Eigenvalue bounds for the signless p-Laplacian

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Abstract

We consider the signless p-Laplacian Q_p of a graph, a generalisation of the quadratic form of the signless Laplacian matrix (the case $p = 2$). In analogy to Rayleigh's principle the minimum and maximum of Q_p on the p-norm unit sphere are called its smallest and largest eigenvalues, respectively. We show a Perron-Frobenius property and basic inequalites for the largest eigenvalue and provide upper and lower bounds for the smallest eigenvalue in terms of a graph parameter related to the bipartiteness. The latter result generalises bounds by Desai and Rao and, interestingly, at $p = 1$ upper and lower bounds coincide.

Mathematics Subject Classifications: 05C50, 05C40, 15A18

1 Introduction

We begin with some notation. All graphs are simple and undirected without loops or multiple edges. For a graph $G = (V, E)$ with vertex set V the edge set E consists of two-element subsets of V and for each edge we write ij rather than $\{i, j\}$. For disjoint subsets $S, T \subseteq V$ define $E_G(S)$ as the edges $ij \in E$ spanned by S, $E_G(S, T)$ the edges with one vertex in S and the other in T; we also define $n = |V|$, $e_G(S) = |E_G(S)|$,

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$e_G(S,T) = |E_G(S,T)|$ and $\mathrm{cut}_G(S) = |E_G(S, V \setminus S)|$. Furthermore $d_i, i \in V$ is the degree of vertex i and $\delta(G), \Delta(G)$ denote the minimum and maximum degree, respectively. We shall drop the subscripts if G is clear from the context. We find it convenient to index vectors and matrices associated with the graph by the vertex set V and write therefore $x = (x_i, i \in V) \in \mathbb{R}^V$ rather than $x \in \mathbb{R}^n$. Accordingly, we denote by $A \in \mathbb{R}^{V \times V}$ the adjacency matrix and by $D \in \mathbb{R}^{V \times V}$ the diagonal matrix of vertex degrees.

The eigenvalues and eigenvectors of the Laplacian $L := D - A$ are well studied, in particular the second smallest eigenvalue $a(G)$, the algebraic connectivity of G. It is non-zero if and only if G is connected and by a well-known inequality due to Mohar [10] it can be upper and lower bounded in terms of the isoperimetric number of G

$$i(G) := \min \left\{ \frac{\mathrm{cut}(S)}{|S|},\ S \subseteq V,\ 0 < |S| \leqslant \frac{n}{2} \right\}.$$

Eigenvectors for $a(G)$ are used in spectral partitioning. Following Amghibech's work [1] Bühler and Hein [4] introduced for $p > 1$ a non-negative functional $L_p(x) = \sum_{ij \in E} |x_i - x_j|^p$ on $\mathbb{R}^V$ which for $p = 2$ yields the quadratic form of L. The p-Laplacian of G is defined as the non-linear operator

$$\frac{1}{p} \nabla_x L_p \colon \mathbb{R}^V \to \mathbb{R}^V, \quad \frac{1}{p} \nabla_x L_p(x)_i = \sum_{j \colon ij \in E} \mathrm{sign}\,(x_i - x_j)\, |x_i - x_j|^{p-1}$$

where

$$\mathrm{sign}\,(x) = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ if } x < 0, \\ 0 \text{ if } x = 0. \end{cases}$$

A vector $x \neq 0$ is called an eigenvector of L_p with corresponding eigenvalue $\mu \in \mathbb{R}$ if the eigenequations

$$\frac{1}{p} \nabla_x L_p(x)_i = \sum_{j \colon ij \in E} \mathrm{sign}\,(x_i - x_j)\, |x_i - x_j|^{p-1} = \mu \mathrm{sign}\,(x_i)\, |x_i|^{p-1},\ i \in V,$$

are satisfied. Observe that in this case $p^{-1} x^\top \nabla_x L_p(x) = L_p(x) = \mu \|x\|_p^p$ and that the eigenequations are necessary conditions for the optimisation problems

$$\min\,(\text{resp. } \max) L_p(x) \text{ s.t. } \|x\|_p^p = 1$$

which for $p = 2$ are the Rayleigh-Ritz characterisations of the smallest (resp. largest) eigenvalue of L. The minimum is always zero, attained by a non-zero multiple of the all ones vector $\mathbf{1}$ and these are the only solutions if and only if G is connected. Observing that $\mathbf{1}^\top \nabla_x L_p(x) = 0$ for any x one is led to characterise the smallest non-zero eigenvalue $a_p(G)$ as

$$a_p(G) = \min L_p(x) \text{ s.t. } \|x\|_p^p = 1 \text{ and } \sum_{i \in V} \mathrm{sign}\,(x_i)\, |x_i|^{p-1} = 0 \tag{1}$$

Bühler and Hein [4] give the following bounds

$$\left(\frac{2}{\Delta}\right)^{p-1}\left(\frac{i(G)}{p}\right)^{p} \leqslant a_p(G) \leqslant 2^{p-1} i(G), \tag{2}$$

where Δ is the maximum degree. Interestingly, in the limit $p \to 1$ upper and lower bounds coincide. Indeed, in [5] spectral properties of the 1-Laplacian are explored. The eigenequations are then a nonlinear system involving set valued functions (gradients are replaced by subdifferentials) and it is found that the smallest non-zero eigenvalue is in fact $i(G)$.

Only recently the signless Laplacian matrix $Q = D + A$ has received greater interest from spectral graph theorists, nevertheless an early remarkable result [6] dates back to 1994. The quadratic form of Q is $x^\top Q x = \sum_{ij\in E}(x_i + x_j)^2$ and hence Q is positive semidefinite. The smallest eigenvalue $q(G)$ is 0 if and only if G has a bipartite component, see Lemma 3. Informally speaking, the results in [6] state that a small $q(G)$ indicates the existence of a nearly bipartite subgraph which is not very well connected to the rest of the graph, and vice versa. More precisely, in [6] they define the graph parameter $\psi(G)$ as

$$\psi(G) := \min\left\{\frac{2e(S) + 2e(T) + \mathrm{cut}(S\cup T)}{|S\cup T|} : S, T \subseteq V,\ S\cap T = \varnothing,\ S\cup T \neq \varnothing\right\}. \tag{3}$$

and prove a lower and upper bound on $q(G)$ reminiscent of (2) for $p = 2$, namely

$$\frac{\psi(G)^2}{2\Delta} \leqslant q(G) \leqslant 2\psi(G). \tag{4}$$

Observe that $\psi(G) = 0$ if and only if G has a bipartite component and, moreover, that for a minimising pair S, T in (3) the number $e(S) + e(T)$ is the smallest number of edges to be removed from the induced subgraph on $S \cup T$ to make it bipartite. Thus the bounds in (4) quite well capture the informal statement. We also remark that the definition of ψ and also the bounds differ (superficially) from those in [6]. We followed the exposition of [7] where bounds on $q(G)$ in terms of edge and vertex bipartiteness and a stronger lower bound in terms of $\psi(G)$ are established. Finding (near) bipartite substructures in graphs is of practical interest in the study of social networks and bioinformatics as pointed out in [8], where the authors devise spectral techniques involving eigenvectors for $q(G)$ to obtain such structures.

