

Semiregular automorphisms in vertex-transitive graphs of order $3p^2$

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Abstract

It has been conjectured that automorphism groups of vertex-transitive (di)graphs, and more generally 2-closures of transitive permutation groups, must necessarily possess a fixed-point-free element of prime order, and thus a non-identity element with all orbits of the same length, in other words, a *semiregular* element. It is the purpose of this paper to prove that vertex-transitive graphs of order $3p^2$, where p is a prime, contain semiregular automorphisms.

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1 Introduction

It is known that every finite transitive permutation group contains a fixed-point-free element of prime power order (see [5, Theorem 1]), but not necessarily a fixed-point-free element of prime order (which is equivalent to existence of a semiregular element) [3, 5]. In 1981 it was asked if every vertex-transitive digraph admits a semiregular automorphism (see [17, Problem 2.4]). The existence of such automorphisms plays an important role in solutions to many important open problems in algebraic graph theory, such as, for example, in the classifications of graphs satisfying certain prescribed symmetry conditions (see [14, 15, 21, 23, 26]). Semiregular automorphisms have also proved useful in a long

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standing hamiltonicity problem for connected vertex-transitive graphs and in a recently explored dichotomy of even/odd automorphisms (see [1, 12, 16]).

In 1997 Klin generalized the semiregularity problem conjecturing that every transitive 2-closed permutation group contains a semiregular element (see [2]) – the term *polycirculant conjecture* is sometimes used for the semiregularity problem in this wider context. (Recall that for a finite permutation group G on a set V the 2-closure $G^{(2)}$ of G is the largest subgroup of the symmetric group $Sym(V)$ containing G and having the same orbits as G in the induced action on $V \times V$.) The problem has spurred a lot of interest in the mathematical community producing several partial results. In particular, Giudici [9] settled the question for quasiprimitive group actions, leaving as one of the main open cases graphs admitting solvable group actions (see [19]). Furthermore, there have also been a number of papers dealing with semiregularity problem for vertex-transitive graphs satisfying certain valency and order restrictions (see, for instance, [3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 20, 22, 24, 25]). For example, it is known that every 2-closed group of square-free degree admits semiregular elements (see [7]). As for composite non-square orders the only positive result is, if we disregard prime power orders, that every vertex-transitive graph of order $2p^2$, p a prime, admits semiregular automorphisms (see [20]). It is the object of this paper to prove the existence of semiregular automorphisms in vertex-transitive graphs of order $3p^2$, where p is a prime. We hope that this will motivate further research, leading eventually to the solution of the semiregularity problem in the case of vertex-transitive (di)graphs of cube-free order.

Theorem 1. *A vertex-transitive graph of order $3p^2$, where p is a prime, admits a semiregular automorphism.*

Theorem 1 is proved in Section 2 after a series of propositions each of which considers vertex-transitive graphs in question with particular (im)primitivity actions of their automorphism groups. A comment is in order. There are two reasons for the restriction to vertex-transitive graphs in the main theorem. First, in the proof of Theorem 1 we use certain results from [19], proved within a restricted setting of vertex-transitive graphs. The second reason is somewhat more philosophical and reflects author's personal bias. If one's goal is a complete solution of the semiregularity problem, then rather than worrying over the distinction between the original question and its generalization to 2-closed groups, one should primarily aim at advancements for groups which are not quasiprimitive – say be it solvable or of particular degrees – even if only in the context of vertex-transitive (di)graphs.

2 Vertex-transitive graphs of order $3p^2$

Let us first recall the concept imprimitive groups. Given a transitive permutation group G on a set V , we say that a partition \mathcal{B} of V is a G -invariant if the elements of G permute the parts, that is, *blocks* of \mathcal{B} , setwise. If the *trivial* partitions $\{V\}$ and $\{\{v\} \mid v \in V\}$ are the only G -invariant partitions of V , then G is said to be *primitive*, and

is said to be *imprimitive* otherwise. In the latter case we shall refer to a corresponding G -invariant partition as to a *complete imprimitivity block system*, in short an *imprimitivity block system*, of G . A transitive permutation group is *quasiprimitive* if each of its non-identity normal subgroups is transitive, and is said to be *genuinely imprimitive* otherwise. Note that in the latter case there exists an imprimitivity block system of G arising from orbits of an intransitive normal subgroup of G . A vertex-transitive graph is *primitive* if its automorphism group is primitive. Otherwise it is called an *imprimitive graph*.

