A Note on the Linear Cycle Cover Conjecture of Gyárfás and Sárközy

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Abstract

A linear cycle in a 3-uniform hypergraph H is a cyclic sequence of hyperedges such that any two consecutive hyperedges intersect in exactly one element and non-consecutive hyperedges are disjoint. Let $\alpha(H)$ denote the size of a largest independent set of H.

We show that the vertex set of every 3-uniform hypergraph H can be covered by at most $\alpha(H)$ edge-disjoint linear cycles (where we accept a vertex and a hyperedge as a linear cycle), proving a weaker version of a conjecture of Gyárfás and Sárközy.
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1 Introduction

A well-known theorem of Pósa [3] states that the vertex set of every graph G can be partitioned into at most $\alpha(G)$ cycles where $\alpha(G)$ denotes the independence number of G (where a vertex or an edge is accepted as a cycle).

Definition 1. A *(linear cycle) linear path* is a (cyclic) sequence of hyperedges such that two consecutive hyperedges intersect in exactly one element and two non-consecutive hyperedges are disjoint.

An independent set of a hypergraph H is a set of vertices that contain no hyperedges of H. Let $\alpha(H)$ denote the size of a largest independent set of H and we call it the independence number of H. Gyárfás and Sárközy [2] conjectured that the following extension of Pósa's theorem holds: One can partition every k-uniform hypergraph H into at most $\alpha(H)$ linear cycles (here, as in Pósa's theorem, vertices and subsets of hyperedges are accepted as linear cycles). In [2] Gyárfás and Sárközy prove a weaker version of their conjecture for *weak* cycles (where only cyclically consecutive hyperedges intersect, but their intersection size is not restricted) instead of linear cycles. Recently, Gyárfás, Győri and Simonovits [1] showed that this conjecture is true for $k = 3$ if we assume there are no linear cycles in H.

In this note, we show their conjecture is true for $k = 3$ provided we allow the linear cycles to be edge-disjoint, instead of being vertex-disjoint.

Theorem 2. *If H is a 3-uniform hypergraph, then its vertex set can be covered by at most $\alpha(H)$ edge-disjoint linear cycles (where we accept a single vertex or a hyperedge as a linear cycle).*

Our proof uses induction on $\alpha(H)$. However, perhaps surprisingly, in order to make induction work, our main idea is to allow the hypergraph H to contain hyperedges of size 2 (in addition to hyperedges of size 3). First we will delete some vertices, and add certain hyperedges of size 2 into the remaining hypergraph so as to ensure the independence number of the remaining hypergraph is smaller than that of H. Then applying induction we will find edge-disjoint linear cycles (which may contain these added hyperedges) covering the remaining hypergraph. It will turn out that the added hyperedges behave nicely, allowing us to construct edge-disjoint linear cycles in H covering all of its vertices. The detailed proof is given in the next section.

2 Proof of Theorem 2

We call a hypergraph *mixed* if it can contain hyperedges of both sizes 2 and 3. A linear cycle in a mixed hypergraph is still defined according to Definition 1. We will in fact prove our theorem for mixed hypergraphs (which is clearly a bigger class of hypergraphs than 3-uniform hypergraphs). More precisely, we will prove the following stronger theorem.

Theorem 3. *If H is a mixed hypergraph, then its vertex set $V(H)$ can be covered by at most $\alpha(H)$ edge-disjoint linear cycles (where we accept a single vertex or a hyperedge as a linear cycle).*

Proof. We prove the theorem by induction on $\alpha(H)$. If $|V(H)| = 1$ or 2, then the statement is trivial. If $|V(H)| \geq 3$ and $\alpha(H) = 1$, then H contains all possible edges of size 2 and there is a Hamiltonian cycle consisting only of edges of size 2, which is of course a linear cycle covering $V(H)$.

Let $\alpha(H) > 1$. If $E(H) = \varnothing$, then $\alpha(H) = V(H)$ and the statement of our theorem holds trivially since we accept each vertex as a linear cycle. If $E(H) \neq \varnothing$, then let P be a longest linear path in H consisting of hyperedges $h_0, h_1, \ldots, h_l$ ($l \geq 0$). If h_i is of size 3, then let $h_i = v_i v_{i+1} u_{i+1}$ and if it is of size 2, then let $h_i = v_i v_{i+1}$. A linear subpath of P starting at v_0 (i.e., a path consisting of hyperedges $h_0, h_1, \ldots, h_j$ for some $j \leq l$) is called an *initial segment* of P. Let C be a linear cycle in H which contains the longest initial segment of P. If there is no linear cycle containing h_0, then we simply let $C = h_0$.

