

Counting compositions over finite abelian groups

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Abstract

We find the number of compositions over finite abelian groups under two types of restriction: (i) each part belongs to a given subset and (ii) small runs of consecutive parts must have given properties. Waring’s problem over finite fields can be converted to type (i) compositions, whereas Carlitz and “locally Mullen” compositions can be formulated as type (ii) compositions. We use the multisection formula to translate the problem from integers to group elements, the transfer matrix method to do exact counting, and finally the Perron-Frobenius theorem to derive asymptotics. We also exhibit bijections involving certain restricted classes of compositions.

Keywords: Integer composition, finite abelian group, transfer matrix, enumeration.

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1 Introduction

Let n and m be positive integers. A *composition* of n is a sequence of positive integers whose sum is n . An m -*composition* is a composition consisting of m terms, also called parts. It is well known that there is a bijection between m -compositions of n and $(m-1)$ -subsets of $\{1, 2, \dots, n-1\}$ and thus there are $\binom{n-1}{m-1}$ m -compositions of n and 2^{n-1} compositions of n . A *weak composition* is the same as a composition except terms equal to 0 are allowed. Using substitution of variables, we can easily see that the number of weak m -compositions of n is equal to the number of m -compositions of $n+m$, which is $\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$.

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Problems on or related to compositions have been studied over algebraic structures beyond the integers. Let \mathbb{F}_q be a finite field of q elements and let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Li and Wan [19, 20] estimated the number

$$N(m, s, S) = |\{\{x_1, x_2, \dots, x_m\} \subseteq S \mid x_1 + x_2 + \dots + x_m = s\}|$$

of m -subsets of $S \subseteq \mathbb{F}_q$ whose sum is $s \in \mathbb{F}_q$. In particular, exact formulas are obtained in cases where $S = \mathbb{F}_q$ or \mathbb{F}_q^* or $\mathbb{F}_q \setminus \{0, 1\}$. A shorter proof was given by Kesters [18] using character theory. Motivated by the study by Li and Wan and the study of polynomials of prescribed ranges over finite fields [13, 24], Muratović and Wang proposed to study

$$c(m, s, S) = |\{(x_1, x_2, \dots, x_m) \in S \times S \times \dots \times S \mid x_1 + x_2 + \dots + x_m = s\}|,$$

that is, the number of ordered m -tuples whose sum is s and where each coordinate belongs to $S \subseteq \mathbb{F}_q$, as well as the number $M(m, s, S)$ which counts the number of m -multisets over $S \subseteq \mathbb{F}_q$ whose sum is $s \in \mathbb{F}_q$. In particular, when $S = \mathbb{F}_q^*$, this essentially gives the concept of compositions and partitions over a finite field, respectively.

A *partition* of $s \in \mathbb{F}_q$ into m parts is a multiset of m elements of \mathbb{F}_q^* whose sum is s . The m nonzero elements are referred to as the parts of the partition. In [26], Muratović and Wang obtain an exact formula for $M(m, s, \mathbb{F}_q^*)$, the number of partitions of an element s into m parts over finite field \mathbb{F}_q .

A *composition* of $s \in \mathbb{F}_q$ with m parts is a solution (x_1, x_2, \dots, x_m) to the equation

$$s = x_1 + x_2 + \dots + x_m, \tag{1}$$

with each $x_i \in \mathbb{F}_q^*$. Similarly, a *weak composition* of $w \in \mathbb{F}_q$ with m parts is a solution (x_1, x_2, \dots, x_m) to Equation (1) with each $x_i \in \mathbb{F}_q$. We denote the number of compositions of s having m parts by $c(m, s, \mathbb{F}_q^*)$. The number of weak compositions of s with m parts is denoted by $c(m, s, \mathbb{F}_q)$. A formula for $c(m, s, \mathbb{F}_p)$ can be found in [6, p. 295]. A general formula for $c(m, s, \mathbb{F}_q^*)$ for arbitrary q and nonzero s can be obtained using a remark on the normalized Jacobi sum of the trivial character given in [11] (see Remark 1 on page 144). A recurrence relation for $c(m, s, \mathbb{F}_q)$ is given in [25].

Counting compositions over finite fields where parts are restricted to a subset is conceptually related to Waring's problem and solutions to diagonal equations. In number theory, Waring's problem asks whether each natural number k has an associated positive integer m such that every natural number is the sum of at most m natural numbers to the power of k . The problem was originally posed in 1770 and answered in the affirmative for integers by Hilbert in 1909. Since then, there has been a good deal of research on estimating the Waring's number $g(k)$ for every k , which denotes the minimum number m of k^{th} powers of naturals needed to represent all positive integers.

Over finite prime fields \mathbb{F}_p , Waring's number $g(k, p)$ is the smallest number m such that for all $a \in \mathbb{F}_p$, the equation

$$x_1^k + x_2^k + \dots + x_m^k \equiv a \pmod{p}$$

has a solution in integers x_i (see for example [10]). Waring's problem over finite fields is to estimate $g(k, p)$ and, when possible, evaluate it. This is equivalent to finding the smallest number m such that

$$x_1 + x_2 + \cdots + x_m \equiv a \pmod{p}$$

has a solution such that $x_i \in S$ where S denotes the subset of \mathbb{F}_p consisting of all k^{th} powers in \mathbb{F}_p . Essentially we need to find the number of solutions (x_1, x_2, \dots, x_m) in $S \times S \times \cdots \times S$ and pick the minimum m so that this number is always positive for any $a \in \mathbb{F}_p$. However, this is a computationally difficult task in general.

There is also extensive study on the number of solutions to “diagonal” equations of the type

$$a_1 x_1^{d_1} + \cdots + a_m x_m^{d_m} = a, \quad (2)$$

where $a_1, \dots, a_m \in \mathbb{F}_p^*$, $a \in \mathbb{F}_p$ and d_1, \dots, d_m are positive integers. The pioneering work has been done by Weil [34], who expressed the number of solutions in terms of Jacobi sums. Explicit formulas for the number of solutions for certain choices of a_1, \dots, a_m, a , and exponents d_1, \dots, d_m can be deduced from Weil's expression; one may consult [1, 2, 8, 29, 30, 35] and the references therein for details. For diagonal equations, the problem can be again viewed as a composition problem over \mathbb{F}_p , such that each part is restricted to lie in a coset of a multiplicative subgroup.

Using the fact that the additive group $(\mathbb{F}_q, +)$ is isomorphic to the additive group $(\mathbb{F}_p^r, +)$, we obtain that the numbers of partitions and compositions of elements over \mathbb{F}_p^r are the same as the numbers of partitions and compositions of corresponding elements over \mathbb{F}_q . More generally, the number of partitions over an arbitrary finite abelian group is given in [26]. However, problems of enumerating compositions over an arbitrary finite abelian group are largely open. We note that there have been some studies on compositions over integer tuples [21, 23], which are also called matrix compositions.

In the present work, we consider two general problems on restricted compositions over finite abelian groups. Let \mathbb{Z}_k denote the additive cyclic group $\{0, 1, 2, \dots, k-1\}$ and $\mathbb{Z}_k^* := \{1, 2, \dots, k-1\}$. Since a finite abelian group G is isomorphic to a direct sum $\bigoplus_{t=1}^r \mathbb{Z}_{k_t}$, in the following we use G to denote such a direct sum. Sometimes it is convenient to add (tuples of) elements of \mathbb{Z}_k as integers, so when performing group addition in G we explicitly speak of modular addition. In the rest of the paper, we also assume $|G| \geq 2$, and consequently $k_t \geq 2$ for each $1 \leq t \leq r$. We also use $\mathbf{0}$ to denote the zero element of G and adopt the notation $G^* := G \setminus \{\mathbf{0}\}$.

Definition 1 (Compositions over a finite abelian group). An m -composition of $\mathbf{s} \in G$ over G is a solution to $\mathbf{x}_1 + \cdots + \mathbf{x}_m = \mathbf{s}$, where each $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,r})$ is a nonzero element of G and addition is taken component-wise and modulo k_t for $1 \leq t \leq r$. If we allow $\mathbf{0}$ as a value for the \mathbf{x}_i , then we speak of *weak compositions* over G .

Firstly we are interested in finding the number of m -compositions over G such that for $1 \leq j \leq m$, part \mathbf{x}_j is restricted to an arbitrary subset $S_j \subseteq G$. Our contribution is Theorem 2, where we obtain an asymptotic formula for the number of such m -compositions as $m \rightarrow \infty$.

Secondly, we enumerate m -compositions over a finite abelian group G such that any d consecutive parts satisfy certain conditions for a given positive integer d . These are called locally restricted compositions. Bender and Canfield [3] studied integer compositions under general local restrictions. For example, Carlitz compositions are those in which adjacent parts are distinct. In a private communication, G. L. Mullen [22] suggested the following problem: Let p be a prime, and let d and m be positive integers with $d \leq m$. Let $N_m^{(d)}$ denote the number of solutions to the congruence $x_1 + \cdots + x_m \equiv 1 \pmod{p}$ where each subsum $\sum x_j$ of at most d parts is nonzero (modulo p). Find $N_m^{(d)}$ when $d \geq 2$. Motivated by Mullen's problem, we consider a related problem which falls within the locally-restricted framework. Namely, we estimate the number of solutions to the congruence $x_1 + \cdots + x_m \equiv 1 \pmod{p}$ where each subsum $\sum_{j=i}^{i+d-1} x_j$ ($1 \leq i \leq m-d+1$) of at most d consecutive parts is nonzero. We call these compositions *locally d -Mullen compositions*. Generating functions for general locally restricted compositions over finite abelian groups are given in Proposition 18. Under moderate conditions, the asymptotic number of locally restricted compositions is given in Theorem 7. As consequences, we obtain asymptotics for a few concrete composition problems including locally d -Mullen compositions (see Corollary 8, Corollary 9, and Theorem 24).

