Subgraphs with large minimum ℓ-degree in hypergraphs where almost all ℓ-degrees are large

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Submitted: Oct 20, 2016; Accepted: Feb 23, 2018; Published: Apr 27, 2018
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Abstract

Let G be an r-uniform hypergraph on n vertices such that all but at most $\varepsilon\binom{n}{\ell}$ ℓ-subsets of vertices have degree at least $p\binom{n-\ell}{r-\ell}$. We show that G contains a large subgraph with high minimum ℓ-degree.

Keywords: r-uniform hypergraphs, ℓ-degree, extremal hypergraph theory

Mathematics Subject Classifications: 05C65, 05D99

1 Introduction

Given $r \in \mathbb{N}$ and a set A, we write $A^{(r)}$ for the collection of all r-subsets of A and $[n]$ for the set $\{1, 2, \ldots n\}$. An *r-graph*, or *r-uniform hypergraph*, is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(r)}$ is a collection of r-subsets, which constitute the edges of G. We say G is *nonempty* if it contains at least one edge and set $v(G) = |V(G)|$ and $e(G) = |E(G)|$. A *subgraph* of G is an r-graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph of G *induced* by a set $X \subseteq V(G)$ is $G[X] = (X, E(G) \cap X^{(r)})$.

Let $\mathcal{F}$ be a family of nonempty r-graphs. If G does not contain a copy of a member of $\mathcal{F}$ as a subgraph, we say that G is *$\mathcal{F}$-free*. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of a family $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices, and its *Turán density* is the limit $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$ (this is easily shown to exist). Let $K_t^{(r)} = ([t], [t]^{(r)})$ denote the complete r-graph on t vertices. Determining $\pi(K_t^{(r)})$ for *any* $t > r \geqslant 3$ is a

*Supported by VR starting grant 2016-03488.
†Supported by EPSRC first grant EP/P002420/1

major problem in extremal combinatorics. Turán [19] famously conjectured in 1941 that $\pi(K_4^{(3)}) = 5/9$, and despite much research effort this remains open [8]. In this paper we shall be interested in some variants of Turán density.

The *neighbourhood* $N(S)$ of an ℓ-subset $S \in V(G)^{(\ell)}$ is the collection of $(r-\ell)$-subsets $T \in V(G)^{(r-\ell)}$ such that $S \cup T$ is an edge of G. The *degree* of S is the number $\deg(S)$ of edges of G containing S, that is, $\deg(S) = |N(S)|$. The minimum ℓ-degree of G, $\delta_\ell(G)$, is defined to be the minimum of $\deg(S)$ over all ℓ-subsets $S \in V(G)^{(\ell)}$. The *Turán ℓ-degree threshold* $\mathrm{ex}_\ell(n, \mathcal{F})$ of a family $\mathcal{F}$ of r-graphs is the maximum of $\delta_\ell(G)$ over all $\mathcal{F}$-free r-graphs G on n vertices. It can be shown [11, 9] that the limit $\pi_\ell(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}_\ell(n, \mathcal{F})/\binom{n-\ell}{r-\ell}$ exists; this quantity is known as the *Turán ℓ-degree density* of $\mathcal{F}$. A simple averaging argument shows that

$$0 \leqslant \pi_{r-1}(\mathcal{F}) \leqslant \ldots \leqslant \pi_2(\mathcal{F}) \leqslant \pi_1(\mathcal{F}) = \pi(\mathcal{F}) \leqslant 1,$$

and it is known that $\pi_\ell(\mathcal{F}) \neq \pi(\mathcal{F})$ in general (for $\ell \notin \{0, 1\}$). In the special case where $(r, \ell) = (r, r-1)$, $\pi_{r-1}(\mathcal{F})$ is known as the *codegree density* of $\mathcal{F}$.

