{2t} for $t > c$, then $e(H) \leqslant (k_1 + k_2 - 2c)c$.*

From the above lemma, we have the following result immediately.

Corollary 9. *Let $0 < \epsilon < 1/4$ and let H be a bipartite graph with bipartition (U, V), where $|U| = k = |V|$. If $e(H) > (\frac{1}{2} - \epsilon)k^2$, then H contains a path P_{2t} with $t > \lfloor(\frac{1}{2} - \sqrt{\epsilon/2})k\rfloor$.*

Proof. Take $c = \lfloor (\frac{1}{2} - \sqrt{\epsilon/2})k \rfloor = (\frac{1}{2} - \sqrt{\epsilon/2})k - \eta$ for some $\eta \geqslant 0$. From Lemma 8, if H contains no path P_{2t} with $t > c$, then

$$\begin{aligned} e(H) &\leqslant (k + k - 2c)c = \left(2k - 2\left[\left(\frac{1}{2} - \sqrt{\frac{\epsilon}{2}}\right)k - \eta\right]\right)\left[\left(\frac{1}{2} - \sqrt{\frac{\epsilon}{2}}\right)k - \eta\right] \\ &= \frac{1}{2}\left[k^2 - \left(2\sqrt{\frac{\epsilon}{2}} \cdot k + 2\eta\right)^2\right] \leqslant \left(\frac{1}{2} - \epsilon\right)k^2, \end{aligned}$$

a contradiction. This completes the proof. □

In the following, for a red-blue edge coloring of $K_{N,N}$, let us denote G_r and G_b by the graphs induced by all red edges and blue edges, respectively.

Lemma 10. *For any $0 < \epsilon < 1/4$ and any integer $k_0 \geqslant 1$, there exists K_0 such that any red-blue edge coloring of $K_{N,N}$ with bipartition $(V^{(1)}, V^{(2)})$ and $N \geqslant K_0$ has an ϵ-regular partition $\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}$ for each $V^{(s)}$ $(s = 1, 2)$ with $k_0 \leqslant k \leqslant K_0$ satisfying the following property. The reduced graph contains a monochromatic path P_{2t} with $t \geqslant (\frac{1}{2} - \sqrt{\epsilon/2})k$ such that all regular pairs corresponding to the edges of P_{2t} are $(\epsilon, 1/2)$-regular on G_r or all of these corresponding regular pairs are $(\epsilon, 1/2)$-regular on G_b.*

Proof. From Lemma 5, there is a partition $\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}$ for each $V^{(s)}$ for $s = 1, 2$, where k is the same for each part $V^{(s)}$ and $k_0 \leqslant k \leqslant K_0$, such that (1) $|V_1^{(s)}| = \cdots = |V_k^{(s)}|$ and $|V_0^{(s)}| \leqslant \epsilon |V^{(s)}|$ for $s = 1, 2$; (2) All but at most ϵk^2 pairs $(V_i^{(1)}, V_j^{(2)})$, $1 \leqslant i, j \leqslant k$, are ϵ-regular.

Let R be the reduced subgraph of $K_{k,k}$ with bipartition $([k], [k])$, in which a pair ij for $i, j \in [k]$ is adjacent in R if and only if $(V_i^{(1)}, V_j^{(2)})$ is ϵ-regular. Thus $e(R) \geqslant (1-\epsilon)k^2$. Color an edge ij of R green if $d_{G_r}(V_i^{(1)}, V_j^{(2)}) \geqslant 1/2$, or white if $d_{G_r}(V_i^{(1)}, V_j^{(2)}) < 1/2$. Denote by R_g and R_w the subgraphs spanned by green edges and white edges, respectively. Without loss of generality, we may assume that

$$e(R_g) \geqslant \frac{(1-\epsilon)k^2}{2} = \left(\frac{1}{2} - \frac{\epsilon}{2}\right)k^2.$$

From Corollary 9, there exists a path P_{2t} in R_g with $t > \lfloor (\frac{1}{2} - \sqrt{\epsilon/2})k \rfloor$, and all regular pairs corresponding to the edge of P_{2t} are $(\epsilon, 1/2)$-regular on G_r as desired. This completes the proof. □

2.3 Locally balanced intervals

Given $\beta > 0$ and integer $\Delta \geqslant 1$, we have known that a balanced (β, Δ)-graph H has a proper 2-coloring $\chi : V(H) \to [2]$ such that the sizes of the color classes are almost equal. This definition focuses on vertices of different colors as a whole. In fact, we can see that the two colors also have approximately the same number of vertices locally.

