Chromatic index, treewidth and maximum degree

Henning Bruhn

Universität Ulm
Ulm, Germany

henning.bruhn@uni-ulm.de

Laura Gellert

Universität Ulm
Ulm, Germany

laura@lk-gellert.de

Richard Lang [*]

University of Birmingham
Birmingham, U.K.

r.lang.1@bham.ac.uk

Abstract

We conjecture that any graph G with treewidth k and maximum degree $\Delta(G) \geqslant k + \sqrt{k}$ satisfies $\chi'(G) = \Delta(G)$. In support of the conjecture we prove its fractional version. We also show that any graph G with treewidth $k \geqslant 4$ and maximum degree $2k - 1$ satisfies $\chi'(G) = \Delta(G)$, extending an old result of Vizing.

Mathematics Subject Classifications: 05C15, 05C72, 05C75

1 Introduction

The least number $\chi'(G)$ of colours necessary to properly colour the edges of a (simple) graph G is either the maximum degree $\Delta(G)$ or $\Delta(G) + 1$. But to decide whether $\Delta(G)$ or $\Delta(G) + 1$ colours suffice is a difficult algorithmic problem [9].

Often, graphs with a relatively simple structure can be edge-coloured with only $\Delta(G)$ colours. This is the case for bipartite graphs (König's theorem) and for cubic Hamiltonian graphs. Arguably, one measure of simplicity is *treewidth*, how closely a graph resembles a tree. (See next section for a definition.)

Vizing [16] (see also Zhou et al. [18]) observed a consequence of his adjacency lemma: any graph with treewidth k and maximum degree at least $2k$ has chromatic index $\chi'(G) = \Delta(G)$.[1] Is this tight? No, it turns out. Using two recent adjacency lemmas we can decrease the required maximum degree:

Proposition 1. *For any graph G of treewidth $k \geqslant 4$ and maximum degree $\Delta(G) \geqslant 2k - 1$ it holds that $\chi'(G) = \Delta(G)$.*

This immediately suggests the question: how much further can the maximum degree be lowered? We conjecture:

---

[1]More generally, the same holds for k-degenerate graphs (see Section 6).

Conjecture 2. Any graph of treewidth k and maximum degree $\Delta \geqslant k + \sqrt{k}$ has chromatic index Δ.

The bound is close to best possible: in Section 5 we construct, for infinitely many k, graphs with treewidth k, maximum degree $\Delta = k + \lfloor \sqrt{k} \rfloor < k + \sqrt{k}$, and chromatic index $\Delta + 1$. For other values k the conjecture (if true) might be off by 1 from the best bound on Δ. This is, for instance, the case for $k = 2$, where the conjecture is known to hold. Indeed, Juvan et al. [11] show that series-parallel graphs with maximum degree $\Delta \geqslant 3$ are even Δ-edge-choosable.

In support of the conjecture we prove its fractional version:

Theorem 3. *Any simple graph of treewidth k and maximum degree $\Delta \geqslant k + \sqrt{k}$ has fractional chromatic index Δ.*

The theorem follows from a new upper bound on the number of edges:

$$2|E(G)| \leqslant \Delta|V(G)| - (\Delta - k)(\Delta - k + 1)$$

The bound is proved in Proposition 4. It implies quite directly that no graph with treewidth k and maximum degree $\Delta \geqslant k + \sqrt{k}$ can be overfull. (A graph G is *overfull* if it has an odd number n of vertices and strictly more than $\Delta(G)\frac{n-1}{2}$ edges; a subgraph H of G is an *overfull subgraph* if it is overfull and satisfies $\Delta(H) = \Delta(G)$.)

Thus, for certain parameters our conjecture coincides with the overfull conjecture of Chetwynd and Hilton [5]:

Overfull conjecture. Every graph G on less than $3\Delta(G)$ vertices can be edge-coloured with $\Delta(G)$ colours unless it contains an overfull subgraph.

Because we can exclude that graphs with treewidth k and maximum degree $\Delta \geqslant k + \sqrt{k}$ are overfull, the overfull conjecture (as well as our conjecture) implies that such graphs on less than 3Δ vertices can always be edge-coloured with Δ colours.

Graphs of treewidth k are in particular k-degenerate (see Section 2 for the definition of treewidth and Section 6 for a discussion on degenerate graphs). Indeed, Vizing [16] originally showed that k-degenerate graphs, rather than treewidth k graphs, of maximum degree $\Delta \geqslant 2k$ have an edge-colouring with Δ colours. We briefly list some related work on edge-colourings and their variants in k-degenerate graphs. Isobe et al. [10] show that any k-degenerate graph of maximum degree $\Delta \geqslant 4k + 3$ has a total colouring with only $\Delta + 1$ colours. For graphs that are not only k-degenerate but also of treewidth k, a maximum degree of $\Delta \geqslant 3k - 3$ already suffices [4]. Noting that they are 5-degenerate, we include some results on planar graphs as well. Borodin, Kostochka and Woodall [2, 3] showed that planar graphs have list-chromatic index $\Delta(G)$ and total chromatic number $\chi''(G) = \Delta(G) + 1$ if $\Delta(G) \geqslant 11$ or if the maximum degree and the girth are at least 5. Vizing [16] proved that a planar graph G has a $\Delta(G)$-edge-colouring if $\Delta(G) \geqslant 8$. Sanders and Zhao [13] and independently Zhang [17] extended this to $\Delta(G) \geqslant 7$.

