

Residual q -Fano Planes and Related Structures

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Abstract

One of the most intriguing problems for q -analogs of designs, is the existence question of an infinite family of q -Steiner systems that are not spreads. In particular the most interesting case is the existence question for the q -analog of the Fano plane, known also as the q -Fano plane. These questions are in the front line of open problems in block design. There was a common belief and a conjecture that such structures do not exist. Only recently, q -Steiner systems were found for one set of parameters. In this paper, a definition for the q -analog of the residual design is presented. This new definition is different from previous known definition, but its properties reflect better the q -analog properties. The existence of a design with the parameters of the residual q -Steiner system in general and the residual q -Fano plane in particular are examined. We construct different residual q -Fano planes for all q , where q is a prime power. The constructed structure is just one step from a construction of a q -Fano plane.

Mathematics Subject Classifications: 05B40, 51E10

1 Introduction

Let \mathbb{F}_q be the finite field with q elements and let \mathbb{F}_q^n be the set of all vectors of length n over \mathbb{F}_q . \mathbb{F}_q^n is a vector space with dimension n over \mathbb{F}_q . For a given integer k , $0 \leq k \leq n$, let $\mathcal{G}_q(n, k)$ denote the set of all k -dimensional subspaces (k -subspaces in short) of \mathbb{F}_q^n . $\mathcal{G}_q(n, k)$ is often referred to as a Grassmannian. It is well known that

$$|\mathcal{G}_q(n, k)| = \begin{bmatrix} n \\ k \end{bmatrix}_q \stackrel{\text{def}}{=} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q -binomial coefficient (known also as the *Gaussian coefficient* [42, pp. 325-332]).

Let Q be a set with n elements. A t - (n, k, λ) *design*, is a collection of k -subsets of V , called *blocks*, such that each t -subset of Q is contained in exactly λ blocks. A t - (n, k, λ) design with $t = \lambda = 1$ is trivial: it is simply a partition of Q into k -subsets, which exists if and only if k divides n . A t - $(n, k, 1)$ design with $t \geq 2$ is known as a *Steiner system*, and usually denoted $S(t, k, n)$. Steiner systems are among the most beautiful and well-studied structures in combinatorics. Their history goes back to the work of Plücker [31], Kirkman [26], Cayley [8], and Steiner [35] in the first half of the 19-th century. Today, the significance of Steiner systems extends well beyond combinatorics — they have found applications in many areas, including group theory, finite geometry, cryptography, and coding theory [2, 12, 17]. For example, a finite projective plane of order q can be characterized as a Steiner system $S(2, q+1, q^2+q+1)$, with lines as blocks. As another example, the Mathieu groups (which played an important role in the classification of finite simple groups) are most naturally understood as automorphism groups of certain Steiner systems.

A long-standing problem in design theory asks whether nontrivial (meaning $t < k < n$) Steiner systems with $t > 5$ exist. Keevash recently announced a resolution of this problem: his breakthrough paper [24] moreover shows that Steiner systems $S(t, k, n)$ exist for all $t < k$ and all sufficiently large integers n that satisfy the necessary divisibility conditions. More recently another (simpler) proof was provided by Glock, Kühn, Lo, and Osthus [22].

The classical theory of q -analogs of mathematical objects and functions has its beginnings in the work of Euler [20, 27]. In 1957, Tits [41] further suggested that combinatorics of sets could be regarded as the limiting case $q \rightarrow 1$ of combinatorics of vector spaces over the finite field \mathbb{F}_q . Indeed, there is a strong analogy between subsets of a set and subspaces of a vector space, expounded by numerous authors—see [11, 23, 43] and references therein. It is therefore natural to ask which combinatorial structures can be generalized from sets (the $q \rightarrow 1$ case) to vector spaces over \mathbb{F}_q . For t -designs and Steiner systems, this question was first studied by Cameron [9, 10] and Delsarte [13] in the early 1970s. Specifically, let \mathbb{F}_q^n be a vector space of dimension n over the finite field \mathbb{F}_q . Then a t - (n, k, λ) *design over \mathbb{F}_q* is defined in [9, 10, 13] as a collection of k -subspaces of \mathbb{F}_q^n , called *blocks*, such that each t -subspace of \mathbb{F}_q^n is contained in exactly λ blocks. Such t -designs over \mathbb{F}_q are the q -analogs of conventional combinatorial designs. By analogy with the $q \rightarrow 1$ case, a t - $(n, k, 1)$ design over \mathbb{F}_q is said to be a q -*Steiner system*, and denoted $\mathbb{S}_q(t, k, n)$.

Remark. We observe that q -analogs of designs and Steiner systems are not only of interest in their own right, but also arise naturally in other areas, such as network coding [16]. The appropriate code in random network coding is a collection of subspaces of \mathbb{F}_q^n that are well-separated according to a metric defined on the Grassmannian. Consequently, a q -Steiner system $\mathbb{S}_q(t, k, n)$ can be thought of as an *optimal code* for error-correction in networks. For more details on this, see [17, 28].

Following the work of Cameron [9, 10] and Delsarte [13], the first examples of nontrivial t -designs over \mathbb{F}_q were found by Thomas [39] in 1987. Today, owing to the efforts of many authors [6, 21, 25, 30, 32, 36, 37, 38, 40], numerous such examples are known.

However, the situation is very different for q -Steiner systems. They are known to exist

in the trivial cases $t = k$ or $k = n$, and in the case where $t = 1$ and k divides n . In the latter case, q -Steiner systems coincide with the classical notion of *spreads* in projective geometry [42, Chapter 24]. Some 40 years ago, Beutelspacher [4] asked whether nontrivial q -Steiner systems with $t \geq 2$ exist, and this question has tantalized mathematicians ever since. The problem has been studied by numerous authors [1, 18, 29, 33, 39, 40], without much progress toward constructing such q -Steiner systems. In particular, Thomas [40] showed in 1996 that certain kinds of $\mathbb{S}_2(2, 3, 7)$ q -Steiner systems (the smallest possible example) cannot exist. Three years later, Metsch [29] conjectured that nontrivial q -Steiner systems with $t \geq 2$ do not exist in general. In contrast to this conjecture, a q -Steiner system $\mathbb{S}_2(2, 3, 13)$ was constructed recently [5]. In fact, once one such system was found, other nonisomorphic systems with the same parameters were found.

Similarly, to Steiner systems, simple necessary divisibility conditions for the existence of a given q -Steiner system were developed [33, 36].

Theorem 1. *If a q -Steiner system $\mathbb{S}_q(t, k, n)$ exists, then for each i , $1 \leq i \leq t - 1$, a q -Steiner system $\mathbb{S}_q(t - i, k - i, n - i)$ exists.*

Corollary 2. *If a q -Steiner system $\mathbb{S}_q(t, k, n)$ exists, then for all $0 \leq i \leq t - 1$,*

$$\frac{\begin{bmatrix} n-i \\ t-i \end{bmatrix}_q}{\begin{bmatrix} k-i \\ t-i \end{bmatrix}_q}$$

must be integers.

Deriving new designs from designs in general and q -Steiner systems in particular is an important direction to find new designs and to exclude the possible existence of other designs. Using q -analog of the derived design and the residual designs it was proved that sometimes the necessary conditions for the existence of a q -Steiner system $\mathbb{S}_q(t, k, n)$ are not sufficient [25]. The first set of parameters (t , k , and n) for which the existence question of q -Steiner systems is not settled is the parameters for the q -analog of the Fano plane, i.e. the q -Steiner systems $\mathbb{S}_q(2, 3, 7)$, which will be called also in this paper the *q -Fano plane*. There was a lot of effort to find whether the q -Fano plane, especially for $q = 2$, exists or does not exist, e.g. [7, 14, 18, 40]. All these attempts did not provide any answer to the existence question. It was proved recently in [7] that if such system exists for $q = 2$, then its automorphism group has a small order. In [15] a different approach to consider q -Steiner systems was given. This approach is based on puncturing a possible existing q -Steiner systems and considering the parameters of the structure derived from the punctured systems. Properties of the q -Fano plane based on this approach were also discussed. This approach led to the results in the current paper.