In this article we consider for real $p \geqslant 1$ the non-negative convex functional

$$Q_p\colon \mathbb{R}^V \to \mathbb{R}, \quad Q_p(x) = \sum_{ij\in E} |x_i + x_j|^p$$

(i.e. $Q_2(x) = x^\top Q x$) and its extremal values on the p-norm unit sphere

$$S_p^V = \left\{x \in \mathbb{R}^V : \sum_{i\in V} |x_i|^p = 1\right\}$$

or, equivalently, extremal values of $R_p(x) := Q_p(x)/\left\|x\right\|_p^p$ on $\mathbb{R}^V \setminus \{0\}$. For $p = 2$ this is the Rayleigh-Ritz characterisation of the smallest and largest eigenvalues. Let

$$q_p(G) = \min\left\{Q_p(x)\colon\ \|x\|_p^p = \sum_{i\in V}|x_i|^p = 1\right\} = \min\left\{R_p(x)\colon x \neq 0\right\} \tag{5}$$

and in an analogous fashion define by $\lambda_p(G)$ the maximum. We call $q_p(G)$ and $\lambda_p(G)$ the smallest and largest eigenvalues of Q_p, respectively and minimising (maximising) vectors eigenvectors. Our study of $q_p(G)$ is strongly motivated by the inequalities (4) and (2) and our main result is an upper and lower bound on $q_p(G)$ in terms of the graph parameter $\psi(G)$.

Theorem 1. *For $p \geqslant 1$ the smallest signless p-Laplacian eigenvalue $q_p(G)$ satisfies*

$$\left(\frac{2}{\Delta}\right)^{p-1}\left(\frac{\psi(G)}{p}\right)^p \leqslant q_p(G) \leqslant 2^{p-1}\psi(G).$$

In particular, we have that $q_1(G) = \psi(G)$.

The proof of Theorem 1 provides actually a stronger lower bound and a method to obtain a "good" pair (S, T) (in view of (3)) by thresholding a solution to the optimisation problem (5):

Corollary 2. *For $x \in \mathbb{R}^V \setminus \{0\}$ and $t \geqslant 0$ define the vertex sets $S_x^t = \{i \in V : x_i > t\}$ and $T_x^t = \{i \in V : x_i < -t\}$ and define*

$$\psi(G, x) := \min\left\{\frac{2e_G\left(S_x^t\right) + 2e_G\left(T_x^t\right) + \mathrm{cut}_G\left(S_x^t \cup T_x^t\right)}{|S_x^t \cup T_x^t|} : 0 \leqslant t < \max_{i\in V}\{|x_i|\}\right\}. \tag{6}$$

If $p \geqslant 1$ and $x^{(p)}$ is an eigenvector for $q_p(G)$ then the bounds in Theorem 1 also hold with $\psi(G)$ replaced by $\psi(G, x^{(p)})$. In particular, $\psi(G, x^{(p)}) \to \psi(G)$ as $p \to 1$ and for $p = 1$, $\psi(G) = \psi(G, x^{(1)})$.

For practical matters as raised in [8] it would be interesting to know if the computation or approximamtion of $q_p(G)$ and a corresponding eigenvector can be carried out efficiently. We are currently investigating this.

Outline. In the next section we provide a proof of Theorem 1 and Corollary 2. In the subsequent section we discuss the case $p = \infty$ and in the final section some basic properties of the largest eigenvalue.

2 Proof of Theorem 1

In this section we assume throughout that $p \geqslant 1$. First we shall see that Theorem 1 is trivially true for graphs with a bipartite connected component:

Lemma 3. *The smallest eigenvalue $q_p(G)$ is zero if and only if G has a bipartite component.*

Proof. If there is a bipartite component with partition $S \cup T$ put $x_i = 1$, if $i \in S$, $x_i = -1$, if $i \in T$ and $x_i = 0$ elsewhere. Then $Q_p(x) = 0$. Conversely, $Q_p(x) = 0$ implies $x_i = -x_j$ for every $ij \in E$ and thus a connected component containing a vertex i with $x_i \neq 0$ is bipartite. $\square$

The next lemma ([6, 7] for $p = 2$) yields the upper bound in Theorem 1.

Lemma 4. *Let $S, T \subseteq V$, $S \cap T = \varnothing$. Then we have*

$$q_p(G)|S \cup T| \leqslant 2^p e(S) + 2^p e(T) + \operatorname{cut}(S \cup T)$$

and in particular $q_p(G) \leqslant 2^{p-1}\psi(G)$.

Proof. Let $S \cup T \neq \varnothing$, otherwise the assertion is trivial. Define x by $x_i = 1$, if $i \in S$, $x_i = -1$, if $i \in T$ and $x_i = 0$ elsewhere. Then a computation shows

$$q_p(G) \leqslant \frac{Q_p(x)}{\|x\|_p^p} = \frac{2^p e(S) + 2^p e(T) + \operatorname{cut}(S \cup T)}{|S \cup T|}.$$

If S, T are chosen as optimal sets in the definition (3) of $\psi(G)$ this last expression is $\leqslant 2^{p-1}\psi(G)$ with equality only if $\operatorname{cut}(S \cup T) = 0$ or $p = 1$. $\square$

For the proof of the lower bound we combine techniques from [6] and [4]. For a graph $G' = (V', E')$ fix a subset $U \subseteq V'$. For a vector $g \in \mathbb{R}^{V'}$ with $g_i > 0$ on U and $g_i = 0$ elsewhere define $C_g^t = \{i \in U : g_i > t\}$ and

$$h_g(U) := \min\left\{\frac{\operatorname{cut}_{G'}\left(C_g^t\right)}{\left|C_g^t\right|} : 0 \leqslant t < \max\{g_i, i \in U\}\right\}. \tag{7}$$

We have the following lemma.

Lemma 5. *With the above notation we have*

$$\left(\frac{2}{\Delta(G')}\right)^{p-1}\left(\frac{h_g(U)}{p}\right)^p \|g\|_p^p \leqslant \sum_{ij \in E'} |g_i - g_j|^p \tag{8}$$

where $\Delta(G')$ is the maximum degree of G'.

We postpone the proof of the lemma and first apply it to prove Theorem 1. The idea is from [6], however, at some places we must be a bit more careful to get a lower bound which, as $p \to 1$, coincides with the upper bound.