There has been much research on Turán ℓ-degree threshold for r-graphs when $(r, \ell) = (3, 2)$. In the late 1990s, Nagle [12] and Nagle and Czygrinow [2] conjectured that $\pi_2(K_4^{(3)-}) = 1/4$ and $\pi_2(K_4^{(3)}) = 1/2$, respectively. Here $K_4^{(3)-}$ denotes the 3-graph obtained by removing one edge from $K_4^{(3)}$. Falgas-Ravry, Pikhurko, Vaughan and Volec [6, 7] recently proved $\pi_2(K_4^{(3)-}) = 1/4$, settling the conjecture of Nagle, and showed all near-extremal constructions are close (in edit distance) to a set of quasirandom tournament constructions of Erdős and Hajnal [3]. The lower bound $\pi_2(K_4^{(3)}) \geqslant 1/2$ also comes from a quasirandom construction, which is due to Rödl [17]. For $t > r \geqslant 3$, the codegree density $\pi_{r-1}(K_t^{(r)})$ has been studied by Falgas-Ravry [4], Lo and Markström [9] and Sidorenko [18]. Recently, Lo and Zhao [10] showed that $1 - \pi_{r-1}(K_t^{(r)}) = \Theta(\ln t/t^{r-1})$ for $r \geqslant 3$.

One variant of ℓ-degree Turán density is to study r-graphs in which almost all ℓ-subsets have large degree. To be precise, given $\varepsilon > 0$, let $\delta_\ell^\varepsilon(G)$ be the largest integer d such that all but at most $\varepsilon\binom{v(G)}{\ell}$ of the ℓ-subsets $S \in V(G)^{(\ell)}$ satisfy $\deg(S) \geqslant d$. Note that r-graphs with large $\delta_\ell^\varepsilon(G)$ but with small $\delta_\ell(G)$ arise naturally. For instance, the reduced graphs R obtained from r-graphs with large minimum ℓ-degree after an application of hypergraph regularity lemma have large $\delta_\ell^\varepsilon(R)$.

Definition 1 (**(r, ℓ)-sequence**)**.** Let $1 \leqslant \ell < r$. We say that a sequence $\mathbf{G} = (G_n)_{n\in\mathbb{N}}$ of r-graphs is an *(r, ℓ)-sequence* if

(i) $v(G_n) \to \infty$ as $n \to \infty$ and

(ii) there is a constant $p \in [0, 1]$ and a sequence of nonnegative reals $\varepsilon_n \to 0$ as $n \to \infty$ such that $\delta_\ell^{\varepsilon_n}(G_n) \geqslant p\binom{v(G_n)-\ell}{r-\ell}$ for each n.

We refer to the supremum of all $p \geqslant 0$ for which (ii) is satisfied as the *density* of the sequence $\mathbf{G}$ and denote it by $\rho(\mathbf{G})$.

We can define the analogue of Turán density for (r, ℓ)-sequences.

Definition 2. Let $1 \leqslant \ell < r$. Let $\mathcal{F}$ be a family of nonempty r-graphs. Define

$$\pi_\ell^\star(\mathcal{F}) := \sup\Big\{\rho(\mathbf{G}) : \ \mathbf{G} \text{ is an } (r,\ell)\text{-sequence of } \mathcal{F}\text{-free } r\text{-graphs}\Big\}.$$

Our main result show that every large r-graph G contains a 'somewhat large' subgraph H with minimum ℓ-degree satisfying $\delta_\ell(H)/\binom{v(H)-\ell}{r-\ell} \approx \delta_\ell^\varepsilon(G)/\binom{v(G)-\ell}{r-\ell}$. Here 'somewhat large' means $v(H) = \Omega(\varepsilon^{1/\ell})$.

Theorem 3. *Let $1 \leqslant \ell < r$. For any fixed $\delta > 0$, there exists $m_0 > 0$ such that any r-graph G on $n \geqslant m \geqslant m_0$ vertices with $\delta_\ell^\varepsilon(G) \geqslant p\binom{n-\ell}{r-\ell}$ for some $\varepsilon \leqslant m^{-\ell}/2$ contains an induced subgraph H on m vertices with*

$$\delta_\ell(H) \geqslant (p-\delta)\binom{m-\ell}{r-\ell}.$$

This immediate implies the $\pi_\ell^\star(\mathcal{F}) = \pi_\ell(\mathcal{F})$ for all families $\mathcal{F}$ of r-graphs.