In this paper we present a construction for a design with the same parameters as the design derived from a q -Fano plane, the residual q -Fano plane. The constructed design will be also called the residual q -Fano plane. The construction has many places in which there is flexibility for many choices which lead to a construction of many such designs. Our definition for the residual q -Steiner system and the derived q -Steiner system result

in two structures whose union has the same size as the related q -Steiner system, which is not the case for the definition given in [25] and other possible definitions. This makes the residual q -Fano plane obtained by our construction to be a design which is almost as close as possible to a q -Fano plane. This definition of residual q -Steiner system and the construction of the residual q -Fano plane is a new direction for a research to solve the existence question of q -Steiner systems in general and q -Fano planes in particular.

The rest of this paper is organized as follows. In Section 2 we present a definition for a residual q -Steiner system, explain why this definition represents the appropriate q -analog definition, and compare it to the other definitions. In Section 3 a few combinatorial structures which are used in the construction are defined and some of their properties are discussed. In Section 4 we will discuss representation of subspaces for our construction. In Section 5 it will be explained how to extend and expand the subspaces in \mathbb{F}_q^4 to subspaces in \mathbb{F}_q^6 . The construction of the residual q -Fano plane is presented in Section 6, where its correctness is also proved. Conclusions and future research are discussed in Section 7. In particular we indicate on the points in the construction in which there is flexibility to construct many different residual q -Fano planes.

2 Derived and Residual Designs

For a design S on a set Q , and an element $x \in Q$, the derived design is defined by

$$\{B \setminus \{x\} : B \in S, x \in B\},$$

and the residual "design" is defined by

$$\{B : B \in S, x \notin B\}.$$

In [25] there is a simple definition for a q -analog of the derived design and the residual design. For this definition we choose an element $u \in \mathbb{F}_q^n$ and an $(n-1)$ -subspace $V \subset \mathbb{F}_q^n$ such that $\langle \{u\} \cup V \rangle = \mathbb{F}_q^n$, where $\langle X \rangle$ denote the linear span of X . The derived design of a design \mathbb{S} over \mathbb{F}_q , was defined as

$$\{B \cap V : B \in \mathbb{S}, u \in B\}, \tag{1}$$

and the residual design of \mathbb{S} , was defined as

$$\{B : B \in \mathbb{S}, B \subset V\}. \tag{2}$$

By these definitions, the derived design and residual design of a q -Steiner system are both designs over \mathbb{F}_q . This is on the positive side. On the negative side, the size of their union is significantly smaller than the size of the design from which they were derived.

We present now a different definition for the q -analog of a derived design and a residual design which solves this problem in the definition of [25]. Let u be the unit vector with the unique *one* in the last coordinate, and $V \stackrel{\text{def}}{=} \{(x, 0) : x \in \mathbb{F}_q^{n-1}\}$. Also, for a subspace

$B \subset \mathbb{F}_q^n$, let $\mathcal{Z}(B)$ be the subspace obtained from B , by removing the last coordinate of all the vectors in B . The derived and residual designs are defined by

$$\text{der}(\mathbb{S}) \stackrel{\text{def}}{=} \{ \mathcal{Z}(B \cap V) : B \in \mathbb{S}, u \in B \} . \quad (3)$$

$$\text{res}(\mathbb{S}) \stackrel{\text{def}}{=} \{ \mathcal{Z}(B) : B \in \mathbb{S}, u \notin B \} . \quad (4)$$

The two definitions of the derived design are equivalent, but there is a significant difference in the two definitions of the residual design. For the new definitions given in (3) and (4), we have that $|\mathbb{S}| = |\text{der}(\mathbb{S})| + |\text{res}(\mathbb{S})|$, a property that does not hold for the definitions given in (1) and (2). The fact that the union of the two derived designs has size as the original design is one argument that these definitions serve better as the q -analog of the derived design and the residual design. We continue to examine more properties, but the examination will relate only to Steiner systems $S(t, k, n)$ or only Steiner triple system $S(2, 3, n)$, which are the topic of this paper (but, these properties are also true for other parameters). Another argument is that the uncovered pairs in a residual Steiner triple system $S(2, 3, n)$ form a perfect matching (known also as a 1-factor or $S(1, 2, n-1)$) (see the work of Spencer [34] for the uncovered pairs of triple systems). The q -analog is the uncovered 2-subspaces in a residual design of a q -Steiner system $\mathbb{S}_q(2, 3, n)$. These uncovered pairs form a q -Steiner system $\mathbb{S}_q(1, 2, n-1)$ (known also as a 1-spread). Indeed, the uncovered pairs in the residual q -Steiner system defined in (4) are exactly the q -analog of the uncovered pairs of the residual Steiner system. This property does not exist in the definition given in (2). A third argument is a consequence of the next theorem.

The union of the derived q -Steiner system and the residual q -Steiner system was called in [15], the punctured (or 1-punctured) q -Steiner system. But, no such system was constructed in [15]. In the exposition given in [15] it was proved that

Theorem 3. *If \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$, then the derived system contains exactly $\frac{\binom{n-1}{t-1}_q}{\binom{k-1}{t-1}_q}$ distinct $(k-1)$ -subspaces which form a q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$. Each t -subspace of \mathbb{F}_q^{n-1} which is contained in a $(k-1)$ -subspace of $\text{der}(\mathbb{S})$ is not contained in any of the k -subspaces of $\text{res}(\mathbb{S})$. Each t -subspace of \mathbb{F}_q^{n-1} which is not contained in a $(k-1)$ -subspace of $\text{der}(\mathbb{S})$, appears exactly q^t times in the k -subspaces of $\text{res}(\mathbb{S})$.*

We will now define any two sets of subspaces which satisfy the properties given in Theorem 3 as the derived design and the residual design for a q -Steiner system $\mathbb{S}_q(t, k, n)$ (but do not depend on the existence of a q -Steiner system $\mathbb{S}_q(t, k, n)$). For a q -Steiner system $\mathbb{S}_q(t, k, n)$ these definitions are given as follows:

- A *derived q -Steiner system* for a q -Steiner system $\mathbb{S}_q(t, k, n)$ is a q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$.
- Let $\text{der}(\mathbb{S})$ be a q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$. The *residual q -Steiner system*, $\text{res}(\mathbb{S})$, for a q -Steiner system $\mathbb{S}_q(t, k, n)$ (which might not exist), \mathbb{S} , is a set of distinct k -subspaces from \mathbb{F}_q^{n-1} such that each t -subspace of \mathbb{F}_q^{n-1} which is not contained in $\text{der}(\mathbb{S})$, is contained in exactly q^t k -subspaces of $\text{res}(\mathbb{S})$.

It should be noted that when $q \rightarrow 1$, i.e. for a Steiner system based on an n -set, each t -subset of the $(n - 1)$ -set which is not contained in the derived design, is contained in exactly one k -subset of the derived design. This is another indication that our definition for the q -analog of the residual design reflects the best transformation from subsets to subspaces.

It is interesting to know if there exists a system with the same properties of the residual design in which each t -subspace which is not contained in the derived design, is contained in exactly λ subspaces of the residual design, where $\lambda < q^t$. It is not difficult to prove that this is not possible if λ is not divisible by q (the proof is left for the interested reader), but it is intriguing to know if λ divisible by q is possible.

3 Combinatorial Structures for the Construction

The construction of the residual q -Fano plane given in the Section 6 will make use of a few combinatorial structures which are defined, described, and discussed in this section.

The first object is a *1-spread* (spread in short) in \mathbb{F}_q^n , where n is even. A spread S in \mathbb{F}_q^n is a set of 2-subspaces of \mathbb{F}_q^n , such that each nonzero vector of \mathbb{F}_q^n is contained in exactly one 2-subspace of S . It is well known that such a spread exists whenever n is even.

A *1-parallelism* (parallelism in short) in \mathbb{F}_q^n is a partition of the 2-subspaces of \mathbb{F}_q^n into pairwise disjoint spreads. The number of 2-subspaces in such a spread is $\frac{q^n-1}{q^2-1}$. It was proved by Beutelspacher [3] that such a parallelism exists whenever n is a power of 2.

We will be interested in a parallelism in \mathbb{F}_q^4 , i.e. a partition of the $(q^2 + q + 1)(q^2 + 1)$ 2-subspaces of \mathbb{F}_q^4 into $q^2 + q + 1$ disjoint spreads.

We further partition, for our construction of a residual q -Fano plane, the $q^2 + q + 1$ pairwise disjoint spreads of any given parallelism into three sets \mathcal{A} , \mathcal{B} , and \mathcal{C} . The set \mathcal{A} contains one spread. The set \mathcal{B} contains q spreads, and the set \mathcal{C} contains q^2 spreads. Any partition of the $q^2 + q + 1$ spreads is appropriate for this purpose. Such a partition for \mathbb{F}_2^4 is given in Table 1.