2 Definitions

All graphs in this article are finite and simple. We use standard graph theory notation as found in the book of Diestel [6].

For a graph G a *tree-decomposition* $(T, \mathcal{B})$ consists of a tree T and a collection $\mathcal{B} = \{B_t : t \in V(T)\}$ of *bags* $B_t \subseteq V(G)$ such that

(i) $V(G) = \bigcup\limits_{t \in V(T)} B_t$,

(ii) for each edge $vw \in E(G)$ there exists a vertex $t \in V(T)$ such that $v, w \in B_t$, and

(iii) if $v \in B_s \cap B_t$, then $v \in B_r$ for each vertex r on the path connecting s and t in T.

A tree-decomposition $(T, \mathcal{B})$ has *width* k if each bag has a size of at most $k + 1$. The *treewidth* of G is the smallest integer k for which there is a width k tree-decomposition of G.

A tree-decomposition $(T, \mathcal{B})$ of width k is *smooth* if

(iv) $|B_t| = k + 1$ for all $t \in V(T)$ and

(v) $|B_s \cap B_t| = k$ for all $st \in E(T)$.

All tree decompositions considered in this paper will be smooth. This is possible as a graph of treewidth at most k always has a smooth tree-decomposition of width k; see Lemma 8 in Bodlaender [1].

The fractional chromatic index of a graph G is defined as

$$\chi'_f(G) = \min \left\{ \sum_{M \in \mathcal{M}} \lambda_M \; : \; \lambda_M \in \mathbb{R}_+, \sum_{M \in \mathcal{M}} \lambda_M \mathbb{1}_M(e) = 1 \quad \forall e \in E(G) \right\},$$

where $\mathcal{M}$ denotes the collection of all matchings in G and $\mathbb{1}_M$ the characteristic vector of M. For more details on the fractional chromatic index, see for instance Scheinerman and Ullman [14].

3 A bound on the number of edges

Theorem 3 follows quickly from a bound on the number of edges:

Proposition 4. *A graph G of treewidth k and maximum degree $\Delta(G) \geqslant k$ satisfies*

$$2|E(G)| \leqslant \Delta(G)|V(G)| - (\Delta(G) - k)(\Delta(G) - k + 1). \tag{1}$$

Before proving Proposition 4 we present one of its consequences:

Lemma 5. *Let G be a graph of treewidth at most k and maximum degree $\Delta \geqslant k + \sqrt{k}$. Then G is not overfull.*

Proof. Proposition 4 implies

$$\frac{2|E(G)|}{|V(G)| - 1} \leqslant \frac{\Delta|V(G)| - (\Delta - k)(\Delta - k + 1)}{|V(G)| - 1} = \frac{\Delta|V(G)| - (\Delta - k)^2 - \Delta + k}{|V(G)| - 1}$$

and as $\Delta \geqslant k + \sqrt{k}$ we obtain

$$\frac{2|E(G)|}{|V(G)| - 1} \leqslant \frac{\Delta|V(G)| - k - \Delta + k}{|V(G)| - 1} = \Delta.$$

This finishes the proof. □

It follows from Edmonds' matching polytope theorem that $\chi'_f(G) = \Delta(G)$, if the graph G does not contain any overfull subgraph of maximum degree Δ; see [15, Ch. 28.5]. As the treewidth of a subgraph is never larger than the treewidth of the original graph, Theorem 3 is a consequence of Lemma 5.

The proof of Proposition 4 rests on two lemmas. We defer their proofs to the end of the section. For a tree T we write $|T|$ to denote the number of its vertices. If $st \in E(T)$ is an edge of T then we let $T_{(s,t)}$ be the component of $T - st$ containing s. For any number k we set $[k]^+ = \max(k, 0)$.

Lemma 6. *For a tree T and a positive integer $d \leqslant |T|$ it holds that*

$$\sum_{(s,t):st \in E(T)} [d - |T_{(s,t)}|]^+ \geqslant d(d - 1).$$

If T^* is a subtree of T then let $\delta^+(T^*)$ be the set of ordered pairs (s, t) so that st is an edge of T with $s \in V(T^*)$ but $t \notin V(T^*)$. (That is, $\delta^+(T^*)$ may be seen as the set of oriented edges leaving T^*.)

Lemma 7. *Let T be a tree and let $d \leqslant |T|$ be a positive integer. Then for any subtree $T^* \subseteq T$ it holds that*

$$\sum_{(s,t) \in \delta^+(T^*)} [d - |T_{(s,t)}|]^+ \leqslant [d - |T^*|]^+. \tag{2}$$

We introduce one more piece of notation. If $(T, \mathcal{B})$ is a tree decomposition of the graph G, then for any vertex v of G we denote by $T(v)$ the subtree of T that consists of those vertices corresponding to bags that contain v.

Proof of Proposition 4. Let $(T, \mathcal{B})$ be a smooth tree decomposition of G of width k. First note that for any vertex v of G, the number of vertices in the union of all bags containing v is at most $|T(v)| + k$ since the tree decomposition is smooth. Thus $\deg(v) \leqslant |T(v)| + k - 1$.

Set $d = \Delta - k + 1 \geqslant 1$, and observe that $d \leqslant |V(G)| - k = |T|$ as the tree decomposition is smooth. We calculate

$$\Delta - \deg(v) \geqslant [\Delta - k + 1 - |T(v)|]^+$$
$$= [d - |T(v)|]^+ \geqslant \sum_{(s,t) \in \delta^+(T(v))} [d - |T_{(s,t)}|]^+,$$

where the last inequality follows from Lemma 7.

