An Easy Subexponential Bound for Online Chain Partitioning

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Abstract

Bosek and Krawczyk exhibited an online algorithm for partitioning an online poset of width w into $w^{14 \lg w}$ chains. We improve this to $w^{6.5 \lg w + 7}$ with a simpler and shorter proof by combining the work of Bosek & Krawczyk with work of Kierstead & Smith on First-Fit chain partitioning of ladder-free posets. We also provide examples illustrating the limits of our approach.

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1 Introduction

An online poset $P^{\prec}$ is a triple $(V, \leqslant_P, \prec)$, where $P = (V, \leqslant_P)$ is a poset and $\prec$ is a total order on V, called the *presentation order* of P. Let P^{v_i} be induced by the first i vertices $v_1 \prec \cdots \prec v_i$. An *online chain partitioning algorithm* is a deterministic algorithm $\mathcal{A}$ that assigns the vertices $v_1 \prec \cdots \prec v_n$ of P to disjoint chains $C_1, \ldots, C_t$ so that for each i, the chain C_j to which v_i is assigned, is determined solely by the subposet P^{v_i}. This formalizes the scenario in which the algorithm $\mathcal{A}$ receives the vertices of P one at a time, and when a vertex is received, irrevocably assigns it to one of the chains. Let $\chi_{\mathcal{A}}(P^{\prec})$ denote the number of (nonempty) chains that $\mathcal{A}$ uses to partition $P^{\prec}$, and $\chi_{\mathcal{A}}(P) = \max_{\prec}(\chi(P^{\prec}))$ over all presentation orders $\prec$ for P. For a class of posets $\mathcal{P}$, let $\mathrm{val}_{\mathcal{A}}(\mathcal{P}) = \max_{P \in \mathcal{P}}(\chi_{\mathcal{A}}(P))$ and $\mathrm{val}(\mathcal{P}) = \min_{\mathcal{A}}(\mathrm{val}_{\mathcal{A}}(\mathcal{P}))$ over all online chain partitioning algorithms $\mathcal{A}$. Our goal is to bound $\mathrm{val}(\mathcal{P}_w)$, where $\mathcal{P}_w$ is the class of finite posets of width w (allowing countably infinite posets with w finite in $\mathcal{P}_w$ would not effect results).

By Dilworth's Theorem [8], every poset with finite width w can be partitioned into w chains, and this is best possible. However this bound cannot be achieved online. In 1981, Kierstead proved

Theorem 1 ([15])**.** $4w - 3 \leqslant \mathrm{val}(\mathcal{P}_w) \leqslant \frac{5^w-1}{4}$.

Kierstead asked whether $\mathrm{val}(\mathcal{P}_w)$ is polynomial in w, and noted that his methods also provided a super linear lower bound. Until recently, there was little progress. Szemerédi (see [16]) proved a quadratic lower bound, which was improved to $(2 - o(1))\binom{w+1}{2}$ by Bosek et al. [2]. In 1997 Felsner [12] proved $\mathrm{val}(\mathcal{P}_2) \leqslant 5$, and in 2008 Bosek [1] proved $\mathrm{val}(\mathcal{P}_3) \leqslant 16$. In 2010 Bosek and Krawczyk made a major advance by proving a subexponential bound.

Theorem 2 ([3, 4])**.** $\mathrm{val}(\mathcal{P}_w) \leqslant w^{14 \lg w}$.

Based on [4, 22] we provide a much shorter and simpler proof of a slightly improved bound:

Theorem 3. $\mathrm{val}(\mathcal{P}_w) \leqslant w^{6.5 \lg w + 7}$.

The difference between the proof of Theorem 1 and the proofs of Theorems 2 and 3 is fundamental. In the former relations are *added* to the online poset $P^{\prec}$ to create a new online poset $Q^{\prec}$ with smaller width so that every online chain of Q can be partitioned into 5 online chains of P; then induction is applied. In the latter relations are *deleted* from $P^{\prec}$ to form an online poset $Q^{\prec}$ with the same width; this would seem to make it harder to partition Q, but paradoxically limits the wrong choices an algorithm can make.

The simplest online chain partitioning algorithm is First-Fit, which assigns each new vertex v_i to the chain C_j, with the least index $j \in \mathbb{Z}^+$ such that for all $h < i$ if $v_h \in C_j$ then v_h is comparable to v_i. It was observed in [15] that $\mathrm{val}_{\mathrm{FF}}(\mathcal{P}_w) = \infty$ (see [16] for details) for any $w > 1$. The poset used to show this fact contains substructures that are important to this paper, so we present it.

Lemma 4 ([15]). *For every $n \in \mathbb{Z}^+$ there is an online poset $R_n^{\prec}$ with* $\text{width}(R_n^{\prec}) \leqslant 2$ *and* $\chi_{\text{FF}}(R_n^{\prec}) = n$.

Proof. We define the online poset $R_n^{\prec} = (X, \leqslant_R, \prec)$ as follows. The poset R_n consists of n chains $X^1, \ldots, X^n$ with

$$X^k = x_k^k \leqslant_R x_{k-1}^k \leqslant_R \ldots \leqslant_R x_2^k \leqslant_R x_1^k$$

and the additional comparabilities and incomparabilities given by:

$$x_i^k \geqslant_R X^1 \cup X^2 \cup \cdots \cup X^{k-2} \cup \{x_{k-1}^{k-1}, x_{k-2}^{k-1}, \ldots, x_i^{k-1}\}$$

$$x_i^k \parallel_R \{x_{i-1}^{k-1}, x_{i-2}^{k-1}, \ldots, x_1^{k-1}\}.$$

Note that the superscript of a vertex indicates to which chain X^k it belongs and the subscript is its index within that chain. The example of R_5 is illustrated in Figure 1. The presentation order $\prec$ is given by $X^1 \prec \cdots \prec X^n$, where the order $\prec$ on the vertices of X^k is the same as $\leqslant_R$ on X^k.

Observe that $X^{k-2} \leqslant_R X^k$. Hence, the width of R_n is 2. By induction on k one can show that each vertex x_i^k is assigned to chain C_i. □

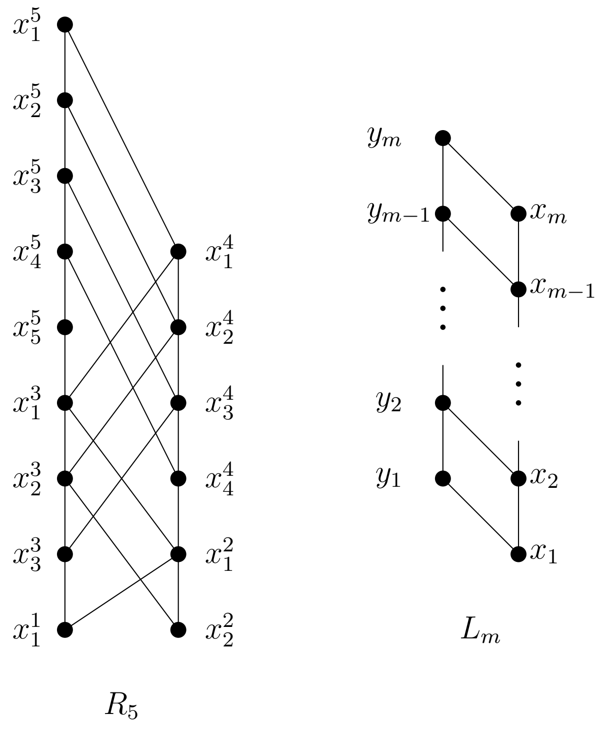

Figure 1: Hasse diagrams of R_5 and L_m.

Despite Lemma 4, the analysis of the performance of First-Fit on restricted classes of posets has been useful and interesting. For posets P and Q, we say P is *Q-free* if P does not contain Q as an induced subposet. Let $\mathrm{Forb}(Q)$ denote the family of Q-free posets, and $\mathrm{Forb}_w(Q)$ denote the family of Q-free posets of width at most w. Abusing notation, we write $\mathrm{val}_{\mathrm{FF}}(Q,w)$ for $\mathrm{val}_{\mathrm{FF}}(\mathrm{Forb}_w(Q))$.

Let $\mathbf{s}$ denote the total order (chain) on s vertices, and $\mathbf{s}+\mathbf{t}$ denote the width 2 poset consisting of disjoint copies of $\mathbf{s}$ and $\mathbf{t}$ with no additional comparabilities or vertices. It is well known [13] that the class of interval graphs is equal to $\mathrm{Forb}(\mathbf{2}{+}\mathbf{2})$. First-Fit chain partitioning of interval orders has applications to polynomial time approximation algorithms [17, 18] and Max-Coloring [25]. The first linear bound $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}{+}\mathbf{2},w) \leqslant 40w$ was proved by Kierstead in 1988 [17]. This was improved later to $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2},w) \leqslant 26w$ in [20]. In 2004 Pemmaraju, Raman, and K. Varadarajan [25] introduced a beautiful new technique to show $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}{+}\mathbf{2},w) \leqslant 10w$, and this was quickly improved to $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2},w) \leqslant 8w$ [7, 24]. In 2010 Kierstead, D. Smith, and Trotter [21, 26] proved $5(1-o(1))w \leqslant \mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2},w)$. In 2010 Bosek, Krawczyk, and Szczypka [6] proved that $\mathrm{val}_{\mathrm{FF}}(\mathbf{t}+\mathbf{t},w) \leqslant 3tw^2$. This result plays an important role in the proof of Theorem 2. Joret and Milans [14] improved this to $\mathrm{val}_{\mathrm{FF}}(\mathbf{s}+\mathbf{t},w) \leqslant 8(s-1)(t-1)w$. Recently, Dujmović, Joret, and Wood [10] proved $\mathrm{val}_{\mathrm{FF}}(\mathbf{t}+\mathbf{t},w) \leqslant 16tw$. In 2010 Bosek, Krawczyk, and Matecki proved:

Theorem 5 ([5])**.** *For every poset Q of width 2 there is a function $f_Q : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{val}_{\mathrm{FF}}(Q,w) \leqslant f_Q(w)$.*

Lemma 4 shows that the theorem cannot be extended to posets Q with width greater than 2.