For a graph $H = (W, E)$ with $W = \{w_1, w_2, \ldots, w_n\}$, where w_i is a labeling of the vertices, let $\chi : W \to [2]$ be a 2-coloring. For $W' \subseteq W$, denote $C_i(W') = |\chi^{-1}(i) \cap W'|$ for

$i = 1, 2$. We know that χ is a β-balanced coloring of W if $1 - \beta \leqslant \frac{C_1(W)}{C_2(W)} \leqslant 1 + \beta$. A set $I \subseteq W$ is called interval if there exists $p < q$ such that $I = \{w_p, w_{p+1}, \ldots, w_q\}$. Finally, let $\sigma : [\ell] \to [\ell]$ be a permutation, and for a partition $\Gamma = \{I_1, I_2, \ldots, I_\ell\}$ of W, where each I_i is an interval, let $C_\tau(\Gamma, \sigma, a, b) = \sum_{j=a}^{b} C_\tau(I_{\sigma(j)})$ for $\tau = 1, 2$, and we always write $C_\tau(\sigma, a, b) = C_\tau(\Gamma, \sigma, a, b)$ for simplicity. The following result by Mota, Sárközy, Schacht and Taraz [17, Lemma 2.11] means that every balanced bipartite graph is also balanced in local.

Lemma 11. *For every $\xi > 0$ and integer $\ell \geqslant 1$ there exists n_0 such that if $H = (W, E)$ is a graph on $W = \{w_1, w_2, \ldots, w_n\}$ with $n \geqslant n_0$, then for every β-balanced 2-coloring χ of W with $\beta \leqslant 2/\ell$, and every partition of W into intervals $I_1, I_2, \ldots, I_\ell$ with $|I_1| \leqslant |I_2| \leqslant \ldots \leqslant |I_\ell| \leqslant |I_1| + 1$ there exists a permutation $\sigma : [\ell] \to [\ell]$ such that for every pair of integers $1 \leqslant a < b \leqslant \ell$ with $b - a \geqslant 7/\xi$,*

$$|C_1(\sigma, a, b) - C_2(\sigma, a, b)| \leqslant \xi C_2(\sigma, a, b).$$

3 Proof of Theorem 1.1

For every $0 < \gamma < 1$ and integer $\Delta \geqslant 1$, we want to prove that there exists a constant $\beta = \beta(\gamma, \Delta) > 0$ and a natural number n_0 such that if H is a balanced (β, Δ)-graph on n vertices for $n \geqslant n_0$ then any red-blue edge coloring of $K_{N,N}$ for $N = (1 + \gamma/3)n$ contains a monochromatic copy of H.

The main idea is as follows. First, we shall apply Lemma 10 to find a monochromatic subgraph, say red graph G_r, of any red/blue edge coloring of $K_{N,N}$ with sufficiently long path P of the reduced graph, and then use Claim 4 to get a subgraph G_P by deleting some vertices from G_r such that G_P contains sufficiently many dense super-regular pairs covering $(1 + o(1))n$ vertices. Second, we partition the vertices of H, and apply Lemma 11 to show that the partition is 2ϵ-compatible with the partition of G_P. Finally, we shall find a copy of H in G_P by using the Embedding Lemma (Lemma 7). The details are as follows.