Consider an edge $st \in E(T)$. Since the tree decomposition is smooth there is exactly one vertex $v \in V(G)$ with $v \in B_s$ and $v \notin B_t$. Setting $\phi((s,t)) = v$ then defines a function from the set of all (s,t) with $st \in E(T)$ into $V(G)$. Note that $\phi((s,t)) = v$ if and only if $(s,t) \in \delta^+(T(v))$. Summing the previous inequality over all vertices, we get

$$\sum_{v \in V(G)} (\Delta - \deg(v)) \geqslant \sum_{v \in V(G)} \sum_{(s,t) \in \phi^{-1}(v)} [d - |T_{(s,t)}|]^+$$

$$= \sum_{(s,t):st \in E(T)} [d - |T_{(s,t)}|]^+ \geqslant d(d-1),$$

where the last inequality is due to Lemma 6. This directly implies (1). $\qquad \square$

It remains to prove Lemma 6 and 7.

Proof of Lemma 6. We proceed by induction on $|T| - d$. The induction starts when $d = |T|$. Then $[d - |T_{(s,t)}|]^+ = d - |T_{(s,t)}|$ and thus

$$\sum_{(s,t):st \in E(T)} [d - |T_{(s,t)}|]^+ = \sum_{st \in E(T)} \left(|T| - |T_{(s,t)}| + |T| - |T_{(t,s)}| \right)$$

$$= \sum_{st \in E(T)} |T_{(t,s)}| + |T_{(s,t)}| = (|T| - 1)|T|.$$

Now, let $d \leqslant |T| - 1$, which implies in particular $|T| \geqslant 2$. Then T has a leaf ℓ. We set $T' = T - \ell$ and note that $d \leqslant |T| - 1 = |T'|$.

Observe that for any edge $st \in E(T')$ we get

$$|T_{(s,t)}| = \begin{cases} |T'_{(s,t)}| + 1 & \text{if } \ell \in V(T_{(s,t)}), \\ |T'_{(s,t)}| & \text{if } \ell \notin V(T_{(s,t)}). \end{cases}$$

We denote by F the set of all (s,t) for which st is an edge in T' with $\ell \in V(T_{(s,t)})$ and with $|T'_{(s,t)}| \leqslant d - 1$. Then

$$[d - |T_{(s,t)}|]^+ = \begin{cases} [d - |T'_{(s,t)}|]^+ - 1 & \text{if } (s,t) \in F, \\ [d - |T'_{(s,t)}|]^+ & \text{if } (s,t) \notin F. \end{cases} \tag{3}$$

Among the $(s,t) \in F$ choose (x,y) such that y maximises the distance to ℓ. This means, that $st \in E(T'_{(x,y)})$ for any $(s,t) \in F \setminus \{(x,y)\}$. Consequently,

$$|T'_{(x,y)}| = |E(T'_{(x,y)})| + 1 \geqslant |F| - 1 + 1 = |F|.$$

Let r be the unique neighbour of the leaf ℓ. Then $|T_{(\ell,r)}| = 1$, and we obtain

$$[d - |T_{(\ell,r)}|]^+ = d - 1 \geqslant |T'_{(x,y)}| \geqslant |F|. \tag{4}$$

We conclude

$$\sum_{(s,t):st\in E(T)} [d - |T_{(s,t)}|]^+ = [d - |T_{(\ell,r)}|]^+ + [d - |T_{(r,\ell)}|]^+$$

$$+ \sum_{(s,t):st\in E(T')} [d - |T_{(s,t)}|]^+$$

$$\overset{(4)}{\geqslant} |F| + 0 + \sum_{(s,t):st\in E(T')} [d - |T_{(s,t)}|]^+$$

$$\overset{(3)}{=} \sum_{(s,t):st\in E(T')} [d - |T'_{(s,t)}|]^+$$

$$\geqslant d(d-1),$$

where the last inequality follows by induction. □

Proof of Lemma 7. We proceed by induction on $|T| - d$. For the induction start, consider the case when $d = |T|$. Then

$$[d - |T_{(s,t)}|]^+ = [|T| - |T_{(s,t)}|]^+ = |T_{(t,s)}|,$$

which yields

$$\sum_{(s,t)\in\delta^+(T^*)} [d - |T_{(s,t)}|]^+ = \sum_{(s,t)\in\delta^+(T^*)} |T_{(t,s)}| = |T| - |T^*| = [d - |T^*|]^+.$$

Now assume $|T| - d \geqslant 1$. If every vertex in $T - V(T^*)$ is a leaf of T then t is a leaf for every $(s,t) \in \delta^+(T^*)$. This implies $|T_{(s,t)}| = |T| - 1 \geqslant d$ and the left hand side of (2) vanishes.

Therefore we may assume that there is a leaf $\ell \notin T^*$ of T such that neither ℓ nor its unique neighbour belongs to $V(T^*)$. Set $T' = T - \ell$, and observe that, by choice of ℓ, the set $\delta^+(T^*)$ of edges leaving T^* is the same in T and in T'. Moreover, $|T_{(s,t)}| \geqslant |T'_{(s,t)}|$ holds for every $(s,t) \in \delta^+(T^*)$. The desired inequality

$$\sum_{(s,t)\in\delta^+(T^*)} [d - |T_{(s,t)}|]^+ \leqslant \sum_{(s,t)\in\delta^+(T^*)} [d - |T'_{(s,t)}|]^+ \leqslant [d - |T^*|]^+$$

now follows by induction. □

4 A lower bound on the maximum degree

Vizing [16] (see also Zhou et al. [18]) proved that every graph of treewidth k and maximum degree $\Delta \geqslant 2k$ has an edge-colouring with Δ colours. Proposition 1 shows that this bound is not tight.