Let $m \in \mathbb{Z}^+$. An *m-ladder* is a poset $L_m = L(x_1 \ldots x_m; y_1 \ldots y_m)$ with vertices $x_1, y_1, \ldots, x_m, y_m$ such that $x_1 <_L \cdots <_L x_m$, $y_1 <_L \cdots <_L y_m$, $x_i <_L y_j$ for $1 \leqslant i \leqslant j \leqslant m$, and $x_i \parallel_L y_j$ for $1 \leqslant j < i \leqslant m$. The vertices $x_1, \ldots, x_m$ are the *lower leg* and the vertices $y_1, \ldots, y_m$ are the *upper leg* of L_m. The vertices x_i, y_i together form the *i-th rung* of L_m. We provide a Hasse diagram of L_m in Figure 1. Notice that for two consecutive chains X^i and X^{i+1} of R_n, the set $X^i \cup (X^{i+1} - x^{i+1}_{i+1})$ induces the ladder L_i in R_n.

Our attack is based on the following observation of Bosek and Krawczyk, first mentioned in [3, 4], but never proved so far.

Lemma 6. $\mathrm{val}(\mathcal{P}_w) \leqslant w\,\mathrm{val}_{\mathrm{FF}}(L_{2w^2+1},w)$ *for* $w \in \mathbb{Z}^+$.

In this paper we provide the first proof of the above-mention lemma. Kierstead and Smith completed this attack with the next lemma.

Lemma 7 ([22])**.** $\mathrm{val}_{\mathrm{FF}}(L_m,w) \leqslant w^{2.5\lg(2w)+2\lg m}$ *for* $m, w \in \mathbb{Z}^+$.

Combining Lemmas 6 and 7 we get $\mathrm{val}(\mathcal{P}_w) \leqslant w^{6.5\lg w+7}$, which completes the proof of Theorem 3. Beside that, the paper presents two new constructions to show that the bounds given in Lemmas 6 and 7 can not be improved substantially and hence a new technique will be needed to prove a polynomial upper bound on $\mathrm{val}(\mathcal{P}_w)$.

This paper is organized as follows. Section 2 introduces some notation and definitions. In Section 3 we present our online algorithm and reduce the proof of its performance bound to proving Lemmas 6 and 7, which are shown in Sections 4 and 5. In Section 6 we present constructions that show limitations of our approach. Section 7 contains some concluding observations.

2 Preliminaries

Let $P = (V, \leqslant_P)$ be a poset with $u, v \in V$. We usually write $u \in P$ for $u \in V(P)$. The *upset of* u *in* P is $U_P(u) = \{v : u <_P v\}$, the *downset of* u *in* P is $D_P(u) = \{v : v <_P u\}$, and the *incomparability set of* u *in* P is $I_P(u) = \{v : v \parallel_P u\}$. The *closed upset* and *closed downset of* u *in* P are, respectively, $U_P[u] = U_P(u) + u$ and $D_P[u] = D_P(u) + u$. Define $[u, v]_P = U_P[u] \cap D_P[v]$. For $U \subseteq V$, define $D_P(U) = \bigcup_{u \in U} D_P(u)$, $U_P(U) = \bigcup_{u \in U} U_P(u)$, $D_P[U] = D_P(U) \cup U$ and $U_P[U] = U_P(U) \cup U$. If $U' \subseteq V$, let $[U, U']_P = U_P[U] \cap D_P[U']$. The *subposet of* P *induced by* U is denoted by $P[U]$, and $P - u$ denotes $P[V - u]$. If $U_P(u) = \varnothing$, then u is *maximal*. If $D_P(u) = \varnothing$, then u is *minimal*. If $D_P[u] = P$, then u is *maximum*. If $U_P[u] = P$, then u is *minimum*. Let $\mathrm{Max}_P(U)$ be the set of maximal vertices in $P[U]$ and $\mathrm{Min}_P(U)$ be the set of minimal vertices in $P[U]$. Let $\mathrm{Max}_P = \mathrm{Max}_P(V)$ and $\mathrm{Min}_P = \mathrm{Min}_P(V)$.

A chain partition $\mathcal{C}$ of P is a *Dilworth partition* if $|\mathcal{C}| = \mathrm{width}(P)$. If vertices u and v are in the same chain of some Dilworth partition then uv is called a *Dilworth edge* of P.

Let $\mathcal{M}_P = (\mathcal{V}_P, \sqsubseteq_P)$, where $\mathcal{V}_P$ is the set of maximum antichains in P and $\sqsubseteq_P$ is defined by

$$A \sqsubseteq_P B \text{ if } A \subseteq D_P[B] \text{ (or equivalently } B \subseteq U_P[A]).$$

If $A \sqsubseteq_P B$ and $A \neq B$, we write $A \sqsubset_P B$. In [9] Dilworth showed that $\mathcal{M}_P$ is a lattice with the meet and the join defined by

$$A \wedge B = \mathrm{Min}_P\{A \cup B\} \text{ and } A \vee B = \mathrm{Max}_P\{A \cup B\}.$$

A poset $P = (V, \leqslant_P)$ is *bipartite* if the set V can be partitioned into two disjoint antichains A, B such that $A \sqsubset_P B$ — such a poset is denoted by $(A, B, \leqslant_P)$. A bipartite poset $P = (A, B, \leqslant_P)$ is a *core* if $|A| = |B|$ and for any comparable pair $x \leqslant_P y$ with $x \in A$ and $y \in B$, xy is a Dilworth edge (see Figure 2). Informally, we think of a core as a bipartite poset whose Hasse diagram is a balanced bipartite graph in which each edge is included in some perfect matching.

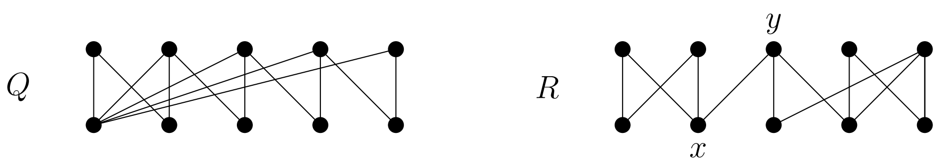

Figure 2: Poset Q is a core of width 5. R is not a core since xy is not a Dilworth edge.

A chain in a poset P corresponds to an independent set in its cocomparability graph. Offline the terms chain partition and coloring are interchangeable, but an online chain partitioning algorithm has more information to use than an online coloring algorithm. This advantage is lost by First-Fit.

The notion of Grundy coloring is useful for analyzing First-Fit.

Definition 8. Let $n \in \mathbb{Z}^+$. A function $\mathfrak{g} : P \to [n]$ is an *n-Grundy coloring of a poset P* if

(G1) for each $i \in [n]$, the set $\{u \in P : \mathfrak{g}(u) = i\}$ is a chain in P;

(G2) for each $i \in [n]$, there is some $u \in P$ so that $\mathfrak{g}(u) = i$ (i.e.: $\mathfrak{g}$ is surjective); and

(G3) if $v \in P$ with $\mathfrak{g}(v) = j$, then for all $i \in [j-1]$ there is some $u \in I_P(v)$ such that $\mathfrak{g}(u) = i$.

Often, we call the elements of $[n]$ as *colors*. If $u \in P$ and $\mathfrak{g}(u) = i$, we say u is *colored with i*. Let $u, v \in P$. If $u \parallel_P v$ and $\mathfrak{g}(u) < \mathfrak{g}(v)$, we say u is a *$\mathfrak{g}(u)$-witness for v under* $\mathfrak{g}$.

The next lemma is folklore. It allows the analysis of a dynamic online process in a static setting.

Lemma 9. *For any poset P, the largest n for which P has an n-Grundy coloring is equal to $\chi_{\mathrm{FF}}(P)$.*

3 The online algorithm

In this section we provide a simple online algorithm $\mathfrak{A}$ for chain partitioning online posets. In the next two sections we show that $\mathfrak{A}$ achieves the performance bound stated in Theorem 3. If W is a subset of P, we set $W^x = W \cap \{y : y \preceq x\}$ and $W^{\prec x} = W \cap \{y : y \prec x\}$.

3.1 Overview

We define $\mathfrak{A}$ using three procedures. Consider an online poset $P^{\prec} = (V, \leqslant_P, \prec)$.

(Pr1) Construct an online partition $V = X_1 \cup \cdots \cup X_{\mathrm{width}(P)}$ by putting every consecutive vertex x of $(V, \prec)$ to the set X_w, where the number w is the least integer such that $\mathrm{width}(P[X_1^{\prec x} \cup \cdots \cup X_w^{\prec x} \cup \{x\}]) = w$. Pick a w-antichain A'_x in $P[X_1^x \cup \cdots \cup X_w^x]$ with $x \in A'_x$.

(Pr2) For every $w \in [\mathrm{width}(P)]$, construct an on-line poset $R_w^{\lll}$, where $R_w = (Z, \leqslant_R)$, together with an injection $\phi : X_w \to Z$ that satisfies the property that $R_w[\phi(X_w)]$ is a subposet of $P[X_w]$. Thus, a partition of $R_w^{\lll}$ into chains yields a partition of $P[X_w]$ into chains. This more complex procedure is explained in Subsection 3.2.

(Pr3) For every $w \in [\mathrm{width}(P)]$, use First-Fit to partition $R_w^{\lll}$ into chains.

The final chain partition consists of all chains produced by procedure (Pr3) for $w = 1, \ldots, \mathrm{width}(P)$.[1]

[1]Kierstead and Trotter [23] used two procedures (Pr1) and (Pr3), without (Pr2), to prove $\mathrm{val}(\mathrm{Forb}_w(\mathbf{2}+\mathbf{2})) = 3w - 2$.