We present these main results in Section 2. Some auxiliary propositions are discussed in Section 3 and the proofs of the main results are given in Section 4. In Section 5 we present bijections which help illuminate locally d -Mullen compositions. Finally we give conclusions in Section 6.

2 Main results

Recall that G denotes the set of integer tuples in the direct sum $\bigoplus_{t=1}^r \mathbb{Z}_{k_t}$. In the following we denote the order of G by $|G| := \prod_{t=1}^r k_t$.

Theorem 2 (Subset restriction). *For each $j \in \{1, 2, \dots\}$, let S_j be the Cartesian product $\prod_{t=1}^r S_{t,j}$ where $S_{t,j}$ is a given subset of \mathbb{Z}_{k_t} . Let $c(\mathbf{s}; S_1, \dots, S_m)$ be the number of m -compositions of \mathbf{s} over G such that the j^{th} part lies in S_j , for each $j \in \{1, \dots, m\}$. Assume that for $1 \leq t \leq r$ and $j \geq 1$ we have*

$$\gcd\{a - b : a, b \in S_{t,j}\} = 1.$$

Then, as $m \rightarrow \infty$,

$$c(\mathbf{s}; S_1, \dots, S_m) = \frac{1}{|G|} \left(\prod_{j=1}^m |S_j| \right) (1 + O(\theta^m)),$$

where $0 < \theta < 1$ is a constant independent of m .

Let $\text{SEQ}_m(G) := \{(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) : \mathbf{x}_j \in G\} = G^m$ denote the class of all possible weak m -compositions over G . The *size* of a composition, denoted $|(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m)|$, is defined to be the sum $\sum_{j=1}^m \mathbf{x}_j$, using component-wise integer addition. Let $\text{SEQ}(G) = \cup_{m \geq 0} \text{SEQ}_m(G)$.

A subclass $\mathcal{A} \subseteq \text{SEQ}(G)$ is called *locally restricted* if any d consecutive parts satisfy certain restrictions for a given positive integer d . General locally restricted integer compositions were first studied in [3] using local restriction functions. They can also be defined using local restriction digraphs. For the purpose of this paper, we use the following definition, which is essentially from [4].

Definition 3 (Locally restricted compositions associated with a digraph). Let σ be a positive integer. We use ε to denote the empty composition, and we use ε_s to denote a distinguished copy of ε . Let $\mathcal{T} = \text{SEQ}_0(G) \cup \cdots \cup \text{SEQ}_{\sigma-1}(G)$, and let \mathcal{R} be a nonempty subset of $\text{SEQ}_\sigma(G)$. Let D be a digraph with vertex set $V(D) = \{\varepsilon_s\} \dot{\cup} \mathcal{R} \dot{\cup} \mathcal{T}$. Assume D satisfies the following conditions.

1. There is at least one arc from ε_s to \mathcal{R} and at least one arc from \mathcal{R} to \mathcal{T} .
2. There are no arcs (including loops) between vertices in $\{\varepsilon_s\} \dot{\cup} \mathcal{T}$.
3. The sub-digraph $D_{\mathcal{R}}$ of D induced by \mathcal{R} is strongly connected and contains at least two vertices.

The vertices in \mathcal{R} are called *recurrent* vertices. Let \mathcal{W} denote the set of directed walks from ε_s to \mathcal{T} , and let $\text{SEQ}(G; D)$ denote the class of compositions obtained by concatenating vertices in \mathcal{W} . Then $\text{SEQ}(G; D)$ is called the class of *locally restricted compositions over G associated with D* . We call σ the *span* of $\text{SEQ}(G; D)$.

Informally, we are defining $\text{SEQ}(G; D)$ by specifying two sets \mathcal{R} and \mathcal{T} of “building block” compositions, and saying how they may be combined sequentially to form the compositions of interest: Two blocks are allowed to be joined if there is an arc between them in D . The structure of the graph implies that sequences of vertices forming a walk from ε_s to \mathcal{T} are made up of one or more elements of \mathcal{R} followed by one element of \mathcal{T} . Defining combinatorial objects by walks in graphs is a standard technique; see e.g. [28, Sec. 4.7].

Example 4 (Nonzero adjacent sum over \mathbb{Z}_k). Take compositions over \mathbb{Z}_k such that the sum of two adjacent parts is nonzero, modulo k . We can represent these compositions using a local restriction digraph as follows. In the digraph D , set $\mathcal{R} = \mathbb{Z}_k^* \subset \text{SEQ}_1(\mathbb{Z}_k)$, and $\mathcal{T} = \text{SEQ}_0(\mathbb{Z}_k) = \{\varepsilon\}$. Include an arc from u to v in \mathcal{R} if and only if $u + v \not\equiv 0 \pmod{k}$. In this case, we have locally restricted compositions over \mathbb{Z}_k with span $\sigma = 1$. The digraph for weak compositions is defined similarly with $\mathcal{R} = \mathbb{Z}_k = \text{SEQ}_1(\mathbb{Z}_k)$.

Example 5 (Carlitz compositions over \mathbb{Z}_k). Recall that Carlitz compositions are those where adjacent parts are distinct. A corresponding local restriction digraph D has vertex sets $\mathcal{R} = \mathbb{Z}_k^* \subset \text{SEQ}_1(\mathbb{Z}_k)$ (or $\mathcal{R} = \mathbb{Z}_k = \text{SEQ}_1(\mathbb{Z}_k)$ for weak compositions), $\mathcal{T} = \{\varepsilon\}$, and there is an arc from any vertex in \mathcal{R} to any different vertex in \mathcal{R} . In this case, we have locally restricted compositions over \mathbb{Z}_k with span $\sigma = 1$.

We note that it is possible for two different digraphs D_1 and D_2 to define equal families $\text{SEQ}(G; D_1)$ and $\text{SEQ}(G; D_2)$ of compositions over G . For example, Carlitz compositions can also be defined using a digraph with span $\sigma = 2$ such that each recurrent vertex is a pair of distinct nonzero elements of G .

Example 6 (Locally Mullen compositions). Recall that locally d -Mullen compositions over G are those such that the sum of at most d consecutive terms is nonzero in G . Example 4 gives locally 2-Mullen compositions. A digraph D with span 3 for locally 3-Mullen compositions can be defined as follows. Let \mathcal{R} be the set of locally 3-Mullen compositions with 3 parts over G . We join ε_s to every vertex in \mathcal{R} , and join a vertex $\mathbf{u} \in \mathcal{R}$ to a vertex $\mathbf{v} \in \mathcal{R} \cup \mathcal{T}$ if the concatenation $\mathbf{u}\mathbf{v}$ is a locally 3-Mullen composition.

Theorem 7 (Local restriction). *Consider a class $\text{SEQ}(G; D)$ of locally restricted compositions with span σ . Suppose the following are satisfied.*

1. *The greatest common divisor of the lengths of all directed cycles in $D_{\mathcal{R}}$ is equal to 1.*
2. *For each $1 \leq t \leq r$, there is a positive integer ℓ , two recurrent vertices \mathbf{u} and \mathbf{v} , and a nonempty set $\mathcal{W}_{\mathcal{R}}$ of directed walks in $D_{\mathcal{R}}$ of length ℓ from \mathbf{u} to \mathbf{v} such that the following hold.*
 - (a) *For each walk $W = \mathbf{u}\mathbf{u}_1 \dots \mathbf{u}_{\ell-1}\mathbf{v}$ in $\mathcal{W}_{\mathcal{R}}$ and each $i \neq t$, we have $u_{i,j} = \phi_{i,j}$, where $u_{i,j}$ denotes the i^{th} coordinate of the size vector $|\mathbf{u}_j|$ and $\phi_{i,j}$ is fixed over all walks in $\mathcal{W}_{\mathcal{R}}$.*
 - (b) *We have $\gcd\{m - n : m, n \in N\} = 1$, where $N = \{n : n = u_{t,1} + \dots + u_{t,\ell-1} \text{ for some walk } \mathbf{u}\mathbf{u}_1 \dots \mathbf{u}_{\ell-1}\mathbf{v} \in \mathcal{W}_{\mathcal{R}}\}$.*

Let $m = a\sigma + b$ for some integers a, b with $a \geq 1$ and $0 \leq b < \sigma$, and let $c_m(\mathbf{s}, D)$ be the number of m -compositions of \mathbf{s} in $\text{SEQ}(G; D)$. Fix b but allow a to vary. Suppose further that there is at least one arc from \mathcal{R} to $\text{SEQ}_b(G)$. Then there are constants $A > 0, B > 1$ and $0 < \theta < 1$ (that is, independent of m but perhaps depending on b) such that

$$c_m(\mathbf{s}, D) = A \cdot B^m (1 + O(\theta^m)), \quad m = a\sigma + b, \quad m \rightarrow \infty.$$

Remark. Assumption 2 in Theorem 2 is a technical condition which allows our proofs to work. It might be possible to relax this technical assumption.