In the construction, we have another set \mathcal{D} which contains all the $q^3 + q^2 + q + 1$ distinct 3-subspaces of \mathbb{F}_q^4 . An example for a basis of the fifteen 3-subspaces of \mathbb{F}_2^4 is given in Table 2.

Let α be a primitive element in \mathbb{F}_q . The next structure that has to be considered is a set of q^2 different matrices of size $2 \times (q + 1)$ over \mathbb{F}_q . These matrices must satisfy the following properties:

1. Let v_1, v_2, \dots, v_{q+1} be the $q + 1$ consecutive columns of such a matrix. For each i , $3 \leq i \leq q + 1$, $v_i = \alpha^{i-3}v_1 + v_2$ (a scalar β is multiplied by each element of a vector v in the product βv , and the vector addition $v_1 + v_2$ is performed element by element in \mathbb{F}_q).
2. The set of q^2 matrices form a linear subspace of dimension two over \mathbb{F}_q .

Table 1: Partition of the 2-subspaces of \mathbb{F}_2^4 into the sets \mathcal{A} , \mathcal{B} , and \mathcal{C}

\mathcal{A}	000	011	011	011	011														
	011	011	101	101	000														
	011	101	011	000	101														
	101	000	011	101	110														
\mathcal{B}	000	011	011	011	011	011	000	011	011	011									
	011	000	101	101	011	101	011	101	011	000									
	101	101	011	101	000	000	101	110	011	101									
	000	101	101	110	101	000	110	101	101	011									
\mathcal{C}	011	011	011	011	000	011	011	011	000	011	011	011	011	000	011	011	000	011	011
	101	101	011	000	011	101	011	000	011	101	101	101	101	000	011	101	011	000	101
	110	011	101	000	101	101	101	011	101	000	110	000	011	101	011	101	101	000	101
	000	110	011	101	101	000	101	101	011	011	011	110	000	101	101	110	011	101	000

Table 2: A basis for each one of the fifteen 3-subspaces of the set \mathcal{D} for \mathbb{F}_2^4

\mathcal{Y}_1	\mathcal{Y}_2	\mathcal{Y}_3	\mathcal{Y}_4	\mathcal{Y}_5	\mathcal{Y}_6	\mathcal{Y}_7	\mathcal{Y}_8	\mathcal{Y}_9	\mathcal{Y}_{10}	\mathcal{Y}_{11}	\mathcal{Y}_{12}	\mathcal{Y}_{13}	\mathcal{Y}_{14}	\mathcal{Y}_{15}
000	001	001	010	010	010	010	010	010	010	010	010	010	010	010
010	010	010	011	010	011	100	100	100	100	100	100	100	000	001
011	010	011	100	100	100	010	011	011	000	001	001	100	100	100
100	100	100	000	001	001	011	011	010	101	100	101	111	110	111

3. For each i , $1 \leq i \leq q + 1$, the q^2 i -th column vectors in the q^2 matrices are all distinct, i.e. they consist of all possible q^2 column vectors of length 2.

Since these q^2 matrices form a linear subspace, it follows that their union is a linear code. In the sequel, this code will be called the *extension code*.

Lemma 4. *For each power of a prime q there exists an extension code.*

Proof. We start with two $2 \times (q + 1)$ matrices over \mathbb{F}_q which will be the basis of the code.

For the first matrix M_1 the first column will be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second column will be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The i -th column, $3 \leq i \leq q + 1$, is $\alpha^{i-3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^{i-3} \\ 1 \end{pmatrix}$.

For the second matrix M_2 the first column will be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the second column will be $\begin{pmatrix} 1 \\ \beta \end{pmatrix}$, where $\beta \in \mathbb{F}_q$. The i -th column, $3 \leq i \leq q + 1$, is $\alpha^{i-3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha^{i-3} \\ 1 + \beta \end{pmatrix}$. We have to prove that there exists a $\beta \in \mathbb{F}_q$ such that the requirements for the extension code are satisfied.

For this proof we form a $(q + 1) \times (q + 1)$ matrix \mathcal{M} whose first row consists of the columns of the matrix M_1 in their given order. The other rows are indexed by the elements of \mathbb{F}_q . The row which are indexed by $\beta \in \mathbb{F}_q$ has $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the first entry and $\begin{pmatrix} 1 \\ \beta \end{pmatrix}$ in the second entry. The i -th entry, $3 \leq i \leq q + 1$, will be $\begin{pmatrix} \alpha^{i-3} \\ 1 + \beta \end{pmatrix}$. It is easy to verify that the $q \times q$ sub-matrix \mathcal{M}' of \mathcal{M} defined by removing the first row and first column of \mathcal{M} is a Latin square (each row and each column is a permutation of the q column vectors $\begin{pmatrix} 1 \\ \beta \end{pmatrix}$, $\beta \in \mathbb{F}_q$). For each i , $3 \leq i \leq q + 1$, the element in the i -th entry of the first row of \mathcal{M}

appears in the linear span of the i -th entry of exactly one row of \mathcal{M}' . Since \mathcal{M}' has q rows, it follows that there exists at least one row which share no linearly dependent entry with the first row of \mathcal{M} . The β of such a row is the required β for M_2 .

The two matrices M_1 and M_2 are linearly independent. In fact, for each i , $1 \leq i \leq q+1$, the i -th columns of the two matrices are linearly independent. Hence, the linear span of M_1 and M_2 form a linear subspace of dimension two and for the each i , $1 \leq i \leq q+1$, the i -th columns of all matrices in the code are distinct. \square

Next, we consider all matrices which are candidates for the extension code. This set of candidates consists of all the q^4 distinct $2 \times (q+1)$ matrices over \mathbb{F}_q . If v_1, v_2, \dots, v_{q+1} are the $q+1$ consecutive columns of such a matrix, then for each i , $3 \leq i \leq q+1$, $v_i = \alpha^{i-3}v_1 + v_2$. This set of matrices is clearly a linear subspace which will be called the *extension space*. Since the entries of the first two column vectors can be chosen arbitrarily, it follows that there are q^4 matrices in the extension space. Moreover, these q^4 matrices form a linear subspace of dimension four over \mathbb{F}_q . Since the extension code is a linear subspace of dimension two of the extension space, it follows that we can partition the q^4 matrices of the extension space into q^2 sets of size q^2 having the following properties:

1. The extension code is the first set.
2. Let v_1, v_2, \dots, v_{q+1} be the $q+1$ consecutive columns of any matrix in any of the codes. For each i , $3 \leq i \leq q+1$, $v_i = \alpha^{i-3}v_1 + v_2$.
3. For each i , $1 \leq i \leq q+1$, the q^2 i -th column vectors in the q^2 matrices, of any of the q^2 sets, are all distinct, i.e. they consist of all possible column vectors of length 2.

An example for an extension space (the extension code and its coset) is given in Table 3.

Table 3: The extension space for $q = 2$ (C is the code and C_i , $i = 1, 2, 3$, are its cosets)

C	000	011	101	110	C_1	000	011	101	110	C_2	110	101	011	000	C_3	110	101	011	000
	000	110	011	101		110	000	101	011		000	110	011	101		110	000	101	011

The construction of the residual q -Fano plane will start from sets of subspaces from \mathbb{F}_q^4 . The subspaces of these sets will be extended in various ways to 3-subspaces of \mathbb{F}_q^6 , in a way that all these extensions will result in the residual q -Fano plane. The extension space will have an important role in these extensions as will be explained in Sections 4 and 6. The methods in which subspaces are extended is explained in Section 4.

We end this section with a connection between the subspaces of \mathcal{A} and the subspaces of the set \mathcal{D} .