In Section 4 we show that R_w is a $(2w^2+1)$-ladder free poset of width w. In Section 5 we show that

$$\mathrm{val}_{FF}(L_m, w) \leqslant w^{2.5 \lg(2w) + 2 \lg m}.$$

Then, since a chain partition of $R_w^{\lll}$ yields a chain partition of $P[X_w]$ with at most the same number of chains, Theorem 3 follows by

$$\begin{aligned} \mathrm{val}(\mathcal{P}_w) &\leqslant \textstyle\sum_{j=1}^{w} \mathrm{val}_{FF}(R_j^{\lll}) \leqslant w \cdot \mathrm{val}_{FF}(L_{2w^2+1}, w) \\ &\leqslant w^{2.5 \lg(2w) + 2 \lg(2w^2+1) + 1} \leqslant w^{6.5 \lg w + 7}. \end{aligned} \tag{1}$$

In the remaining of the paper, we write $R^{\lll}$ and R instead of $R_w^{\lll}$ and R_w whenever w is clear from the context.

3.2 Procedure (Pr2)

Fix $w \in [\mathrm{width}(P)]$. Note that procedure (Pr1) produces a partition of the set V into $X_1 \uplus \ldots \uplus X_{\mathrm{width}(P)}$ such that $\mathrm{width}(P[X_1 \cup \ldots \cup X_w]) = w$. Let $V_w = X_1 \cup \ldots \cup X_w$ and let $\mathcal{M} = \mathcal{M}(P[V_w])$ be the set of all maximum antichains in $P[V_w]$. First, algorithm $\mathfrak{A}$ constructs a chain $\mathcal{A} = \{A_y : y \in X_w\}$ in $(\mathcal{M}, \sqsubseteq_P)$. The antichain A_x is obtained from A'_x and the $\sqsubseteq_P$-chain $\mathcal{A}^{\prec x} = \{A_y : y \in X_w^{\prec x}\}$ when $x \in X_w$ is processed. Put

$$A_x = (A'_x \wedge U_x) \vee D_x,$$

where

$$\mathcal{U}_x = \{A \in \mathcal{A}^{\prec x} : x \in D_P(A)\} \text{ and } U_x = \begin{cases} \bigwedge \mathcal{U}_x & \text{if } \mathcal{U}_x \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

and

$$\mathcal{D}_x = \{A \in \mathcal{A}^{\prec x} : x \in U_P(A)\} \text{ and } D_x = \begin{cases} \bigvee \mathcal{D}_x & \text{if } \mathcal{D}_x \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

see Figure 3. Clearly, $x \in A_x$. Each $A \in \mathcal{A}^{\prec x}$ is a w-antichain contained in $P[V_w^{\prec x}]$, so some $y \in A$ is comparable to x. Thus $\mathcal{A}^{\prec x} = \mathcal{D}_x \cup \mathcal{U}_x$. Let $\mathcal{A}^x = \mathcal{A}^{\prec x} \cup \{A_x\}$. As $\mathcal{A}^{\prec x}$ is a chain and $u <_P x <_P v$ for some $u \in D_x$ and $v \in U_x$ we note that

$$\mathcal{A}^x \text{ is a } \sqsubseteq_P\text{-chain with consecutive elements } D_x, A_x, U_x \text{ (unless } \mathcal{D}_x = \varnothing \text{ or } \mathcal{U}_x = \varnothing). \tag{2}$$

Define $p(x)$ by $A_{p(x)} = D_x$ if $\mathcal{D}_x \neq \varnothing$ and $s(x)$ by $A_{s(x)} = U_x$ if $\mathcal{U}_x \neq \varnothing$.

The maximum antichains A_x in $\mathcal{A}$ are computed in the order in which the elements x are added to the set X_w. So, we may view $(\bigcup \mathcal{A}, \leqslant_P)$ as an on-line poset with the presentation order extended by the elements from $A_x \setminus \bigcup\{A_y : y \in X_w^{\prec x}\}$ each time a new antichain A_x from $\mathcal{A}$ is computed. It is likely that the antichains in $\mathcal{A}$ are not disjoint. In the next step we slightly modify this poset by making these antichains pairwise disjoint.

When x is processed, set $B_x = \{(u, A_x) : u \in A_x\}$. Let $\mathcal{B} = \{B_y : y \in X_w\}$, $Z = \bigcup \mathcal{B}$ and, following our notation, let $\mathcal{B}^x = \{B_y : y \in X_w^x\}$ and $Z^x = \bigcup \mathcal{B}^x$. Let $\leqslant_U$ be the product order defined by

$$(u, A) \leqslant_U (u', A') \iff u \leqslant_P u' \text{ and } A \sqsubseteq A', \quad \text{for } u \in A,\ u' \in A',\ \text{and } A, A' \in \mathcal{A}. \tag{3}$$

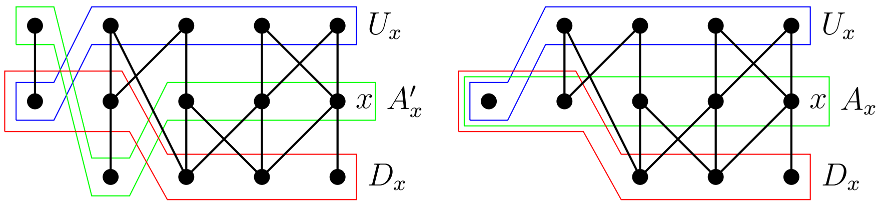

Figure 3: Constructing A_x based on A'_x, D_x, and U_x.

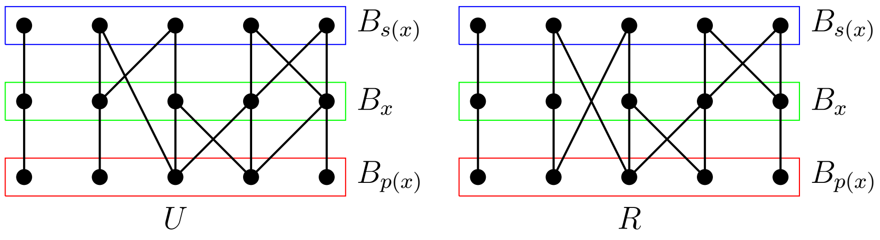

Figure 4: Hasse diagrams of $U[B_{p(x)} \cup B_x \cup B_{s(x)}]$ and $R[B_{p(x)} \cup B_x \cup B_{s(x)}]$.

Define the presentation order $\ll$ of the poset $U = (Z, \leqslant_U)$ by putting $B_x \ll B_y$ if $x \prec y$ for $x, y \in X_w$, and by arbitrarily ordering the elements in B_x for $x \in X_w$.

By (3), B_x is an w-antichain in U. Indeed, if S is a $(w+1)$-subset in Z then there are distinct $y, z \in S$ with comparable (possibly identical) first coordinates. By (3), they are comparable in U. Thus width$(U) = w$, and $\mathcal{B}$ partitions Z into disjoint maximum antichains. Moreover, $\mathcal{B}$ is a $\sqsubseteq_U$-chain. Note that the antichain B_x and the relation $\leqslant_U$ between the elements in B_x and the elements in the set $Z^{\prec x}$ can be computed when x is processed.

In its last step, procedure (Pr2) constructs an online poset $R^{\ll} = (Z, \leqslant_R, \ll)$, where $R = (Z, \leqslant_R)$ is obtained from $(Z, \leqslant_U)$ by deleting some non-Dilworth edges of $(Z, \leqslant_U)$. Suppose $\leqslant_R$ restricted to $Z^{\prec x}$ is already computed. When x is processed, the edges (v, u) of $U[Z^x]$ with $u \in B_x$ are deleted unless there is a Dilworth edge $(u', u) \in U[B_{p(x)} \cup B_x]$ such that $v \leqslant_R u'$ (possibly $v = u'$) in $(Z^{\prec x}, \leqslant_R)$. Dually, the edges (u, v) of $U[Z^x]$ with $u \in B_x$ are deleted unless there is a Dilworth edge $(u, u') \in U[B_x \cup B_{s(x)}]$ such that $u' \leqslant_R v$ (possibly $v = u'$) in $(Z^{\prec x}, \leqslant_R)$. This completes the definition of R. Figure 4 illustrates the derivation of $\leqslant_R$ from $\leqslant_U$.

Note that $R[Z^x]$ can be computed from P^x. Note that if (v, u) is a Dilworth edge in $U[Z^x]$ then $v <_R u$ in R. We prove this by induction on $\prec$. The claim holds for the set Z^y, where y is the first vertex in $(V, \prec)$ added to X_w. Suppose the claim holds for $Z^{\prec x}$. Assume that $v \ll u$ and $u \in B_x$ (the other cases are handled similarly). Then there is a Dilworth partition $\mathcal{C}$ with a chain C such that $v, u \in C$. Thus there is $z \in C \cap B_{p(x)}$. Since $\mathcal{C}$ restricted to $Z^{\prec x}$ is a chain partition of $U[Z^{\prec x}]$, we get $v \leqslant_R z$ by inductive hypothesis. As $v \leqslant_R z$ and (z, u) is Dilworth in $U(B_{p(x)} \cup B_x)$, (v, u) is not deleted, and hence $v \leqslant_R u$.

Let $\phi : X_w \to Z$ be a mapping defined $\phi(x) = (x, A_x)$. Let $x, x' \in X_w$. Clearly, $\phi(x) \leqslant_R \phi(x')$ is equivalent to $(x, A_x) \leqslant_R (x', A_{x'})$, which yields $x \leqslant_P x'$. Hence, a chain partition of $R_w^{\leqslant}$ induces a chain partition of $P[X_w]$ into at most the same number of chains: indeed, it is enough to assign $x \in X_w$ to a chain labeled i if $\phi(x) = (x, A_x)$ is assigned to a chain i.

Lemma 10. *The relation $R = (Z, \leqslant_R)$ is a width w poset, and for all $x, y \in X_w$ with $B_x \sqsubset_R B_y$:*

(R1) $R[B_{p(x)} \cup B_x]$ and $R[B_x \cup B_{s(x)}]$ are cores;

(R2) Suppose $u \in B_x$, $v \in B_y$ and $u <_R v$. If $x \prec y$ then there is $v' \in B_{p(y)}$ with $u \leqslant_R v' <_R v$; else there is $u' \in B_{s(x)}$ with $u <_R u' \leqslant_R v$.