Corollary 8. *Assume the notation and conditions of Theorem 2. Let H be the outdegree of ε_s . Suppose further that there are constants J and K such that*

1. *every recurrent vertex has outdegree K in $D_{\mathcal{R}}$, and*
2. *every recurrent vertex is joined to J vertices in $\text{SEQ}_b(G)$.*

Then Theorem 7 holds with

$$A = \frac{H J}{|G| K^{1+b/\sigma}}, \quad B = K^{1/\sigma}. \quad (3)$$

In the following we adopt the notation $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$.

Corollary 9. *Let $c_m(\mathbf{s})$ be the number of m -compositions of \mathbf{s} for each of the following classes of compositions over G . Then*

$$c_m(\mathbf{s}) = A \cdot B^m (1 + O(\theta^m)), \quad m \rightarrow \infty,$$

where A, B are given below.

1. *For weak compositions such that there is no repeated part among any $d+1$ consecutive parts and $|G| \geq d+2$,*

$$A = \frac{1}{|G|} |G|^{\underline{d}} (|G| - d)^{-d}, \quad B = |G| - d.$$

2. *For compositions such that there is no repeated part among any $d+1$ consecutive parts and $|G| \geq d+3$,*

$$A = \frac{1}{|G|} (|G| - 1)^{\underline{d}} (|G| - 1 - d)^{-d}, \quad B = |G| - 1 - d.$$

3. *For weak compositions such that there is no repeated part among any $d+1$ consecutive parts and the first d parts are nonzero, and $|G| \geq d+2$,*

$$A = \frac{1}{|G|} (|G| - 1)^{\underline{d}} (|G| - d)^{-d}, \quad B = |G| - d.$$

4. *For weak compositions such that the sum of any d consecutive parts is nonzero and $|G| \geq 3, d \geq 2$,*

$$A = |G|^{d-2} (|G| - 1)^{1-d}, \quad B = |G| - 1.$$

5. *For compositions over \mathbb{F}_q such that the product of any d consecutive parts is not equal to $\mathbf{1}$ and $q \geq 4, d \geq 2$,*

$$A = \frac{1}{q} (q-1)^{d-1} (q-2)^{1-d}, \quad B = q-2.$$

Next we give an example to which Corollary 8 does not apply.

Example 10. Consider the class of compositions over G such that the sum of any three consecutive parts is nonzero. A corresponding restriction digraph D is defined as follows. The set \mathcal{R} consists of all ordered triples of nonzero elements of G whose sum is nonzero. We note that the sum of two consecutive parts might be zero. The vertex ε_s is joined to every vertex in \mathcal{R} , and every recurrent vertex is joined to the vertex ε in \mathcal{T} . A recurrent vertex \mathbf{u} is joined to a vertex $\mathbf{v} \in \mathcal{R} \cup \text{SEQ}_1(G^*) \cup \text{SEQ}_2(G^*)$ if the sum of any three consecutive parts in the concatenation \mathbf{uv} is nonzero. To compute the outdegree of each recurrent vertex, we need to distinguish two cases.

Case 1: the recurrent vertex $\mathbf{u} := (u_1, u_2, u_3)$ where $u_1, u_2, u_3 \neq \mathbf{0}$ and $u_2 + u_3 = \mathbf{0}$. Such a vertex is joined to a recurrent vertex $\mathbf{v} := (v_1, v_2, v_3)$ if $v_1, v_2, v_3 \neq \mathbf{0}$, $u_3 + v_1 + v_2 \neq \mathbf{0}$ and $v_1 + v_2 + v_3 \neq \mathbf{0}$. The outdegree of \mathbf{u} in \mathcal{R} is equal to

$$\begin{aligned} & |\{(v_1, -v_1, v_3) : v_1, v_3 \neq \mathbf{0}\}| \\ & + |\{(v_1, v_2, v_3) : v_1 \neq \mathbf{0}; v_2 \neq -v_1, -v_1 - u_3, \mathbf{0}; v_3 \neq -v_1 - v_2, \mathbf{0}\}| \\ & = (|G| - 1)^2 + (|G| - 1)(|G| - 3)(|G| - 2). \end{aligned}$$

The outdegree of \mathbf{u} in $\text{SEQ}_1(G)$ is clearly equal to $|G| - 1$. The outdegree of \mathbf{u} in $\text{SEQ}_2(G)$ is equal to

$$\begin{aligned} & |\{(-u_3, v_2) : v_2 \neq \mathbf{0}\}| + |\{(v_1, v_2) : v_1 \neq -u_3, \mathbf{0}; v_2 \neq -v_1 - u_3, \mathbf{0}\}| \\ & = (|G| - 1) + (|G| - 2)^2. \end{aligned}$$

Case 2: the recurrent vertex $\mathbf{u} := (u_1, u_2, u_3)$ where $u_1, u_2, u_3 \neq \mathbf{0}$ and $u_2 + u_3 \neq \mathbf{0}$. The outdegree of \mathbf{u} in \mathcal{R} is equal to

$$\begin{aligned} & |\{(v_1, -v_1, v_3) : v_1 \neq -u_2 - u_3, \mathbf{0}; v_3 \neq \mathbf{0}\}| \\ & + |\{(v_1, v_2, v_3) : v_1 \neq -u_2 - u_3, \mathbf{0}; v_2 \neq -v_1, -v_1 - u_3, \mathbf{0}; v_3 \neq -v_1 - v_2, \mathbf{0}\}| \\ & = (|G| - 2)(|G| - 1) + (|G| - 2)(|G| - 3)(|G| - 2). \end{aligned}$$

The outdegree of \mathbf{u} in $\text{SEQ}_1(G)$ is clearly equal to $|G| - 2$. The outdegree of \mathbf{u} in $\text{SEQ}_2(G)$ is equal to

$$\begin{aligned} & |\{(-u_3, v_2) : v_2 \neq \mathbf{0}\}| + |\{(v_1, v_2) : v_1 \neq -u_3, -u_2 - u_3, \mathbf{0}; v_2 \neq -u_3 - v_1, \mathbf{0}\}| \\ & = (|G| - 1) + (|G| - 3)(|G| - 2). \end{aligned}$$

We show later that in fact Theorem 7 can still apply to this class.

3 Propositions

In this section we present results which are used to prove our main theorems.

The following multivariate “multisection formula” might be known (the univariate case can be found in [15, Ex. 1.1.9]), but we are unable to find a reference. So we also include a proof.

For two vectors $\mathbf{z} = (z_1, z_2, \dots, z_r)$ and $\mathbf{n} = (n_1, n_2, \dots, n_r)$, we use $\mathbf{z}^{\mathbf{n}}$ to denote the product $\prod_{j=1}^r z_j^{n_j}$. In the rest of the paper, we use the Iverson bracket $[P]$ which is equal to 1 if the statement P is true and 0 otherwise. We also use $\omega_k := \exp(2\pi i/k)$ to denote the k^{th} primitive root of unity which has the property

$$\sum_{j=0}^{k-1} \omega_k^{sj} = k [s \equiv 0 \pmod{k}].$$

Proposition 11 (Multivariate multisection formula). *Let $A(\mathbf{z}) = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$ be a multivariate generating function, where the indeterminate is $\mathbf{z} := (z_1, \dots, z_r)$ and the sum is over all $\mathbf{n} := (n_1, \dots, n_r)$. For any $\mathbf{s} = (s_1, \dots, s_r) \in G$, we have*

$$\sum_{\mathbf{n} \equiv \mathbf{s} \pmod{\mathbf{k}}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} = \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) A(z_1 \omega_{k_1}^{j_1}, \dots, z_r \omega_{k_r}^{j_r}),$$

where $\omega_{k_t} = \exp(2\pi i/k_t)$ is a primitive k_t^{th} root of unity for $1 \leq t \leq r$.