A graph G of maximum degree Δ is Δ-*critical*, if $\chi(G) = \Delta + 1$ and all proper subgraphs can be edge-coloured using not more than Δ colours. For the proof of Proposition 1 we use Vizing's adjacency lemma, as well as two adjacency lemmas that involve the second neighbourhood.
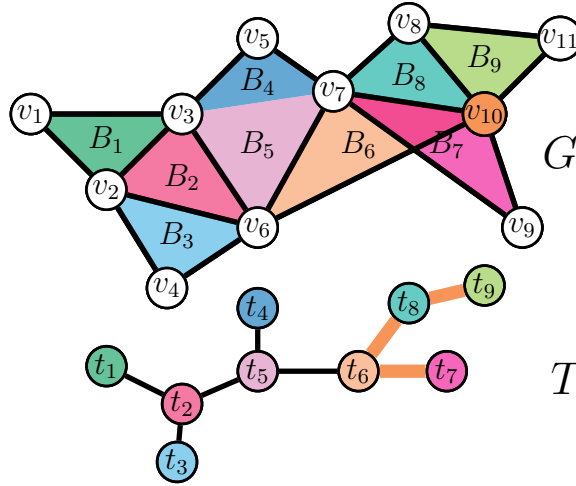
Figure 1: A graph G with a smooth tree-decomposition $(T, \mathcal{B})$ of width 2. Here, $L = \{v_2, v_3, v_6, v_7, v_{10}\}$. If T is rooted in t_2 then $v^* = v_{10}$, $q = t_6$ and $B_q = \{v_7, v_9, v_{10}\}$. So, $T(v^*)$ consists of vertices t_6, t_7, t_8, t_9, and $S = \{t_7, t_8\}$ and $X = \{v_7, v_8, v_9, v_{10}, v_{11}\}$.

Vizing's adjacency lemma. Let uv be an edge in a Δ-critical graph. Then v has at least $\Delta - \deg(u) + 1$ neighbours of degree Δ.

Theorem 8 (Zhang [17])**.** *Let G be a Δ-critical graph, and let uwv be a path in G. If $\deg(u) + \deg(w) = \Delta + 2$ then all neighbours of v but u and w have degree Δ.*

Theorem 9 (Sanders and Zhao [13])**.** *Let G be a Δ-critical graph, and let v be a common neighbour of u and w such that $\deg(u) + \deg(v) + \deg(w) \leqslant 2\Delta + 1$. Then there are at most $\deg(u) + \deg(v) - \Delta - 3$ common neighbours $x \neq u$ of v and w.*

The rest of this subsection is dedicated to the proof of Proposition 1. To this end, let us assume Proposition 1 to be wrong. Then there is a Δ-critical graph G of treewidth at most k for $\Delta = 2k - 1$. (Note that the case $\Delta \geqslant 2k$ is covered by the above mentioned result of Vizing.) Let $(T, \mathcal{B})$ be a smooth tree-decomposition of G of width $\leqslant k$. By picking an arbitrary root, we may consider T as a rooted tree. For any $s \in V(T)$, we denote by $\lceil s \rceil$ the subtree of T rooted at s, that is, the subtree of T consisting of the vertices $t \in V(T)$ for which s is contained in the path between t and the root of T.

Recall the definition of $T(v)$ after Lemma 7. Set $L = \{v \in V(G) : \deg(v) \geqslant k + 2\}$, and choose a vertex $v^* \in L$ that maximises the distance of $T(v^*)$ to the root (among the vertices in L). Let q be the vertex of $T(v^*)$ that achieves this distance. For $S := N(q) \cap T(v^*)$ and any $s \in S$, define $X_s = \bigcup_{t \in V(\lceil s \rceil)} B_t$, and let $X = B_q \cup \bigcup_{s \in S} X_s$. (See Figure 1 for an illustration.) Note that by the definition of v^* and q

$$N(v^*) \subseteq X \text{ and } X \cap L \subseteq B_q. \tag{5}$$

Claim 10. *All vertices of $X \setminus B_q$ have degree at most k.*

Proof of Claim 10. Suppose the statement to be false. Then there is an $s \in S$ for which $X_s \setminus B_q$ contains a vertex of degree at least $k + 1$. Fix a vertex $w^* \in \{w \in X_s \setminus B_q : \deg(w) \geqslant k + 1\} =: L'$ that maximises the distance of $T(w^*)$ to s. Let p be the vertex of $T(w^*)$ that achieves this distance. Set $Y = \bigcup_{t \in V(\lceil p \rceil)} B_t$. As in (5) we have $N(w^*) \subseteq Y$ and $Y \cap L' \subseteq B_p$.

By (5), the vertex $w^* \in X_s \setminus B_q$ has degree $k + 1$. Thus w^* has a neighbour u^* outside B_p, which then has degree at most k (by choice of w^*).