Proof of Theorem 1, lower bound. Consider a connected graph $G = (V, E)$ and its signless p-Laplacian Q_p. Let x be a normalised eigenvector ($\|x\|_p = 1$) for the smallest eigenvalue $q_p = q_p(G)$ such that $Q_p(x) = q_p$. Define $S = \{i \in V : x_i > 0\}$ and $T = \{i \in V : x_i < 0\}$ and a graph $G' = (V', E')$ as follows. Let $V' = V \dot{\cup} S' \dot{\cup} T'$ where $S' = \{i' : i \in S\}$ and $T' = \{i' : i \in T\}$ are disjoint copies of S and T, respectively, and define

$$E' = E_G(S, T) \cup E_G(S \cup T, V \setminus (S \cup T)) \cup \{i'j, ij' : ij \in E_G(S)\} \cup \{i'j, ij' : ij \in E_G(T)\}$$

i.e. E' is obtained from E by deleting every edge ij with both endpoints in S (resp. T) and adding two edges ij' and $i'j$. Define $g \in \mathbb{R}^{V'}$ by $g_i = |x_i|$ if $i \in S \cup T$ and $g_i = 0$ if $i \in V' \setminus (S \cup T)$. Then we have $\|g\|_p = \|x\|_p = 1$ and

$$\sum_{ij \in E'} |g_i - g_j|^p \leqslant \sum_{ij \in E} |x_i + x_j|^p = q_p. \tag{9}$$

To see this, first consider an edge $ij \in E_G(S) \cup E_G(T)$ on the right hand side. We have two corresponding edges in $i'j, ij' \in E'$ on the left side and

$$|g_{i'} - g_j|^p + |g_i - g_{j'}|^p = |g_j|^p + |g_i|^p \leqslant ||x_i| + |x_j||^p = |x_i + x_j|^p$$

because x_i and x_j have the same sign. The remaining summands on both sides correspond to edges $ij \in E \cap E'$. A case by case inspection yields $|g_i - g_j|^p = |x_i + x_j|^p$ and (9) follows.[1]

Now we show $h_g(S \cup T) \geqslant \psi(G)$. To that end consider an optimal $t \geqslant 0$ in the definition (7) of $h_g(S \cup T)$ and a corresponding cut set $C_g^t \subseteq S \cup T$. Define $S^t = C_g^t \cap S$ and $T^t = C_g^t \cap T$. Recall the vertex sets S_x^t and T_x^t of G and the parameter $\psi(G, x)$ defined in Corollary 2 and observe that actually $S^t = S_x^t$ and $T^t = T_x^t$ (viewed as vertex subsets of G) by the definitions of g and C_g^t. Therefore, in G, we have a chain of inequalities

$$\frac{2e_G(S^t) + 2e_G(T^t) + \mathrm{cut}_G(S^t \cup T^t)}{|S^t \cup T^t|} \geqslant \psi(G, x) \geqslant \psi(G). \tag{10}$$

On the other hand we observe that

$$2e_G(S^t) + 2e_G(T^t) + \mathrm{cut}_G(S^t \cup T^t) = \mathrm{cut}_{G'}(S^t \cup T^t) \tag{11}$$

because every edge $ij \in E_G(S^t) \cup E_G(T^t)$ corresponds to two edges counted in $\mathrm{cut}_{G'}(S^t \cup T^t)$, namely ij' and $i'j$. Moreover, every edge in $E_G(S^t \cup T^t, V \setminus (S^t \cup T^t))$ contributes exactly one edge counted in $\mathrm{cut}_{G'}(S^t \cup T^t)$: for example, an edge $ij \in E$ in G with $i \in S^t$ and $j \in S \setminus S^t$ is accounted for in $\mathrm{cut}_{G'}(S^t \cup T^t)$ by the edge $ij' \in E'$ but not by the edge $i'j \in E'(V' \setminus (S^t \cup T^t))$.

Now (10), (11) and the optimality of $C_g^t = S^t \cup T^t$ for (7) imply

$$h_g(S \cup T) \geqslant \psi(G, x) \geqslant \psi(G). \tag{12}$$

Combine Lemma 5 (applied to G' with $U = S \cup T$) with (9) and (12) and observe that $\Delta(G') = \Delta(G)$ to complete the proof of Corollary 2 and Theorem 1. $\square$

[1] Inequality (9) is sharper than the original [6, inequality (23)] where there is a factor 2 on the right hand side which can be omitted. This was also pointed out in [7].

We now prove Lemma 5, where we follow [4] up to minor modifications.

Proof of Lemma 5. We first show

$$h_g(U)\,\|g\|_p^p \leqslant \sum_{ij\in E'} |g_i^p - g_j^p|. \tag{13}$$

To that end write

$$\sum_{ij\in E', g_i>g_j} (g_i^p - g_j^p) = \sum_{ij\in E', g_i>g_j} p\int_{g_j}^{g_i} t^{p-1}dt = p\int_0^\infty t^{p-1} \sum_{ij\in E'\colon\, g_i>t\geqslant g_j} 1dt$$

and observe that $\sum_{ij\in E'\colon\, g_i>t\geqslant g_j} 1 = |\{ij\in E'\colon i\in C_g^t, j\notin C_g^t\}| = \mathrm{cut}(C_g^t)$. By the definition of $h_g(U)$ we have

$$\mathrm{cut}(C_g^t) \geqslant h_g(U)|C_g^t| = h_g(U) \sum_{i\in V'\colon\, g_i>t} 1.$$

Hence we can give a lower bound

$$\begin{aligned}\sum_{ij\in E', g_i>g_j} (g_i^p - g_j^p) &= p\int_0^\infty t^{p-1} \sum_{ij\in E'\colon\, g_i>t\geqslant g_j} 1dt \\ &\geqslant p\int_0^\infty t^{p-1} h_g(U) \sum_{i\in V'\colon\, g_i>t} 1dt \\ &= h_g(U) \sum_{i\in V'\colon\, g_i>0} \int_0^{g_i} pt^{p-1}dt \\ &= h_g(U) \sum_{i\in V'\colon\, g_i>0} g_i^p \\ &= h_g(U)\,\|g\|_p^p\end{aligned}$$

and obtain (13). Observe that this completes the proof of the lemma if $p=1$.
For $p>1$ we proceed with Hölder's inequality $\sum|x_iy_i| \leqslant (\sum|x_i^p|)^{1/p}(\sum|y_i^q|)^{1/q}$ where $q=p/(p-1)$.

$$\begin{aligned}\sum_{ij\in E'} |g_i^p - g_j^p| &= \sum_{ij\in E'} |g_i-g_j| \frac{g_i^p-g_j^p}{g_i-g_j} \\ &\leqslant \left(\sum_{ij\in E'} |g_i-g_j|^p\right)^{1/p} \left(\sum_{ij\in E'} \left(\frac{g_i^p-g_j^p}{g_i-g_j}\right)^q\right)^{1/q}\end{aligned} \tag{14}$$