Proof. We have

$$\begin{aligned} & \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) A(z_1 \omega_{k_1}^{j_1}, \dots, z_r \omega_{k_r}^{j_r}) \\ &= \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \prod_{t=1}^r \sum_{j_t=0}^{k_t-1} \omega_{k_t}^{j_t(n_t - s_t)} \\ &= \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \prod_{t=1}^r k_t [n_t \equiv s_t \pmod{k_t}] \\ &= \prod_{t=1}^r k_t \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} [\mathbf{n} \equiv \mathbf{s} \pmod{\mathbf{k}}]. \square \end{aligned}$$

Corollary 12. *In this corollary compositions may be weak. Fix some class $\mathcal{A} \subseteq \text{SEQ}(G)$ of restricted compositions over G . For $\mathbf{s} \in G$, let*

$$c_m(\mathbf{s}) = |\{\mathbf{x} : \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) \in \mathcal{A}, |\mathbf{x}| \equiv \mathbf{s} \pmod{\mathbf{k}}\}|.$$

For $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r$, we define the integer composition counting sequence

$$\hat{c}_m(\mathbf{n}) = |\{\mathbf{x} : \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) \in \mathcal{A}, |\mathbf{x}| = \mathbf{n}\}|,$$

and the generating function $\hat{C}_m(\mathbf{z}) = \sum_{\mathbf{n}} \hat{c}_m(\mathbf{n}) \mathbf{z}^{\mathbf{n}}$. Then

$$c_m(\mathbf{s}) = \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \hat{C}_m(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}).$$

Proof. Immediate from Proposition 11. □

Example 13 (Unrestricted \mathbf{m} -compositions over \mathbf{G}). The corresponding generating function for integer m -compositions is

$$\hat{C}_m(\mathbf{z}) = \left(\sum_{\mathbf{j} \in G} \mathbf{z}^{\mathbf{j}} - 1 \right)^m = \left(\prod_{t=1}^r \sum_{j_t=0}^{k_t-1} z_t^{j_t} - 1 \right)^m = \left(\prod_{t=1}^r \frac{1 - z_t^{k_t}}{1 - z_t} - 1 \right)^m.$$

It follows from Corollary 12 that the number of m -compositions of \mathbf{s} over G is

$$\begin{aligned}
c_m(\mathbf{s}) &= \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \hat{C}_m(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}) \\
&= \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \left([\mathbf{j} = \mathbf{0}] \prod_{t=1}^r k_t - 1 \right)^m \\
&= \frac{1}{|G|} \left((|G| - 1)^m + (-1)^m \sum_{\mathbf{j} \in G^*} \prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \\
&= \frac{1}{|G|} \left((|G| - 1)^m + (-1)^m \left(\prod_{t=1}^r \sum_{j_t=0}^{k_t-1} \omega_{k_t}^{-j_t s_t} - 1 \right) \right) \\
&= \frac{1}{|G|} \left((|G| - 1)^m + (-1)^m ([\mathbf{s} = \mathbf{0}] |G| - 1) \right).
\end{aligned}$$

Thus

$$\begin{aligned}
c_m(\mathbf{s}) &= \frac{1}{|G|} \left((|G| - 1)^m - (-1)^m \right), \quad \text{if } \mathbf{s} \neq \mathbf{0}, \\
c_m(\mathbf{0}) &= \frac{1}{|G|} \left((|G| - 1)^m + (-1)^m (|G| - 1) \right).
\end{aligned}$$

If $r = 1$, this is the result given in [25]. We also note that

$$c_m(\mathbf{s}) \sim \frac{1}{|G|} (|G| - 1)^m$$

for each $\mathbf{s} \in G$ as $m \rightarrow \infty$, which agrees with the result given by Theorem 1.

Example 14 (Unrestricted weak \mathbf{m} -compositions over \mathbf{G}). The corresponding generating function for integer compositions is

$$\hat{C}_m(\mathbf{z}) = \left(\prod_{t=1}^r \frac{1 - z_t^{k_t}}{1 - z_t} \right)^m.$$

Applying Corollary 12, we have

$$\begin{aligned}
c_m(\mathbf{s}) &= \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \left(\prod_{t=1}^r \frac{1 - \omega_{k_t}^{j_t k_t}}{1 - \omega_{k_t}^{j_t}} \right)^m \\
&= \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \left([\mathbf{j} = \mathbf{0}] \prod_{t=1}^r k_t \right)^m \\
&= \left(\prod_{t=1}^r \frac{1}{k_t} \right) \left(\prod_{t=1}^r k_t \right)^m \\
&= \left(\prod_{t=1}^r k_t \right)^{m-1}.
\end{aligned}$$

This result can also be obtained by a direct counting argument: the first $m - 1$ parts can be constructed in $(\prod_{t=1}^r k_t)^{m-1}$ ways, and the last part is then uniquely determined by the equation $\mathbf{x}_1 + \cdots + \mathbf{x}_{m-1} + \mathbf{x}_m = \mathbf{s}$ in G .

Example 15. Let c_m be the number of solutions to Equation (2). Let $y_j = a_j x_j^{d_j}$ and $S_j = \{a_j x_j^{d_j} : x_j \in \mathbb{F}_p^*\}$. Then c_m is the number of m -compositions y_1, \dots, y_m of a over \mathbb{F}_p such that $y_j \in S_j$. We have

$$|S_j| = |\{a_j x_j^{d_j} : x_j \in \mathbb{F}_p^*\}| = |\{x_j^{d_j} : x_j \in \mathbb{F}_p^*\}| = \frac{p-1}{\gcd(d_j, p-1)}.$$

Viewing elements of $S_j, S_j - S_j$ as integers less than p , the condition

$$\gcd\{a_j s^{d_j} - a_j t^{d_j} : s, t \in \mathbb{F}_p^*\} = 1,$$

is satisfied when $S_j - S_j$ contains two relatively prime integers.

Definition 16 (Transfer matrix, weights, start and finish vectors). Let $\text{SEQ}(G; D)$ be a class of locally restricted compositions from Definition 3. Fix an order on the vertices in \mathcal{R} . We define the *transfer matrix* $T(\mathbf{z})$ as the square matrix whose rows and columns are indexed by the vertices in \mathcal{R} such that $T_{\mathbf{u}, \mathbf{v}}(\mathbf{z}) = \mathbf{z}^{|\mathbf{v}|}$ if there is an arc from \mathbf{u} to \mathbf{v} , and $T_{\mathbf{u}, \mathbf{v}}(\mathbf{z}) = 0$ otherwise. We say that $\mathbf{z}^{|\mathbf{v}|}$ is the *weight* of an arc (\mathbf{u}, \mathbf{v}) . The weight of a directed walk in D is defined to be the product of the weights of the arcs of the walk. The *start vector* $\boldsymbol{\alpha}(\mathbf{z})$ is defined as the row vector whose j^{th} entry is the weight of the arc from ε_s to the j^{th} recurrent vertex (this entry is zero if such an arc does not exist). For each integer $0 \leq b < \sigma$, we define the *finish vector* $\boldsymbol{\beta}_b(\mathbf{z})$ as the column vector whose j^{th} entry is the sum of weights of all arcs from the j^{th} recurrent vertex to a vertex in $\text{SEQ}_b(G)$.

Example 17 (Nonzero adjacent sum, continues Example 4). For weak compositions, the transfer matrix $T(z)$ has size $k \times k$ where $T_{i,j}(z) = z^j$ if $i+j \not\equiv 0 \pmod{k}$ and $T_{i,j}(z) = 0$ if $i+j \equiv 0 \pmod{k}$. The start and finish vectors are, respectively, $\boldsymbol{\alpha}(z) = [1 \ z \ \cdots \ z^{k-1}]$ and $\boldsymbol{\beta}_0(z) = [1 \ 1 \ \cdots \ 1]^\top$. For example, if $k = 3$, we have

$$T(z) = \begin{bmatrix} 0 & z & z^2 \\ 1 & z & 0 \\ 1 & 0 & z^2 \end{bmatrix}, \quad \boldsymbol{\alpha}(z) = [1 \ z \ z^2], \quad \boldsymbol{\beta}_0(z) = [1 \ 1 \ 1]^\top.$$

Now we are ready to use the transfer matrix to enumerate locally restricted compositions in $\text{SEQ}(G; D)$.

Proposition 18. Let m, σ be positive integers and define integers a, b such that $m = a\sigma + b$ where $0 \leq b < \sigma$. Let $\mathbf{s} = (s_1, \dots, s_r)$ be a member of G ($= \bigoplus_{j=1}^r \mathbb{Z}_{k_j}$), and let $\text{SEQ}(G; D)$ be a class of locally restricted compositions with span σ . As in Theorem 7, we let $c_m(\mathbf{s}, D)$ denote the number of m -compositions of \mathbf{s} in $\text{SEQ}(G; D)$. Let $T(\mathbf{z}), \boldsymbol{\alpha}(\mathbf{z}), \boldsymbol{\beta}_b(\mathbf{z})$ be the corresponding transfer matrix, start vector, and finish vector, and define

$$\hat{C}_m(\mathbf{z}) = \boldsymbol{\alpha}(\mathbf{z}) T^{a-1}(\mathbf{z}) \boldsymbol{\beta}_b(\mathbf{z}).$$

Then we have

$$c_m(\mathbf{s}, D) = \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{j} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-j_t s_t} \right) \hat{C}_m(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}).$$

Proof. We note that the (\mathbf{u}, \mathbf{v}) entry in $T(\mathbf{z})^{a-1}$ is the sum of weights of all directed walks of length $a-1$ (that is, containing a vertices) in $D_{\mathcal{R}}$ from vertex \mathbf{u} to vertex \mathbf{v} . Since each vertex in $D_{\mathcal{R}}$ is a sequence of parts from G of length σ , it follows from the definition of α and β_b that $\hat{C}_m(\mathbf{z})$ is the generating function of all directed walks from ε_s to \mathcal{T} totaling $m = a\sigma + b$ parts. The result follows by applying Corollary 12. \square

Example 19 (Carlitz compositions over \mathbb{Z}_k , continues Example 5). For Carlitz weak compositions, it is clear that every recurrent vertex has in-degree and out-degree equal to $k-1$ in $D_{\mathcal{R}}$. The transfer matrix $T(z)$ is given by $T_{i,i}(z) = 0$ and $T_{i,j}(z) = z^j$ if $i \neq j$. The start and finish vectors are, respectively, $\alpha(z) = [1 \ z \ \dots \ z^{k-1}]$ and $\beta_0(z) = [1 \ 1 \ \dots \ 1]^\top$. So the generating function $\hat{C}_m(z)$ from Proposition 18 for Carlitz weak compositions is given by

$$\hat{C}_m(z) = [1 \ z \ \dots \ z^{k-1}] T^{m-1}(z) [1 \ 1 \ \dots \ 1]^\top, \quad m \geq 1.$$

For Carlitz compositions, the transfer matrix $T(z)$ is obtained from the one above by deleting row 1 and column 1. The start and finish vectors are, respectively, $\alpha(z) = [z \ z^2 \ \dots \ z^{k-1}]$ and $\beta_0(z) = [1 \ 1 \ \dots \ 1]^\top$. This gives

$$\hat{C}_m(z) = [z \ z^2 \ \dots \ z^{k-1}] T^{m-1}(z) [1 \ 1 \ \dots \ 1]^\top, \quad m \geq 1.$$

The following two propositions are used later to derive the asymptotic number of compositions over G from their generating functions. Proposition 20 below shows that, under an aperiodicity condition, a polynomial attains a unique maximum absolute value on the unit disc at 1. Its proof can be found in [14, Lemma 1]. Proposition 21 is essentially the Perron-Frobenius theorem and its proof can be found in [5].