Lemma 5. *A 2-subspace X of \mathbb{F}_q^4 can be expanded in $q+1$ distinct ways to a 3-subspace of \mathbb{F}_q^4 .*

Proof. A 2-subspace X has $q + 1$ pairwise linearly independent vectors. \mathbb{F}_q^4 has $\frac{q^4-1}{q-1} = q^3+q^2+q+1$ pairwise linearly independent vectors. Each one of the q^3+q^2 pairwise linearly independent vectors not in X can be used to form a 3-subspace of \mathbb{F}_q^4 . Each 3-subspace contains $\frac{q^3-1}{q-1} = q^2 + q + 1$ pairwise linearly independent vectors, i.e. q^2 additional vectors to X . Each one of them will form the same 3-subspace when appended to X . Hence X can be expanded in $\frac{q^3+q^2}{q^2} = q + 1$ distinct ways to a 3-subspace of \mathbb{F}_q^4 . \square

Lemma 6. *Each 3-subspace of \mathbb{F}_q^4 (also of \mathcal{D}) contains a unique 2-subspace of the set \mathcal{A} .*

Proof. If $X \in \mathcal{A}$ and $v \in \mathbb{F}_q^4$ is a vector such that $v \notin \mathcal{A}$, then $Y \stackrel{\text{def}}{=} \langle X \cup \{v\} \rangle$ is clearly a 3-subspace of \mathbb{F}_q^4 . Since Y is a 3-subspace and all the 2-subspaces of \mathcal{A} are pairwise disjoint, it follows that Y cannot contain two 2-subspaces of \mathcal{A} .

There are $q + 1$ different 3-subspaces which contain X , $q^2 + 1$ different 2-subspaces in \mathcal{A} , and hence there are $(q^2 + 1)(q + 1) = \frac{q^4-1}{q-1}$ 3-subspaces which contain 2-subspaces from \mathcal{A} . The total number of different 3-subspaces of \mathbb{F}_q^4 is $\frac{q^4-1}{q-1}$. It implies that each 3-subspace of \mathbb{F}_q^4 contains a unique 2-subspace of the set \mathcal{A} . \square

4 Representation of Subspaces

The construction of the derived q -Fano plane and the residual q -Fano plane will be presented in Section 6. The construction will start with subspaces from \mathbb{F}_q^4 which consists of the unique 0-subspace of \mathbb{F}_q^4 and the subspaces of the sets \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} . These subspaces will be extended and/or expanded to 2-subspaces in \mathbb{F}_q^6 for the derived q -Fano plane, and to 3-subspaces in \mathbb{F}_q^6 for the residual q -Fano plane. Most of these extensions will be performed with the extension space and hence the representations of these subspaces and the matrices of the extension space must be matched in their representation to make sure that the outcome will be subspaces with the required properties. To make these extensions and/or expansions simple to explain we will use certain representations of 2-subspaces and 3-subspaces of \mathbb{F}_q^4 , and 2-subspaces and 3-subspaces of \mathbb{F}_q^6 . These representations will also help to verify the correctness of the construction. For these representations we form an order between the vectors of length 4 of \mathbb{F}_q^4 . For simplicity we will use the standard lexicographic order from the smallest to the largest element.

In the representations which follows we will take only one of the $q - 1$ different vectors from which any two are linearly dependent, i.e., $q + 1$ vectors for a 2-subspace and $q^2 + q + 1$ vectors for a 3-subspace. W.l.o.g. (without loss of generality) the vectors which will be taken will always be those whose first nonzero element is a *one*.

Representation of 2-Subspaces of \mathbb{F}_q^r , $r \in \{4, 6\}$:

A 2-subspace X of \mathbb{F}_q^r will be presented by an $r \times (q + 1)$ matrix M and an *expanded representation* by an $r \times (q^2 + q + 1)$ matrix $E(M)$ (or $E(X)$) as follows. The first $q + 1$ columns of the matrices (M and $E(M)$) will be the $q + 1$ vectors of length r of X , where each two columns are linearly independent (let us denote these $q + 1$ columns by Y), with the following two properties:

- Any two columns of the $4 \times (q + 1)$ matrix defined by the first 4 rows and the first $q + 1$ columns of Y are linearly independent, and hence form a basis for Y .
- Let $v_1 v_2 \dots v_{q+1}$ be the consecutive columns of the matrix defined by the first four rows and the first $q + 1$ columns of Y . The first two columns are the smallest among the $q + 1$ columns in the given lexicographic order and $v_1 < v_2$. Furthermore, $v_i = \alpha^{i-3} v_1 + v_2$, $3 \leq i \leq q + 1$.

This completes the definition of M . For the definition of $E(M)$, the next column (the $(q + 2)$ -th column) will be an all-zero column. The next (and last) $(q - 1)(q + 1)$ columns will consist of $q - 1$ identical copies of Y .

Any 2-subspace which cannot be represented in this way will not be considered for this representation (These are 2-subspaces of \mathbb{F}_q^6 which have vectors starting with four zeroes.).

The 2-subspaces in Table 1 are represented by this definition.

Representation of 3-Subspaces of \mathbb{F}_q^r , $r \in \{4, 6\}$:

A 3-subspace X of \mathbb{F}_q^r (obtained from \mathcal{D}) will be presented by an $r \times (q^2 + q + 1)$ matrix M as follows. The first $q + 1$ columns of M will be the $q + 1$ vectors of length r of a 2-subspace of X , where each two columns are linearly independent (let us denote these $q + 1$ columns by Y), with the following two properties:

- The first $q + 1$ columns of the $4 \times (q + 1)$ matrix defined by the first 4 rows and the first $q + 1$ columns of Y represent a 2-subspace of \mathcal{A} , whose existence is guaranteed by Lemma 6.
- Let $v_1 v_2 \dots v_{q+1}$ be the consecutive columns of the matrix defined by the first four rows and the first $q + 1$ columns of Y . The first two columns are the smallest among the $q + 1$ columns in the given order and $v_1 < v_2$. Furthermore, $v_i = \alpha^{i-3} v_1 + v_2$, $3 \leq i \leq q + 1$.

The next column of M (the $(q + 2)$ -th column) will be a non-zero column vector v of length r linearly independent of the first $q + 1$ columns of M (or Y). It will be taken as the smallest vector, in the lexicographic order, among the other columns of X . The next $(q - 1)(q + 1)$ columns of M will consist of $q - 1$ $r \times (q + 1)$ matrices, where the i -th matrix, $0 \leq i \leq q - 2$, is $\alpha^i v + Y$ (the addition of a column vector v of length r to an $r \times m$ matrix Y is done by adding v to each column of Y). Hence, any two of the first $q + 1$ columns with the $(q + 2)$ -th column form a basis for the 3-subspace.

After describing the representations of 2-subspaces and 3-subspaces, we are in a position to describe how we extend and expand a subspace in \mathbb{F}_q^4 to a subspace in \mathbb{F}_q^6 , while keeping these representations. To make these extensions and expansions simple, we will give a few properties of our representations whose proofs are trivial. First let u_i (u'_i), $1 \leq i \leq q^2 + q + 1$, be the i -th column in the representation of two distinct subspaces.

Lemma 7. *In the representation of a 3-subspace u_1, u_2, u_{q+2} are linearly independent.*

Lemma 8. *If for a given 3-subspace and $1 \leq i < j < k \leq q^2 + q + 1$ we have $\gamma_i u_i + \gamma_j u_j + \gamma_k u_k = 0$, where $\gamma_i, \gamma_j, \gamma_k \in \mathbb{F}_q$, then for another subspace (of dimension two or three) we have $\gamma_i u'_i + \gamma_j u'_j + \gamma_k u'_k = 0$.*

Lemma 9. *Any 2-subspace X of a 3-subspace Y contains either all the $q + 1$ first columns of Y or exactly one of the first $q + 1$ columns of Y .*

Lemma 10. *There exists a set \mathcal{P} which contains $q^2 + q + 1$ subsets of $\{1, 2, \dots, q^2 + q + 1\}$, each subset is of size $q + 1$, such that the columns of the $q^2 + q + 1$ 2-subspaces of any $r \times (q^2 + q + 1)$ matrix M , $r \in \{4, 6\}$, which represents a 3-subspace, are exactly on the coordinates of the subsets of \mathcal{P} .*

5 Extensions and Expansions of Subspaces

The construction of the derived q -Fano plane and the residual q -Fano plane will start with 2-subspaces and 3-subspaces of \mathbb{F}_q^4 . They will be extended and possibly expanded to 3-subspaces of \mathbb{F}_q^6 . We start with a formal definition of the expansion, which was mentioned before in the representation $E(X)$ of a 2-subspace X .