Corollary 4. *For any $1 \leqslant \ell < r$ and any family $\mathcal{F}$ of nonempty r-graphs, $\pi_\ell^\star(\mathcal{F}) = \pi_\ell(\mathcal{F})$.*

We note that the (tight) upper bounds for codegree densities $\pi_2(F)$ for 3-graphs F obtained by flag algebraic methods in [5, 6, 7] actually relied on giving upper bounds for $\pi_\ell^\star(F)$. Corollary 4 provides theoretical justification for why this strategy could give optimal bounds.

1.1 Quasirandomness in 3-graphs

One of the main motivations for this note comes from recent work of Reiher, Rödl and Schacht [13, 14, 15, 16] on extremal questions for quasirandom hypergraphs. These authors studied the following notion of quasirandomness for 3-graphs.

Definition 5 ((1,2)-quasirandomness)**.** A 3-graph G is $(p, \varepsilon, (1,2))$-*quasirandom* if for every set of vertices $X \subseteq V$ and every set of pairs of vertices $P \subseteq V^{(2)}$, the number $e_{1,2}(X, P)$ of pairs $(x, uv) \in X \times P$ such that $\{x\} \cup \{uv\} \in E(G)$ satisfies:

$$\Big| e_{1,2}(X,P) - p|X| \cdot |P| \Big| \leqslant \varepsilon v(G)^3.$$

We define a $(1,2)$-quasirandom sequence and the corresponding extremal density, denoted by $\pi_{(1,2)-qr}(\mathcal{F})$, analogously to the way we defined (r,ℓ)-sequences and $\pi_\ell^\star(\mathcal{F})$ in Definitions 1 and 2. It is not difficult to see that $\pi_{(1,2)-qr}(\mathcal{F}) \leqslant \pi(\mathcal{F})$ for all families $\mathcal{F}$ of 3-graphs. Moreover, a $(p, \varepsilon, (1,2))$-quasirandom 3-graph G satisfies $\delta_2^{\sqrt{\varepsilon}}(G) \geqslant (p - 4\sqrt{\varepsilon})v(G)$. Hence, Theorem 3 and Corollary 6 imply the following.

Corollary 6. *For any family of nonempty 3-graphs $\mathcal{F}$, $\pi_{(1,2)-qr}(\mathcal{F}) \leqslant \pi_2(\mathcal{F})$.*

Consider a $(p, \varepsilon, (1,2))$-quasirandom 3-graph G for some $p > 4\sqrt{\varepsilon} > 0$. As noted above, $\delta_2^{\sqrt{\varepsilon}}(G) \geqslant (p - 4\sqrt{\varepsilon})v(G)$. Thus provided $v(G)$ is sufficiently large, Theorem 3 tells us we can find a subgraph H of G on $m = \Omega(\varepsilon^{-1/4})$ vertices with strictly positive minimum codegree (at least $(p - 4\sqrt{\varepsilon})m$).

However, as we show below, we cannot guarantee the existence of any subgraph with strictly positive codegree on more than $2/\varepsilon + 1$ vertices: our lower bound on m above in terms of an inverse power of the error parameter ε is thus sharp up to the value of the exponent.

Proposition 7. *For every $p \in (0,1)$ and every $\varepsilon > 0$, there exists n_0 such that for all $n \geqslant n_0$ there exist $(p, 2\varepsilon, (1,2))$-quasirandom 3-graphs in which every subgraph on $m \geqslant \lfloor \varepsilon^{-1} \rfloor + 1$ vertices has minimum codegree equal to zero.*

Proof. Let $G = (V, E)$ be a $(p, \varepsilon, (1,2))$-quasirandom 3-graph on n vertices. Such a 3-graph can be obtained for example by taking a typical instance of an Erdős–Rényi random 3-graph with edge probability p. Consider a balanced partition of V into $N = \lfloor \varepsilon^{-1} \rfloor$ sets $V = \bigcup_{i=1}^{N} V_i$ with $\lfloor n/N \rfloor \leqslant |V_1| \leqslant |V_2| \leqslant \ldots \leqslant |V_N| \leqslant \lceil n/N \rceil$. Now let G' be the 3-graph obtained from G by deleting all triples that meet some V_i in at least two vertices for some i: $1 \leqslant i \leqslant N$.