In order to bound the last term from above we use an inequality from [1] (an alternative proof is given at the end of this section) which states that

$$\left(\frac{g_i^p-g_j^p}{g_i-g_j}\right)^q \leqslant p^q \left(\frac{g_i^p+g_j^p}{2}\right). \tag{15}$$

The second factor in the last term of (14) can thus be bounded from above by (with d'_i the degree of vertex i in G')

$$\left(\sum_{ij\in E'}\left(\frac{g_i^p-g_j^p}{g_i-g_j}\right)^q\right)^{1/q} \leqslant p\left(\sum_{ij\in E'}\frac{g_i^p+g_j^p}{2}\right)^{1/q} = \frac{p}{2^{1/q}}\left(\sum_{i\in V'}d'_i g_i^p\right)^{1/q}$$
$$\leqslant \frac{p}{2^{1/q}}\left(\Delta(G')\,\|g\|_p^p\right)^{1/q} = p\left(\frac{\Delta(G')}{2}\right)^{(p-1)/p}\|g\|_p^{p-1}$$

Substitute this last term into (14) combine with (13) and regroup terms to obtain the assertion of the Lemma.

For a self-contained exposition we show inequality (15). Recall the power mean inequality: for $x_1,\dots,x_n>0$ and $r<s$ we have $\left(\frac{1}{n}\sum_{k=1}^n x_k^r\right)^{1/r} \leqslant \left(\frac{1}{n}\sum_{k=1}^n x_k^s\right)^{1/s}$. Using Riemann sum approximations we obtain for $0\leqslant a<b$ a continuous version

$$\left(\frac{1}{b-a}\int_a^b t^r dt\right)^{1/r} = \lim_{n\to\infty}\left(\sum_{k=1}^n\frac{1}{n}\left(a+k\cdot\frac{b-a}{n}\right)^r\right)^{1/r}$$
$$\leqslant \lim_{n\to\infty}\left(\sum_{k=1}^n\frac{1}{n}\left(a+k\cdot\frac{b-a}{n}\right)^s\right)^{1/s} = \left(\frac{1}{b-a}\int_a^b t^s dt\right)^{1/s}$$

An application of this to (15) yields (assuming $g_i>g_j$)

$$\left(\frac{g_i^p-g_j^p}{g_i-g_j}\right)^q = p^q\left(\frac{1}{g_i-g_j}\int_{g_j}^{g_i}t^{p-1}dt\right)^{p/(p-1)}$$
$$\leqslant p^q\frac{1}{g_i-g_j}\int_{g_j}^{g_i}t^p dt \leqslant p^q\left(\frac{g_i^p+g_j^p}{2}\right)$$

where the last "$\leqslant$" follows from the convexity of t^p for $p\geqslant 1$: in the interval $g_j\leqslant t\leqslant g_i$ we have $t^p\leqslant s(t):=g_j^p+\frac{g_i^p-g_j^p}{g_i-g_j}(t-g_j)$ and the last inequality is obtained upon replacing t^p by $s(t)$ in the integral. □

3 The case $p=\infty$

If we equivalently minimise $Q_p(x)^{1/p}$ in (5) the case $p=\infty$ is also meaningful with

$$q_\infty(G)=\min_{\|x\|_\infty=1}\max_{ij\in E}|x_i+x_j|.$$

Lemma 3 holds accordingly, that is, q_∞ is zero if and only if G has a bipartite component.

Theorem 6. *For $u\in V$ let $l(u)$ be the smallest odd integer $2k+1$ ($k\in\mathbb{N}$) such that there exists a closed walk $ui_1, i_1i_2,\dots,i_{2k-1}i_{2k}, i_{2k}u$ in G starting and ending in u and let*

$l(u) = \infty$ if no such walk exists (i.e. u lies in a bipartite component of G). Then we have (with the convention $1/\infty = 0$)

$$q_\infty(G) = \frac{2}{\max_{u \in V} l(u)}.$$

Proof. We assume that G does not have a bipartite component because otherwise the assertion is true. Observe that

$$q_\infty = \min_{u \in V} p_u$$

where p_u is the optimal value of the linear program

$$\min_{\substack{\|x\|_\infty = 1 \\ x_u = 1}} \max_{ij \in E} |x_i + x_j| = \text{minimise } \mu \text{ s.t. } \begin{cases} -\mu \leqslant x_i + x_j \leqslant \mu & (ij \in E), \\ -1 \leqslant x_i \leqslant 1 & (i \in V \setminus \{u\}), \\ x_u = 1. \end{cases}$$

Since $p_u > 0$ for an optimal solution we can assume $\mu > 0$ and divide the constraints by μ, introduce new variables $y_i = x_i/\mu$ and obtain an equivalent program whose optimum is $1/p_u$.

$$\text{maximise } y_u \text{ s.t. } \begin{cases} -1 \leqslant y_i + y_j \leqslant 1 & (ij \in E) \\ -y_u \leqslant y_i \leqslant y_u & (i \in V \setminus \{u\}) \end{cases} \tag{16}$$

Consider a closed walk $ui_1, i_1i_2, \ldots, i_{2k-1}i_{2k}, i_{2k}u$ of odd length $2k+1$ in G and sum up the corresponding inequalities in an alternating fashion to obtain

$$-2k - 1 \leqslant y_u + y_{i_1} - y_{i_1} - y_{i_2} + y_{i_2} + y_{i_3} - \ldots - y_{i_{2k-1}} - y_{i_{2k}} + y_{i_{2k}} + y_u \leqslant 2k + 1,$$

in particular we have $y_u \leqslant l(u)/2$. We show that this inequality can be attained with equality. For $v, w \in V$ let $d(v, w)$ be the usual graph distance (length of shortest v-w path) and let $S_i = \{v \in V : d(u, v) = i\}$, $i \geqslant 0$, be the i-th level in a breadth first search tree rooted at u. Recall that a vertex in S_{i+1} has a neighbour in S_i and that any edge in E has either both endpoints in the same level or connects two consecutive levels. Now $l(u) = 2k_0 + 1$ where k_0 is the smallest integer such that S_{k_0} contains two vertices $v \neq w$ with $vw \in E$. We define a feasible vector y by

$$y_v = \begin{cases} (-1)^i \left(k_0 - i + \frac{1}{2}\right) & \text{if } v \in S_i \text{ and } i \leqslant k_0, \\ 0, & \text{otherwise}, \end{cases}$$

in particular $y_u = l(u)/2$. This y is feasible: for an edge vw we have $|y_v + y_w| \in \{0, 1/2, 1\}$ if $v \in S_i$ and $w \in S_{i+1}$ for some $i \geqslant 0$, $|y_v + y_w| = 1$ if $v, w \in S_{k_0}$ and $|y_v + y_w| = 0$ if $v, w \in S_i$, $i > k_0$. □