Proposition 20. *Let $F(z) = \sum_{j \geq 0} f_j z^j$ be a polynomial with nonnegative coefficients. Define $J = \{j : f_j > 0\}$. Suppose J is not empty and $\gcd\{j - k : j, k \in J\} = 1$. Then $|F(z)| < F(1)$ for all $|z| \leq 1, z \neq 1$.*

Proposition 21. *Let $T(\mathbf{z})$ be a transfer matrix as in Definition 16, and let $\rho(\mathbf{z})$ denote the spectral radius of $T(\mathbf{z})$. Suppose the greatest common divisor of all directed cycle lengths of the digraph $D_{\mathcal{R}}$ is equal to 1. Then we have the following.*

1. *The value $\rho := \rho(\mathbf{1})$ is a simple eigenvalue of $T(\mathbf{1})$ and the corresponding eigenspace is spanned by a positive vector.*
2. *All other eigenvalues of $T(\mathbf{1})$ are smaller, in modulus, than ρ .*
3. *Suppose $|\mathbf{z}| \neq 1, |z_t| \leq 1$ for all t , and $T(\mathbf{z}) \neq T(\mathbf{1})$. Then $\rho(\mathbf{z}) < \rho$.*

4 Proofs

Proof of Theorem 2. The generating function for integer compositions is

$$\hat{C}_m(\mathbf{z}) = \prod_{j=1}^m \sum_{\mathbf{n} \in S_j} \mathbf{z}^{\mathbf{n}} = \prod_{j=1}^m \prod_{t=1}^r \sum_{n_t \in S_{t,j}} z_t^{n_t}.$$

We note $\hat{C}_m(\mathbf{1}) = \prod_{j=1}^m |S_j|$. It follows from Corollary 12 that

$$\begin{aligned} c(\mathbf{s}; S_1, \dots, S_m) &= \left(\prod_{t=1}^r \frac{1}{k_t} \right) \sum_{\mathbf{g} \in G} \left(\prod_{t=1}^r \omega_{k_t}^{-g_t s_t} \right) \hat{C}_m(\omega_{k_1}^{g_1}, \dots, \omega_{k_r}^{g_r}) \\ &= \frac{1}{|G|} \left(\prod_{j=1}^m |S_j| + \sum_{\mathbf{g} \in G^*} \left(\prod_{t=1}^r \omega_{k_t}^{-g_t s_t} \right) \hat{C}_m(\omega_{k_1}^{g_1}, \dots, \omega_{k_r}^{g_r}) \right). \end{aligned}$$

For a subset A of \mathbb{Z}_{k_t} , define $P(z; A) = \sum_{a \in A} z^a$. Let $\mathcal{I} := \{A \subseteq \mathbb{Z}_{k_t} : |P(z; A)| < |A| \text{ when } |z| = 1 \text{ and } z \neq 1\}$ and define

$$\theta = \max \left\{ \frac{|P(\omega_{k_t}^{g_t}; A)|}{|A|} : 1 \leq g_t \leq k_t - 1, 1 \leq t \leq r, A \in \mathcal{I} \right\}.$$

Then $0 \leq \theta < 1$. It follows from Proposition 20 that $S_{t,j} \in \mathcal{I}$ and consequently $|P(z; S_{t,j})| < \theta |S_{t,j}|$. Let $\mathbf{g} \in G$ satisfy $g_{t^*} > 0$ for some $1 \leq t^* \leq r$. It follows that

$$\begin{aligned} |\hat{C}_m(\omega_{k_1}^{g_1}, \dots, \omega_{k_r}^{g_r})| &= \left(\prod_{j=1}^m \left| \sum_{a \in S_{t^*,j}} \omega_{k_{t^*}}^{a g_{t^*}} \right| \right) \left(\prod_{t=1, t \neq t^*}^r \prod_{j=1}^m \left| \sum_{n_t \in S_{t,j}} \omega_{k_t}^{n_t g_t} \right| \right) \\ &= \left(\prod_{j=1}^m |P_{t^*,j}(\omega_{k_{t^*}}^{g_{t^*}})| \right) \left(\prod_{t=1, t \neq t^*}^r \prod_{j=1}^m |P_{t,j}(\omega_{k_t}^{g_t})| \right) \\ &< \left(\prod_{j=1}^m \theta |S_{t^*,j}| \right) \left(\prod_{t=1, t \neq t^*}^r \prod_{j=1}^m |S_{t,j}| \right) \\ &= \theta^m \prod_{t=1}^r \prod_{j=1}^m |S_{t,j}| \\ &= \theta^m \prod_{t=1}^r |S_t|. \end{aligned}$$

Now Theorem 1 follows immediately. \square

Proof of Theorem 7. As in Proposition 21, let $T(\mathbf{z})$ be the transfer matrix of $\text{SEQ}(G; D)$ and define $\rho(\mathbf{z})$ to be the absolute value of the dominant eigenvalue of $T(\mathbf{z})$. We abbreviate

$\rho(\mathbf{1})$ simply as ρ . Define

$$\theta_1 = \max \left\{ \frac{|\lambda|}{\rho} : \lambda \text{ is any eigenvalue of } T(\mathbf{1}) \text{ other than } \rho \right\},$$

$$\theta_2 = \max \left\{ \frac{\rho(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r})}{\rho} : 0 \leq j_t \leq k_t - 1, 1 \leq t \leq r, \mathbf{j} \neq \mathbf{0} \right\}.$$

It follows from item 2 of Proposition 21 that $0 \leq \theta_1 < 1$. Recall that the (\mathbf{u}, \mathbf{v}) -entry of $T^\ell(\mathbf{z})$, denoted by $(T^\ell(\mathbf{z}))_{\mathbf{u}, \mathbf{v}}$, is equal to the sum of the weights of all directed walks from vertex \mathbf{u} to vertex \mathbf{v} of length ℓ . Let $\mathbf{j} = (j_1, \dots, j_r)$ satisfy $0 \leq j_i \leq k_i - 1$ and $j_t \neq 0$ for some $1 \leq t \leq r$. Let $\mathcal{W}_{\mathcal{R}}$ be the set of directed walks given in condition 2. Dividing the set of all directed walks of length ℓ from \mathbf{u} to \mathbf{v} into $\mathcal{W}_{\mathcal{R}}$ and its complement $\bar{\mathcal{W}}_{\mathcal{R}}$, we obtain

$$\begin{aligned} & \left| (T^\ell(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}))_{\mathbf{u}, \mathbf{v}} \right| \\ & \leq \left| \sum_{W \in \mathcal{W}_{\mathcal{R}}} \prod_{i=1}^r \omega_{k_i}^{j_i(v_i + \sum_{h=1}^{\ell-1} u_{i,h})} \right| + \left| \sum_{W \in \bar{\mathcal{W}}_{\mathcal{R}}} \prod_{i=1}^r \omega_{k_i}^{j_i(v_i + \sum_{h=1}^{\ell-1} u_{i,h})} \right| \\ & \leq \left| \prod_{i=1, i \neq t}^r \omega_{k_i}^{j_i(v_i + \sum_{h=1}^{\ell-1} \phi_{i,h})} \right| \left| \omega_{k_t}^{j_t v_t} \right| \left| \sum_{W \in \mathcal{W}_{\mathcal{R}}} \omega_{k_t}^{j_t \sum_{j=1}^{\ell-1} u_{t,j}} \right| + \sum_{W \in \bar{\mathcal{W}}_{\mathcal{R}}} 1 \\ & \leq \left| \sum_{W \in \mathcal{W}_{\mathcal{R}}} \omega_{k_t}^{j_t \sum_{h=1}^{\ell-1} u_{t,h}} \right| + \sum_{W \in \bar{\mathcal{W}}_{\mathcal{R}}} 1. \end{aligned}$$

Applying condition 2 and Proposition 20, we obtain

$$\left| \sum_{W \in \mathcal{W}_{\mathcal{R}}} \omega_{k_t}^{j_t \sum_{h=1}^{\ell-1} u_{t,h}} \right| < \sum_{W \in \bar{\mathcal{W}}_{\mathcal{R}}} 1.$$

It follows that

$$\left| (T^\ell(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}))_{\mathbf{u}, \mathbf{v}} \right| < \sum_{W \in \mathcal{W}_{\mathcal{R}}} 1 + \sum_{W \in \bar{\mathcal{W}}_{\mathcal{R}}} 1 = (T^\ell(\mathbf{1}))_{\mathbf{u}, \mathbf{v}}.$$

Applying item 3 of Proposition 21 to the matrix $T^\ell((\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}))$, we obtain

$$\rho(T(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}))^\ell < \rho(T(\mathbf{1}))^\ell,$$

and hence $0 \leq \theta_2 < 1$.