(R3) $\mathcal{B}$ is a partition of Z into maximum antichains.

(R4) $\mathcal{B}$ is a chain in $\sqsubseteq_R$.

(R5) $R[B_x, B_y]$ is a core.

(R6) Let $z \in X_w$ be such that $B_x \sqsubseteq_R B_z \sqsubseteq_R B_y$ and suppose that for every $z' \in X_w$ such that $B_z \sqsubseteq_R B_{z'} \sqsubseteq_R B_y$ ($B_x \sqsubseteq_R B_{z'} \sqsubseteq_R B_z$) we have $z \preceq z'$. Then, for all $u \in B_x$ and $v \in B_y$ with $u \leqslant_R v$, there is $z'' \in B_z$ with $u \leqslant_R z'' \leqslant_R v$.

Proof. Conditions (R1) and (R2) follow immediately from the definition of R.

First we prove that R is a poset of width w. As R is obtained from the poset U by removing some non-loops, R is reflexive and antisymmetric. For transitivity, argue by induction on $\prec$. Suppose $u <_R v <_R w$. Then there are distinct $x, y, z \in X^w$ with $u \in B_x$, $v \in B_y$, $w \in B_z$, and $B_x \sqsubset_R B_y \sqsubset_R B_z$. Let $s = \prec\text{-max}\{x, y, z\}$. If $s = y$ then using (R2) there are $v' \in B_{p(y)}$ and $v'' \in B_{s(y)}$ such that $u \leqslant_R v' \leqslant_R v$, $v \leqslant_R v'' \leqslant_R w$, (v', v) is Dilworth in $U[B_{p(y)} \cup B_y]$ and (v, v'') is Dilworth in $U[B_y \cup B_{s(y)}]$; thus (v', v'') is Dilworth in $U[B_{p(y)} \cup B_{s(y)}]$, $v' <_R v''$, and $u <_R w$ by induction. The other two cases are similar, but easier. Thus R is a poset. As no Dilworth edges are removed from U to form R, $\text{width}(R) = \text{width}(U) = w$. Thus (R3) and (R4) also hold.

We prove (R5) by induction on $\prec$. Assume $x \prec y$. The case $y \prec x$ is dual. By (R1), $R[B_{p(y)} \cup B_y]$ is a core. If $x = p(y)$ we are done. Otherwise, $R[B_x \cup B_{p(y)}]$ is a core by induction. Thus $R[B_x, B_y]$ is a core by definition of $\leqslant_R$. So, (R5) holds.

We prove (R6) by induction on $\prec$. Suppose $u \in B_x$ and $v \in B_y$ with $u \leqslant_R v$. The claim holds if $z = x$ or $z = y$. Suppose $z \in X_w$ is such that $B_x \sqsubset_R B_z \sqsubset_R B_y$ and $z \preceq z'$ for any $B_z \sqsubseteq_R B_{z'} \sqsubseteq_R B_y$. Suppose $x \prec y$. By (R2), there is $w' \in B_{p(y)}$ with $u \leqslant_R w' <_R v$. If $z = p(y)$ we are done. Otherwise, as $z \prec p(y)$, there is $w \in B_z$ with $u <_R w <_R w' <_R y$ by induction, and hence (R6) holds. Suppose $y \prec x$. By (R2), there is $w' \in B_{s(x)}$ with $u \leqslant_R w' <_R v$. If $z = s(x)$ we are done. Otherwise, as $B_{s(x)} \sqsubset_R B_z \sqsubseteq_R B_y$, there is $w \in B_z$ with $u <_R w' <_R w <_R y$ by induction. Thus (R6) holds. ☐

In [3, 4] a width w poset R is defined to be *regular* if it has a partition $\mathcal{B}$ satisfying (R1)–(R4). An example of a regular poset is given in Figure 5.

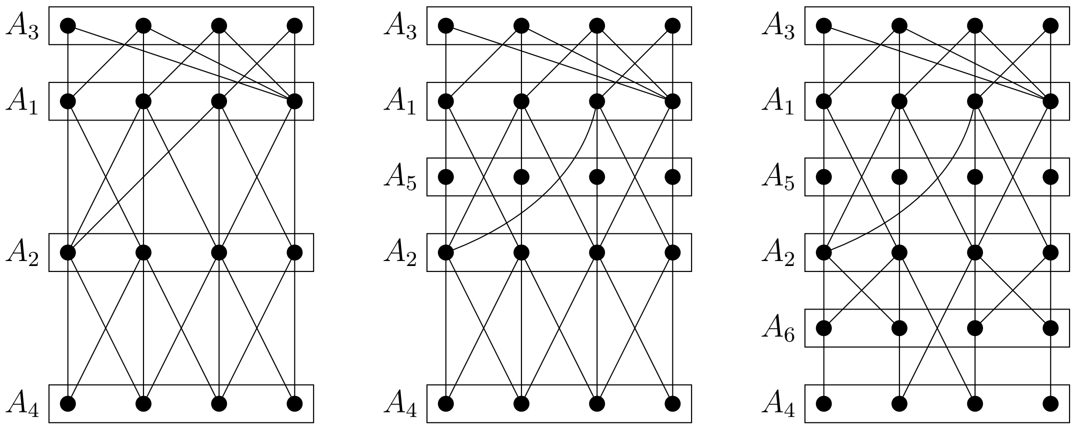

Figure 5: Regular poset $P^{\ll}$ with $P = (\bigcup_{i=1}^{6} A_i, \leqslant_P)$ and the presentation order given by $A_1 \ll A_2 \ll \ldots \ll A_6$ (the order $\ll$ inside each A_i is arbitrary).

4 Regular posets do not induce large ladders

In this section we prove that $R \in \mathrm{Forb}_w(L_{2w^2+1})$, which yields Lemma 6 as a corollary. For any $u \in R$, let $B(u)$ be the antichain with $u \in B(u) \in \mathcal{B}$. Consider an arbitrary m-ladder $L = L(x_1 \ldots x_m; y_1 \ldots y_m)$ in R. Call L *canonical* if $B(y_i) \sqsubseteq_R B(x_{i+1})$ for all $i \in [m-1]$.

Proposition 11. *If $L = L(x_1 \ldots x_m; y_1 \ldots y_m) \subseteq R$ is canonical then $m \leqslant w$.*

Proof. See Figure 6. As $\mathrm{width}(R) \leqslant w$, it suffices to show by induction on i that

$$|U_R[y_1] \cap B(y_i)| \geqslant i \text{ for } i \in [m]. \tag{4}$$

The base step $i = 1$ holds, since $y_1 \in U_R[y_1] \cap B(y_1)$, so assume $1 < i \leqslant m$. As L is canonical, $B(y_{i-1}) \sqsubseteq_R B(x_i)$. Thus there is $z \in B(y_{i-1})$ such that $z \leqslant_R x_i \leqslant_R y_i$. Since $y_1 \parallel_R x_i$, we have $y_1 \parallel_R z$. Thus $z \notin S := U_R[y_1] \cap B(y_{i-1})$. By induction, $|S| \geqslant i-1$. By (R5), $R[B(y_{i-1}) \cup B(y_i)]$ is a core with Dilworth edge zy_i. Let $\mathcal{C}$ be a Dilworth partition of $R[B(y_{i-1}) \cup B(y_i)]$ with z and y_i in the same chain. Each vertex of S is matched in $\mathcal{C}$ to a distinct vertex of $B(y_i)$, different than y_i (see Figure 6) as $z \notin S$. Consequently, $|U_R[y_1] \cap B(y_i)| \geqslant |S| + 1 \geqslant i - 1 + 1 = i$. This proves (4). $\square$

Proposition 12. *If $L(x_1 \ldots x_m; y_1 \ldots y_m) \subseteq R$ with $m \geqslant 2w+1$ then $B(y_1) \sqsubseteq_R B(x_{2w+1})$.*

Proof. See Figure 7. Assume to the contrary that $B(x_{2w+1}) \sqsubset_R B(y_1)$. It follows that

$$B(x_1) \sqsubset_R \ldots \sqsubset_R B(x_{2w+1}) \sqsubset_R B(y_1) \sqsubset_R \ldots \sqsubset_R B(y_{2w+1}).$$

Let $z \in X_w$ be the $\prec$-least index with $B(x_{w+1}) \sqsubseteq_R B_z \sqsubseteq_R B(y_{w+1})$. If $B_z \sqsubset_R B(y_1)$ then set $I = [w+1]$; else set $I = \{w+1, \ldots, 2w+1\}$. Regardless, $|I| = w+1$, and by (R6), there are z_i with $x_i <_R z_i <_R y_i$ for all $i \in I$ (see Figure 7). As $|B_z| = w$ there are $i, j \in I$ with $i < j$ and $z_i = z_j$. Then $x_j <_R z_j = z_i <_R y_i$, a contradiction with $x_j \parallel_R y_i$. $\square$

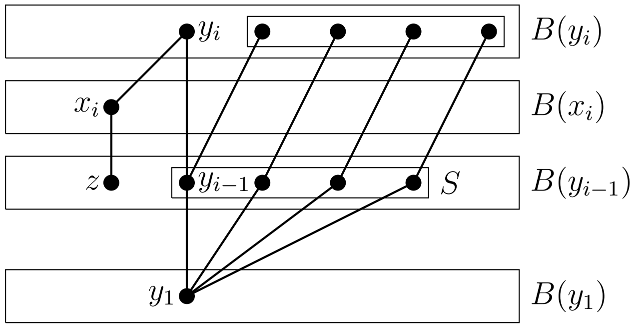

Figure 6: The intersections of $U_P[y_1]$ with $B(y_{i-1})$ and $B(y_i)$.

Lemma 13. $R \in Forb(L_{2w^2+1})$.