Remark. Following the formulation of problem (1) in [4] one also has $p = \infty$ version which reads

$$a_\infty(G) = \min \max_{ij \in E} |x_i - x_j| \text{ s.t. } \|x\|_\infty = 1 \text{ and } \max_{i \in V} x_i + \min_{i \in V} x_i = 0. \tag{17}$$

The optimum value is $2/\operatorname{diam}(G)$ where $\operatorname{diam}(G) = \max\{d(s,t)\colon s,t \in V\}$ is the diameter of G. Similarly as for q_∞, one has to solve for every *pair* of vertices $s \neq t$ a linear program

$$\text{minimise } \mu \text{ s.t. } \begin{cases} -\mu \leqslant x_i - x_j \leqslant \mu & (ij \in E), \\ -1 \leqslant x_i \leqslant 1 & (i \in V), \\ \left.\begin{array}{l} x_t = 1, \\ x_s = -1, \end{array}\right\} \Leftrightarrow & \left\{\begin{array}{l} x_t + x_s = 0, \\ x_t - x_s = 2, \end{array}\right. \end{cases}$$

which is equivalent to

$$\text{maximise } \frac{1}{2}(y_t - y_s) \text{ s.t. } \begin{cases} -1 \leqslant y_i - y_j \leqslant 1 & (ij \in E), \\ -y_t \leqslant y_i \leqslant y_t & (i \in V), \\ y_s + y_t = 0, \end{cases}$$

and whose optimum can be shown to be half the length of a shortest s-t path.

4 Basic properties of the largest eigenvalue

For $p > 1$ a necessary condition for an $x \in S_p^V$ to yield the minimum (resp. maximum) in (5) is the existence of a Lagrange multiplier μ such that the *eigenequations*

$$\frac{1}{p}\nabla_x Q_p(x)_i = \sum_{j\colon ij \in E} \operatorname{sign}(x_i + x_j)\,|x_i + x_j|^{p-1} = \mu \operatorname{sign}(x_i)\,|x_i|^{p-1},\ i \in V, \tag{18}$$

are satisfied. If a pair $(x, \mu) \in (\mathbb{R}^V \setminus \{0\}) \times \mathbb{R}$ satisfies (18) we call x an eigenvector and μ an eigenvalue of Q_p. Then (18) implies $p^{-1}x^\top \nabla_x Q_p(x) = Q_p(x) = \mu\,\|x\|_p^p$ and hence eigenvalues are non-negative and $q_p(G)$ and $\lambda_p(G)$ are the smallest, resp. largest eigenvalues.

We first observe that $q_p(G)$ and $\lambda_p(G)$ do not increase when passing to subgraphs.

Lemma 7. *Let H be a subgraph of G. Then $q_p(H) \leqslant q_p(G)$ and $\lambda_p(H) \leqslant \lambda_p(G)$.*

Proof. Since adding isolated vertices does not affect $\lambda_p(H)$ we assume that $H = (V, F)$ with $F \subseteq E$. Let $x \in S_p^V$ be a normalised eigenvector for $\lambda_p(H)$, then

$$\lambda_p(H) = \sum_{ij \in F} |x_i + x_j|^p \leqslant \sum_{ij \in E} |x_i + x_j|^p \leqslant \lambda_p(G).$$

The proof for q_p is similar, starting with an eigenvector for $q_p(G)$. $\square$

The standard basis of $\mathbb{R}^V$ yields easy bounds for $q_p(G)$ and $\lambda_p(G)$ in terms of the maximum and minimum degrees.

Lemma 8. *We have for $p \geqslant 1$,*

$$q_p(G) \leqslant \delta(G) \text{ and } \lambda_p(G) \geqslant \Delta(G)$$

($\delta(G), \Delta(G)$ the minimum/maximum degree). If $p > 1$ equality holds in the former if and only if G has an isolated vertex and equality in the latter if and only if G has no edges.

The standard basis vector e_i ($i \in V$) of $\mathbb{R}^V$ is an eigenvector of Q_p if and only if $d_i = 0$, that is i is an isolated vertex.

Proof. By definition, $q_p(G) \leqslant Q_p(e_i) = d_i \leqslant \lambda_p(G)$. If equality holds in either of the two then e_i is an eigenvector for the eigenvalue d_i. If $d_i > 0$ then i has a neighbour j and the j-th eigenequation (see (18)) reads $1 = d_i \cdot 0$, and so $d_i = 0$. □

If G is a connected graph then the signless Laplacian matrix Q is a non-negative, irreducible, aperiodic matrix. By the Perron-Frobenius Theorem its largest eigenvalue is simple and a corresponding eigenvector has only strictly positive or only stricly negative components. The next theorem shows that this property is shared by eigenvectors corresponding to $\lambda_p(G)$.

Theorem 9. *Let $p > 1$, G be connected and $x \in S_p^V$ be an eigenvector for $\lambda_p(G)$. Then x has only stricly positive or only strictly negative components and is unique up to sign.*

Proof. Denote by C^+, C^- and C^0 the vertex sets on which x_i is > 0, < 0 and $= 0$, respectively, and assume without loss of generality that $C^+ \neq \varnothing$, otherwise consider $-x$. Define $u = (|x_i|, i \in V) \in S_p^V$ and observe that $|x_i + x_j| \leqslant ||x_i| + |x_j|| = |u_i + u_j|$ with equality if and only if x_i and x_j have the same sign or at least one of them is 0. Thus $E(C^+, C^-) = \varnothing$ and u is another eigenvector for λ_p because otherwise we had a contradiction $\lambda_p = Q_p(x) < Q_p(u) \leqslant \lambda_p$. Since G is connected there is an edge ab with $a \in C^+$ and $b \in C^0$. The b-th eigenequation (see (18)) for the eigenpair (u, λ_p) then yields a contradiction

$$0 < |u_a + u_b|^{p-1} \leqslant \sum_{j\,:\, bj \in E} |u_b + u_j|^{p-1} = \lambda_p \operatorname{sign}(u_b)\, |u_b|^{p-1} = 0.$$

So C^0 must be empty. Because G is connected and $E(C^+, C^-) = \varnothing$ it follows that also C^- is empty and $C^+ = V$. As for the uniqueness assume that $x, y \in S_p^V$ are both stricly positive eigenvectors and define $z \in S_p^V$ by

$$z_i = \left(\frac{x_i^p + y_i^p}{2}\right)^{1/p}, \, i \in V.$$

The triangle inequality for the p-norm (Minkowski's inequality) yields

$$
\begin{aligned}
(x_i+x_j)^p+(y_i+y_j)^p &= \left\| \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right\|_p^p \\
&\leqslant \left(\left\| \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\|_p + \left\| \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right\|_p \right)^p \\
&= \left((x_i^p+y_i^p)^{1/p} + \left(x_j^p+y_j^p\right)^{1/p} \right)^p \\
&= 2(z_i+z_j)^p
\end{aligned} \tag{19}
$$

and thus $2\lambda_p = Q_p(x)+Q_p(y) \leqslant 2Q_p(z) \leqslant 2\lambda_p$ and equality must hold in (19) for every edge $ij \in E$. Since the p-norm for $p>1$ is strictly convex (x_i, y_i) must be a positive multiple of (x_j, y_j) whenever $ij \in E$. Therefore, by connectedness of G, the rank of (x, y) is one and hence $x = y$. □

Remark: The theorem is false for $p = 1$. The unit ball is the convex hull of $\{\pm e_i, i \in V\}$ and so by the convexity of Q_1 we have that $\lambda_1 = \max_{i\in V} Q_1(e_i) = \Delta$, the maximum degree of G. The solution is neither strictly positive nor is it unique, unless the maximum degree vertex is unique.