Using the Jordan normal form of $T(\mathbf{z})$, it is easy to see that, for each $\mathbf{z} = (\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r})$ with $(j_1, \dots, j_r) \neq (0, \dots, 0)$, all entries of $T^{a-1}(\mathbf{z})$ are of the order $O(a^N \rho^a \theta_2^a)$, where N is the size of $T(\mathbf{z})$. Since N is fixed (i.e., independent of m or a), it follows that, for $(j_1, \dots, j_r) \neq (0, \dots, 0)$,

$$\hat{C}_m(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}) = \boldsymbol{\alpha}(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}) T^{a-1}(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}) \boldsymbol{\beta}_b(\omega_{k_1}^{j_1}, \dots, \omega_{k_r}^{j_r}) = O(a^N \theta_2^a) \rho^a.$$

Next we estimate $\hat{C}_m(\mathbf{1})$. By Proposition 4(1), ρ is a simple eigenvalue of $T(\mathbf{1})$ (and $T^\top(\mathbf{1})$), so we may write the Jordan normal form of $T(\mathbf{1})$ in the following block form

$$T(\mathbf{1}) = [\mathbf{g} \ L] \begin{bmatrix} \rho & \mathbf{0} \\ \mathbf{0} & \Lambda \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ H \end{bmatrix}$$

such that \mathbf{g} is a positive eigenvector of $T(\mathbf{1})$ corresponding to ρ , \mathbf{h}^\top is a positive eigenvector of $T^\top(\mathbf{1})$ corresponding to ρ , the matrix Λ corresponds to other eigenvalues of $T(\mathbf{1})$ which are all smaller, in absolute value, than ρ , and the matrix $\begin{bmatrix} \mathbf{h} \\ H \end{bmatrix}$ is the inverse of $[\mathbf{g} \ L]$.

Since all entries of Λ^{a-1} are of the order $O(a^N \rho^a \theta_1^a)$, it follows that

$$\begin{aligned} \hat{C}_m(\mathbf{1}) &= \boldsymbol{\alpha}(\mathbf{1}) T^{a-1}(\mathbf{1}) \boldsymbol{\beta}_b(\mathbf{1}) \\ &= (\boldsymbol{\alpha}(\mathbf{1}) \cdot \mathbf{g})(\mathbf{h} \cdot \boldsymbol{\beta}_b(\mathbf{1})) \rho^{a-1} + \boldsymbol{\alpha}(\mathbf{1}) L \Lambda^{a-1} H \boldsymbol{\beta}_b(\mathbf{1}) \\ &= (\boldsymbol{\alpha}(\mathbf{1}) \cdot \mathbf{g})(\mathbf{h} \cdot \boldsymbol{\beta}_b(\mathbf{1})) \rho^{a-1} + O(a^N \rho^a \theta_1^a). \end{aligned}$$

Since there is at least one arc from ε_s to \mathcal{R} and at least one arc from \mathcal{R} to $\text{SEQ}_b(G)$, both $\boldsymbol{\alpha}(\mathbf{1})$ and $\boldsymbol{\beta}_b(\mathbf{1})$ are non-negative vectors with at least one positive entry. Consequently both $\boldsymbol{\alpha}(\mathbf{1}) \cdot \mathbf{g}$ and $\mathbf{h} \cdot \boldsymbol{\beta}_b(\mathbf{1})$ are positive. Now Theorem 2 follows from Proposition 18 with θ being any constant satisfying $\max\{\theta_1^{1/\sigma}, \theta_2^{1/\sigma}\} < \theta < 1$, and

$$A = \frac{1}{|G|} (\boldsymbol{\alpha}(\mathbf{1}) \cdot \mathbf{g})(\mathbf{h} \cdot \boldsymbol{\beta}_b(\mathbf{1})) \rho^{-1-b/\sigma}, \quad B = \rho^{1/\sigma}. \quad \square$$

Example 22 (Continues Example 10). We again consider the class of compositions over G such that the sum of any three consecutive parts is nonzero. To make matrices as convenient as possible, we use a span of $\sigma = 2$. We specialize to the case $G = \mathbb{Z}_k$, where $k \geq 4$. An argument similar to that used below in the proof of Corollary 2 Part 4 shows that $D_{\mathcal{R}}$ is strongly connected. The first condition of Theorem 7 is satisfied by the loop at the vertex $(1, 1)$. The second condition is satisfied by setting $\mathcal{W}_{\mathcal{R}}$ to contain the walks $(1, 1), (1, k-1), (2, 1), (k-1, 1), (k-1, k-1)$ and $(1, 1), (1, k-1), (2, 2), (k-1, 1), (k-1, k-1)$.

We further specialize, taking $k = 4$, and in this case, with the help of the Computer Algebra System *MAPLE*, we can apply Proposition 2 to derive an exact formula as well as a simple approximation formula for $c_m(\mathbf{s})$. The elements of \mathcal{R} are $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)$. Using this ordering, we get

$$\begin{aligned} \boldsymbol{\alpha}(z) &= [z^2 \quad z^3 \quad z^4 \quad z^3 \quad z^4 \quad z^5 \quad z^4 \quad z^5 \quad z^6] \\ \boldsymbol{\beta}_0(z) &= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^\top \\ \boldsymbol{\beta}_1(z) &= [z + z^3 \quad z^2 + z^3 \quad z + z^2 + z^3 \quad z^2 + z^3 \quad z + z^2 + z^3 \quad z + z^2 \quad z + z^2 + z^3 \quad z + z^2 \quad z + z^3]^\top \end{aligned}$$

$$T(z) = \begin{bmatrix} z^2 & 0 & z^4 & 0 & 0 & 0 & z^4 & z^5 & z^6 \\ 0 & 0 & 0 & z^3 & z^4 & z^5 & z^4 & z^5 & 0 \\ z^2 & z^3 & z^4 & z^3 & z^4 & 0 & z^4 & 0 & z^6 \\ 0 & 0 & 0 & 0 & z^4 & z^5 & z^4 & z^5 & z^6 \\ 0 & z^3 & z^4 & z^3 & z^4 & z^5 & z^4 & z^5 & 0 \\ z^2 & z^3 & z^4 & z^3 & z^4 & 0 & 0 & 0 & 0 \\ z^2 & 0 & z^4 & 0 & z^4 & z^5 & z^4 & z^5 & z^6 \\ 0 & z^3 & z^4 & z^3 & z^4 & z^5 & 0 & 0 & 0 \\ z^2 & z^3 & z^4 & 0 & 0 & 0 & z^4 & 0 & z^6 \end{bmatrix}$$

Using *MAPLE*, we find that the four matrices $T(1), T(-1), T(i), T(-i)$ are all diagonalizable and

$$\begin{aligned} \hat{C}_m(1) &= \frac{3}{2}(1 + \sqrt{2})^m + \frac{3}{2}(1 - \sqrt{2})^m, \\ \hat{C}_m(-1) &= \hat{C}_m(i) = \hat{C}_m(-i) \\ &= (-1)^m f(r_1)r_1^{m/2} + f(r_2)r_2^{m/2} + (-1)^m f(r_3)r_3^{m/2} + 2\Re\left(f(r_4)r_4^{m/2}\right), \end{aligned}$$

where

$$f(r) = \frac{r^3 - 5r^2 + 6r - 1}{r(4r^4 - 11r^3 - 13r^2 - 19r + 17)},$$

$r_1 \doteq 3.848862156$, $r_2 \doteq 0.4736256091$, $r_3 \doteq 0.3639455409$, and $r_4 \doteq -0.8432166528 + 0.892341334i$ are eigenvalues of $T(-1), T(i), T(-i)$ satisfying

$$r^5 - 3r^4 - 3r^3 - 2r^2 + 4r - 1 = 0.$$

It follows from Proposition 2 that

$$\begin{aligned} c_m(0) &= \frac{3}{8}(1 + \sqrt{2})^m + \frac{3}{8}(1 - \sqrt{2})^m + \frac{3}{4}(-1)^m f(r_1)r_1^{m/2} + \frac{3}{4}f(r_2)r_2^{m/2} \\ &\quad + \frac{3}{4}(-1)^m f(r_3)r_3^{m/2} + \frac{3}{2}\Re\left(f(r_4)r_4^{m/2}\right) \\ &= \frac{3}{8}(1 + \sqrt{2})^m + \frac{3}{4}(-1)^m f(r_1)r_1^{m/2} + \frac{3}{2}\Re\left(f(r_4)r_4^{m/2}\right) + O\left(r_2^{m/2}\right), \\ c_m(1) &= c_m(2) = c_m(3) \\ &= \frac{3}{8}(1 + \sqrt{2})^m + \frac{3}{8}(1 - \sqrt{2})^m - \frac{1}{4}(-1)^m f(r_1)r_1^{m/2} - \frac{1}{4}f(r_2)r_2^{m/2} \\ &\quad - \frac{1}{4}(-1)^m f(r_3)r_3^{m/2} - \frac{1}{2}\Re\left(f(r_4)r_4^{m/2}\right) \\ &= \frac{3}{8}(1 + \sqrt{2})^m - \frac{1}{4}(-1)^m f(r_1)r_1^{m/2} - \frac{1}{2}\Re\left(f(r_4)r_4^{m/2}\right) + O\left(r_2^{m/2}\right). \end{aligned}$$