The expansion $E(M, u)$ of an $r \times (q + 1)$ matrix M , having columns v_1, v_2, \dots, v_{q+1} , with a column vector u of length r to an $r \times (q^2 + q + 1)$ matrix as follows. The next column v_{q+2} is u , and the next $q^2 - 1$ columns consists of $q - 1$ $r \times (q + 1)$ matrices, where the i -th matrix is $\alpha^i u + M$. We note that if M represent a 2-subspace X and u is linearly independent in the columns of X (i.e. M) then $E(M, u)$ represent a 3-subspace. If M represents a 2-subspaces we can write $E(X, u)$ instead of $E(M, u)$.

The following simple lemmas which were also proved in [15] provide some of the foundations for the extensions (with possible expansions).

Lemma 11. *Each 2-subspace in \mathbb{F}_q^r has exactly q^2 distinct extensions to a 2-subspace in \mathbb{F}_q^{r+1} .*

Lemma 12. *Each 2-subspace in \mathbb{F}_q^r has a unique extension (with expansion) to a 3-subspace in \mathbb{F}_q^{r+1} .*

Lemma 13. *Each 3-subspace in \mathbb{F}_q^r has exactly q^3 distinct extensions to a 3-subspace in \mathbb{F}_q^{r+1} .*

Lemma 14. *Each 2-subspace in \mathbb{F}_q^4 has exactly q^4 distinct extensions to a 2-subspace in \mathbb{F}_q^6 . Each such extension is done by a different $2 \times (q + 1)$ matrix of the extension space.*

In the extensions with possible expansions required in our construction, these lemmas are implemented as follows.

Extension of a 2-subspace from \mathbb{F}_q^4 to a 2-subspace of \mathbb{F}_q^6 :

Let X be any 2-subspace of \mathbb{F}_q^4 which is going to be extended to a 2-subspace of \mathbb{F}_q^6 . This extension can be done in two steps:

1. Choose a $2 \times (q + 1)$ matrix Z from the extension space.
2. Form a $6 \times (q + 1)$ representation matrix for a 2-subspace whose first four rows is the $4 \times (q + 1)$ matrix representation of X and last two rows is Z .

Lemma 15. *A 2-subspace X of \mathbb{F}_q^4 can be extended in q^4 distinct ways to a 2-subspace of \mathbb{F}_q^6 .*

Proof. There are q^4 distinct ways to choose an extension matrix Z from the extension space. Each one yields a different 2-subspace of \mathbb{F}_q^6 and all extensions can be formed in this way. \square

Extension of a 2-subspace from \mathbb{F}_q^4 to a 3-subspace of \mathbb{F}_q^6 :

There are two distinct ways to extend a 2-subspace X of \mathbb{F}_q^4 to a 3-subspace of \mathbb{F}_q^6 .

One way is to extend X first to one of the q^2 distinct 2-subspaces of \mathbb{F}_q^5 and then use a unique extension (with expansion) to a 3-subspace of \mathbb{F}_q^6 . This is done by extending X to a 2-subspace \tilde{X} of \mathbb{F}_q^6 by appending to X any one of the q^2 matrices of the extension space whose second row is an all-zero row. The unique 3-subspace of \mathbb{F}_q^6 is obtain by expanding X with e_6 , the unit vector of length 6 with the *one* in the last position. Hence, the final 3-subspace is $E(\tilde{X}, e_6)$. Therefore, there are q^2 distinct ways for this extension (with expansion).

The second way is to extend X in a unique way (with expansion) to a 3-subspace of \mathbb{F}_q^5 . The 3-subspace can be extended in q^3 distinct ways to a 3-subspace of \mathbb{F}_q^6 . This is done first by appending to X an all-zero row and expand is with e_5 , the unit vector of length 5 with the *one* in the last position. There are q^3 ways to extend the 3-subspace of \mathbb{F}_q^5 to a 3-subspace of \mathbb{F}_q^6 . This is done either by using any of the q^3 linear combinations of the first five rows to form the 6-th row, or by taking any of the q^3 assignments from \mathbb{F}_q to positions 1, 2, and $q + 2$, and the other positions are fixed by the linear combinations of the other columns.

Lemma 16. *A 2-subspace X of \mathbb{F}_q^4 can be extended in $q^3 + q^2$ distinct ways to a 3-subspace of \mathbb{F}_q^6 .*

Extension of a 3-subspace from \mathbb{F}_q^4 to a 3-subspace of \mathbb{F}_q^6 :

Let Y be any 3-subspace of \mathbb{F}_q^4 which is going to be extended to a 3-subspace of \mathbb{F}_q^6 . This extension can be done in four steps:

1. Choose a $2 \times (q + 1)$ matrix Z from the extension space.
2. Choose a column vector u of length two over \mathbb{F}_q .
3. Form the $2 \times (q^2 + q + 1)$ expansion $E(Z, u)$.
4. Form a $6 \times (q^2 + q + 1)$ representation matrix for a 3-subspace whose first four rows is the matrix representation of Y and last two rows is $E(Z, u)$.

Lemma 17. *A 3-subspace Y of \mathbb{F}_q^4 can be extended in q^6 distinct ways to a 3-subspace of \mathbb{F}_q^6 .*

Proof. There are q^4 distinct ways to choose an extension matrix Z from the extension space and q^2 way to choose the vector u for $E(Z, u)$. Each such choice will yield a different 3-subspace of \mathbb{F}_q^6 since the process starts with a 3-subspace. To complete the proof we note that each extension can be formed in this way. \square

6 Construction of Residual q -Fano Planes

The construction of the derived q -Fano plane and the residual q -Fano plane is based on extensions and possible expansion of the subspaces in the sets \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} , which contain 2-subspaces and 3-subspaces of \mathbb{F}_q^4 into 3-subspaces in \mathbb{F}_q^6 . These extensions and/or expansions and the extension of the null-subspace of \mathbb{F}_q^4 will form the residual q -Fano plane.

There is one possible way to form a 2-subspace of \mathbb{F}_q^6 whose first four rows in the matrix representation corresponds to the 0-subspace of \mathbb{F}_q^4 . The set of size one which contains this 2-subspace will be denoted by \mathbb{S}_0 .

Extension of Type A:

The set \mathcal{A} of 2-subspaces of \mathbb{F}_q^4 contains $q^2 + 1$ subspaces. Each one is extended in the q^2 possible distinct ways, based on the extension code C , to a 2-subspace in \mathbb{F}_q^6 . The result is a set with $q^4 + q^2$ distinct 2-subspaces of \mathbb{F}_q^6 . This set will be denoted by $\mathbb{S}_{\mathcal{A}}$.

Lemma 18. *The set $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$ is a spread in \mathbb{F}_q^6 .*

Proof. The set \mathcal{A} is a spread in \mathbb{F}_q^4 by definition. The extension based on C is a 2-subspace in \mathbb{F}_q^6 . A spread in \mathbb{F}_q^6 contains $\frac{q^6-1}{q^2-1} = q^4 + q^2 + 1$ disjoint 2-subspaces. \mathbb{S}_0 has one 2-subspace and $\mathbb{S}_{\mathcal{A}}$ contains $q^4 + q^2$ 2-subspaces. Hence, to complete the proof it is sufficient to prove that no nonzero vector of \mathbb{F}_q^6 appears more than once in a subspaces of $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$. Assume a vector $v \in \mathbb{F}_q^6$ appears in two such subspaces. Let $v' \in \mathbb{F}_q^4$ be the prefix vector of length 4 obtained from v . By the definition of \mathcal{A} we have that v' is either the all-zero vector or it is contained in a unique 2-subspace of \mathcal{A} . If v' is the all-zero vector then v is contained only in the unique subspace of \mathbb{S}_0 . If v' is contained in a unique 2-subspace X of \mathcal{A} , then by the definition of the extension code C , each one of the q^2 extensions of X with the extension code C appends a different suffix of length two to v' and hence v cannot appear more than once. \square

Table 4 presents the 21 2-subspaces of $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$ for $q = 2$. The first four rows in the matrix representation is a 2-subspace of $\mathbb{S}_{\mathcal{A}}$ and the last two rows are taken from the extension code.