Vizing's adjacency lemma implies that w^* has at least $\Delta - \deg(u^*) + 1 \geqslant 2k - 1 - k + 1 = k$ neighbours of degree Δ. By (5), all vertices of degree Δ of Y have to be in $B_q \cap B_s$. Since by smoothness of the tree decomposition $B_q \cap B_s$ is a cutset of size at most k, the vertex w^* is adjacent to all vertices in $B_q \cap B_s$. As w^* is therefore adjacent to at most k vertices of degree Δ it holds $\deg(u^*) = k$. By definition of S, the set B_s contains v^*, which implies that v^* is adjacent to w^* and of degree Δ. As $k \geqslant 4$, it follows that v^* has degree $\Delta = 2k - 1 \geqslant k + 3$, which means by (5) that v^* has at least three neighbours of degree $\leqslant k + 1$. Thus, v^* has a neighbour of degree $\leqslant k + 1$, which is neither u^* nor w^*. This, however, contradicts Theorem 8 (applied to v^*, w^*, u^*). $\qquad\square$

By (5) and since v^* has degree at least $k + 2$, the vertex v^* has a neighbour $u \notin B_q$. (In fact, v^* has at least two such neighbours.) By Vizing's adjacency lemma, applied to uv^*, it follows that v^* has at least $\Delta - \deg(u) + 1 \geqslant k$ neighbours of degree Δ. In particular, it follows from $X \cap L \subseteq B_q$, see (5), that each of these neighbours lies in B_q. Since $|B_q| \leqslant k + 1$ we get:

$$v^* \text{ is adjacent to every vertex in } B_q, \text{ each of which has degree } \Delta. \qquad (6)$$

We also observe that $\Delta - \deg(u) + 1 > k$ contradicts $|B_q| \leqslant k + 1$, which means that

$$\text{every } u \in N(v^*) \setminus B_q \text{ has degree exactly } k. \qquad (7)$$

Claim 11. *Every $u \in N(v^*) \setminus B_q$ has exactly k neighbours, all of which are contained in B_q.*

Proof of Claim 11. By (7), u has exactly k neighbours. Since B_q is a separator, u has all its neighbours in X. However, u cannot be adjacent to any vertex w of degree $\leqslant k$; otherwise we could extend any Δ-edge-colouring of $G - uw$ to G. It follows from Claim 10 that all of the k neighbours of u are in B_q. $\qquad\square$

Since the vertex v^* has degree at least $k + 2$ and since $N(v^*) \subseteq X$, by (5), it follows that v^* has two neighbours u, w that are contained in $X \setminus B_q$. By Claim 11, the degree of u and w is k. Thus, $\deg(u) + \deg(v^*) + \deg(w) \leqslant k + \Delta + k = 2\Delta + 1$. Moreover, by Claim 11 and (6), the vertices v^* and w have $k - 1$ common neighbours in B_q. As $k - 1 > \deg(u) + \deg(v^*) - \Delta - 3$, we obtain a contradiction to Theorem 9. This finishes the proof of Proposition 1.

5 Discussion

Proposition 4 bounds the number of edges in a graph G of fixed treewidth and maximum degree. A simpler bound – only considering the treewidth – is easily shown by induction (see Rose [12]):

$$2|E(G)| \leqslant 2k|V(G)| - k(k+1) \tag{8}$$

For $\Delta < 2k$ and $|V(G)| > \Delta + 1$ a straightforward computation shows that the bound of Proposition 4 is strictly better than (8). The bounds are the same if $\Delta = 2k$ or if $|V(G)| = \Delta + 1$. For $\Delta = 2k$ this is illustrated by the kth power P^k of a long path P.

The bound in Proposition 4 is tight. There are simple examples that show this: take the complete graph K_k on k vertices and add $r \geqslant 1$ further vertices each adjacent to each vertex of K_k. These graphs also demonstrate that Conjecture 2 (if true) would be tight or almost tight. Indeed, if $k + \lfloor\sqrt{k}\rfloor$ is even, and k not a square, then we obtain for $r = \lfloor\sqrt{k}\rfloor + 1$ an overfull graph with maximum degree $\Delta = k + \lfloor\sqrt{k}\rfloor$. If $k + \lfloor\sqrt{k}\rfloor$ is odd, then, by setting $r = \lfloor\sqrt{k}\rfloor$, we obtain an overfull graph with $\Delta = k + \lfloor\sqrt{k}\rfloor - 1$.

These tight graphs, however, have a very special structure. In particular, they all satisfy $|V(G)| = \Delta(G) + 1$. Both, Conjecture 2 and Proposition 4, stay tight for an arbitrarily large number of vertices compared to Δ:

Proposition 12. *For every $k_0 \geqslant 4$ there is a $k \in \{k_0, k_0 + 1, \ldots, k_0 + 8\}$ such that for every $n \geqslant 4k$ there exists a graph G on n vertices with treewidth at most k and maximum degree $\Delta = k + \lfloor\sqrt{k}\rfloor < k + \sqrt{k}$ such that*

$$2|E(G)| = \Delta n - (\Delta - k)(\Delta - k + 1).$$

In particular, the graph G is overfull whenever n is odd.

We need the following lemma.

Lemma 13. *Let $c, r \in \mathbb{N}$. Then there is a graph with degree sequence*

$$\mathbf{d} = \big(\underbrace{c \ldots, c}_{r+1}, c - 1, c - 2, \ldots, 1\big) \in \mathbb{Z}^{c+r}$$

if and only if 4 divides $c(2r + c + 1)$ and if $r^2 \geqslant c$.

We defer the proof of Lemma 13 until the end of the section and only show sufficiency. A closer look at the arguments in the proof yields necessity.