Table 1 shows numerical values for relevant sequences, where

$$a_m(0) = \frac{3}{8}(1 + \sqrt{2})^m + \frac{3}{4}(-1)^m f(r_1)r_1^{m/2} + \frac{3}{2}\Re\left(f(r_2)r_2^{m/2}\right)$$

is evaluated up to one decimal place.

m	$c_m(1)$	$c_m(0)$	$a_m(0)$
2	2	3	2.7
3	7	0	-0.1
4	10	21	20.8
5	35	18	17.9
6	64	105	104.9
7	199	120	119.9
8	396	543	542.9
9	1119	822	821.9
10	2376	2961	2960.9
11	6373	5238	5237.9
12	14142	16377	16376.9
13	36589	32196	32195.9
14	83532	92133	92132.9
15	211075	194196	194195.9
16	491110	524241	524240.9
17	1221885	1156908	1156908.0
18	2878806	3006279	3006279.0
19	7089517	6839406	6839406.0
20	16841988	17332647	17332647.0
21	41196941	40234356	40234356.0

Table 1: Values of $c_m(1)$, $c_m(0)$ and $a_m(0)$ from Example 10.

Proof of Corollary 8. We first note that the matrix $T(\mathbf{1})$ is a 0-1 matrix. Condition 1 implies that each row of $T(\mathbf{1})$ contains exactly K 1's. Hence the dominant eigenvalue of $T(\mathbf{1})$ is $\rho = K$ and $[1 \ 1 \ \cdots \ 1]^\top$ is a corresponding eigenvector. Conditions 2 and 3 imply that $\boldsymbol{\alpha}(\mathbf{1})$ contains exactly H 1's and $\boldsymbol{\beta}_b(\mathbf{1}) = J [1 \ 1 \ \cdots \ 1]^\top$. Since $\boldsymbol{\beta}_b(\mathbf{1})$ is an eigenvector of $T(\mathbf{1})$ corresponding to the dominant eigenvalue ρ , $\hat{C}_m(\mathbf{1})$ can be evaluated exactly without using the Jordan normal form. And we have

$$\hat{C}_m(\mathbf{1}) = \boldsymbol{\alpha}(\mathbf{1})T^{a-1}(\mathbf{1})\boldsymbol{\beta}_b(\mathbf{1}) = \boldsymbol{\alpha}(\mathbf{1})K^{a-1}J [1 \ 1 \ \cdots \ 1]^\top = H \cdot J \cdot K^{a-1},$$

where we used the fact $H = \boldsymbol{\alpha}(\mathbf{1}) [1 \ 1 \ \cdots \ 1]^\top$. This establishes (3). \square

Proof of Corollary 9. We first define the digraph D for each class and compute the values of H , J and K . For Part 1, we let \mathcal{R} be the set of all $(d+1)$ -tuples of distinct elements of G . Hence $\sigma = d+1$ and $|\mathcal{R}| = |G|^{\underline{d+1}}$. The vertex ε_s is joined to all recurrent vertices and so $H = |G|^{\underline{d+1}}$. A recurrent vertex $\mathbf{u} = (u_1, u_2, \dots, u_{d+1})$ is joined to a recurrent vertex $\mathbf{v} = (v_1, v_2, \dots, v_{d+1})$ if v_j is different from $v_1, \dots, v_{j-1}, u_{j+1}, \dots, u_{d+1}$ for each $1 \leq j \leq d+1$. Hence \mathbf{u} is joined to $K = (|G| - d)^{\underline{d+1}}$ recurrent vertices. Similarly $\mathbf{u} \in \mathcal{R}$ is joined to a vertex $\mathbf{v} = (v_1, v_2, \dots, v_b)$ if v_j is different from $v_1, \dots, v_{j-1}, u_{j+1}, \dots, u_{d+1}$ for each $1 \leq j \leq b$. Hence $J = (|G| - d)^{\underline{b}}$.

For Part 2, we let \mathcal{R} be the set of all $(d+1)$ -tuples of distinct elements of G^* . So the remaining argument is exactly the same as for Part 1 with G being replaced by G^* .

For Part 3, the corresponding digraph D is defined as in Part 1 except that there is an arc from ε_s to a vertex $\mathbf{v} \in \mathcal{R}$ only if the first d parts of \mathbf{v} are all nonzero. Thus we have $\sigma = d+1$, $J = (|G| - d)^b$, $K = (|G| - d)^{d+1}$, and $H = (|G| - 1)^d(|G| - d)$.

For Part 4, we let $\mathcal{R} = \{(u_1, \dots, u_d) : u_j \in G, \sum_{j=1}^d u_j \neq \mathbf{0}\}$. So we have $\sigma = d$. We note that for every choice of (u_1, \dots, u_{d-1}) , there are $|G| - 1$ choices of u_d such that the total sum is nonzero (we need the assumption $d \geq 2$ here). Hence $|\mathcal{R}| = (|G| - 1)|G|^{d-1}$. The vertex ε_s is joined to all recurrent vertices and so $H = (|G| - 1)|G|^{d-1}$. A recurrent vertex $\mathbf{u} = (u_1, u_2, \dots, u_d)$ is joined to a vertex $\mathbf{v} = (v_1, v_2, \dots, v_k) \in \mathcal{R} \cup \text{SEQ}_b(G)$ if $v_1 + \dots + v_j + u_{j+1} + \dots + u_d$ is not zero for each $1 \leq j \leq k$. Hence $K = (|G| - 1)^d$ and $J = (|G| - 1)^b$.

For Part 5, we take \mathcal{R} to be the set of all d -tuples x_1, \dots, x_d of nonzero elements of \mathbb{F}_q such that $x_d \neq (x_1 \cdots x_{d-1})^{-1}$. If $q = p^n$, each field element is represented as an n -tuple over \mathbb{Z}_p . This gives $\sigma = d$ and $H = |\mathcal{R}| = (q - 1)^{d-1}(q - 2)$. We have $K = (q - 2)^d$ since to join x_1, \dots, x_d to y_1, \dots, y_d , each y_i is constrained to be nonzero and not the multiplicative inverse of the previous $d - 1$ parts. Similarly, $J = (q - 2)^b$.

Next we verify that the digraph $D_{\mathcal{R}}$ is strongly connected by showing that there is a directed walk from any recurrent vertex to any other recurrent vertex.

For Part 1, Let $\mathbf{b} = (b_1, b_2, \dots, b_{d+1})$ and $\mathbf{a} = (a_1, a_2, \dots, a_{d+1})$ be any two distinct recurrent vertices. Let j be the smallest integer such that $a_{j+1} \neq b_{j+1}$. Thus we can write $\mathbf{a} = (b_1, \dots, b_j, a_{j+1}, \dots, a_{d+1})$, where $0 \leq j \leq d$ and $a_{j+1} \neq b_{j+1}$. We use induction on $d - j$ to show that there is a directed walk from \mathbf{b} to \mathbf{a} . The basis case $j = d$ is obvious since \mathbf{b} is joined to \mathbf{a} . Now we move to the inductive step by finding a recurrent vertex $(b_1, \dots, b_j, b_{j+1}, x_{j+2}, \dots, x_{d+1})$ which is joined to \mathbf{a} . If $b_{j+1} \notin \{a_{j+2}, \dots, a_{d+1}\}$, then $(b_1, \dots, b_j, b_{j+1}, a_{j+2}, \dots, a_{d+1})$ is joined to \mathbf{a} . If $b_{j+1} = a_k$ for some $k \in \{j + 2, \dots, d + 1\}$, then for any $y \notin \{b_1, \dots, b_{j+1}, a_{j+1}, \dots, a_{d+1}\}$, it is easy to check that $(b_1, \dots, b_j, b_{j+1}, a_{j+2}, \dots, a_{k-1}, y, a_{k+1}, \dots, a_{d+1})$ is joined to \mathbf{a} .

The argument for Parts 2 and 3 is similar to above.

For Part 4, we show that there is a directed walk from a recurrent vertex $\mathbf{b} = (b_1, b_2, \dots, b_d)$ to another recurrent vertex $\mathbf{a} = (b_1, \dots, b_j, a_{j+1}, \dots, a_d)$ using induction on $d - j$ as for Part 1. When $j = d - 1$, it is clear that \mathbf{b} is joined to \mathbf{a} . For the inductive step, let $y \in G$ satisfy

$$\begin{aligned} y + (b_1 + \dots + b_{j+1} + a_{j+3} + \dots + a_d) &\neq 0, \\ y + (b_1 + \dots + b_j + a_{j+1} + a_{j+3} + \dots + a_d) &\neq 0. \end{aligned}$$

Then it is easy to check that $(b_1, \dots, b_j, b_{j+1}, y, a_{j+3}, \dots, a_d)$ is joined to \mathbf{a} .