Proof. Suppose $L(x_1 \dots x_{2w^2+1}; y_1 \dots y_{2w^2+1}) \subseteq R$. By Proposition 12, we must have $B(y_i) \sqsubseteq_R B(x_j)$ for any $i, j \in [2w^2+1]$ with $j - i \geqslant 2w$. Thus, the subposet induced by the vertices

$$\bigcup_{0 \leqslant i \leqslant w} \{x_{2wi+1}, y_{2wi+1}\}$$

is a canonical ladder with $w+1$ rungs, which contradicts Proposition 11. $\square$

5 First-Fit on ladder-free posets

In this section we prove $\mathrm{val}_{\mathrm{FF}}(L_m, w) \leqslant w^{2.5 \lg(2w) + 2 \lg m}$ for $m, w \in \mathbb{Z}^+$, which shows Lemma 7. This proof was already published in [22], here we present its shortened version to keep the paper self-contained. Consider a Grundy coloring $\mathfrak{g}$ of a poset P. Let $C = \{x_1, \dots, x_k\}$ be a chain of P such that $\mathfrak{g}(x_1) < \dots < \mathfrak{g}(x_k)$; we call C *ascending* if $x_1 <_P x_2 <_P \dots <_P x_k$ (see Figure 8) and we call C *descending* if $x_1 >_P x_2 >_P \dots >_P x_k$.

The next two propositions are the combinatorial tools for the upcoming arguments in the proof of Lemma 7. The first one is just a restatement of the Erdős-Szekeres Theorem and the second one presents conditions for ascending and descending chains in a poset with forbidden ladder.

Proposition 14. *Consider a poset P and its Grundy coloring $\mathfrak{g}$. Let C be a chain in P such that all $\mathfrak{g}(c)$ for $c \in C$ are distinct. If the length of every ascending subchain of C is at most s and the length of every descending subchain C is at most t, then $|C| \leqslant st$.*

Proposition 15. *Suppose $P \in \mathrm{Forb}(L_m, w)$ and $w \geqslant 2$. Let $x_1 < \dots < x_k$ be an ascending (resp. let $x_1 > \dots > x_k$ be a descending) chain in P and for each $i \in [k-1]$ let y_i be a $\mathfrak{g}(x_i)$-witness for x_{i+1}. Then for all i, j with $1 \leqslant i < j \leqslant k$,*

(C1) $x_i <_P y_i$ (resp. $x_i >_P y_i$);

(C2) $y_i \not>_P x_j$ (resp. $y_i \not<_P x_j$); and

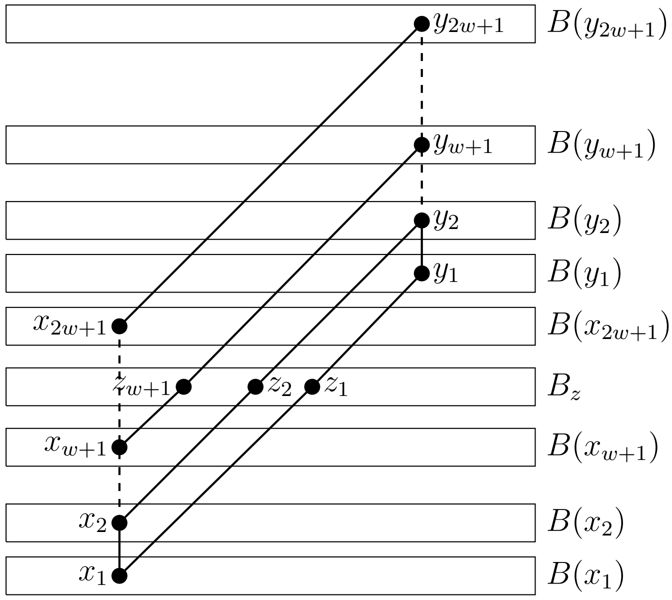

Figure 7: The ladder L and the antichain B_z.

(C3) if $y_h \parallel_P x_k$ for all $h \in [k-1]$, then $k \leqslant m(w-1)$.

Proof. We consider the case that $x_1 < \ldots < x_k$ is an ascending chain; the other case can be proved analogically. Observe that (C1) and (C2) follow immediately from definitions (see Figure 8). To show (C3) assume to the contrary that $k > m(w-1)$. Then $k \geqslant (m-1)(w-1)+2$ as $w \geqslant 2$. The subposet $P_0 := P[\{y_1, \ldots, y_{k-1}\}]$ has width at most $w-1$ as $y_h \parallel_P x_k$ for all $h \in [k-1]$ by hypothesis. Since $|P_0| \geqslant (m-1)(w-1)+1$, there is a chain $y_{i_1} <_P \ldots <_P y_{i_m}$ in P_0, by Dilworth's Theorem. By (C2), we have $i_1 < \ldots < i_m$. Thus by (C1) and hypothesis, $P[\{x_{i_1}, y_{i_1}, \ldots, x_{i_m}, y_{i_m}\}]$ is an m-ladder, contradicting $P \in \mathrm{Forb}(L_m, w)$. □

Proof of Lemma 7. We argue by induction on $w = \mathrm{width}(P)$. The base step $w = 1$ is trivial. Now fix w, and assume the lemma holds for all smaller values of w.

Let $P = (V, \leqslant_P)$ be a poset of width w such that $P \in \mathrm{Forb}(L_m)$, let $\mathfrak{g} : V \to [n]$ be an n-Grundy coloring of P with $n = \mathrm{val}_{\mathrm{FF}}(L_m, w)$, and let $C_i = g^{-1}(i)$. We must show that $n \leqslant w^{2.5 \lg(2w) + 2 \lg m}$.

Pick a maximum antichain $A \in \mathcal{V}_P$ with $N := \min_{a \in A} \mathfrak{g}(a)$ maximum, i.e.,

$$N = \min_{a \in A} \mathfrak{g}(a) = \max_{B \in \mathcal{V}_P} \min_{b \in B} \mathfrak{g}(b).$$

Then $H := P[C_{N+1} \cup \cdots \cup C_n]$ has width at most $w-1$. As $\mathfrak{g} - N$ is a Grundy coloring of H,

$$n \leqslant N + \mathrm{val}_{\mathrm{FF}}(L_m, w-1). \tag{5}$$

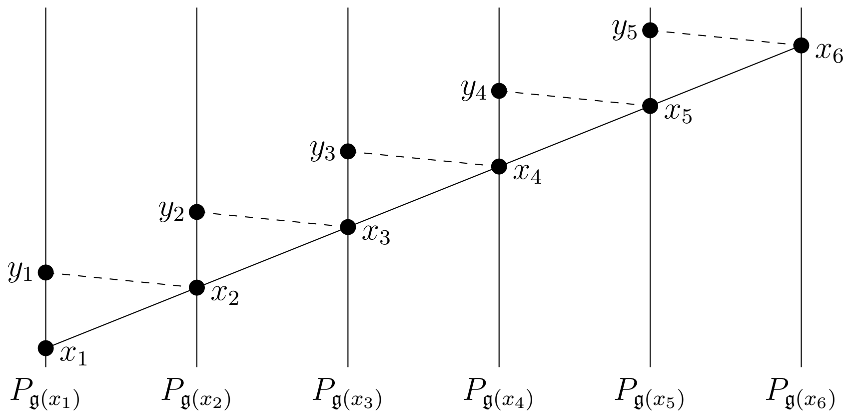

Figure 8: The ascending chain $x_1 <_P \ldots <_P x_6$. The point y_i is a $\mathfrak{g}(x_i)$-witness of x_{i+1}.

Let $D = D(A)$ and $U = U(A)$. As A is a maximum antichain, $P = D \cup A \cup U$. Call a vertex x *special* if $|I_P(x) \cap A| \geqslant w/2$. For $i \in [N-1]$, set $q_i^- = \max(C_i \cap D)$ and $q_i^+ = \min(C_i \cap U)$. Hence q_i^- and q_i^+ are consecutive on C_i. Each $a \in A$ has an i-witness, and so satisfies $q_i^- \parallel_P a$ or $q_i^+ \parallel_P a$. Thus, q_i^- or q_i^+ is special. Pick a special vertex $q_i \in \{q_i^-, q_i^+\}$ and pick $r_i \in C_i$ so that r_i is a minimal special vertex in C_i if $q_i \in D$, and r_i is a maximal special vertex in C_i if $q_i \in U$ (it might happen that $r_i = q_i$). Call q_i the *near witness* and r_i the *far witness*. Set $R = \{r_1, \ldots, r_{N-1}\}$. The next claim completes our recursion for $\mathrm{val}_{FF}(L_m, w)$.

Claim 15.A. $|S| \leqslant \frac{1}{2}m^2w(w-1)^2 \,\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$ *for all chains S with $S \subseteq R \cap D$ or $S \subseteq R \cap U$.*

Proof. Let $S \subseteq R \cap U$; the case $S \subseteq R \cap D$ is dual. Recall that all $\mathfrak{g}(r_i)$ are distinct. It gives that also all $\mathfrak{g}(s)$, for $s \in S$, are distinct. Therefore, by Proposition 14, it suffices to show:

(T1) the size of any ascending chain in S is at most $m(w-1)$; and

(T2) the size of any descending chain in S is at most $\frac{w}{2}m(w-1)\,\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$.

For (T1), let $x_1 <_P \ldots <_P x_k$ be any ascending chain in S. For each $i \in [k-1]$, pick a $\mathfrak{g}(x_i)$-witness y_i for x_{i+1}. Using Proposition 15, (C2) implies $y_i \not>_P x_k$. Suppose $y_i <_P x_k$. By (C1), $x_i <_P y_i$. Then $y_i \in U$ and y_i is special as x_k is special. As $\mathfrak{g}(x_i) = \mathfrak{g}(y_i)$ this contradicts the choice of x_i as a far witness. Thus $y_i \parallel_P x_k$. By (C3), $|S| = k \leqslant m(w-1)$.