By construction, every set of $N + 1$ vertices in G' must contain at least two vertices from the same V_i, and thus must induce a subgraph of G' with minimum codegree zero. Note that $e(G) - e(G') \leqslant Nn\binom{\lceil n/N \rceil}{2} \leqslant n^3/N \leqslant \varepsilon n^3$. Since G is $(p, \varepsilon, (1,2))$-quasirandom, it follows that G' is $(p, 2\varepsilon, (1,2))$-quasirandom. □

2 Finding high minimum ℓ-degree subgraphs in r-graphs with large δ_ℓ^ε

In this section we show how we can extract arbitrarily large subgraphs with high minimum ℓ-degree from sufficiently large r-graphs with sufficiently small error ε. To do so, we will need Azuma's inequality (see e.g. [1]).

Lemma 8 (Azuma's inequality)**.** *Let $\{X_i : i = 0, 1, \ldots\}$ be a martingale with $|X_i - X_{i-1}| \leqslant c_i$ for all i. Then for all positive integers N and $\lambda > 0$,*

$$\mathbb{P}(X_N \leqslant X_0 - \lambda) \leqslant \exp\left(\frac{-\lambda^2}{2\sum_{i=1}^{N} c_i^2}\right).$$

Proof of Theorem 3. We may assume without loss of generality that $\delta > 0$ is small enough to ensure $\delta^{-1} \geqslant 26\ell(r-\ell)^2 \log(1/\delta)$ and $\ell \log(1/\delta) \geqslant \log 2$ as this only makes our task harder. Set $m_0 = \lceil 26\ell(r-\ell)^2\delta^{-2}\log(1/\delta) \rceil$. Note that this implies that

$$2\ell \log m_0 \leqslant 4\ell \log\left(26\ell(r-\ell)^2\delta^{-2}\log(1/\delta)\right) \leqslant 12\ell \log(1/\delta). \qquad (1)$$

Fix $m \geqslant m_0$. Let $n \geqslant m \geqslant m_0$ and $\varepsilon = m^{-\ell}/2$.

Suppose $G=(V,E)$ is an r-graph on n vertices with $\delta_\ell^\varepsilon(G) \geqslant p\binom{n-\ell}{r-\ell}$. We claim that it contains an induced subgraph on m vertices with minimum ℓ-degree at least $(p-\delta)\binom{m-\ell}{r-\ell}$. For $p \leqslant \delta$, we have nothing to prove, so we may assume that $1 \geqslant p > \delta$.

Call an ℓ-subset $S \in V^{(\ell)}$ *poor* if $\deg(S) < p\binom{n-\ell}{r-\ell}$, and *rich* otherwise. Let $\mathcal{P}$ be the collection of all poor ℓ-subsets. By our assumption on $\delta_\ell^\varepsilon(G)$, $|\mathcal{P}| \leqslant \varepsilon\binom{n}{\ell}$. As each poor ℓ-subset is contained in $\binom{n-\ell}{m-\ell}$ m-subsets, it follows that there are at least

$$\binom{n}{m} - |\mathcal{P}|\binom{n-\ell}{m-\ell} > \left(1-\varepsilon m^\ell\right)\binom{n}{m} = \frac{1}{2}\binom{n}{m} \tag{2}$$

m-subsets of vertices which do not contain any poor ℓ-subsets.

Given an ℓ-subset $S \in V^{(\ell)} \setminus \mathcal{P}$, we call an m-subset T of V *bad for* S if $S \subseteq T$ and $\left|N(S) \cap T^{(r-\ell)}\right| \leqslant (p-\delta)\binom{m-\ell}{r-\ell}$. Let ϕ_S be the number of bad m-subsets for S. We claim that

$$\phi_S \leqslant \binom{n-\ell}{m-\ell} \exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right). \tag{3}$$

Observe that

$$\phi_S = \left|\left\{T \in (V \setminus S)^{(m-\ell)} \colon \left|N(S) \cap T^{(r-\ell)}\right| \leqslant (p-\delta)\binom{m-\ell}{r-\ell}\right\}\right|.$$