With the positive eigenvector at hand we can prove upper and lower bounds on $\lambda_p(G)$ in terms of vertex degrees.

Lemma 10. *Let $p > 1$, $G = (V, E)$ be connected with maximum vertex degree Δ and minimum degree δ. Then*

$$
2^{p-1}\delta \leqslant 2^{p-1}\frac{2|E|}{|V|} \leqslant \lambda_p(G) \leqslant 2^{p-1}\Delta
$$

with equality in either place if and only if G is regular. In particular, the all ones vector is an eigenvector if and only if G is regular.

Proof. The first "$\leqslant$" is trivially true because the minimum degree is less than or equal to the average degree with equality only for regular graphs. For the second inequality observe that for the all ones vector $\mathbf{1}$ we have

$$
\lambda_p(G) \geqslant \frac{Q_p(\mathbf{1})}{\|\mathbf{1}\|_p^p} = \frac{2^p|E|}{n}
$$

with equality if and only if $\mathbf{1}$ is an eigenvector for $\lambda_p(G)$. More generally, if $\mathbf{1}$ is an eigenvector for some eigenvalue μ then the eigenequations read $2^{p-1}d_i = \mu, i \in V$ (see (18)). Thus G is regular and $\mu = 2^{p-1}\Delta \leqslant \lambda_p$.

For the last inequality let x be the positive eigenvector for λ_p and assume w.l.o.g. $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n > 0$. The first eigenequation reads

$$
\lambda_p x_1^{p-1} = \sum_{j:\, 1j\in E} |x_1+x_j|^{p-1} \leqslant 2^{p-1}x_1^{p-1}d_1 \leqslant 2^{p-1}\Delta x_1^{p-1}.
$$

If equality holds then $d_1 = \Delta$ and $x_a = x_1$ for every neighbour a of 1. Then the eigenequation for a yields $2^{p-1}\Delta x_a^{p-1} \leqslant 2^{p-1}x_a^{p-1}d_a$ and so $d_a = \Delta$ and $x_b = x_a$ for any neighbour b of a. Continuing in this "breadth first search" fashion shows that G is regular and $x_1 = x_2 = \ldots = x_n$. □

Remark: For $p = 1$ the upper bound is always attained and equality holds in the lower bounds if and only if G is regular.

A better upper bound is the following (cf. [2]).

Lemma 11. *Let $p > 1$, $q = p/(p-1)$ and G be connected. Then*

$$\lambda_p(G) \leqslant 2^{p-1} \max_{ij\in E} \left(\frac{d_i^q + d_j^q}{2} \right)^{1/q}$$

with equality if and only if G is regular.

Proof. Let x be a positive eigenvector for λ_p and choose an edge ij such that $x_i + x_j$ is maximal. The i-th eigenequation yields the estimate

$$\lambda_p x_i^{p-1} = \sum_{k\colon\, ik\in E} (x_i + x_k)^{p-1} \leqslant d_i(x_i + x_j)^{p-1}. \tag{20}$$

Take the q-th power on both sides and use the convexity of $t \mapsto t^p$ to obtain

$$\lambda_p^q x_i^p \leqslant d_i^q(x_i + x_j)^p \leqslant 2^{p-1} d_i^q(x_i^p + x_j^p). \tag{21}$$

The same computation for i replaced by j yields $\lambda_p^q x_j^p \leqslant 2^{p-1}d_j^q(x_i^p + x_j^p)$ and adding up the two yields

$$\lambda_p^q(x_i^p + x_j^p) \leqslant 2^p \frac{d_i^q + d_j^q}{2}(x_i^p + x_j^p)$$

and hence the bound. If the bound and thus (21) hold with equality then $x_i = x_j$ by the strict convexity of $t \mapsto t^p$ and therefore $d_i = d_j$. From (20) we get $x_k = x_j = x_i$ for every k with $ik \in E$ and we can argue similarly as in the proof of Lemma 10 that G is regular. □

Before we generalise some more known inequalities involving the chromatic number we make some remarks on odd cycles and complete graphs.

Lemma 12. *Let $C_n = (V = \{1, \ldots, n\}, E = \{12, 23, \ldots, (n-1)n, n1\})$ be the cycle on n vertices. The largest eigenvalue is $\lambda_p(C_n) = 2^p$. For the smallest eigenvalue we have $q_p(C_n) < 1$ if $n \geqslant 4$, and $q_p(C_3) \leqslant 1$ with equality if and only if $p = 2$.*

Proof. By the regularity of C_n Lemma 10 implies $\lambda_p(C_n) = 2^p$. The assertion on q_p is trivially true for even n because then C_n is bipartite and thus $q_p = 0$ by Lemma 3. Therefore assume that n is odd. Lemma 4 with $S = \{1, 3, \ldots, n-2\}$ and $T = \{2, 4, \ldots, n-1\}$ yields $q_p(C_n) \leqslant 2/(n-1)$ which is < 1 if $n \geqslant 5$. For $n = 3$ consider the vector $x = (2, -1, -1)$. We have $Q_p(x)/\left\|x\right\|_p = 1$. If $q_p = 1$ then the pair $(x, 1)$ must satisfy the eigenequations (18); the one for the vertex 2 reads $(2-1)^{p-1} - |-1-1|^{p-1} = -1$. This is satisfied if and only if $p = 2$. □

Lemma 13. *Let $n \geqslant 2$ and K_n be the complete graph on the vertex set $\{1, \dots, n\}$.*

1. *$\lambda_p(K_n) = 2^{p-1}(n-1)$.*

2. *With $\psi(G)$ defined as in (3) we have* $\psi(K_n) = \begin{cases} \frac{n-2}{2}, & \text{if } n \text{ is even}, \\ \frac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$

3. *The number* $\nu_p(n) := \begin{cases} 2^{p-2}(n-2), & \text{if } n \text{ is even}, \\ 2^{p-2}(n-3)+1, & \text{if } n \text{ is odd} \end{cases}$ *is an eigenvalue, in particular we have $q_p(K_n) \leqslant \nu_p(n)$.*

4. *For the smallest eigenvalue $q_p(K_n)$ we have the upper bound*

$$q_p(K_n) \leqslant \mu_p(n) := (n-2)\frac{(n-2)^{p-1} + 2^{p-1}}{(n-1)^{p-1}+1}. \tag{22}$$

   Equality holds for $n = 2$. If $p = 2$ equality holds for any $n \geqslant 3$. If $p < 2$ the inequality is strict for every $n \geqslant 3$. If $p > 2$ the inequality is strict for $n \in \{3, 4\}$ and $n \geqslant 7$. If $n \in \{5, 6\}$ there is at most one $p = p(n) > 2$ for which equality can hold ($p(5) \approx 3.3618$, $p(6) \approx 2.3490$).