For (T2), let $S' = \{z_1 >_P \ldots >_P z_k\}$ be a descending chain in S, and set $P' = P[D_P(z_1) \cap U]$. If B is an antichain in P' then $(A - D(z_1)) \cup B$ is an antichain in P. As z_1 is special, $|B| \leqslant w/2$. So

$$\mathrm{width}(P') \leqslant w/2. \tag{6}$$

For $i \in [k]$, let w_i be the near witness for color $\mathfrak{g}(z_i)$. Note that $w_i \in P'$ since $w_i \leqslant z_i \leqslant z_1$. By Dilworth's Theorem, there is a chain $T \subseteq \{w_1, \ldots, w_k\}$ with $k \leqslant \frac{w}{2}|T|$. Each w_i has different Grundy color, thus by Proposition 14, it suffices to prove:

(T3) the size of any descending sequence in T is at most $m(w-1)$,

(T4) the size of any ascending sequence in T is at most $\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor \frac{w}{2} \rfloor)$.

For (T3), let $s_1 >_P \ldots >_P s_l$ be a descending chain in T. For $1 \leqslant i \leqslant l-1$, pick a $\mathfrak{g}(s_i)$-witness t_i of s_{i+1}. Using Proposition 15, (C2) implies $t_i \not<_P s_l$. Suppose $s_l <_P t_i$. Then $t_i \in U$ and t_i is also special as $t_i <_P s_i$ by (C1). As $\mathfrak{g}(t_i) = \mathfrak{g}(s_i)$ this contradicts the choice of s_i as a near witness. Thus $t_i \parallel_P s_l$. By (C3), $|T| = l \leqslant m(w-1)$.

For (T4), let $u_1 <_P \ldots <_P u_l$ be an ascending chain in T, and for $i \in [l]$ let $v_i \in S'$ be the far $\mathfrak{g}(u_i)$-witness. Then $u_l \leqslant_P v_l$. Note that $u_1 <_P \cdots <_P u_l \leqslant_P v_l <_P \cdots <_P v_1$ as S' is descending. Set $U_i = [u_i, v_i] \cap C_{\mathfrak{g}(u_i)}$, $U' = \bigcup_{i=1}^{l} U_i$ and $P'' = P'[U']$. Define $\mathfrak{g}' : U' \to [l]$ by $\mathfrak{g}'(x) = i$ iff $x \in U_i$. Then $\mathfrak{g}'$ is an l-Grundy coloring of P'': as $\mathfrak{g}$ is a Grundy coloring, if $i < j$ and $y \in U_j$ then there is $x \in C_i$ with $x \parallel_P y$; as $u_i <_P u_j \leqslant_P y \leqslant_P v_j <_P u_i$, we have $x \in U_i$. Since $P'' \subset P$ is L_m-free, $l \leqslant \mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$. □

Consider a Dilworth chain decompositions of R and let S be a chain with a maximum size in this decomposition. Since the width of R is at most w, we have

$$N - 1 = |R| \leqslant w|S| = w|S \cap D| + w|S \cap U|.$$

After applying Claim 15.A we get

$$N \leqslant 1 + m^2 w^2 (w-1)^2 \, \mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor) \leqslant m^2 w^4 \, \mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor).$$

The equation (5) with $n = \mathrm{val}_{\mathrm{FF}}(L_m, w)$ can be now rewrite into the following recursion

$$\mathrm{val}_{\mathrm{FF}}(L_m, w) \leqslant m^2 w^4 \, \mathrm{val}_{\mathrm{FF}}\left(L_m, \lfloor w/2 \rfloor\right) + \mathrm{val}_{\mathrm{FF}}(L_m, w-1).$$

Applying this recursion repeatedly to the second term, with $\mathrm{val}_{\mathrm{FF}}(L_m, 1) = 1$, we obtain

$$\mathrm{val}_{\mathrm{FF}}(L_m, w) \leqslant 1 + \sum_{2 \leqslant k \leqslant w} m^2 k^4 \, \mathrm{val}_{\mathrm{FF}}\left(L_m, \lfloor k/2 \rfloor\right) \leqslant w m^2 w^4 \, \mathrm{val}_{\mathrm{FF}}\left(L_m, \lfloor w/2 \rfloor\right).$$

Arguing by induction yields:

$$\mathrm{val}_{\mathrm{FF}}(L_m, w) \leqslant m^{2 \lg w} w^{2.5 \lg(2w)} \leqslant w^{2.5 \lg(2w) + 2 \lg m},$$

which completes the proof of the lemma. □

6 Limitations of Our Methods

Loosely speaking, two major parts of the proof of our main theorem rely on limiting the number of rungs in a ladder within a regular poset and the performance of First-Fit on the family $\mathrm{Forb}(L_m)$. Here, we will show that our general upper bound for the online coloring problem cannot be greatly improved with our current methods.

In the first part of the section we show that the assertion of Lemma 13 can not be improved. Although L_{2w^2+1} is not a subposet of any width w regular poset, we show that there are regular posets of width w that contain ladders whose number of rungs is quadratic in w.

Lemma 16. *For each integer $w \geqslant 2$, there is a regular poset $P^{\ll}$ so that* $\mathrm{width}(P) = w$ *and P contains $L_{w\lfloor (w+2)/2 \rfloor}$ as an induced subposet.*

Proof. Consider two antichains $A = \{u_1, \ldots, u_w\}$ and $B = \{v_1, \ldots, v_w\}$, where u_i and v_i are the i-th elements of A and B, respectively. We say $(A, B, \leqslant)$ is a core of:

- *type I* if for all $i, j \in [w]$
$$u_i \leqslant v_j \text{ iff } i = j,$$
- *type S_k* for $k \in [w]$ if for all $i, j \in [w]$
$$u_i \leqslant v_j \text{ iff } i = j \text{ or } (i = 1 \text{ and } j \in [k]) \text{ or } (i \in [2, k] \text{ and } j \in [i-1, i]),$$
- *type T_k* for $k \in [w]$ if for all $i, j \in [w]$
$$u_i \leqslant v_j \text{ iff } i = j \text{ or } (i \in [w-k+1, w] \text{ and } j = w) \text{ or } (i \in [w-k] \text{ and } j \in [i-1, i]).$$

It is straightforward to verify that bipartite posets of types I, S_k and T_k are cores. See Figure 9 for examples.

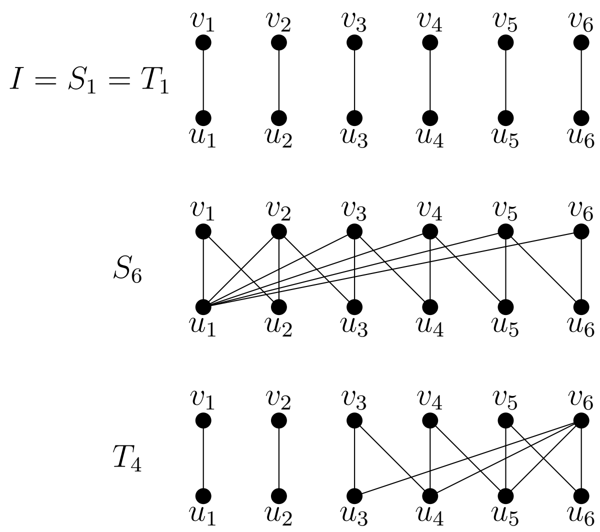

Figure 9: Hasse diagrams of I, S_6, and T_4 for $w = 6$.

Now we construct an auxiliary regular poset $Q^{\ll}$, based on which the regular poset $P^{\ll}$ is built. Let $V = \bigcup_{i=1}^{2w+1} A_i$, and let the presentation order of $Q^{\ll}$ be defined by $A_1 \ll A_2 \ll \ldots \ll A_{2w+1}$. The poset $Q = (V, \leqslant_Q)$ is defined as follows. At every step in the presentation of Q, every two $\sqsubseteq_Q$-consecutive antichains induce a core, one of type: I, S_k or T_k for $k \in [w]$. Suppose that the antichains $A_{s(i)}$ and $A_{p(i)}$, if exist, denote the antichains that are respectively just above and just below A_i at the moment A_i is presented. To define the relation $\leqslant_Q$ in Q we need only to determine the relation $\leqslant_Q$ between A_i and $A_{s(i)}$ and between $A_{p(i)}$ and A_i at the moment A_i is presented; the other comparabilities will follow by transitivity – see (R2). Below are the rules how $\leqslant_Q$ is determined for the successively presented antichains $A_1, \ldots, A_{2w+1}$:

1. A_2 is set so that
   - $s(2) = 1$ and $(A_2, A_1, \leqslant_Q)$ is of type S_w,
   - $p(2)$ is not defined.
2. For $i \in [3, w+1]$ the antichain A_i is set so that
   - $s(i) = 1$ and $(A_i, A_1, \leqslant_Q)$ is of type S_{w-i+2},
   - $p(i) = i-1$ and $(A_{i-1}, A_i, \leqslant_Q)$ is of type I.
3. A_{w+2} is set so that
   - $s(w+2)$ is not defined,
   - $p(w+2) = 1$ and $(A_1, A_{w+2}, \leqslant_Q)$ is of type T_w.
4. For $i \in [w+3, 2w+1]$ the antichain A_i is set so that
   - $s(i) = i-1$ and $(A_i, A_{i-1}, \leqslant_Q)$ is of type I,
   - $p(i) = 1$ and $(A_1, A_i, \leqslant_Q)$ is of type T_{2w-i+2}.

The above rules imply the following relations between the antichains $A_1, \ldots, A_{2w+1}$ in the poset Q (see Figure 10):

$$A_2 \sqsubset_Q A_3 \sqsubset_Q \ldots \sqsubset_Q A_{w+1} \sqsubset_Q A_1 \sqsubset_Q A_{2w+1} \sqsubset_Q A_{2w} \sqsubset_Q \ldots \sqsubset_Q A_{w+2}.$$

Although it is tedious to verify that $Q^{\ll}$ is indeed a width w regular poset, it is straightforward and we leave it to the reader.
Let $\perp = A_2$, $\top = A_{w+2}$. For every $i \in [w]$ we denote by:

- x_i – the first point in A_{i+1},
- y_i – the w-th point in A_{2w+2-i},
- b_i – the i-th point in $\perp$,
- t_i – the i-th point in $\top$,

and finally we let $X = \{x_1, \ldots, x_w\}$ and $Y = \{y_1, \ldots, y_w\}$. By inspection we may easily check the following properties of Q.