For $0 < \gamma < 1$, $\Delta \geqslant 1$ be given and $d = 1/3$, we have $\varepsilon_0 = \varepsilon_0(d, \Delta)$ by Lemma 7. Set

$$\epsilon = \min\left\{\frac{\varepsilon_0}{2}, \frac{\gamma^2}{25}\right\}.$$

For such $\epsilon > 0$ defined above and integer $k_0 \geqslant 1$, K_0 is determined by k_0 and ϵ from Lemma 10. Fix

$$\xi = \frac{\gamma}{60},$$

and let n_0 be obtained from Lemma 11 dependent on ξ and K_0. Set

$$\beta = \frac{\epsilon\xi(1 + 2\xi)}{36\Delta^2 K_0^2}.$$

For sufficiently small $\gamma > 0$, we have

$$\beta \ll \epsilon \ll \xi < \gamma.$$

Let

$$c = c(\gamma) = 1 + \frac{\gamma}{3},$$

and let $H = (W, E_H)$ be a balanced (β, Δ)-graph on n vertices with $n \leqslant N$, where $N = cn \geqslant \max\{n_0, K_0\}$.

3.1 Preparing the host graph G_P

From Lemma 10, any red-blue edge coloring of $K_{N,N}$ with bipartition $(V^{(1)}, V^{(2)})$ and $N \geqslant K_0$ has an ϵ-regular partition $\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}$ for each $V^{(s)}$ $(s = 1, 2)$ with $k_0 \leqslant k \leqslant K_0$ satisfying the following property. The reduced graph contains a monochromatic path P_{2t} with $t \geqslant (\frac{1}{2} - \frac{\sqrt{\epsilon}}{2})k$, and all regular pairs corresponding to the edges of P_{2t} are $(\epsilon, 1/2)$-regular on G_r or G_b, say G_r.

Without loss of generality, we may relabel those $V_i^{(s)}$'s for $s = 1, 2$ such that P_{2t} defined on $([t], [t])$ that corresponding to $V_i^{(1)}$ and $V_i^{(2)}$ for $i \in [t]$, where ii (corresponds to the regular pair $V_i^{(1)} V_i^{(2)}$) for $1 \leqslant i \leqslant t$ and $(i+1)i$ (corresponds to the regular pair $V_{i+1}^{(1)} V_i^{(2)}$) for $1 \leqslant i \leqslant t-1$ are those edges of P_{2t}, and suppose that all edges of type ii consist of the edges of the matching M of P_{2t}.

Applying Claim 4 to the regular pair $(V_i^{(1)}, V_i^{(2)})$ for $1 \leqslant i \leqslant t$, we have $A_i \subseteq V_i^{(1)}$ and $B_i \subseteq V_i^{(2)}$ with $|A_i| = |B_i| \geqslant (1 - \epsilon)m$ such that

$$(A_i, B_i) \text{ is } (2\epsilon, 1/2 - 2\epsilon)\text{-super regular.}$$

Note that $\gamma > 0$, $\epsilon \leqslant \gamma^2/25$, $c = 1 + \gamma/3$ and $t \geqslant (1/2 - \sqrt{\epsilon}/2)k$, so we have

$$|A_i| = |B_i| \geqslant (1 - \epsilon)m \geqslant (1 - \epsilon)\frac{(1 - \epsilon)N}{k} \geqslant \left(1 + \frac{\gamma}{30}\right)\frac{n}{2t}. \tag{1}$$

Slightly abusing the notations, we also denote P_{2t} by the path defined on $([t], [t])$ corresponding to the pairs (A_i, B_i) for $1 \leqslant i \leqslant t$ and (A_{i+1}, B_i) for $1 \leqslant i \leqslant t-1$, and denote M by the matching corresponding to the pairs (A_i, B_i) for $1 \leqslant i \leqslant t$ that are $(2\epsilon, 1/2 - 2\epsilon)$-super regular. Note that all the other pairs (A_{i+1}, B_i) corresponding to edges of $P_{2t} \setminus M$ are $(2\epsilon, 1/2 - 2\epsilon)$-regular since the pairs $(V_{i+1}^{(1)}, V_i^{(2)})$ are $(\epsilon, 1/2)$-regular for $1 \leqslant i \leqslant t-1$. Let G_P be the subgraph induced by all of the red edges from $(\cup_{i=1}^t A_i) \cup (\cup_{i=1}^t B_i)$ of G_r.