Extension of Type B:

The set \mathcal{B} of 2-subspaces of \mathbb{F}_q^4 contains q spreads with a total of $q(q^2 + 1)$ subspaces. Each one is extended in all possible q^2 distinct ways to a 2-subspace in \mathbb{F}_q^6 . Each such

Table 4: the 2-subspaces of $\mathbb{S}_0 \cup \mathbb{S}_A$ for $q = 2$

000	000	011	011	011	011	000	011	011	011	011	000	011	011	011	011	000	011	011	011	011
000	011	011	101	101	000	011	011	101	101	000	011	011	101	101	000	011	011	101	101	000
000	011	101	011	000	101	011	101	011	000	101	011	101	011	000	101	011	101	011	000	101
000	101	000	011	101	110	101	000	011	101	110	101	000	011	101	110	101	000	011	101	110
011	000	000	000	000	000	011	011	011	011	011	101	101	101	101	101	110	110	110	110	110
101	000	000	000	000	110	110	110	110	110	011	011	011	011	011	101	101	101	101	101	101

2-subspace of \mathbb{F}_q^5 is extended in a unique way to a 3-subspace in \mathbb{F}_q^6 . The result is a set with $q^3(q^2 + 1)$ distinct 3-subspaces of \mathbb{F}_q^6 . This set will be denoted by \mathbb{S}_B . Table 5 presents the forty 3-subspaces of \mathbb{S}_B for $q = 2$. Note that the third vector in all the subspaces is the same.

Table 5: A basis for each one of the forty 3-subspaces of \mathbb{S}_B for $q = 2$

000	000	000	000	010	010	010	010	010	010	010	010	010	010	010	010	010	010	010	010	010
010	010	010	010	000	000	000	000	100	100	100	100	100	100	100	100	100	100	100	100	100
100	100	100	100	100	100	100	100	010	010	010	010	100	100	100	100	000	000	000	000	000
000	000	000	000	100	100	100	100	100	100	100	100	110	110	110	110	100	100	100	100	100
000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110	000
001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001
010	010	010	010	000	000	000	000	010	010	010	010	010	010	010	010	010	010	010	010	010
100	100	100	100	010	010	010	010	100	100	100	100	010	010	010	010	000	000	000	000	000
000	000	000	000	100	100	100	100	110	110	110	110	010	010	010	010	100	100	100	100	100
000	000	000	000	110	110	110	110	100	100	100	100	100	100	100	100	010	010	010	010	010
000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110	000
001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001

Extension of Type C:

The set \mathcal{C} of 2-subspaces of \mathbb{F}_q^4 contains q^2 spreads, each one has $q^2 + 1$ subspaces. We further partition \mathcal{C} into q subsets \mathcal{C}_ξ , $\xi \in \mathbb{F}_q$, where \mathcal{C}_ξ contains q spreads.

For each ξ , $\xi \in \mathbb{F}_q$, the set \mathcal{C}_ξ of 2-subspaces of \mathbb{F}_q^4 contains $q(q^2 + 1)$ subspaces. Each one is extended in a unique way to a 3-subspace in \mathbb{F}_q^5 . Each such 3-subspace of \mathbb{F}_q^5 has q^3 extensions to a 3-subspace in \mathbb{F}_q^6 . Let $\mathbb{S}_{\mathcal{C}_\xi}$ be the set of these 3-subspaces which have ξ in the 6-th row of the $(q + 2)$ -th column of the matrix representation. This set $\mathbb{S}_{\mathcal{C}_\xi}$ contains $q^3(q^2 + 1)$ distinct 3-subspaces of \mathbb{F}_q^6 since there are q^2 distinct ways to choose the pair of symbols in the sixth row for the first two linearly independent vectors of the 3-subspace. If $\mathbb{S}_{\mathcal{C}} \stackrel{\text{def}}{=} \cup_{\xi \in \mathbb{F}_q} \mathbb{S}_{\mathcal{C}_\xi}$ then clearly $\mathbb{S}_{\mathcal{C}}$ contains $q^4(q^2 + 1)$ distinct 3-subspaces.

Table 6 presents the eighty 3-subspaces of $\mathbb{S}_{\mathcal{C}}$ for $q = 2$, where the first two spreads in Table 1 are taken as \mathcal{C}_0 and the other two spreads form \mathcal{C}_1 . Note, that the third vector in the basis of the subspaces from $\mathbb{S}_{\mathcal{C}_0}$ and the one from $\mathbb{S}_{\mathcal{C}_1}$ differ exactly in the last entry.

Table 6: A basis for each one of the eighty 3-subspaces of $\mathbb{S}_{\mathcal{C}}$ for $q = 2$

$\mathbb{S}_{\mathcal{C}_0}$	010	010	010	010	010	010	010	010	010	010	010	010	010	010	010	000	000	000	000	
	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	010	010	010	010	
	110	110	110	110	010	010	010	010	100	100	100	100	000	000	000	000	100	100	100	100
	000	000	000	000	110	110	110	110	010	010	010	010	100	100	100	100	100	100	100	100
	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001
	000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110
$\mathbb{S}_{\mathcal{C}_0}$	010	010	010	010	010	010	010	010	010	010	010	010	010	010	010	000	000	000	000	
	100	100	100	100	010	010	010	010	000	000	000	000	010	010	010	010	100	100	100	100
	100	100	100	100	100	100	100	100	010	010	010	010	100	100	100	100	000	000	000	000
	000	000	000	000	100	100	100	100	100	100	100	100	100	010	010	010	010	010	010	010
	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001
	000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110	000	010	100	110
$\mathbb{S}_{\mathcal{C}_1}$	010	010	010	010	010	010	010	010	010	010	010	010	010	010	010	000	000	000	000	
	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	000	000	000	000	
	110	110	110	110	000	000	000	000	010	010	010	010	100	100	100	100	010	010	010	010
	010	010	010	010	110	110	110	110	000	000	000	000	100	100	100	100	100	100	100	100
	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001
	001	011	101	111	001	011	101	111	001	011	101	111	001	011	101	111	001	011	101	111
$\mathbb{S}_{\mathcal{C}_1}$	010	010	010	010	010	010	010	000	000	000	000	010	010	010	010	010	010	010	010	
	010	010	010	010	100	100	100	100	010	010	010	010	000	000	000	000	100	100	100	100
	100	100	100	100	100	100	100	100	000	000	000	000	100	100	100	100	110	110	110	110
	110	110	110	110	010	010	010	010	100	100	100	100	000	000	000	000	110	110	110	110
	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001
	001	011	101	111	001	011	101	111	001	011	101	111	001	011	101	111	001	011	101	001

Extension of Type D:

First, we partition the cosets of the extension code C (all the extension space excluding C) into $q + 1$ parts, C_1, C_2, \dots, C_{q+1} , each one contains $q - 1$ cosets with q^2 matrices, i.e. C_j , $1 \leq j \leq q + 1$, contains $q^2(q - 1)$ matrices.

The set \mathcal{D} of 3-subspaces of \mathbb{F}_q^4 has size $q^3 + q^2 + q + 1$. By Lemma 6 each 3-subspace of \mathcal{D} contains a unique 2-subspace from \mathcal{A} . By Lemma 5 for a given such 2-subspace $X \in \mathcal{A}$ there are $q + 1$ different 3-subspaces of \mathbb{F}_q^4 which contain X (expanded from X , the first vector is defined by the lexicographic order). Let Y_1, Y_2, \dots, Y_{q+1} be the $q + 1$ subspaces of \mathcal{D} which contain X , where $Y_j \stackrel{\text{def}}{=} E(X, u)$ for a column vector $u \in \mathbb{F}_q^4$.

For any j , $1 \leq j \leq q + 1$, we extend the 3-subspace Y_j using C_j as follows. For each $2 \times (q + 1)$ matrix Z from the $q^2(q - 1)$ matrices of C_j and for each column vector v of length 2 from \mathbb{F}_q^2 we form the expanded representation $E(Z, v)$. Y_j is extended with $E(Z, v)$, i.e. the new 3-subspace is represented by a $6 \times (q^2 + q + 1)$ matrix whose first four rows is the matrix representation of Y_j and the last two rows are $E(Z, v)$. The result is a set $\mathbb{S}_{X,j}$ which contains $q^4(q - 1)$ distinct 3-subspaces ($q^2(q - 1)$ matrices in C_j , where each matrix is expanded with q^2 vectors of length 2).

The set of all 3-subspaces formed from the 2-subspace $X \in \mathcal{A}$ will be denoted by \mathbb{S}_X and its size is $q^4(q - 1)(q + 1) = q^4(q^2 - 1)$. The set of all 3-subspaces formed from \mathcal{D} will

be denoted by $\mathbb{S}_{\mathcal{D}}$ and its size is $q^4(q^2 - 1)(q^2 + 1) = q^4(q^4 - 1)$ since the size of \mathcal{A} (from which X was taken) is $q^2 + 1$.