(P1) $x_1 <_Q \ldots <_Q x_w$ and $y_1 <_Q \ldots <_Q y_w$.

Moreover, for any $i, j \in [w]$:

(P2) If $i \leqslant j$ then $x_i <_Q y_j$, otherwise $x_i \parallel_Q y_j$.

(P3) If $j \leqslant i \leqslant j+2$ or $i = 1$ or $j = w$ then $b_i \leqslant_Q t_j$, otherwise $b_i \parallel_Q t_j$.

(P4) If $j = w$ then $y_i <_Q t_j$, otherwise $y_i \parallel_Q t_j$.

(P5) If $i = 1$ then $b_i <_Q x_j$, otherwise $b_i \parallel_Q x_j$.

Now, we are ready to describe the regular poset $P^{\ll}$. The poset P will consists of $h = \lfloor (w+2)/2 \rfloor$ copies of Q. We will use the same variable names to denote elements (sets) in the copies of Q in P as these introduced for Q; however, we add the superscript i to specify that a variable describes an element (a set) from the i-th copy of Q. Formally, the poset $P = (V, \leqslant_P)$ is defined such that $V = \bigcup_{i=1}^{h} V^i$ and $\leqslant_P$ is the transitive closure of

$$(\leqslant_{Q^1} \cup \ldots \cup \leqslant_{Q^h}) \cup \{(t_i^j, b_i^{j+1}) : i \in [w], j \in [h-1]\}.$$

The presentation order $\ll$ of P is set so as:

(i) $V^i \ll V^j$ for any $1 \leqslant i < j \leqslant h$,

(ii) the order of the elements within every copy of Q is the same as in Q.

Again, checking that $P^{\ll}$ is a regular poset of width w is straightforward; an example of $P^{\ll}$ is shown in Figure 10.

To finish the proof of the lemma we show that

$$\text{the set } \bigcup_{j=1}^{h} (X^j \cup Y^j) \text{ induces an } (w \cdot h)\text{-ladder in } P, \tag{7}$$

with $x_i^j y_i^j$ being its $((j-1)h + i)$-th rung. Clearly, we have

$$X^1 <_P \ldots <_P X^h \text{ and } Y^1 <_P \ldots <_P Y^h \tag{8}$$

by the definition of $\leqslant_P$. Finally, we will show that for all $i, j \in [h]$:

$$X^i <_P Y^j \text{ if } i < j \text{ and } X^i \parallel_Q Y^j \text{ if } i > j. \tag{9}$$

Note that the relation between X^i and Y^j in the case when $i = j$ is handled by (P2). Clearly, if we prove (9), (7) follows by (P1), (8), (9), and (P2). Assume that $i < j$. Clearly, by (P2) it follows that X^i is less than the greatest element in Y^i. Consequently, $X^i <_P Y^j$ by (8). Assume $i > j$. We consider only the case $i = h$ and $j = 1$; the remaining ones are even easier to prove. First note that every comparability between a point in Y_1 and a point in X_h needs to be implied by transitivity on some point from $\perp^h$. Note that $D_P(X_h) \cap \perp^h$ contains only the first element of $\perp^h$ by (P5). By (P3) and (P4), note that the set $U_P(Y_1) \cap A_2^i$ contains exactly $2i - 3$ last elements in $\perp^i$ for $i \in [2, h]$. Plugging $h = \lfloor (w+2)/2 \rfloor$ to the last observation we get $U_P(Y_1) \cap \perp^h$ contains not more than $2\lfloor (w+2)/2 \rfloor - 3 \leqslant w - 1$ last elements from $\perp^h$. In particular, $U_P(Y_1) \cap \perp^h$ does not contain the first element of $\perp^h$. It follows that $X^h \parallel_P Y^1$. □

In the last part of this section we give the lower bound on $\mathrm{val}_{\mathrm{FF}}(L_m, w)$, which shows that the upper bound from Lemma 7 can not be substantially improved. For the upcoming construction we remind the definition of the *lexicographical product* of two posets. For

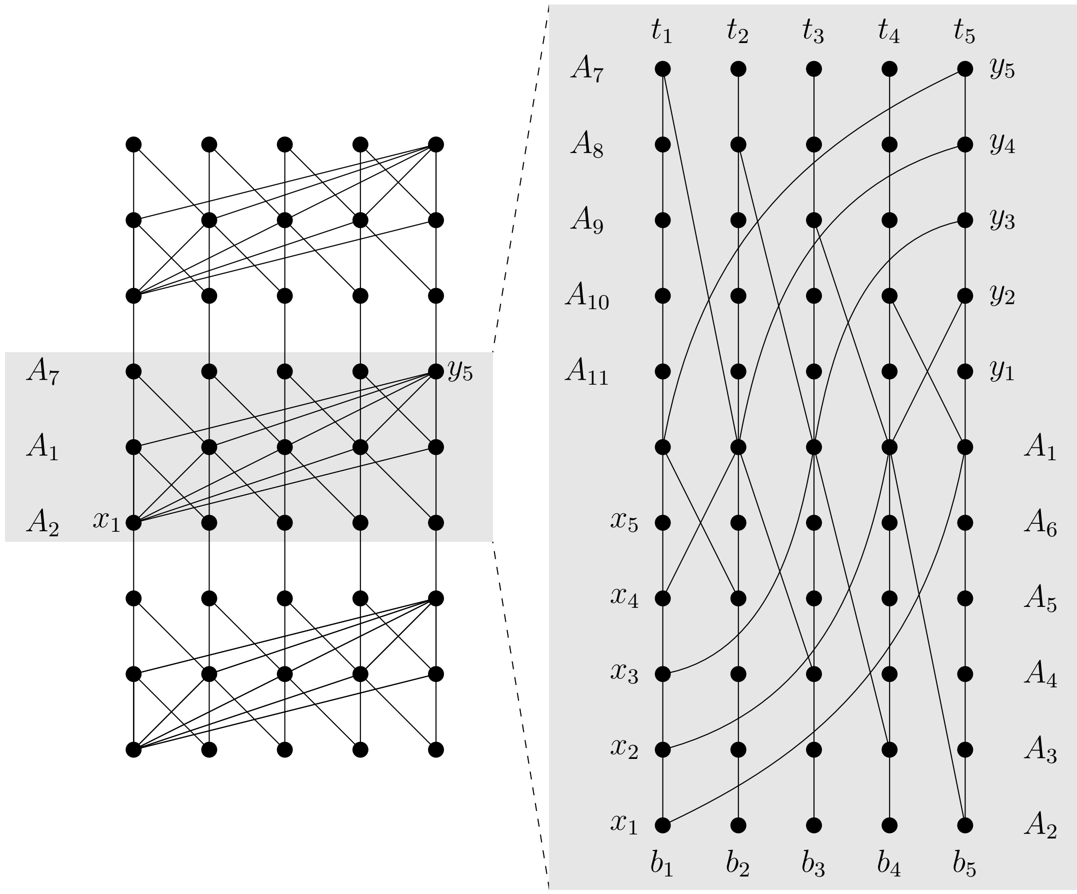

Figure 10: The width 5 poset Q is shown on the right. The sketch of the construction of the width 5 poset P is shown on the left. It consists of 3 copies of Q (the middle copy of Q in P is depicted with gray background) joined as shown in the figure.

posets P and Q, the lexicographical product $P \cdot Q$ is the poset with vertices $\{(p, q) : p \in P, q \in Q\}$ and order $\leqslant_{P \cdot Q}$, where

$$(p_1, q_1) \leqslant_{P \cdot Q} (p_2, q_2) \text{ if either } p_1 <_P p_2 \text{ or } (p_1 = p_2 \text{ and } q_1 \leqslant_Q q_2).$$

Informally, we may think of $P \cdot Q$ as the poset P where each vertex has been "inflated" to a copy of Q. It is well know that

$$\text{width}(P \cdot Q) = \text{width}(P)\,\text{width}(Q). \tag{10}$$

The following two simple properties (we left the proof for the reader) are the key in the proof of the upcoming lemma. For $p, r \in P$ and $u, v, s \in Q$ we have:

$$\text{If } ((p, u) \leqslant_{P \cdot Q} (r, s) \text{ or } (p, u) \geqslant_{P \cdot Q} (r, s)) \text{ and } (r, s) \parallel_{P \cdot Q} (p, v), \text{then } p = r. \tag{11}$$

$$\text{If } (p, u) \leqslant_{P \cdot Q} (r, s) \leqslant_{P \cdot Q} (p, v), \text{then } p = r. \tag{12}$$

Lemma 17. *For $m, w \in \mathbb{Z}^+$ with $m > 1$, we have $w^{\lg(m-1)}/(m-1) \leqslant \mathrm{val}_{\mathrm{FF}}(L_m, w)$.*

Proof. Fix $m \in \mathbb{Z}^+$ with $m > 1$. Let R be the width 2 poset R_{m-1} as defined in the proof of Lemma 4. For technical reasons we would like R to have the least and the greatest element. Vertex x_1^{m-1} is already the greatest in R, but there is no least element in R. Therefore we extend R to P by adding a new element $\hat{0}$ which is below entire R. The greatest element in P is still x_1^{m-1}, which we denote by $\hat{1}$.