Let X be the random variable $\left|N(S) \cap T^{(r-\ell)}\right|$, where T is an $(m-\ell)$-subset of $V \setminus S$ picked uniformly at random. We consider the vertex exposure martingale on T. Let Z_i be the ith exposed vertex in T. Define $X_i = \mathbb{E}(X|Z_1,\dots,Z_i)$. Note that $\{X_i : i=0,1,\dots,m-\ell\}$ is a martingale and $X_0 \geqslant p\binom{m-\ell}{r-\ell}$. Moreover, $|X_i - X_{i-1}| \leqslant \binom{m-\ell-1}{r-\ell-1} < \binom{m-1}{r-\ell-1}$. Thus, by Lemma 8 applied with $\lambda = \delta\binom{m}{r-\ell}$ and $c_i = \binom{m-1}{r-\ell-1}$, we have

$$\begin{aligned}\mathbb{P}\left(X_m \leqslant (p-\delta)\binom{m-\ell}{r-\ell}\right) \leqslant \mathbb{P}(X_m \leqslant X_0 - \lambda) \leqslant \exp\left(\frac{-\delta^2\binom{m}{r-\ell}^2}{2m\binom{m-1}{r-\ell-1}^2}\right) = \left(\frac{-\delta^2\binom{m}{r-\ell}}{2(r-\ell)}\right)\\ \leqslant \exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right).\end{aligned}$$

Hence (3) holds.

An m-subset T of V is called *bad* if it is bad for some $S \in V^{(\ell)} \setminus \mathcal{P}$. The number of bad m-subsets is at most

$$\begin{aligned}\sum_{S \in V^{(\ell)} \setminus \mathcal{P}} \phi_S &\leqslant \binom{n}{\ell}\binom{n-\ell}{m-\ell}\exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right) = \binom{n}{m}\binom{m}{\ell}\exp\left(-\frac{\delta^2 m}{2(r-\ell)^2}\right)\\ &\leqslant \binom{n}{m} m_0^\ell \exp\left(-\frac{\delta^2 m_0}{2(r-\ell)^2}\right) \leqslant \binom{n}{m}\exp\left(2\ell\log m_0 - 13\ell\log(1/\delta)\right)\\ &\leqslant \binom{n}{m}\exp\left(-\ell\log(1/\delta)\right) \leqslant \frac{1}{2}\binom{n}{m},\end{aligned}$$

where the last three inequalities hold by our choice of m_0, by inequality (1), and by our assumption on δ, respectively. Together with (2), this shows there exists an m-subset inside which there is no poor ℓ-subsets and in which every rich ℓ-subset has degree at least $(p-\delta)\binom{m-\ell}{r-\ell}$. Such a set clearly gives us an induced subgraph of G on m vertices with minimum ℓ-degree at least $(p-\delta)\binom{m-\ell}{r-\ell}$. $\square$

3 Concluding remarks

A 3-graph G is $(p, \varepsilon, (1,1,1))$-*quasirandom* if for every triple of sets of vertices X, Y and $Z \subseteq V$, the number $e_{1,1,1}(X, Y, Z)$ of triples $(x, y, z) \in X \times Y \times Z$ such that $xyz \in E(G)$ satisfies $\left| e_{1,1,1}(X, Y, Z) - p|X| \cdot |Y| \cdot |Z| \right| \leqslant \varepsilon v(G)^3$. Define $\pi_{(1,1,1)-qr}(\mathcal{F})$ analogously to $\pi_{(1,2)-qr}(\mathcal{F})$. Note that $\pi_{(1,2)-qr}(\mathcal{F}) \leqslant \pi_{(1,1,1)-qr}(\mathcal{F}) \leqslant \pi(\mathcal{F})$ for all 3-graph families $\mathcal{F}$. An obvious open question is whether we have

$$\pi_{(1,1,1)-qr}(\mathcal{F}) \leqslant \pi_2(\mathcal{F}).$$

Even more: can one always extract subgraphs with large minimum codegree from $(1,1,1)$-quasirandom graphs? Even obtaining large subgraphs with non-zero minimum codegree remains an open problem for this weaker notion of quasirandomness.

Acknowledgements

The authors are grateful for a Scheme 4 grant from the London Mathematical Society which allowed Victor Falgas-Ravry to visit Allan Lo in Birmingham in July 2016, when this research was done.

Further, the authors would like to thank the anonymous referees for their careful work and helpful suggestions, which led to considerable improvements in the paper — in particular, their comments led us to state and prove a much more general form of Theorem 3 than we had in the first version of this paper.

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