The argument for Part 5 is similar to that for Part 4.

Finally we verify the two conditions in Theorem 7 for the five classes of compositions. The verification proceeds as follows. For each $1 \leq t \leq r$, we find two distinct recurrent vertices $\mathbf{u} = (u_1, u_2, \dots, u_\sigma)$ and $\mathbf{v} = (v_1, v_2, \dots, v_\sigma)$ such that \mathbf{u} is joined to itself and to

$s \backslash m$	1	2	3	4	5	6	7	8	9	10
0	0	0	12	24	48	204	624	1680	5196	16008
1	1	3	6	21	69	192	573	1767	5262	15681

Table 2: Counts of locally 2-Mullen compositions over \mathbb{Z}_5 .

$s \backslash m$	2	3	4	5	6	7	8	9	10
0	4	24	88	320	1248	5120	20728	82284	326296
1	6	18	72	320	1284	5120	20232	81738	329064
2	4	18	88	320	1236	5120	20728	81738	326296
3	6	24	72	320	1392	5120	20232	82284	329064
4	4	18	88	320	1236	5120	20728	81738	326296
5	6	18	72	320	1284	5120	20232	81738	329064

Table 3: Counts of 2-Carlitz weak compositions over \mathbb{Z}_6 .

\mathbf{v} , \mathbf{v} is joined to itself, and $|\mathbf{v}| - |\mathbf{u}| = \mathbf{e}_t$ where \mathbf{e}_t is the t^{th} unit vector of dimension r . Hence condition 1 is satisfied because of the loop at \mathbf{u} , and $\mathcal{W}_{\mathcal{R}} = \{\mathbf{u}\mathbf{u}\mathbf{v}, \mathbf{u}\mathbf{v}\mathbf{v}\}$ is a set of directed walks of length 2 from \mathbf{u} to \mathbf{v} which satisfies condition 2.

For Part 1, we choose d distinct elements u_1, \dots, u_d from $G \setminus \{\mathbf{0}, \mathbf{e}_t\}$ and let $\mathbf{u} = (u_1, \dots, u_d, \mathbf{0})$ and $\mathbf{v} = (u_1, \dots, u_d, \mathbf{e}_t)$. It is easy to see that there are $(|G| - 2)^d > 0$ choices of u_1, \dots, u_d .

For Part 2, we distinguish two cases. If $r = 1$, then $G = \mathbb{Z}_k$ for some $k \geq d + 3$. In this case we simply let $\mathbf{u} = (3, 4, \dots, d + 2, 1)$ and $\mathbf{v} = (3, 4, \dots, d + 2, 2)$. If $r \geq 2$, we let t and t' be two distinct integers in $\{1, 2, \dots, r\}$. We then choose d distinct elements u_1, \dots, u_d from $G \setminus \{\mathbf{0}, \mathbf{e}_{t'}, \mathbf{e}_{t'} + \mathbf{e}_t\}$ and let $\mathbf{u} = (u_1, \dots, u_d, \mathbf{e}_{t'})$ and $\mathbf{v} = (u_1, \dots, u_d, \mathbf{e}_{t'} + \mathbf{e}_t)$. It is clear that there are $(|G| - 3)^d > 0$ choices of u_1, \dots, u_d .

For Part 3, we may use the same \mathbf{u} and \mathbf{v} as in Part 1.

For Part 4, again we discuss two cases. The case $G = \mathbb{Z}_k$ is treated as in Part 2. If $r \geq 2$, we let t and t' be two distinct integers in $\{1, 2, \dots, r\}$, and choose $d - 1$ elements u_1, \dots, u_{d-1} from G such that $u_1 + \dots + u_{d-1} + \mathbf{e}_{t'} \neq \mathbf{0}$ and $u_1 + \dots + u_{d-1} + \mathbf{e}_{t'} + \mathbf{e}_t \neq \mathbf{0}$. We then let $\mathbf{u} := (u_1, \dots, u_{d-1}, \mathbf{e}_{t'})$, $\mathbf{v} := (u_1, \dots, u_{d-1}, \mathbf{e}_{t'} + \mathbf{e}_t)$. There are $(|G| - 2)|G|^{d-2}$ choices of u_1, \dots, u_{d-1} .

Part 5 is handled in a similar manner to Part 4. □

5 Bijections and exact values

Here we first provide numerical values for the number of locally Mullen compositions and Carlitz compositions. Table 2 shows initial values of $c_m(\mathbf{s})$ for locally 2-Mullen compositions; Table 3 is similar for 2-Carlitz weak compositions; and Table 4 is similar for 2-Carlitz compositions.

$s \backslash m$	2	3	4	5	6	7	8	9	10
0	4	12	32	80	280	812	2572	6644	23460
1	4	6	34	82	284	748	2498	7372	21522
2	2	12	32	80	274	866	2266	7484	21642
3	4	12	16	136	224	820	2480	7384	21432
4	2	12	32	80	274	866	2266	7484	21642
5	4	6	34	82	284	748	2498	7372	21522

Table 4: Counts of 2-Carlitz compositions over \mathbb{Z}_6 .

Next we give bijections between different families of compositions. The following observation establishes a connection between Carlitz compositions and locally d -Mullen compositions.

Proposition 23. *For each m -composition $\mathbf{u} = u_1, u_2, \dots, u_m$ over G , let $\mathbf{v} = \phi(\mathbf{u})$ be an m -composition defined by $v_j = u_1 + \dots + u_j, 1 \leq j \leq m$. Then ϕ is a bijection between locally d -Mullen m -compositions and d -Carlitz weak m -compositions over G such that the first d parts are nonzero.*

Proof. It follows from the definition of the mapping ϕ that $v_{i+j} = v_i + (u_{i+1} + \dots + u_{i+j})$. It is clear that $v_i \neq v_{i+j}$ if and only if $u_{i+1} + \dots + u_{i+j} \neq 0$. It is important to note that v_k might be zero when $k > d$, however $v_k \neq 0$ when $1 \leq k \leq d$ when \mathbf{u} is locally d -Mullen. Also \mathbf{u} and \mathbf{v} generally do not have the same sum. \square

Using Corollary 2 Part 3 and the above proposition, we immediately obtain the following.

Theorem 24. *Let G be a finite abelian group, and d be a positive integer such that $|G| \geq d + 1$. Let $c_m(\mathbf{s})$ be the number of locally d -Mullen m -compositions of $\mathbf{s} \in G$. Then there is a positive constant $\theta < 1$ such that*

$$c_m(\mathbf{s}) = \frac{1}{|G|} (|G| - 1)^d (|G| - d)^{m-d} (1 + O(\theta^m)), \text{ as } m \rightarrow \infty.$$

Proposition 25. 1. *Let \mathcal{A} be a family of compositions over a finite abelian group G . Suppose \mathcal{A} is closed under multiplication, that is, if $\mathbf{x} := (x_1, \dots, x_m)$ belongs to \mathcal{A} then $a\mathbf{x} := (ax_1, \dots, ax_m)$ also belongs to \mathcal{A} for every $a \in G^*$. Let $c_m(\mathbf{s}; \mathcal{A})$ be the number of m -compositions of \mathbf{s} in \mathcal{A} . If $\mathbf{s} \in G^*$ has a multiplicative inverse \mathbf{s}^{-1} in G^* , then $c_m(\mathbf{s}; \mathcal{A}) = c_m(\mathbf{1}; \mathcal{A})$.*

2 *Let $c_m(\mathbf{s})$ be the number of locally d -Mullen m -compositions of $\mathbf{s} \in G$. Then $c_m(\mathbf{s}) = c_m(\mathbf{1})$ for every $\mathbf{s} \in G^*$.*

Proof. 1. It is clear that $\mathbf{x} \mapsto \mathbf{s}^{-1}\mathbf{x}$ is a bijection between compositions of \mathbf{s} and compositions of $\mathbf{1}$ in \mathcal{A} .

2. The claim follows from Part 1 if \mathbf{s}^{-1} exists. For general $\mathbf{s} \in G^*$, let π be a permutation of G such that $\pi(\mathbf{s}) = \mathbf{1}$ and $\pi(\mathbf{0}) = \mathbf{0}$. Then it is easy to verify that $\phi\pi\phi^{-1}$ is a bijection from locally d -Mullen m -compositions of \mathbf{s} to those of $\mathbf{1}$. \square

Remark. The above proposition implies that the number of Carlitz m -compositions of s over a finite field is equal to that of 1 when $s \neq 0$. And the same is true for Carlitz weak compositions.

6 Conclusion

The structure of finite abelian groups naturally leads us to use Corollary 12 to go from a generating function for restricted integer compositions to the number of restricted compositions over group elements. Assuming some aperiodicity conditions, asymptotic analysis of this expression gives us the dominant term, as the number of parts goes to infinity. Exact counts are available as well, by taking powers of the transfer matrix. A couple of potential extensions to this work are apparent. It would be interesting to get analogous asymptotic counting results for non-abelian groups. A similar level of generality is unlikely but one could be curious about what techniques are necessary to generalize at least somewhat beyond the restriction of commutativity. It may also be interesting to consider further types of restriction such as pattern avoidance.

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