It is a simple exercise to see that P also satisfies the statement of Lemma 4, i.e., $\mathrm{width}(P) = 2$ and $\chi_{\mathrm{FF}}(P) \geqslant \chi_{\mathrm{FF}}(R) \geqslant m-1$. As R is an induced subposet of P we have $I_P(\hat{0}) = \varnothing$ and $\left|I_P(x_i^k)\right| = k < m-1$ for $1 \leqslant i \leqslant k < m-1$ and $\left|I_P(x_i^{m-1})\right| = i-1 < m-1$ for $i \in [m-1]$. Observe that in a ladder L_m, the lowest vertex of the upper leg is always incomparable to $m-1$ vertices. Hence, there is no vertex in P that can serve as the lowest vertex of the upper leg of an m-ladder and thus

$$P \in \mathrm{Forb}(L_m). \tag{13}$$

We are prepared to build a poset $Q_k \in \mathrm{Forb}(L_m)$ with n-Grundy coloring so that $\mathrm{width}(Q_k) = 2^k$ and $n \geqslant (m-1)^k$. Poset Q_k is defined by the following rules:

(Q1) Q_0 is a single vertex z.

(Q2) $Q_{k+1} = P \cdot Q_k$.

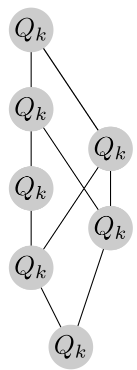

Figure 11: Simplified Hasse diagram of Q_{k+1} with $m = 4$.

Note that Q_1 and P are isomorphic and so we will treat Q_1 as P. The next two properties are the consequence of the definition of Q_k, equation (10) and the fact that P has the least and the greatest element with $\mathrm{width}(P) = 2$. For each $k \in \mathbb{N}$

(Q3) Q_k has a minimum vertex and a maximum vertex,

(Q4) $\mathrm{width}(Q_k) = 2^k$.

Claim 17.A. *For each $k \in \mathbb{N}$, $Q_k \in \mathrm{Forb}(L_m)$.*

Proof. We will use induction on k. For our bases, we see $k = 0$ is trivial and $k = 1$ is established by (13). Take $k > 1$ and suppose the inductive hypothesis holds for all smaller cases. Assume L is an m-ladder in Q_k with the lower leg $(a_1, u_1) <_{Q_k} (a_2, u_2) <_{Q_k} \ldots <_{Q_k} (a_m, u_m)$ and the upper leg $(b_1, v_1) <_{Q_k} (b_2, v_2) <_{Q_k} \ldots <_{Q_k} (b_m, v_m)$. If all vertices of L are pairwise different in the first coordinate, then these vertices would induce an m-ladder in P, which violates (13). Hence, at least two vertices of L share a first coordinate, say $p \in P$. Let $Q' = \{(p, q) : q \in Q_{k-1}\}$ and note that Q' and Q_{k-1} are isomorphic. Let $\hat{0}$ and $\hat{1}$ to be the minimum and the maximum, respectively, vertices of Q' (which exist by (Q3)).

Assume for a while, Q' contains two vertices of the lower leg of L, i.e., there are $i < j \in [m]$ so that $(a_i, u_i), (a_j, u_j) \in Q'$ with $a_i = a_j = p$. From the definition of a ladder, we know $(a_i, u_i) \leqslant_{Q_k} (b_i, v_i) \parallel_{Q_k} (a_j, u_j)$. By (11) we have $b_i = p$ and thus Q' contains (b_i, v_i), a vertex of the upper leg of L. For similar reasons, if Q' contains two vertices of the upper leg of L, then it has to have one of the lower leg of L. Therefore, there are $(a_i, u_i), (b_j, v_j) \in Q'$, vertices of the lower and the upper leg of L, respectively. We see $(a_i, u_i) \leqslant_{Q_k} (a_m, u_m) \parallel_{Q_k} (b_j, v_j)$ (if $j < m$) or $(a_i, u_i) \leqslant_{Q_k} (a_m, u_m) \leqslant_{Q_k} (b_j, v_j)$ (if $j = m$). In the former case we use (11) and in the latter case (12) to show $(a_m, u_m) \in Q'$. Similarly, $(a_i, u_i) \parallel_{Q_k} (b_1, v_1) \leqslant_{Q_k} (b_j, v_j)$ (if $i > 1$) or $(a_i, u_i) \leqslant_{Q_k} (b_1, v_1) \leqslant_{Q_k} (b_j, v_j)$ (if $i = 1$). Again, using (11) or (12), we have $(b_1, v_1) \in Q'$.

For any vertex (r, s) in L so that $(r, s) \notin \{(a_1, u_1), (b_m, v_m)\}$, we have either $(b_1, v_1) \leqslant_{Q_k} (r, s) \parallel_{Q_k} (a_m, u_m)$ or $(b_1, v_1) \parallel_{Q_w} (r, s) \leqslant_{Q_k} (a_m, u_m)$. By (11) we deduce $(r, s) \in Q'$. Finally, the vertices

$$\{\hat{0}, (a_2, u_2), (a_3, u_3), \ldots, (a_m, u_m), (b_1, v_1), (b_2, v_2), \ldots, (b_{m-1}, v_{m-1}), \hat{1}\} \subseteq Q'$$

induce an m-ladder in Q', which contradicts the inductive hypothesis, proving the claim. □

Claim 17.B. $\chi_{\mathrm{FF}}(Q_{k+1}) \geqslant (m-1)\chi_{\mathrm{FF}}(Q_k)$.

Proof. We already know P has an $(m-1)$-Grundy coloring, say $\mathfrak{f}$. Let $\mathfrak{g}$ be a n-Grundy coloring of Q_k. Define $\mathfrak{h} : Q_{k+1} \to [(m-1)n]$ by $\mathfrak{h}((p, q)) = (\mathfrak{f}(p) - 1)n + \mathfrak{g}(q)$. We will show $\mathfrak{h}$ is an $((m-1)n)$-Grundy coloring of Q_{k+1}. For that we need to prove (G1)-(G3) of Definition 8.

It is easy to check that a function $(f, g) \to (f-1)n + g$ is a bijection between $[m-1] \times [n]$ and $[(m-1)n]$. Since $\mathfrak{f}$ and $\mathfrak{g}$ are surjective, then also $\mathfrak{h}$ must be surjective. Thus, $\mathfrak{h}$ satisfies (G2). To show (G1) suppose $\mathfrak{h}((p, q)) = \mathfrak{h}((r, s))$. This implies that $\mathfrak{f}(p) = \mathfrak{f}(r)$ and $\mathfrak{g}(q) = \mathfrak{g}(s)$. By (G1) of $\mathfrak{f}$ and $\mathfrak{g}$, two pairs of vertices p, r and q, s are comparable respectively in P and in Q_k. Therefore, by the definition of the lexicographical product, vertices (p, q) and (r, s) are comparable in Q_{k+1} and condition (G1) holds for $\mathfrak{h}$.

Consider $(r, s) \in Q_{k+1}$ so that $\mathfrak{h}((r, s)) = j > 1$ and take any $i < j$. We will show (r, s) has an i-witness in Q_{k+1} which will prove (G3). There are unique integers $c \in [m-1]$ and $d \in [n]$ so that $j = (c-1)n + d$ and $\mathfrak{f}(r) = c$, $\mathfrak{g}(s) = d$. Similarly, we can find $a \in [m-1]$ and $b \in [k]$ so that $i = (a-1)n + b$. As $i < j$, we must have $a \leqslant c$.

Suppose $a = c$, then $b < d$. As $\mathfrak{g}$ satisfies (G3), there is some $q \in Q_k$ so that $\mathfrak{g}(q) = b$ and $q \parallel_{Q_k} s$. By the definition of lexicographical product, $(r, q) \parallel_{Q_{k+1}} (r, s)$. Observe $\mathfrak{h}((r, q)) = i$ and then (r, q) is the desired witness.

The case $a < c$ is similar. This time we use (G3) of $\mathfrak{f}$ to get $p \in P$ so that $\mathfrak{f}(p) = a$ and $p \parallel_P r$. Take any $q \in Q_k$ so that $\mathfrak{g}(q) = b$ (q exists by (G2) of $\mathfrak{g}$). Again, by the definition of lexicographical product, $(p, q) \parallel_{Q_{k+1}} (r, s)$. Finally, as $\mathfrak{h}((p, q)) = i$, we deduce (p, q) is the desired witness in this case. □

Claim 17.B with $\chi_{\mathrm{FF}}(Q_0) = 1$ implies $\chi_{\mathrm{FF}}(Q_k) \geqslant (m-1)^k$. Note that $\mathrm{width}(Q_{\lfloor \lg w \rfloor})$ could be less then w. But we can always add some isolated vertices to $Q_{\lfloor \lg w \rfloor}$ to get width w poset Q' so that $\chi_{\mathrm{FF}}(Q') \geqslant \chi_{\mathrm{FF}}(Q_{\lfloor \lg w \rfloor})$. This finally shows

$$\mathrm{val}_{\mathrm{FF}}(L_m, w) \geqslant \chi_{\mathrm{FF}}(Q_{\lfloor \lg w \rfloor}) \geqslant (m-1)^{\lfloor \lg w \rfloor} \geqslant \frac{w^{\lg(m-1)}}{m-1}. \qquad \square$$

Lemmas 16 and 17 show that the upper bound of $\mathrm{val}(\mathcal{P}_w)$ cannot be pushed below $w^{\lg w}$ using our current methods.

7 Concluding Remarks

Although we have improved the upper bound for $\mathrm{val}(\mathcal{P}_w)$, our current methods cannot bring it down to a polynomial bound without some major changes. Perhaps improvements in the understanding of regular posets could lead us to a subfamily of more interesting forbidden substructures. We could also examine online coloring algorithms other than First-Fit to reduce the number of colors used on the family $\mathrm{Forb}(L_m)$.

We may look beyond the scope of $\mathrm{val}(\mathcal{P}_w)$. So far, the reduction to regular posets has only been studied on general posets. We might ask what the results of procedures (Pr1) and (Pr2) are when we start with a poset from $\mathrm{Forb}(Q)$ (for some poset Q). It is interesting to ask what analogues of the inequality (1) could be built. For instance, could an analogue for cocomparability graphs be created? Already, Kierstead, Penrice, and Trotter [19] have shown that a cocomparability graph can be colored online using a bounded number of colors. However, this bound is so large that it was not computed. Perhaps methods similar to the reduction to regular posets could be created.

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