Proof of Proposition 12. We start by showing with a case distinction that there is a $k \in \{k_0, k_0 + 1, \ldots, k_0 + 8\}$ such that

$$k \equiv \left\lfloor\sqrt{k}\right\rfloor \pmod{8} \text{ and } \left\lfloor\sqrt{k}\right\rfloor < \sqrt{k}. \tag{9}$$

To this end, let i such that $\lfloor\sqrt{k_0}\rfloor \equiv k_0 + i \pmod 8$ and $0 \leqslant i \leqslant 7$.

Firstly, let us assume that $i = 0$. If k_0 is not a square, then $k = k_0$ satisfies (9). Otherwise $k = k_0 + 8$ satisfies (9) as $k_0 \geqslant 4 > 1$, and consequently $\lfloor\sqrt{k_0 + 8}\rfloor = \sqrt{k_0}$.

Secondly, we consider the case that $i \neq 0$. If $\lfloor\sqrt{k_0 + i}\rfloor = \lfloor\sqrt{k_0}\rfloor$, then $k = k_0 + i$ satisfies $\lfloor\sqrt{k_0}\rfloor \equiv k \pmod 8$ and $\sqrt{k} > \sqrt{k_0} \geqslant \lfloor\sqrt{k_0}\rfloor = \lfloor\sqrt{k}\rfloor$, which shows (9). If, on the other hand, $\lfloor\sqrt{k_0 + i}\rfloor > \lfloor\sqrt{k_0}\rfloor$, then $\lfloor\sqrt{k_0 + i}\rfloor = \lfloor\sqrt{k_0}\rfloor + 1 = \lfloor\sqrt{k_0 + i + 1}\rfloor$ as $k_0 \geqslant 4$. Set $k = k_0 + i + 1$. By choice of i, we have $\lfloor\sqrt{k_0 + 1}\rfloor \equiv k \pmod 8$. Thus, we obtain $\lfloor\sqrt{k}\rfloor \equiv k \pmod 8$ as desired. Moreover, $\sqrt{k} > \sqrt{k_0 + i} \geqslant \lfloor\sqrt{k_0 + i}\rfloor = \lfloor\sqrt{k_0}\rfloor + 1 = \lfloor\sqrt{k}\rfloor$.

In all cases an element of $\{k_0, k_0 + 1, \ldots, k_0 + 8\}$ satisfies (9).

Next we show that for any $n \geqslant 4k$, there is a graph G of treewidth k whose degree sequence $(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n))$ equals

$$(k, k + 1, \ldots, \Delta - 1, \Delta, \ldots, \Delta, \Delta - 1, \ldots, k + 1, k) \qquad (10)$$

with $\Delta = k + \lfloor\sqrt{k}\rfloor$. A computation similar to Lemma 5 shows that G is overfull if $|V(G)|$ is odd.

We construct G in three steps. First we take a power of a path, where all but the outer vertices have the right degree. We increase the degree of the outer vertices by connecting them to vertices towards the middle of the path. This will create some degree excess for the used vertices. We balance this by deleting a subgraph H provided by Lemma 13. The construction is illustrated in Figure 2. Note that for ease of exposition the parameters k and Δ are not as in this proof.
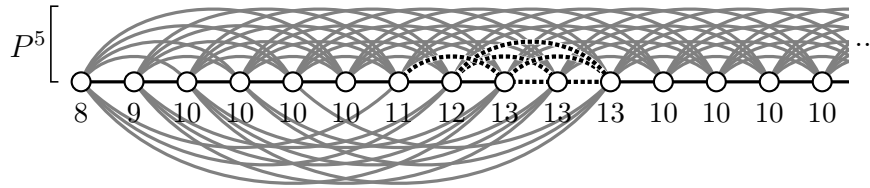

Figure 2: Extreme example for $k = 8$ and $\Delta = 10$. The graph H is dotted.

Let P be a $\Delta/2$-th power of a path on vertices $v_1, \ldots, v_n$. This means, v_i and v_j are adjacent if and only if $0 < |i - j| \leqslant \Delta/2$. As P is symmetric, and as G will be symmetric as well, we concentrate on the part of P on the vertices $v_1, \ldots, v_{\lceil n/2\rceil}$. We tacitly agree that any additions and deletions of edges are also applied to the other half of P.

Comparing the degrees of P to (10) we see that all vertices have the target degree except for the initial vertices $v_1, \ldots, v_{\Delta/2}$, whose degree is too small. For $i = 1, \ldots, \Delta - k$ the vertex v_i has degree $\Delta/2 - 1 + i$ but should have degree $k - 1 + i$. We fix this by connecting v_i to $v_{i+\Delta/2+1}, \ldots, v_{i+k+1}$. For $i = \Delta - k + 1, \ldots, \Delta/2$, the vertex v_i should have degree Δ but has degree $\Delta/2 - 1 + i$. We make v_i adjacent to each of $v_{i+\Delta/2+1}, \ldots, v_{\Delta+1}$.