The following proposition, proved by Giudici in [9], implies the existence of semiregular automorphisms in a vertex-transitive graph in case its automorphism group is quasiprimitive. (A finite transitive permutation group is said to be *elusive* if it has no semiregular element.)

Proposition 2. [9] *A 2-closed quasiprimitive group is not elusive.*

Proposition 2 implies that only those graphs with genuinely imprimitive automorphism groups need to be considered. In particular, let X be a vertex-transitive graph of order $3p^2$, where p is a prime. We may assume that there exists an intransitive normal subgroup N of the automorphism group $\text{Aut}(X)$ of X . In fact, we may, without loss of generality, assume that N is a minimal normal subgroup of $\text{Aut}(X)$. The size of the blocks arising from the orbits of N divides the order of X , and is therefore 3, p , p^2 or $3p$. The proposition below was recently proved in [19], where semiregularity in vertex-transitive graphs with a solvable automorphism group is considered. Note, however, that this particular result does not require the permutation group to be solvable. The proposition implies the existence of semiregular automorphisms in X in case the blocks arising from the orbits of N are of prime size.

Proposition 3. [19, Corollary 2.2] *Let G be a permutation group acting transitively on a set V and let M be a minimal normal subgroup of G having orbits of prime length q on V . Then $G^{(2)}$ contains a semiregular element of order q .*

The remaining two cases, that is graphs whose automorphism groups admit an intransitive normal subgroup giving rise to imprimitivity block system consisting of blocks of size p^2 and $3p$, are considered in Propositions 6 and 7. For the sake of completeness we first state the following classical result which will be used in the proofs. It implies that in a vertex-transitive graph of order $3p^2$, where p is a prime, the orbits of a Sylow p -subgroup of the automorphism group are of length p^2 .

Proposition 4. [27, Theorem 3.4] *Let p be a prime and let P be a Sylow p -subgroup of a permutation group G acting on a set Ω . Let $\omega \in \Omega$. If p^m divides the length of the G -orbit containing ω , then p^m also divides the length of the P -orbit containing ω .*

Before considering the remaining two cases let us recall a recent result about existence of semiregular automorphisms in a vertex-transitive graph with solvable automorphism group of order mp^2 , where m satisfies certain conditions. This result implies that only vertex-transitive graphs of order $3p^2$ with non-solvable automorphism groups need to be considered.

Proposition 5. [19, Theorem 2.4] *Let X be a connected vertex-transitive graph of order p^2q , where p and q are primes, and either $q \leq p$ or $p^2 < q$. Then either*

- (i) *X admits a semiregular automorphism, or*
- (ii) *$2 < q < p$ and $\text{Aut}(X)$ is nonsolvable with an intransitive non-abelian minimal normal subgroup whose orbits are either of length p^2 or of length pq .*

In the next proposition we consider the case where the blocks of imprimitivity are of size p^2 .

Proposition 6. *Let p be a prime and let X be a vertex-transitive graph of order $3p^2$, where $p > 3$ is a prime, admitting an imprimitivity block system consisting of three blocks of size p^2 arising from orbits of an intransitive normal subgroup N of $\text{Aut}(X)$. Then X admits a semiregular automorphism.*

Proof. We may assume that X is connected as otherwise a semiregular automorphism in X can be easily constructed via semiregular automorphisms in the connected components. Namely, if X is disconnected then its connected components are vertex-transitive graphs of order 3 , p , p^2 or $3p$, and it is well known that such graphs admit semiregular automorphisms.

Let $\mathcal{B} = \{A, B, C\}$ be the imprimitivity block system arising from the orbits of N , each of length p^2 . Clearly, in view of Proposition 4, the orbits A , B and C coincide with the orbits of a Sylow p -subgroup P of $\text{Aut}(X)$. Observe also that there must exist an automorphism $\pi \in \text{Aut}(X)$ which cyclically permutes the three blocks in \mathcal{B} . We may, without loss of generality, assume that $\pi|_{\mathcal{B}} = (A B C)$.