3.2 Preparing H

Since $H = (W, E_H)$ is a balanced (β, Δ)-graph, there is a proper 2-coloring $\chi : V(H) \to [2]$ such that $||\chi^{-1}(1)| - |\chi^{-1}(2)|| \leqslant \beta|\chi^{-1}(2)|$. Label the vertices of W as $w_1, w_2, \ldots, w_n$ such that $|g - h| \leqslant \beta n$ for every edge $w_g w_h \in E_H$, and let ℓ be the smallest integer divisible by t with $\ell \geqslant 7(K_0/\xi) + t \geqslant t(7/\xi + 1)$. Moreover, we may choose n suitably such that n is divisible by ℓ, and let $\Gamma = \{I_1, I_2, \ldots, I_\ell\}$ be the partition of W with $|I_1| = |I_2| = \cdots = |I_\ell| = \frac{n}{\ell}$ in order, i.e., for $1 \leqslant i \leqslant \ell$,

$$I_i = \left\{w_{(i-1)\frac{n}{\ell}+1}, w_{(i-1)\frac{n}{\ell}+2}, \ldots, w_{i\frac{n}{\ell}}\right\}.$$

Let $a_i = (i-1)\frac{\ell}{t} + 1$ and $b_i = i\frac{\ell}{t}$. Then $b_i - a_i = \frac{\ell}{t} - 1 \geqslant 7/\xi$. Since $\beta = \frac{\epsilon\xi(1+2\xi)}{36\Delta^2 K_0^2} < 1/\ell$, Lemma 11 implies that there exists a permutation $\sigma : [\ell] \to [\ell]$ such that

$$|C_1(\sigma, a_i, b_i) - C_2(\sigma, a_i, b_i)| \leqslant \xi C_2(\sigma, a_i, b_i),$$

where $C_\tau(\sigma, a_i, b_i) = \sum_{j=a_i}^{b_i} C_\tau(I_{\sigma(j)})$ for $\tau = 1, 2$. Denote

$$J_i = I_{\sigma(a_i)} \cup I_{\sigma(a_i+1)} \cup \cdots \cup I_{\sigma(b_i)}, \quad C_\tau(J_i) = C_\tau(\sigma, a_i, b_i)$$

for $\tau = 1, 2$. Clearly $|J_i| = (b_i - a_i + 1)\frac{n}{\ell} = \frac{n}{t}$, and

$$(1-\xi)\frac{n}{2t} \leqslant C_1(J_i), C_2(J_i) \leqslant (1+\xi)\frac{n}{2t}.$$

Note that

$$W = \bigcup_{i=1}^{\ell} I_i = \bigcup_{i=1}^{\ell} I_{\sigma(i)} = \bigcup_{i=1}^{t} J_i,$$

and we will partition W into disjoint subsets $X_1, Y_1, X_2, Y_2, \ldots, X_t, Y_t$ as follows.

Noticing that the edges of H can only belong to two successive intervals I_i and I_{i+1} since H is a graph with bandwidth at most $\beta|W| < |W|/\ell$ from the definition of ℓ.

For $i = \ell$, if $I_\ell \in J_j$ for some $1 \leqslant j \leqslant t$, then put $I_\ell \cap \chi^{-1}(1)$ into X_j and put $I_\ell \cap \chi^{-1}(2)$ into Y_j. For $1 \leqslant i \leqslant \ell - 1$,

(i) if I_i and I_{i+1} belong to the same J_j for some $1 \leqslant j \leqslant t$, then put $I_i \cap \chi^{-1}(1)$ into X_j and put $I_i \cap \chi^{-1}(2)$ into Y_j.