Denote the obtained graph by P' and observe that the vertices of P' in the range of

$1, \ldots, \lceil n/2 \rceil$ have the following degrees

$$\underbrace{k, k+1, \ldots, \Delta,}_{1, \ldots, \Delta-k+1} \underbrace{\Delta, \ldots, \Delta}_{\Delta-k+2, \ldots, \Delta/2+1}, \underbrace{\Delta+1, \ldots, k+\tfrac{\Delta}{2}}_{\tfrac{\Delta}{2}+2, \ldots, k+1}, \underbrace{k+\tfrac{\Delta}{2}, \ldots, k+\tfrac{\Delta}{2}}_{k+2, \ldots, \Delta+1}, \underbrace{\Delta, \ldots, \Delta}_{\Delta+2, \ldots, \lceil n/2 \rceil}$$

Hence all but the vertices v_i with index i between $\Delta/2+2$ and $\Delta+1$ have the correct degree. The difference between their degree in P' and the desired degree is

$$\mathbf{d} = \Big(1, 2, \ldots, k - \tfrac{\Delta}{2} - 1, \underbrace{k - \tfrac{\Delta}{2}, \ldots, k - \tfrac{\Delta}{2}}_{\Delta-k+1}\Big). \tag{11}$$

Set $c = k - \tfrac{\Delta}{2} = \tfrac{1}{2}\big(k - \lfloor \sqrt{k} \rfloor\big)$ and $r = \Delta - k$. Note that k is chosen in such a way (see (9)) that c is divisible by 4. As furthermore $r^2 = (\Delta - k)^2 = \lfloor \sqrt{k} \rfloor^2 \geqslant \tfrac{1}{2}\big(k - \lfloor \sqrt{k} \rfloor\big) = c$, Lemma 13 yields that there is a graph H with degree sequence $\mathbf{d}$. Since the vertices $v_{\Delta/2+2}, \ldots, v_{\Delta+1}$ induce a complete graph in P' there is a copy of H in P', such that deleting its edges results in a graph G of the desired degree sequence. Note that for any two adjacent vertices v_i, v_j in P' it holds that $|i - j| \leqslant k$. This implies that P' is a subgraph of a k-th power of a path. Thus the subgraph G of P' has treewidth at most k. This finishes the proof. $\qquad \square$

To prove Lemma 13 we use the Erdős-Gallai-criterion:

Theorem 14 (Erdős and Gallai [7]). *There is a graph with degree sequence $d_1 \geqslant \cdots \geqslant d_n$ if and only if $\sum_{i=1}^{n} d_i$ is even and if for all $\ell = 1, \ldots, n$*

$$\sum_{i=1}^{\ell} d_i \leqslant \ell(\ell - 1) + \sum_{i=\ell+1}^{n} \min(d_i, \ell). \tag{12}$$

Proof of Lemma 13. We check the conditions of Theorem 14 for the degree sequence $\mathbf{d}$. The parity condition holds as 4 divides $c(2r + c + 1)$ and

$$\sum_{i=1}^{c+r} d_i = cr + \frac{c(c+1)}{2} = \frac{c}{2}(2r + c + 1).$$

Let of us now verify (12). If $\ell > c$, then

$$\sum_{i=1}^{\ell} d_i \leqslant c\ell \leqslant \ell(\ell - 1) \leqslant \ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} \min(d_i, \ell).$$

Thus we can assume that $\ell \leqslant c$. Two remarks: Firstly, $\min(d_i, \ell) = \ell$ for $i = 1, \ldots, \leqslant c + r - \ell + 1$. Consequently, if $2\ell \leqslant c + r$ then

$$\ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} \min(d_i, \ell) = \ell(\ell - 1) + (c + r - 2\ell + 1)\ell + \frac{\ell(\ell - 1)}{2}$$

$$= \tfrac{\ell}{2}(2r - 1 - \ell) + c\ell. \tag{13}$$

Secondly, if $\ell > r$, then

$$\sum_{i=1}^{\ell} d_i = c\ell - \frac{(\ell - r - 1)(\ell - r)}{2} = c\ell + \tfrac{\ell}{2}(2r + 1 - \ell) - \tfrac{1}{2}(r^2 + r). \tag{14}$$

Now suppose that $2\ell \leqslant c + r$. For $\ell \leqslant r$, we have $\sum_{i=1}^{\ell} d_i = c\ell$ and hence (12) is easily seen to be satisfied in light of (13). On the other hand, for $\ell > r$ the assumption of $r^2 \geqslant c$ together with a comparison of (13) and (14) gives (12).

So let $2\ell > c + r$. This implies that $\ell > r$. Consequently, the right hand side of (12) is

$$\ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} \min(d_i, \ell) = \ell(\ell - 1) + \sum_{i=\ell+1}^{c+r} d_i$$
$$= \ell(\ell - 1) + \tfrac{1}{2}(c + r - \ell)(c + r - \ell + 1).$$

It follows from equation (14) that (12) is satisfied if the following expression is non-negative.

$$2\ell(\ell - 1) + (c + r - \ell)(c + r - \ell + 1) - (2c\ell + \ell(2r + 1 - \ell) - (r^2 + r))$$
$$= 4\ell^2 - 4\ell(c + r) + (c + r)^2 + (c + 2r + r^2) - 4\ell$$
$$= (2\ell - (c + r))^2 + (c + r) + (r + r^2) - 4\ell$$
$$= (2\ell - (c + r))^2 - 2\left(2\ell - \frac{(c + r) + (r + r^2)}{2}\right) \tag{15}$$

First, let $r^2 = c$. Then (15) equals

$$(2\ell - (c + r))^2 - 2(2\ell - (c + r)) \tag{16}$$

The term (16) is negative only if $2\ell - (c + r) = 1$. As $c + r = r^2 + r$ is even (for any integer r), (16) and thus (15) is non-negative.

Now let $r^2 > c$. Then (15) is strictly greater than (16) and hence non-negative. This shows that (12) is satisfied.