Tables 7 and 8 present first forty eight 3-subspaces of $\mathbb{S}_{\mathcal{D}}$ for $q = 2$, where X is taken as the first 2-subspace of \mathcal{A} in Table 1 and the cosets of the extension code are taken from Table 3. The 3-subspaces $Y_1 = \mathcal{Y}_1$, $Y_2 = \mathcal{Y}_2$, and $Y_3 = \mathcal{Y}_3$ are presented first with their basis as in Table 2 and after that with their matrix representation. Note, that the first four rows of the 3-subspaces in Table 8 form the matrix representation of Y_1 , Y_2 , and Y_3 . The other 192 3-subspaces of $\mathbb{S}_{\mathcal{D}}$ are presented in Tables 9, 10, 11, and 12.

Table 7: The three 3-subspaces of the set \mathcal{D} which contain the first 2-subspace of \mathcal{A}

\mathcal{Y}_1	\mathcal{Y}_2	\mathcal{Y}_3	Y_1	Y_2	Y_3
000	001	001	0000000	0001111	0001111
010	010	010	0110011	0110011	0110011
011	010	011	0111100	0110011	0111100
100	100	100	1010101	1010101	1010101

Let

$$\mathbb{S} \stackrel{\text{def}}{=} \mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}} \cup \mathbb{S}_{\mathcal{B}} \cup \mathbb{S}_{\mathcal{C}} \cup \mathbb{S}_{\mathcal{D}} .$$

A simple algebraic computation leads to

Lemma 19.

$$|\mathbb{S}| = |\mathbb{S}_0| + |\mathbb{S}_{\mathcal{A}}| + |\mathbb{S}_{\mathcal{B}}| + |\mathbb{S}_{\mathcal{C}}| + |\mathbb{S}_{\mathcal{D}}| = \frac{\begin{bmatrix} 7 \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q} .$$

By Lemma 19, the number of subspaces in the sets \mathbb{S} is the same as the number of 3-subspaces in a q -Fano plane. Recall, that by Lemma 18 we have that $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$ is a spread. Hence, to show that $\mathbb{S} \setminus (\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}})$ is a residual q -Fano plane it is sufficient to prove that either each 2-subspace of \mathbb{F}_q^6 which is not contained in $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$ is contained in at least q^2 subspaces of \mathbb{S} , or each 2-subspace of \mathbb{F}_q^6 which is not contained in $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$ is contained in at most q^2 subspaces of \mathbb{S} .

Lemma 20. *Each 2-subspace of \mathbb{F}_q^6 which can be extended from a 2-subspace of \mathcal{A} , but not extended to a 2-subspace of the spread $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$, is contained q^2 times in the 3-subspaces of $\mathbb{S}_{\mathcal{D}}$.*

Proof. Since the 2-subspaces of \mathcal{A} are extended only to the spread of $\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$, it follows that any 2-subspaces of \mathbb{F}_q^6 which can be extended from a 2-subspace of \mathcal{A} was formed by extending the 3-subspaces of \mathcal{D} . By Lemma 6 each 2-subspace of \mathcal{A} is contained in $q + 1$ distinct 3-subspaces of \mathcal{D} . By the definition for the representation of 3-subspaces, this 2-subspace appear in the first four rows and the first $q + 1$ column of the 3-subspace representation. Let $X \in \mathcal{A}$ and let Y_1, Y_2, \dots, Y_{q+1} be the $q + 1$ subspaces of \mathcal{D} which contain X . By the extensions of \mathcal{D} , each matrix Z of the extension space, which is not part of the extension code, is used q^2 times to extend X , using the $2 \times (q^2 + q + 1)$ matrices

Table 8: Extensions of Type D with $Y_1 = \mathcal{Y}_1$, $Y_2 = \mathcal{Y}_2$, and $Y_3 = \mathcal{Y}_3$

Y_1, C_1	0000000	0000000	0000000	0000000	0000000	0000000	0000000	0000000
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	0111100	0111100	0111100	0111100	0111100	0111100	0111100	0111100
	1010101	1010101	1010101	1010101	1010101	1010101	1010101	1010101
	0000000	0000000	0001111	0001111	0110011	0110011	0111100	0111100
	1100110	1101001	1100110	1101001	0000000	0001111	0000000	0001111
Y_1, C_1	0000000	0000000	0000000	0000000	0000000	0000000	0000000	0000000
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	0111100	0111100	0111100	0111100	0111100	0111100	0111100	0111100
	1010101	1010101	1010101	1010101	1010101	1010101	1010101	1010101
	1010101	1010101	1011010	1011010	1100110	1100110	1101001	1101001
	1010101	1011010	1010101	1011010	0110011	0111100	0110011	0111100
Y_2, C_2	0001111	0001111	0001111	0001111	0001111	0001111	0001111	0001111
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	1010101	1010101	1010101	1010101	1010101	1010101	1010101	1010101
	1100110	1100110	1101001	1101001	1010101	1010101	1011010	1011010
	0000000	0001111	0000000	0001111	1100110	1101001	1100110	1101001
Y_2, C_2	0001111	0001111	0001111	0001111	0001111	0001111	0001111	0001111
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	1010101	1010101	1010101	1010101	1010101	1010101	1010101	1010101
	0110011	0110011	0111100	0111100	0000000	0000000	0001111	0001111
	0110011	0111100	0110011	0111100	1010101	1011010	1010101	1011010
Y_3, C_3	0001111	0001111	0001111	0001111	0001111	0001111	0001111	0001111
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	0111100	0111100	0111100	0111100	0111100	0111100	0111100	0111100
	1010101	1010101	1010101	1010101	1010101	1010101	1010101	1010101
	1100110	1100110	1101001	1101001	1010101	1010101	1011010	1011010
	1100110	1101001	1100110	1101001	0000000	0001111	0000000	0001111
Y_3, C_3	0001111	0001111	0001111	0001111	0001111	0001111	0001111	0001111
	0110011	0110011	0110011	0110011	0110011	0110011	0110011	0110011
	0111100	0111100	0111100	0111100	0111100	0111100	0111100	0111100
	1010101	1010101	1010101	1010101	1010101	1010101	1010101	1010101
	0110011	0110011	0111100	0111100	0000000	0000000	0001111	0001111
	1010101	1011010	1010101	1011010	0110011	0111100	0110011	0111100

$E(Z, u)$, where any column vector of length 2 over \mathbb{F}_q is used once as u . By lemma 16, these are all the possible extensions of 2-subspaces from \mathcal{A} (note, that the extensions of subspaces from \mathcal{A} with the extension code are exactly the 2-subspaces of $\mathbb{S}_{\mathcal{A}}$). \square

Lemma 21. *Each 2-subspaces of \mathbb{F}_q^6 extended from a 2-subspace of \mathcal{B} is contained exactly once in the 3-subspaces of $\mathbb{S}_{\mathcal{B}}$.*

Proof. Any 2-subspace X of \mathcal{B} is first extended in all the q^2 possible distinct ways to a 2-subspace of \mathbb{F}_q^5 . Each 2-subspace Y of these q^2 subspaces is extended in a unique way to a 3-subspace Z . Such a 3-subspace Z contains all the q^2 distinct 2-subspaces of \mathbb{F}_q^6 , extended from Y . \square

Lemma 22. *Each 2-subspaces of \mathbb{F}_q^6 extended from a 2-subspace of \mathcal{C} is contained exactly once in the 3-subspaces of $\mathcal{S}_{\mathcal{C}}$.*

Proof. Any 2-subspace X of \mathcal{C} is first extended in a unique way to a 3-subspace Z of \mathbb{F}_q^5 . Such a 3-subspace Z contains all the q^2 distinct 2-subspaces of \mathbb{F}_q^5 , extended from X . Each such 3-subspace Y is extended to q^2 (out of the q^3) distinct 3-subspaces of \mathbb{F}_q^6 . All these q^2 distinct 3-subspaces have the same symbol in the last row of the $(q+2)$ -th column, in the matrix representation, which implies that each distinct 2-subspace of \mathbb{F}_q^5 is extended in q^2 distinct ways to all possible distinct 2-subspaces of \mathbb{F}_q^6 extended from X . \square

For the next set of 2-subspaces we need one property of the extension space.