5. *$q_p(K_n) \leqslant n-2$ with equality if and only if $p = 2$.*

Proof. 1. This is Lemma 10 because K_n is regular.
2 and 3. Let the vertex set of K_n be denoted by $\{1, \dots, n\}$ and let $k = \lfloor n/2 \rfloor$. Observe that if $|S \cup T|$ in (3) is fixed then $2e(S) + 2e(T) = |S|(|S|-1) + |T|(|T|-1)$ is minimum for pairs (S,T) with $||S| - |T|| \leqslant 1$. Checking all pairs (S,T), $S \cap T = \varnothing$, with $|S| = m$ and $|T| \in \{m, m+1\}$ and $m \leqslant k$ it turns out that all pairs with $|S| = k = |T|$ are optimal. For assertion 3 consider such a pair (S,T), and the vector $x \in \mathbb{R}^V$ with $x_i = 1$ if $i \in S$, $x_i = -1$ if $i \in T$, and $x_i = 0$ otherwise and observe that the pair $(x, \nu_p(n))$ satisfies the eigenequations (18) for K_n.
4. First observe that $e_i - e_j, i \neq j$, is an eigenvector of Q_p with eigenvalue $\mu = n-2$, regardless of the value of p, and for $p = 2$ we see by a dimension argument that the only eigenvalues are $q_2(K_n) = n-2$ and $\lambda_2(K_n) = 2n-2$, the latter afforded by $\mathbf{1}$.

By evaluating $\frac{Q_p(x)}{\|x\|_p^p}$ at the vector $x = (n-1, -1, \dots, -1)^\top$ we find that $\mu_p(n) = \frac{Q_p(x)}{\|x\|_p^p} \geqslant q_p(K_n)$ and hence the bound (22). Equality holds for $p = 2$ because $\mu_2(n) = n-2$, and also for $n = 2$ because K_2 is bipartite and therefore $q_p(K_n) = 0 = \mu_p(2)$. For $n = 3$ we have $\mu_p(3) = 1 > q_p(C_3) = q_p(K_3)$ unless $p = 2$ by Lemma 12. So from now on we can assume that $n \geqslant 4$. For equality to hold in (22) it is necessary that the eigenequations (18) are satisfied for x and $\lambda = \mu_p(n) = q_p(K_n)$. They read in this case

$$\begin{aligned} (n-1)(n-2)^{p-1} &= \lambda(n-1)^{p-1}, \\ (n-2)^{p-1} - (n-2)2^{p-1} &= -\lambda. \end{aligned} \tag{23}$$

From the first equation it follows that

$$\lambda = (n-2)\left(\frac{n-2}{n-1}\right)^{p-2}. \tag{24}$$

If $p < 2$ it follows on the one hand that $q_p(K_n) = \lambda > n-2$. On the other hand $\nu_p(n)$ from item 3 is an eigenvalue with $\nu_p(n) < n-2$ if $n \geqslant 4$ which is a contradiction. So $\mu_p(n) = q_p(K_n)$ cannot hold for $p < 2$.

For $p > 2$ we substitute (24) into the second equation of (23) to obtain the equation

$$0 = 1 + \left(\frac{1}{n-1}\right)^{p-2} - 2\left(\frac{2}{n-2}\right)^{p-2} =: f(p).$$

If $f(p) \neq 0$ then the eigenequations cannot hold and consequently (22) is strict. Clearly, $f(2) = 0$. For $n = 4$ we see directly that $f(p) < 0$ if $p > 2$. So let $n \geqslant 5$ and observe that $\lim_{p\to\infty} f(p) = 1$. The derivative of f with respect to p is

$$f'(p) = \log\left(\frac{(n-2)^2}{4}\right)\left(\frac{2}{n-2}\right)^{p-2} - \log(n-1)\left(\frac{1}{n-1}\right)^{p-2}.$$

If $n \geqslant 7$ then $2/(n-2) > 1/(n-1)$ and $(n-2)^2/4 > n-1$ and therefore we have $f'(p) > 0$ on the intervall $[2,\infty)$ and $f(p) > 0$ on $(2,\infty)$. If $n \in \{5,6\}$ observe that $f'(2) < 0$ and that there is a unique $r > 2$ such that $f'(r) = 0$. Consequently, $f(r) < 0$ and there is a unique $s = s(n) > r$ with $f(s) = 0$. In this case x is an eigenvector of Q_s and the equality $q_s(K_n) = \mu_s(n)$ could possibly hold.

5. We have already seen that the assertion is true if $n = 3$ and that for $n \geqslant 4$ we have $\nu_p(n) < n-2$ if $p < 2$. It remains to show that $q_p(K_n) < n-2$ if $p > 2$. We show that $\mu_p(n) < n-2$ or, equivalently,

$$\frac{(n-2)^{p-1} + 2^{p-1}}{(n-1)^{p-1} + 1} < 1 \tag{25}$$

To that end observe that the function $t \mapsto f(t) = |t|^{p-1}$ is strictly convex if $p > 2$ and that we can write $2 = (1-\lambda)1 + \lambda\frac{n}{2}$ and $n-2 = (1-\lambda)(n-1) + \lambda\frac{n}{2}$ with $\lambda = 2/(n-2)$. More generally let f be strictly convex, $a < b$ and $c = (1-\lambda)a + \lambda\frac{a+b}{2}$ and $d = (1-\lambda)b + \lambda\frac{a+b}{2}$ with $0 < \lambda \leqslant 1$. Then we have $f(c) + f(d) \leqslant (1-\lambda)(f(a)+f(b)) + 2\lambda f\left(\frac{a+b}{2}\right)$ and hence

$$\frac{f(c)+f(d)}{f(a)+f(b)} \leqslant (1-\lambda) + \lambda\frac{2f\left(\frac{a+b}{2}\right)}{f(a)+f(b)} < (1-\lambda) + \lambda = 1.$$

This yields (25). □

The following eigenvalue inequalities are simple generalisations of known ones in the case $p = 2$. The proofs carry over almost verbatim and we refer to the original sources. Wilf's bound is originally for the adjacency matrix A and true for Q by the relation $\lambda_2(G) \geqslant 2\mu$ where μ denotes the largest eigenvalue of A. The proof is not long so we decided to include it here.