The center $Z(P)$ of the Sylow p -subgroup P is non-trivial and thus there exists a central element $\alpha \in Z(P)$ of order p . Clearly, for each $Y \in \{A, B, C\}$ either α^Y is trivial or α^Y is semiregular of order p . If α is not semiregular then there are essentially only two possibilities that need to be considered, depending on the number of orbits $Y \in \{A, B, C\}$ for which the restriction α^Y is trivial.

CASE 1. α^A is semiregular and $\alpha^B = \alpha^C = 1$.

Then $(\pi\alpha\pi^{-1})^B$ and $(\pi^2\alpha\pi^{-2})^C$ are semiregular, and

$$(\pi\alpha\pi^{-1})^A = (\pi\alpha\pi^{-1})^C = (\pi^2\alpha\pi^{-2})^A = (\pi^2\alpha\pi^{-2})^B = 1,$$

implying that $\alpha \cdot \pi\alpha\pi^{-1} \cdot \pi^2\alpha\pi^{-2} = (\alpha\pi)^3\pi^{-3}$ is the desired semiregular automorphism.

CASE 2. $\alpha^A = 1$, and α^B and α^C are semiregular of order p .

Recall that $\mathcal{B} = \{A, B, C\}$ consists of orbits of the normal subgroup N . For each $Y \in \mathcal{B}$ let $K_{(Y)} = \text{Ker}(N \rightarrow N^Y)$. Then $\alpha \in K_{(A)}$, and there exists $\beta \in K_{(B)}$ (without loss of generality we may assume that $\beta = \alpha^\pi$) such that β^A and β^C are semiregular of order p . Now consider $\alpha\beta$. Clearly, $(\alpha\beta)^A$ and $(\alpha\beta)^B$ are semiregular of order p . We need to consider the action of $\alpha\beta$ on C . Observe that either the orbits of $\langle\alpha\rangle$ on C are blocks of imprimitivity for N^C – as $K_{(A)}$ is normal in N and so $K_{(A)}^C$ is normal in N^C – or

$K_{(A)}$ is transitive on C . In the latter case the bipartite subgraphs $X[A, B]$, $X[B, C]$ and $X[A, C]$ are all isomorphic to the complete bipartite graph K_{p^2, p^2} , and X clearly admits a semiregular automorphism. (For disjoint subsets U, W of the vertex set $V(X)$ the subgraph of X induced by the set U is denoted by $X[U]$, and similarly, the bipartite subgraph of X induced by the edges having one endvertex in U and the other endvertex in W is denoted by $X[U, W]$.) Hence, we may assume that the orbits of $\langle \alpha \rangle$ on C are blocks of imprimitivity for N^C . Note that $(\alpha\beta)^C$ either fixes the orbits of α or cyclically permutes them. We deal with these two cases in the two subcases below.

SUBCASE 2.1. $(\alpha\beta)^C$ fixes the orbits of α .

It follows that the orbits of α and β on C coincide. Denote these orbits by C_i , $i \in \mathbb{Z}_p$. If all of the restrictions $(\alpha\beta)^{C_i}$, $i \in \mathbb{Z}_p$, are of order p then $\alpha\beta$ is a semiregular automorphism of X . If not, then there exists $r \in \mathbb{Z}_p$ such that the restrictions $(\alpha\beta)^{C_j}$ are of order p for $j \in \{r+1, \dots, p-1\}$ and are not of order p for $j \in \{0, 1, \dots, r\}$. We now define a semiregular automorphism σ of X in the following way:

$$\sigma(u) = \begin{cases} \alpha\beta(u), & u \in A \cup B \cup C_{r+1} \cup C_{r+2} \cup \dots \cup C_{p-1} \\ \alpha(u), & u \in C_0 \cup C_1 \cup \dots \cup C_r \end{cases}.$$

To show that σ is indeed an automorphism of X observe first that the bipartite graph $X[C_i, C_j]$, where $i \in \{0, 1, \dots, r\}$ and $j \in \{r+1, \dots, p-1\}$, is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. Combining this with the fact that $(\alpha\beta)^B = \alpha^B$ we obtain that $\sigma^{B \cup C}$ is an automorphism of the subgraph of X induced on $B \cup C$. To complete the proof we need to check the edges of the induced bipartite graph $X[A, C]$. Since for each $j \in \{0, 1, \dots, r\}$ there exists k_j coprime with p such that $((\alpha\beta)^{k_j})^{C_j} = 1$ it follows that each of the bipartite graphs $X[A_i, C_j]$, where $A = \{A_i \mid i \in \mathbb{Z}_p\}$ is a partition of A into the orbits of β and $j \in \{0, 1, \dots, r\}$, is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. It follows that σ preserves the edges of $X[A, C]$, and consequently σ is an automorphism of X .