As (12) holds for all ℓ, there is a graph with degree sequence $\mathbf{d}$. $\qquad\square$

6 Degenerate graphs

Recall that a graph G is *k-degenerate* if there is an enumeration $v_n, \ldots, v_1$ of the vertices such that v_{i-1} has degree at most k in $G - \{v_n, \ldots, v_i\}$ for every i. By simple induction following the elimination order (or recalling the formula $1 + 2 + \ldots + N = N(N + 1)/2$), we can obtain a bound with half the degree loss of (1):

$$2|E(G)| \leqslant \Delta|V(G)| - \tfrac{1}{2}(\Delta - k)(\Delta - k + 1). \tag{17}$$

The bound in (17) turns out to be tight for some Δ, k as the construction below shows. Moreover, by (17), Theorem 3 can easily be transferred: any simple k-degenerate graph
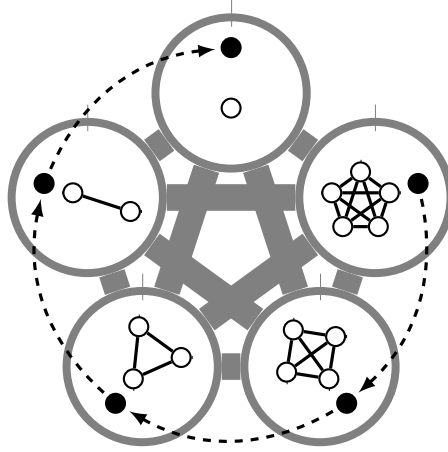
Figure 3: The graph G_5 with the vertices v_i drawn in black; thick gray edges indicate that two vertex sets are complete to each other; elimination order of the v_i is shown in dashed lines

of maximum degree $\Delta \geqslant k + 1/2 + \sqrt{2k + 1/4}$ is not overfull and therefore has fractional chromatic index $\chi'_f(G) = \Delta$.

Consider a positive integer p and let G_p be the complement of the disjoint union of p stars $K_{1,1}, K_{1,2}, \ldots, K_{1,p}$; see Figure 3. Denote the centre of the ith star by v_i, and let W be the union of all leaves. The graph G_p has $n = p(p+1)/2 + p$ vertices and satisfies $\deg(v_i) = n - 1 - i$ for $i = 1, \ldots, p$ and $\deg(w) = n - 2$ for $w \in W$. In particular, the maximum degree of G_p is $\Delta = n - 2$. Setting $k = n - 1 - p$, we note that G_p is k-degenerate as $v_p, v_{p-1}, \ldots, v_1$ followed by an arbitrary enumeration of W is an elimination order. Finally, we observe that G_p satisfies (17) with equality.

References

[1] H. L. Bodlaender, *A partial k-arboretum of graphs with bounded treewidth*, Theoretical Computer Science **209** (1998), no. 1–2, 1 – 45.

[2] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, *List edge and list total colourings of multigraphs*, Journal of Combinatorial Theory, Series B **71** (1997), no. 2, 184 – 204.

[3] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, *Total colorings of planar graphs with large maximum degree*, Journal of Graph Theory **26** (1997), no. 1, 53 – 59.

[4] H. Bruhn, R. Lang, and M. Stein, *List edge-coloring and total coloring in graphs of low treewidth*, Journal of Graph Theory **81** (2016), no. 3, 272 – 282.

[5] A. G. Chetwynd and A. J. W. Hilton, *Star multigraphs with three vertices of maximum degree*, Mathematical Proceedings of the Cambridge Philosophical Society **100** (1986), 303 – 317.

[6] R. Diestel, *Graph theory*, Graduate Texts in Mathematics, Springer, 2000.

[7] P. Erdős and T. Gallai, *Graphs with given degrees of vertices [in Hungarian]*, Matematikai Lapok **11** (1960), 264 – 274.

[8] L. M. Favrholdt, D. Scheide, M. Stiebitz, and B. Toft and, *Graph edge coloring: Vizing's theorem and Goldberg's conjecture*, John Wiley & Sons, 2012.

[9] I. Holyer, *The NP-completeness of edge-coloring*, SIAM Journal on Computing **10** (1981), no. 4, 718 – 720.

[10] S. Isobe, X. Zhou, and T. Nishizeki, *Total colorings of degenerate graphs*, Combinatorica **27** (2007), no. 2, 167 – 182.

[11] M. Juvan, B. Mohar, and R. Thomas, *List edge-colorings of series-parallel graphs*, Electronic Journal of Combinatorics **6** (1999).

[12] D. J. Rose, *On simple characterizations of k-trees*, Discrete Mathematics **7** (1974), no. 34, 317 – 322.

[13] D. P. Sanders and Y. Zhao, *Planar graphs of maximum degree seven are class I*, Journal of Combinatorial Theory, Series B **83** (2001), no. 2, 201 – 212.

[14] E. R. Scheinerman and D. H. Ullman, *Fractional graph theory*, Dover Publications, 2013, http://www.ams.jhu.edu/ers.

[15] A. Schrijver, *Combinatorial optimization: Polyhedra and efficiency*, Algorithms and Combinatorics, Springer, 2003.

[16] V. G. Vizing, *Critical graphs with given chromatic class [in Russian]*, Metody Diskretnogo Analiza **5** (1965), 9 – 17, English translation in *Graph Edge Coloring* by Favrholdt et al.

[17] L. Zhang, *Every planar graph with maximum degree 7 is of class 1*, Graphs and Combinatorics **16** (2000), no. 4, 467 – 495.

[18] X. Zhou, S.-I. Nakano, and T. Nishizeki, *Edge-coloring partial k-trees*, Journal of Algorithms **21** (1996), no. 3, 598 – 617.