(ii) if I_i and I_{i+1} belong to different J_j and $J_{j'}$ for $j < j'$, then we divide I_i into two disjoint subsets L_i (we call it a link) and K_i, where $L_i \subseteq I_i$ consists all of the last $[2(j'-j)+1]\beta n$ vertices of I_i, and $K_i = I_i \setminus L_i$. For K_i, put $K_i \cap \chi^{-1}(1)$ into X_j and put $K_i \cap \chi^{-1}(2)$ into Y_j.

For L_i, let $p = 2(j'-j)$, and denote

$$L_i = \bigcup_{q=1}^{p+1} L_i(q),$$

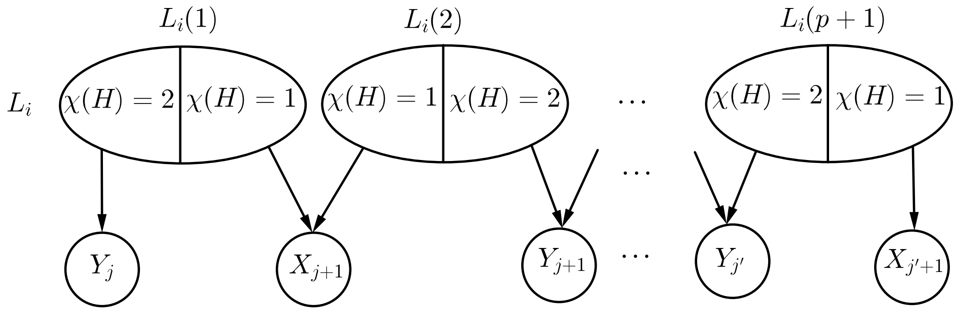
where

$$L_i(q) = \left\{ w_{[i-(p+2-q)\beta\ell]\frac{n}{\ell}+1}, w_{[i-(p+2-q)\beta\ell]\frac{n}{\ell}+2}, \ldots, w_{[i-(p+1-q)\beta\ell]\frac{n}{\ell}} \right\}$$

with $|L_i(q)| = \beta n$ for $1 \leqslant q \leqslant p+1$.

For $q = 1$, put $L_i(1) \cap \chi^{-1}(2)$ into Y_j. For odd q, put $\big(L_i(q) \cup L_i(q+1)\big) \cap \chi^{-1}(1)$ into $X_{j+\frac{q+1}{2}}$. For even q, put $\big(L_i(q) \cup L_i(q+1)\big) \cap \chi^{-1}(2)$ into $Y_{j+\frac{q}{2}}$. And for $q = p+1$, put $L_i(p+1) \cap \chi^{-1}(1)$ into $X_{j'+1}$. See the figure below.

Now, we have put the vertices of I_i for $1 \leqslant i \leqslant \ell$ into subsets $X_1, Y_1, X_2, Y_2, \ldots, X_t, Y_t$ that form a vertex partition of W. From the above construction, we can see that each



X_i induces an independent set that consists of most vertices from $C_1(J_i)$ together with at most two pieces of a fixed link, and each Y_i induces an independent set that consists of most vertices from $C_2(J_i)$ together with at most two pieces of a fixed link. Hence, by noting that $\xi = \gamma/60$, $\beta = \epsilon\xi(1+2\xi)/(36\Delta^2 K_0^2)$, $t \leqslant K_0$ and $\ell \leqslant (7K_0 + 2K_0\xi)/\xi$. We have

$$|X_i| \leqslant C_1(J_i) + 2\ell \cdot \beta n \leqslant (1+\xi)\frac{n}{2t} + 2\ell\beta n = (1+\xi+4t\ell\beta)\frac{n}{2t} < |A_i|,$$

and similarly $|Y_i| < |B_i|$.