SUBCASE 2.2. $(\alpha\beta)^C$ cyclically permutes the orbits of α .

Then either $(\alpha\beta)^C$ is of order p and clearly semiregular, in which case $\alpha\beta$ is a semiregular automorphism of X . Alternatively, $(\alpha\beta)^C$ is of order p^2 in which case $(\alpha\beta)^p$ is trivial on $A \cup B$ and semiregular of order p on C . In this case take $(\alpha\beta)^p \alpha^{\pi^2}$ to get the desired semiregular automorphism. \square

In the next proposition we deal with blocks of size $3p$.

Proposition 7. *Let X be a vertex-transitive graph of order $3p^2$, where $p > 3$ is a prime, admitting an imprimitivity block system consisting of p blocks of size $3p$ arising from orbits of an intransitive normal subgroup N of $\text{Aut}(X)$. Then X admits a semiregular automorphism.*

Proof. We may again assume that X is connected. Let \mathcal{B} be the imprimitivity block system arising from orbits of N . Let P be a Sylow p -subgroup of $\text{Aut}(X)$ with orbits

A , B and C . Observe that each block in \mathcal{B} intersects each of A , B and C in exactly p vertices. We have

$$\begin{aligned} A &= A_0 \cup A_1 \cup \dots \cup A_{p-1}, \\ B &= B_0 \cup B_1 \cup \dots \cup B_{p-1}, \\ C &= C_0 \cup C_1 \cup \dots \cup C_{p-1}, \end{aligned}$$

and $\mathcal{B} = \{Y_i \mid i \in \mathbb{Z}_p\}$ where $Y_i = A_i \cup B_i \cup C_i$.

The center $Z(P)$ of P is non-trivial and thus there exists a central element $\alpha \in Z(P)$ of order p such that for each $Y \in \{A, B, C\}$ either α^Y is trivial or α^Y is semiregular of order p . If α is not semiregular then there are essentially only two possibilities depending on the number of orbits $Y \in \{A, B, C\}$ for which the restriction α^Y is trivial.

CASE 1. α^A is semiregular of order p , and $\alpha^B = \alpha^C = 1$.

First observe that, since \mathcal{B} is an imprimitivity block system, the set of orbits of α^A is equal to the set $\{A_i \mid i \in \mathbb{Z}_p\}$. By [18, Proposition 3.2], every transitive group of degree p^2 contains a regular (abelian) subgroup, and so there exists $Q \leq P$ such that Q^B is either a cyclic or an elementary abelian subgroup acting regularly on B . Thus we distinguish two subcases.

SUBCASE 1.1. $Q^B \cong \mathbb{Z}_{p^2}$.

There exists $\rho \in Q$ of order p^2 such that ρ^B is also of order p^2 and maps according to the rule

$$\rho: B_i \rightarrow B_{i+1}, \quad i \in \mathbb{Z}_p.$$

Further, since $A_i \cup B_i \cup C_i$ are blocks of imprimitivity we have that $\rho(A_i) = A_{i+1}$ and $\rho(C_i) = C_{i+1}$ for every $i \in \mathbb{Z}_p$. Let e , f , and g denote the respective orders of ρ^A , ρ^B , and ρ^C . Then (e, f, g) is one of the following ordered triples:

$$(p^2, p^2, p^2), (p, p^2, p^2), (p^2, p^2, p) \text{ or } (p, p^2, p).$$

In the first case ρ is semiregular. In the second case $\alpha\rho^p$ is semiregular. In the third and the fourth case, the existence of automorphisms α and σ^p implies that each of the bipartite subgraphs $X[A_i, B_j]$, $X[B_i, C_j]$, and $X[A_i, C_j]$ ($i, j \in \mathbb{Z}_p$) is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. Consequently, any permutation ω fixing each of A_i , B_i and C_i , $i \in \mathbb{Z}_p$, set-wise and satisfying the property that ω^A , ω^B , and ω^C is, respectively, an automorphism of $X[A]$, $X[B]$, and $X[C]$, is in fact an automorphism of X . As in the case of the orbit B and the subgroup $Q \leq P$ there exists a subgroup $R \leq P$ such that R^C is a regular abelian group. Since C_i , $i \in \mathbb{Z}_p$, are the intersections of the blocks Y_i , $i \in \mathbb{Z}_p$, with the orbit C there must exist an automorphism $\tau \in R$ fixing these blocks and such that τ^C is semiregular (and of order p on each C_i). We now define ω as follows

$$\omega(u) = \begin{cases} \alpha(u), & u \in A \\ \sigma^p(u), & u \in B \\ \tau(u), & u \in C \end{cases}.$$

Clearly, ω is the desired semiregular automorphism of X .

SUBCASE 1.2. $Q^B \cong \mathbb{Z}_p^2$.

There exist $\rho, \sigma \in Q$ such that both ρ^B and σ^B are of order p . Furthermore, we may assume that the orders of ρ^C and σ^C are either p or 1 , for otherwise an argument analogous to the one used in Subcase 1. 1, with B replaced by C , applies. As for the orders of ρ^A and σ^A they can be 1 , p or p^2 .

We may assume that ρ^B permutes the sets B_i and that σ^B fixes the sets B_i . Consequently, ρ^C permutes the sets C_i , and so ρ is semiregular on both B and C . If ρ^A is of order p then it permutes the sets A_i , and so ρ^A is semiregular, and thus ρ is semiregular. Hence we may assume that ρ^A is of order p^2 . Consider now σ . Clearly, σ^B is semiregular. If σ^C is of order p and semiregular then we are done because we can construct the desired automorphism ω as follows:

$$\omega(u) = \begin{cases} \alpha(u), & u \in A \\ \sigma(u), & u \in B \\ \sigma(u), & u \in C \end{cases}.$$

The mapping ω is an automorphism of X since each of the bipartite subgraphs $X[A_i, B_j]$ and $X[A_i, C_j]$ is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected.

Finally, suppose that σ^C is not semiregular. In this case apart from the bipartite subgraphs $X[A_i, B_j]$ and $X[A_i, C_j]$ also any of the induced bipartite subgraphs $X[B_i, C_j]$ is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. Recall that σ^B is semiregular with orbits B_i . Analogously, we may assume that there exists $\tau \in P$ such that τ^C is semiregular with orbits C_i . Hence the permutation defined by the rule

$$\omega(u) = \begin{cases} \alpha(u), & u \in A \\ \sigma(u), & u \in B \\ \tau(u), & u \in C \end{cases}$$

is a semiregular automorphism of X .

CASE 2. $\alpha^A = 1$, and α^B and α^C are semiregular of order p .

There exists $Q \leq P$ such that Q^A is abelian and regular, and so either cyclic or elementary abelian. Observe also that the set of orbits of α^B is equal to the set $\{B_i \mid i \in \mathbb{Z}_p\}$ and that the set of orbits of α^C coincides with the set $\{C_i \mid i \in \mathbb{Z}_p\}$.

SUBCASE 2.1. $Q^A \cong \mathbb{Z}_{p^2}$.

Note that non-identity elements of P are all of order p or p^2 . There exists $\rho \in Q$ of order p^2 such that ρ^A is also of order p^2 . Let $\sigma = \rho^p$. Hence σ^A is semiregular of order p . We now analyze possibilities for σ^B and σ^C : they are either trivial or semiregular of order p . If σ^B and σ^C are both semiregular then σ is the desired automorphism. If $\sigma^B = \sigma^C$ are both trivial then $\sigma\alpha$ is the desired automorphism. Finally, without loss of generality, assume that σ^B is semiregular and $\sigma^C = 1$. Then $\langle \rho \rangle^B \cong \mathbb{Z}_{p^2}$ and being contained in $\langle \alpha, \rho \rangle^B$

which is abelian (since $\alpha \in Z(P)$), it follows that $\langle \rho \rangle^B = \langle \alpha, \rho \rangle^B$. Therefore $\alpha^B \in \langle \rho \rangle^B$. In particular $\alpha^B = (\rho^{pj})^B$, for some $j \in \mathbb{Z}_p^*$. It follows that $\alpha \rho^{pj}$ is the desired automorphism.