Proposition 14. *Let $G = (V, E)$ be connected, $n = |V|$, $m = |E|$ and $\chi = \chi(G)$ be the chromatic number.*

1. *$2^{p-1}(\chi-1) \leqslant \lambda_p(G)$ with equality if and only if G is complete or an odd cycle (Wilf's bound [11]).*

2. *$q_p(G) \leqslant \frac{2m}{n} \cdot \frac{\chi-2}{\chi-1} \cdot \frac{(\chi-2)^{p-1}+2^{p-1}}{(\chi-1)^{p-1}+1}$. If $p = 2$ and G is complete then equality holds [9, Theorem 2.11].*

3. *$\lambda_p(G) - q_p(G) \geqslant 2^{p-1}(\chi - 1) - (\chi - 2)\frac{(\chi-2)^{p-1}+2^{p-1}}{(\chi-1)^{p-1}+1}$. If $p = 2$ and $G = K_\chi$ equality holds. Conversely, if equality holds for some $p > 1$ then $G = K_\chi$ is a complete graph. If $p < 2$ equality is impossible; likewise if $p > 2$ and $\chi \notin \{5, 6\}$. If $\chi \in \{5, 6\}$ equality can hold for at most one $p > 2$. [9, Corollary 2.12].*

4. *Denote by $\nu(G)$ the vertex bipartiteness of G, i.e. the minimum cardinality of a set $W \subseteq V$ such that $G[V \setminus W]$ is bipartite. Then $q_p(G) \leqslant \nu(G)$. [7, Theorem 2.1]*

Proof. 1. Remove vertices from G to obtain a $\chi-$critical subgraph $H = (V(H), E(H))$ (i.e. $\chi(H) = \chi$ and $H - i$ is $\chi - 1$-colourable for every $i \in V(H)$). Then H is connected and the minimum degree $\delta(H)$ is at least $\chi - 1$. So by Lemmas 10 and 7 we have the desired bound:

$$2^{p-1}(\chi - 1) \leqslant 2^{p-1}\delta(H) \leqslant \lambda_p(H) \leqslant \lambda_p(G).$$

If equality holds in item 1 then $\lambda_p(H) = \lambda_p(G)$ and, again by Lemma 10, H is $(\chi - 1)$-regular and the all ones vector $\mathbf{1} \in \mathbb{R}^{V(H)}$ is an eigenvector for $\lambda_p(H)$. We show that $V(H) = V$. Define $x \in \mathbb{R}^V$ by $x_i = 1$ if $i \in V(H)$ and $x_i = 0$ otherwise. Then, in G we get

$$\lambda_p(G) \geqslant \frac{Q_p(x)}{\|x\|_p^p} = \frac{2^p|E(H)| + \mathrm{cut}_G(V(H))}{|V(H)|} = \lambda_p(H) + \frac{\mathrm{cut}_G(V(H))}{|V(H)|} \geqslant \lambda_p(G)$$

and so $\mathrm{cut}_G(V(H)) = 0$. Connectedness of G implies $V = V(H)$ and thus G is a χ-chromatic, $\chi - 1$-regular graph. So G is a complete graph or an odd cycle by the theorem of Brooks [3] which states that $\chi(G) \leqslant \Delta(G)$ unless G is complete or an odd cycle.
2. The second assertion is (22) with $p = 2$. The bound is shown as in [9, Theorem 2.11]. Let $V_1, \ldots, V_\chi$ be the colour classes. For each class V_k define a vector $x^{(k)}$ as

$$x_i^{(k)} = \begin{cases} \chi - 1, & \text{if } i \in V_k \\ -1 & \text{otherwise,} \end{cases} \qquad k = 1, \ldots, \chi.$$

Evaluate both sides of the inequality $q_p \left\|x^{(k)}\right\|_p^p \leqslant Q_p(x^{(k)})$ to get

$$\begin{aligned} q_p((\chi - 1)^p|V_k| + n - |V_k|) &\leqslant e(V_k, V \setminus V_k)(\chi - 2)^p + e(V \setminus V_k)2^p \\ &= e(V_k, V \setminus V_k)((\chi - 2)^p - 2^p) + 2^p m \end{aligned}$$

where for the second equality we used that V_k is an independent set, i.e. $e(V_k) = 0$. Summation of these inequalities over $k = 1, \ldots, \chi$ gives

$$q_p((\chi - 1)^p n + \chi n - n) \leqslant 2m((\chi - 2)^p - 2^p) + 2^p \chi m$$

and thus item 2 after some rearrangements.
3. Here we slightly deviate from [9] but get the same result for $p = 2$. With item 2 we have

$$\begin{aligned}\lambda_p - q_p &\overset{\text{item 2}}{\geqslant} \lambda_p - \frac{2m}{n} \cdot \frac{\chi-2}{\chi-1} \cdot \frac{(\chi-2)^{p-1}+2^{p-1}}{(\chi-1)^{p-1}+1} \\ &\overset{\text{Lem. 10}}{\geqslant} \lambda_p - \frac{\lambda_p}{2^{p-1}} \cdot \frac{\chi-2}{\chi-1} \cdot \frac{(\chi-2)^{p-1}+2^{p-1}}{(\chi-1)^{p-1}+1} \\ &\overset{\text{item 1}}{\geqslant} 2^{p-1}(\chi-1) - (\chi-2)\frac{(\chi-2)^{p-1}+2^{p-1}}{(\chi-1)^{p-1}+1}.\end{aligned}$$

If equality holds in the last step then G is the complete graph K_n or an odd cycle C_n by item 1. If $G = C_n$, $n \geqslant 5$, then the first inequality is strict because the bound in item 2 for an odd cycle reads $q_p(C_n) \leqslant 1$ and actually $q_p(C_n) < 1$ by Lemma 12. Therefore $G = K_\chi$ and item 2 yields precisely the bound (22) in Lemma 13. This shows the remaining statements.
4. Let W be a vertex set with $|W| = \nu(G)$ such that $G[V \setminus W]$ is bipartite and let $S \cup T = V \setminus W$ be a corresponding bipartition. The assertion follows from Lemma 4 because $e(S) = e(T) = 0$ and therefore

$$q_p \leqslant \frac{\operatorname{cut}(S \cup T)}{|S \cup T|} = \frac{\operatorname{cut}(V \setminus W)}{|V \setminus W||W|}|W| \leqslant |W| = \nu(G). \qquad \square$$

Remark: In the limit $p \to 1$ item 1 becomes Brooks' Theorem. With $\psi = \psi(G)$ defined in (3) items 2–4 become $\psi \leqslant \frac{2m}{n} \cdot \frac{\chi-2}{\chi-1}$, $\psi \leqslant \Delta - 1$ and $\psi \leqslant \nu$, respectively.

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