3.3 Embedding H into G_P

Now, let G_P be the graph with the vertex partition $\{A_1, B_1, \ldots, A_t, B_t\}$, and H be the balanced (β, Δ)-vertex of order $n \leqslant |V(G_P)|$ with the vertex partition $\{X_1, Y_1, \ldots, X_t, Y_t\}$ as above. Note that the matching M whose edges corresponds to (A_i, B_i) for $1 \leqslant i \leqslant t$ are $(2\epsilon, 1/2 - 2\epsilon)$-super regular, and (A_{i+1}, B_i) are $(2\epsilon, 1/2 - 2\epsilon)$-regular. We will apply the Embedding Lemma, i.e. Lemma 7, to the host graph G_P with reduced graph P_{2t}. It suffices to prove that the partition $\{X_1, Y_1, \ldots, X_t, Y_t\}$ of H is 2ϵ-compatible with the partition $\{A_1, B_1, \ldots, A_t, B_t\}$ of G_P. To this end, we shall check all of the four conditions of Definition 6 as follows.

Note that from the partition of $V(H)$, we have

(1) $|X_i| \leqslant |A_i|$ and $|Y_i| \leqslant |B_i|$ for all $1 \leqslant i \leqslant t$.

(2) $xy \in E_H$ for $x \in X_i, y \in Y_j$ ($j = i$ or $j = i-1$) implies that $(i,j) \in E_{P_{2t}}$.

(3) Note that all edges of $E_{P_{2t}} \setminus E_M$ if of type $(i, i-1)$ for $2 \leqslant i \leqslant t$. From the vertex partition of W, the vertices of X_i can be adjacent to at most two pieces of each link. Moreover, Y_{i-1} contains at most ℓ links. Denote $S_i \subseteq X_i$ by the vertex set that has some neighbours in Y_{i-1}, i.e. $S_i = N_H(Y_{i-1}) \cap X_i$. Therefore, by noting that each piece of a link is of size at most βn, we have

$$|S_i| \leqslant \Delta \cdot (2\ell\beta n) \leqslant \frac{\epsilon}{\Delta}(1+\gamma/30)\frac{n}{2t} \leqslant \frac{\epsilon}{\Delta}|A_i|.$$

Similarly, denote $S'_{i-1} \subseteq Y_{i-1}$ by the vertex set that has some neighbours in X_i, we have $|S'_{i-1}| \leqslant \frac{\epsilon}{\Delta}|B_{i-1}|$.

(4) Set $S = \cup_{i=1}^{t}(S_i \cup S_i')$. Let $T_i = N_H(S) \cap (X_i \setminus S)$ and $T_i' = N_H(S) \cap (Y_i \setminus S)$. For $T_i = N_H(S) \cap (X_i \setminus S)$, noticing that the neighbors of vertices of X_i can only locate in Y_{i-1} and Y_i, hence $N_H(S) \cap (X_i \setminus S)$ consists of vertices of $N_H(S \cap Y_i) \cap (X_i \setminus S)$ and $N_H(S \cap Y_{i-1}) \cap (X_i \setminus S)$. Since $N_H(Y_{i-1}) \cap X_i = S_i$, we have $N_H(S \cap Y_{i-1}) \cap (X_i \setminus S) = S_i \cap (X_i \setminus S) = \varnothing$, and hence

$$|T_i| = |N_H(S \cap Y_i) \cap (X_i \setminus S)| < \Delta |S \cap Y_i| = \Delta |S_i'| \leqslant \epsilon |B_i|.$$

Similarly, for $T_i' = N_H(S) \cap (Y_i \setminus S)$, we have $|T_i'| < \epsilon |A_i|$.

Therefore, the partition $\{X_1, Y_1, \ldots, X_t, Y_t\}$ of H is indeed 2ϵ-compatible with the partition $\{A_1, B_1, \ldots, A_t, B_t\}$ of G_P. This completes the proof of Theorem 1. □

Acknowledgments

The authors are grateful to the anonymous referees for their careful reading of the manuscript and invaluable comments which have improved the manuscript greatly.

References

[1] L. Beineke and A. Schwenk. On a bipartite form of the Ramsey problem. Proc. 5th British Combin. Conf. 1975, Utilitas Math. Winnipeg, 17–22, 1976.

[2] J. Böttcher. Embedding large graphs–The Bollobás-Komlós conjecture and beyond. Ph.D. thesis, Technischen Universität München, 2009.