Let us denote the subhypergraph of H induced on $V(H) \setminus V(C)$ by $H \setminus C$. Let $R = \{v_k u_k \mid \{v_k, u_k\} \subseteq V(P) \setminus V(C) \text{ and } v_0 v_k u_k \in E(H)\}$ be the set of *red* edges. Let us construct a new hypergraph H' where $V(H') = V(H) \setminus V(C)$ and $E(H') = E(H \setminus C) \cup R$. We will show that $\alpha(H') < \alpha(H)$ and any linear cycle cover of H' can be extended to a linear cycle cover of H by adding C and extending the red edges by v_0.

The following claim shows that the independence number of H' is smaller than the independence number of H. This fact will later allow us to apply induction.

Claim 4. *If I is an independent set in H', then $I \cup v_0$ is an independent set in H.*

Proof. Suppose by contradiction that $h \subseteq (I \cup v_0)$ for some $h \in E(H)$. Then, clearly $v_0 \in h$ because otherwise I is not an independent set in H'. Now let us consider different cases depending on the size of $h \cap (V(P) \setminus V(C))$. If $|h \cap (V(P) \setminus V(C))| = 0$ then, by adding h to P, we can produce a longer path than P, a contradiction. If $|h \cap (V(P) \setminus V(C))| = 1$, let $h \cap (V(P) \setminus V(C)) = \{x\}$. Then the linear subpath of P between v_0 and x together with h forms a linear cycle which contains a larger initial segment of P than C, a contradiction. If $|h \cap (V(P) \setminus V(C))| = 2$, then let $h \cap (V(P) \setminus V(C)) = \{x, y\}$. Let us take smallest i and j such that $x \in h_i$ and $y \in h_j$ (i.e., if $x \in h_i \cap h_{i+1}$ then let us take h_i). If $i \neq j$, say $i < j$ without loss of generality, then the linear subpath of P between v_0 and x together with h forms a linear cycle with longer initial segment of P than C, a contradiction. Therefore, $i = j$ but in this case, $\{x, y\}$ is a red edge and so at most one of them can be contained in I, contradicting the assumption that $h = v_0 xy \subseteq (I \cup v_0)$. Hence, $I \cup v_0$ is an independent set in H, as desired. □

The following claim will allow us to construct linear cycles in H from red edges.

Claim 5. *The set of hyperedges of every linear cycle in H' contains at most one red edge.*

Proof. Suppose by contradiction that there is a linear cycle C' in H' containing at least two hyperedges which are red edges. Then there is a linear subpath P' of C' consisting of hyperedges $h'_0, h'_1, \ldots, h'_m$ such that $h'_0 := v_s u_s$ and $h'_m := v_t u_t$ (where $s > t$) are red edges but h'_k is not a red edge for any $1 \leq k \leq m-1$. Let us first take the smallest i such that $V(P') \cap h_i \neq \varnothing$ and then the smallest j such that $h'_j \cap h_i \neq \varnothing$. It is easy to see that $|V(P') \cap h_i| \leq 2$ (since i was smallest). If $\left|h'_j \cap h_i\right| = 1$, then the linear cycle consisting of hyperedges $h'_1, \ldots, h'_j$ and $h_i, h_{i-1}, \ldots, h_0$ and $v_0 v_s u_s$ contains a larger initial segment of P than C (as $h'_j \cap h_i \in V(P) \setminus V(C)$), a contradiction. If $\left|h'_j \cap h_i\right| = 2$, then notice that $\left|h'_{j+1} \cap h_i\right| = 1$. Now the linear cycle consisting of the hyperedges $h'_{m-1}, h'_{m-2}, \ldots, h'_{j+1}$ and $h_i, h_{i-1}, \ldots, h_0$ and $v_0 v_t u_t$ contains a larger initial segment of P than C, a contradiction. □

By Claim 4, $\alpha(H') \leq \alpha(H) - 1$. So by induction hypothesis, $V(H')$ can be covered by at most $\alpha(H) - 1$ edge-disjoint linear cycles (where we accept a single vertex or a hyperedge as a linear cycle). Now let us replace each red edge $\{x, y\}$ with the hyperedge xyv_0 of H. Claim 5 ensures that in each of these linear cycles, at most one of the hyperedges is a red edge. Therefore, it is easy to see that after the above replacement, linear cycles of H' remain as *linear* cycles in H and they cover $V(H') = V(H) \setminus V(C)$. Now the linear cycle C, together with these linear cycles give us at most $\alpha(H) - 1 + 1 = \alpha(H)$ edge-disjoint linear cycles covering $V(H)$, completing the proof. □

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