Lemma 23. *Let C' be a coset of the extension code, let M_1 and M_2 two $2 \times (q+1)$ matrices of C' , and let u_1 and u_2 two column vectors of \mathbb{F}_q^2 . Let $\{i_1, i_2, \dots, i_{q+1}\} \in \mathcal{P}$, defined in Lemma 10, and let X be a 3-subspace. If Y_1 and Y_2 are extensions of X with $E(M_1, u_1)$ and $E(M_2, u_2)$, respectively, then columns i_1, i_2, \dots, i_{q+1} of Y_1 and Y_2 define two different 2-subspaces of \mathbb{F}_q^6 unless $M_1 = M_2$ and $\{i_1, i_2, \dots, i_{q+1}\} = \{1, 2, \dots, q+1\}$ or $M_1 = M_2$ and $u_1 = u_2$.*

Proof. By the definition of the extension code, the columns of M_1 and M_2 are distinct in pairs unless $M_1 = M_2$. Hence, by Lemma 9 we infer the result in the case that $M_1 \neq M_2$. If $M_1 = M_2$ then $u_1 \neq u_2$ implies that except for the first $q+1$ columns all the columns of $E(M_1, u_1)$ and $E(M_2, u_2)$ are different in pairs and hence the result follows from Lemma 9. \square

Since each 2-subspace of either \mathcal{B} or \mathcal{C} is contained in $q^2 - 1$ 3-subspaces of \mathcal{D} , it follows as a consequence of Lemma 23 that

Lemma 24. *Each 2-subspaces of \mathbb{F}_q^6 extended from a 2-subspace of either \mathcal{B} or \mathcal{C} is contained exactly $q^2 - 1$ times in the 3-subspaces of $\mathcal{S}_{\mathcal{D}}$.*

Lemma 25. *Each 2-subspace of \mathbb{F}_q^6 which contains a vector which start with five zeroes is contained either in \mathcal{S}_0 or contained q^2 times in $\mathcal{S}_{\mathcal{B}}$.*

Proof. The unique 2-subspace in which all vectors start with four or five zeroes is contained in \mathcal{S}_0 .

Vectors which start with five zeroes are contained in \mathcal{S}_0 and in the extensions of 2-subspaces from \mathcal{B} . The reason is that the 2-subspaces of \mathcal{B} are first extended to 2-subspaces of \mathbb{F}_q^5 in q^2 distinct ways. Since \mathcal{B} contains q spreads, it follows that each nonzero vector of length 4 is contained q times in the 2-subspaces of \mathcal{B} . Since there are q^2 distinct extensions of a 2-subspace of \mathbb{F}_q^4 to a 2-subspace of \mathbb{F}_q^5 it follows that each vector of length 4 is extended with a symbol $\xi \in \mathbb{F}_q$ to a vector of length 5 exactly q times.

Hence, each nonzero vector of length 5 appears in the extensions of \mathcal{B} to 2-subspaces of \mathbb{F}_q^5 exactly q^2 times. Thus, each vector of length 6 appears exactly q^2 times in the extensions (with expansions) of \mathcal{B} to 3-subspaces in a unique way. Thus, each 2-subspace which contains a vector of length 6 starting with 5 zeroes is contained in q^2 distinct 3-subspaces of $\mathbb{S}_{\mathcal{B}}$. \square

Lemma 26. *Each 2-subspace of \mathbb{F}_q^6 which contains a vector which start with four zeroes and the 5-th symbol is nonzero, is contained either in \mathbb{S}_0 or contained q^2 times in $\mathbb{S}_{\mathcal{C}}$.*

Proof. The unique 2-subspace in which all vectors start with four zeroes is contained in \mathbb{S}_0 .

Vectors which start with four zeroes and the 5-th symbol is nonzero, are contained in \mathbb{S}_0 and in the extensions of subspaces from \mathcal{C} . Since \mathcal{C} contains q^2 spreads, it follows that each nonzero vector of length 4 is contained q^2 times in the 2-subspaces of \mathcal{C} . Hence, each vector of length 5 appears exactly q^2 times in the extensions (with expansions) of \mathcal{C} to 3-subspaces in a unique way. Thus, each 2-subspace which contains a vector of length 5 starting with 4 zeroes and 5-th nonzero, is contained in q^2 distinct 3-subspaces of \mathbb{F}_q^5 extended (and expanded) from \mathcal{C} . For each 2-subspace of \mathbb{F}_q^5 which contains a vector which starts with 4 zeroes there are q^2 distinct extensions to a 2-subspace of \mathbb{F}_q^6 . Each one is considered in the extensions of \mathcal{C}_{ξ} , $\xi \in \mathbb{F}_q$, and since each 2-subspace of \mathbb{F}_q^5 was contained q^2 times, it follows that the same is true for the 2-subspaces of \mathbb{F}_q^6 which contain a vector which start with four zeroes and the 5-th symbol is nonzero. \square

A consequence of Lemmas 19, 20, 21, 22, 24, 25, 26, we have the concluding result.

Theorem 27. *$\mathbb{S}_0 \cup \mathbb{S}_{\mathcal{A}}$ is a derived q -Fano plane and $\mathbb{S}_{\mathcal{B}} \cup \mathbb{S}_{\mathcal{C}} \cup \mathbb{S}_{\mathcal{D}}$ is a residual q -Fano plane.*

7 Conclusions and Future Research

We have presented a new definition for the residual q -design which reflects better the relations between the design on one side and its derived design and residual designs on the other hand. We have constructed designs with the parameters of the residual design of the q -Fano plane for each power of a prime q . This is the closest as was achieved until today towards a construction of infinite family of q -Steiner systems, arguably, the most intriguing open problem in block design today. Our construction is flexible which enable to construct many residual q -Fano planes for each q . The number of different residual q -Fano planes is increased with the increase of q . The first point with flexibility is the number of parallelisms in \mathbb{F}_q^4 which are generally increasing as q get larger. The number of partitions of the spreads in such a parallelism into the sets \mathcal{A} , \mathcal{B} , and \mathcal{C} , is clearly increasing as q get larger. Similarly, \mathcal{C} can be partitions in a few different ways to $\{\mathcal{C}_{\xi} : \xi \in \mathbb{F}_q\}$ and the number of such partitions is clearly increasing with q . The extension code can be chosen in a few different ways and the number of different ways is also increasing when q increases. Finally, there are many different ways to make the extensions of Type D. First, the cosets

of the extension code (the extension space without the extension code) can be partitioned in a few different ways (with an exception for $q = 2$) to C_1, C_2, \dots, C_{q+1} and these number of different ways is clearly increasing with the increase of q . The matching of the pairs (Y_i, C_i) , for the extension of Type D, can be done in $(q + 1)!$ different ways and this can be done for each spread in \mathcal{A} . Hence, we have many different residual q -Fano planes for each q and each one might have different properties and can be used for different purpose. This is a subject for future research. In particular one can find different residual q -Fano planes which differ in a small number of subspaces (by using pairs in the extensions of Type D which differ only in one transposition). One can easily verify that the structure obtained from the dual subspaces of the subspaces in a residual q -Fano plane is also a residual q -Fano plane. This can lead to other interesting properties of the q -Fano plane and this is a topic for future research. Finally, an applications of the new structure in network coding is presented in [19].

The new construction and the new structure open also a sequence of other directions for future research, for which we list a few:

- Provide more constructions for residual q -Steiner systems with other parameters.
- Can a residual q -Steiner system exists, while a related q -Steiner system does not exist? We conjecture that the answer is positive.
- Prove that the residual q -Fano plane constructed can be extended or cannot be extended to a q -Fano plane. For $q = 2$ it cannot be extended, while for some $q > 2$ such an extension might be possible.
- For $q = 2$ the stabiliser of the the constructed 3-dimensional subspaces is trivial. Is this is the same for all q ?
- Examine the properties of the residual q -Steiner systems with respect to the possible existence of a related q -Steiner systems.

Finally, we note that the subspaces used throughout the construction can be represented by their basis and the same is true for the construction. We believe that with such more natural representation the proof of the main result and its verification will be more complicated and less intuitive. But, based on our exposition the construction can be easily and readily given with basis for subspaces.

Appendix

In the appendix we present the extensions of Type D to the 3-subspaces $\mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6, \mathcal{Y}_7, \mathcal{Y}_8, \mathcal{Y}_9, \mathcal{Y}_{10}, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{13}, \mathcal{Y}_{14}$, and \mathcal{Y}_{15} .

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