SUBCASE 2.2. $Q^A \cong \mathbb{Z}_p^2$.

There are elements $\sigma, \rho \in P$ such that $\langle \sigma, \rho \rangle^A \cong \mathbb{Z}_p^2$. Of course, both ρ^A and σ^A are semiregular. Moreover, since the sets $\{A_i \cup B_i \cup C_i\}$, $i \in \mathbb{Z}_p$ are blocks, we may assume that ρ^A maps A_i to A_{i+1} , and similarly ρ^B maps B_i to B_{i+1} and ρ^C maps C_i to C_{i+1} , whereas σ^A , σ^B and σ^C fix these sets. In particular, this means that σ^B and σ^C fix the orbits of α on B and C . Consider now the action of the conjugates $\epsilon_k = \rho^{-k} \sigma \rho^k$ ($k \in \mathbb{Z}_p$) on B and C . Clearly, $\epsilon_k^A = \sigma^A$. If σ is semiregular on B and C then we are done. If σ is not semiregular on B then there exists B_i such that $\sigma^{B_i} = 1$. Consequently, every bipartite subgraph $X[A_j, B_i]$, $i, j \in \mathbb{Z}_p$, is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. Applying the automorphisms ϵ_k we see that each of the bipartite subgraphs $X[A_j, B_k]$ is isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. An analogous argument holds for the case when σ is not semiregular on C , implying that $X[A_j, C_k]$ is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. Then the permutation ω mapping according to the rule:

$$\omega(u) = \begin{cases} \sigma(u), & u \in A \\ \alpha(u), & u \in B \\ \alpha(u), & u \in C \end{cases}$$

is a semiregular automorphism of X . We are now left with the case where σ is semiregular on one of the two orbits B and C and not semiregular on the other. Without loss of generality we assume that σ^B is semiregular and σ^C is not semiregular. Then applying the same argument as above it follows that each of the bipartite subgraphs $X[A_j, C_i]$ and $X[B_j, C_i]$, $j \in \mathbb{Z}_p$, is either isomorphic to the complete bipartite graph $K_{p,p}$ or is totally disconnected. Applying then the automorphisms ϵ_k it follows that the same holds for all of the subgraphs $X[A_j, C_k]$ and $X[B_j, C_k]$, $j, k \in \mathbb{Z}_p$. Then the permutation ω mapping according to the rule:

$$\omega(u) = \begin{cases} \sigma(u), & u \in A \\ \sigma(u), & u \in B \\ \alpha(u), & u \in C \end{cases}$$

is the desired semiregular automorphism of X . □

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1: Let X be a vertex-transitive graph of order $3p^2$, where p is a prime, and let $\text{Aut}(X)$ be its automorphism group. If $p \in \{2, 3\}$ then X is of order 12 or 3^3 , and the existence of semiregular automorphisms follows from the fact that X is a Cayley graph in both of these two cases (see [18]). We may therefore assume that $p > 3$.

If $\text{Aut}(X)$ is quasiprimitive then Proposition 2 implies the existence of semiregular automorphisms in $\text{Aut}(X)$. We may thus assume that $\text{Aut}(X)$ is genuinely imprimitive. Let N be an intransitive minimal normal subgroup of $\text{Aut}(X)$, and let \mathcal{B} be an $\text{Aut}(X)$ -invariant partition of $V(X)$ arising from the orbits of N . Then the blocks in \mathcal{B} are of

size 3, p , p^2 or $3p$. If the blocks in \mathcal{B} are of prime size then the existence of semiregular automorphisms is assured by Proposition 3. If the blocks in \mathcal{B} are of prime-squared size then the existence of semiregular automorphisms follows from Proposition 6. If, however, the blocks in \mathcal{B} are of size $3p$ then semiregular automorphisms in $\text{Aut}(X)$ exist by Proposition 7. \square