[3] J. Böttcher, P. Heinig and A. Taraz. Embedding into bipartite graphs. *SIAM J. Discrete Math.*, 24(4):1215–1233, 2010.

[4] W.A. Carnielli and E. L. Monte Carmelo. $K_{2,2} - K_{1,n}$ and $K_{2,n} - K_{2,n}$ bipartite Ramsey numbers. *Discrete Math.*, 223(1–3):83–92, 2000.

[5] Y. Caro and C. Rousseau. Asymptotic bounds for bipartite Ramsey numbers. *Electron. J. Combin.*, 8(1):#R17, 2001.

[6] D. Conlon. A new upper bound for the bipartite Ramsey problem. *J. Graph Theory*, 58(4):351–356, 2008.

[7] D. Conlon, J. Fox, and B. Sudakov. Recent developments in graph Ramsey theory. Surveys in Combinatorics 2015. Edited by Artur Czumaj, Agelos Georgakopoulos, Daniel Král, Vadim Lozin, Oleg Pikhurko. Cambridge University Press, pp 49–118.

[8] R. Faudree, R. Schelp. Path-path Ramsey-type numbers for the complete bipartite graphs. *J. Combin. Theory Ser. B*, 19(2):161–173, 1975.

[9] R. Graham, B. Rothschild and J. Spencer. Ramsey Theory. Wiley, New York, 1980.

[10] A. Gyárfás, C. C. Rousseau and R. H. Schelp. An extremal problem for paths in bipartite graphs. *J. Graph Theory*, 8(1):83–95, 1984.

[11] J. H. Hattingh, M. A. Henning. Bipartite Ramsey theory. *Utilitas Math.*, 53:217–230, 1998.

[12] R. Irving. A bipartite Ramsey problem and the Zarankiewicz number. *Glasgow Math. J.*, 19(1):13–26, 1978.

[13] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. in: Combinatorics, Paul Erdős is eighty, vol. 2 (Miklós, Sós, and Szőnyi eds.), Bolyai Math. Soc., Budapest, 295–352, 1996.

[14] Y. Li, X. Tang and W. Zang. Ramsey functions involving $K_{m,n}$ with n large. *Discrete Math.*, 300(1–3):120–128, 2005.

[15] Q. Lin and Y. Li. A Folkman Linear Family. *SIAM J. Discrete Math.*, 29(4):1988–1998, 2015.

[16] Q. Lin and Y. Li. Bipartite ramsey numbers involving large $K_{n,n}$. *European J. Combin.*, 30(4):923–928, 2009.

[17] G. Mota, G. N. Sárközy, M. Schacht and A. Taraz. Ramsey number for bipartite graphs with small bandwidth. *European J. Combin.*, 48:165–176, 2015.

[18] K. Roth. On certain sets of integers. *J. Lond. Math. Soc.*, 28:104–109, 1953.

[19] K. Roth. On certain sets of integers. II. *J. Lond. Math. Soc.*, 29:20–26, 1954.

[20] J. H. Spencer. Ramsey's theorem-A new lower bound. *J. Combin. Theory Ser. A*, 18:108–115, 1975.

[21] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arithmetica*, 27:199–245, 1975.

[22] E. Szemerédi. Regular partitions of graphs. in: Problémes Combinatories et théorie des graphs, Colloque Inter. CNRS, Univ. Orsay, Orsay, 1976, J. Bermond, J. Fournier, M. Las Vergnas, and D. Scotteau, Eds., 399–402, 1978.

[23] A. Thomason. On finite Ramsey numbers. *European J. Combin.*, 3(3):263–273, 1982.

[24] R. Zhang and Y. Sun. The bipartite Ramsey numbers $b(C_{2m}; K_{2,2})$. *Electron. J. Combin.*, 18(1):#P51, 2011.

[25] R. Zhang, Y. Sun and Y. Wu. The bipartite Ramsey numbers $b(C_{2m}; C_{2n})$. *Int. J. Math. Comput. Nat. Phys. Eng.*, 1:80–83, 2013.