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References

- [1] B. Alspach, Lifting Hamilton cycles of quotient graphs, *Discrete Math.* **78** (1989), 25–36.
- [2] P. J. Cameron (ed.), Problems from the Fifteenth British Combinatorial Conference, *Discrete Math.* **167/168** (1997), 605–615.
- [3] P. J. Cameron, M. Giudici, W. M. Kantor, G. A. Jones, M. H. Klin, D. Marušič and L. A. Nowitz, Transitive permutation groups without semiregular subgroups, *J. London Math. Soc.* **66** (2002), 325–333.
- [4] P. J. Cameron, J. Sheehan and P. Spiga, Semiregular automorphisms of vertex-transitive cubic graphs, *European J. Combin.* **27** (2006), 924–930.
- [5] B. Fein, W. M. Kantor and M. Schacher, Relative Brauer groups II, *J. Reine Angew. Mat.* **328** (1981), 39–57.
- [6] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, *J. Combin. Theory, Ser. B* **97** (2007), 371–380.
- [7] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Minimal normal subgroups of transitive permutation groups of square-free degree, *Discrete Math.* **307** (2007), 373–385.
- [8] E. Dobson and D. Marušič, On semiregular elements of solvable groups, *Comm. Algebra* **39** (2011), 1413–1426.
- [9] M. Giudici, Quasiprimitive groups with no fixed point free elements of prime order, *J. London Math. Soc.* **67** (2003), 73–84.
- [10] M. Giudici, New constructions of groups without semiregular subgroups, *Comm. Algebra* **35** (2007), 2719–2730.
- [11] M. Giudici and J. Xu, All vertex-transitive locally-quasiprimitive graphs have a semiregular automorphism, *J. Algebr. Combin.* **25** (2007), 217–232.
- [12] A. Hujdurović, K. Kutnar and D. Marušič, Odd automorphisms in vertex-transitive graphs, *Ars Math. Contemp.* **10** (2016), 427–437.

- [13] K. Kutnar and D. Marušič, Recent trends and future directions in vertex-transitive graphs, *Ars Math. Contemp.* **1** (2008), 112–125.
- [14] K. Kutnar and D. Marušič, A complete classification of cubic symmetric graphs of girth 6, *J. Combin. Theory Ser. B* **99** (2009), 162–184.
- [15] K. Kutnar, D. Marušič, P. Šparl, R.-J. Wang and M.-Y. Xu, Classification of half-arc-transitive graphs of order $4p$, *European J. Combin.* **34** (2013), 1158–1176.
- [16] L. Lovász, “Combinatorial structures and their applications”, (Proc. Calgary Internat. Conf., Calgary, Alberta, 1969), pp. 243–246, Problem 11, Gordon and Breach, New York, 1970.
- [17] D. Marušič, On vertex symmetric digraphs, *Discrete Math.* **36** (1981), 69–81.
- [18] D. Marušič, Vertex transitive graphs and digraphs of order p^k , *Ann. Discrete Math.* **27** (1985), 115–128.
- [19] D. Marušič, Semiregular automorphisms in vertex-transitive graphs with a solvable group of automorphisms, *Ars Math. Contemp.* **13** (2017), 461–468.
- [20] D. Marušič and R. Scapellato, Permutation groups, vertex-transitive digraphs and semiregular automorphisms, *European J. Combin.* **19** (1998), 707–712.
- [21] A. Ramos Rivera and P. Šparl, The classification of half-arc-transitive generalizations of Bouwer graphs, *European J. Combin.* **64** (2017), 88–112.
- [22] P. Spiga, Semiregular elements in cubic vertex-transitive graphs and the restricted Burnside problem, *Math. Proc. Cambridge Philos. Soc.* **157** (2014), 45–61.
- [23] P. Šparl, A classification of tightly attached half-arc-transitive graphs of valency 4, *J. Combin. Theory Ser. B* **98** (2008), 1076–1108.
- [24] G. Verret, Arc-transitive graphs of valency 8 have a semiregular automorphism, *Ars Math. Contemp.* **8** (2015), 29–34.
- [25] J. Xu, Semiregular automorphisms of arc-transitive graphs with valency pq , *European J. Combin.* **29** (2008), 622–629.
- [26] S. Zhou, Classification of a family of symmetric graphs with complete 2-arc-transitive quotients, *Discrete Math.* **309** (2009), 5404–5410.
- [27] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.