The Extremal Function and Colin de Verdière Graph Parameter

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Abstract

The Colin de Verdière parameter $\mu(G)$ is a minor-monotone graph parameter with connections to differential geometry. We study the conjecture that for every integer t, if G is a graph with at least t vertices and $\mu(G) \leqslant t$, then $|E(G)| \leqslant t|V(G)| - \binom{t+1}{2}$. We observe a relation to the graph complement conjecture for the Colin de Verdière parameter and prove the conjectured edge upper bound for graphs G such that either $\mu(G) \leqslant 7$, or $\mu(G) \geqslant |V(G)| - 6$, or the complement of G is chordal, or G is chordal.

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1 Introduction

We consider only finite, simple graphs without loops. Let $\mu(G)$ denote the Colin de Verdière parameter of a graph G introduced in [7] (cf. [8]). We give a formal definition of $\mu(G)$ in Section 2. The Colin de Verdière parameter is minor-monotone; that is, if H is a minor of G, then $\mu(H) \leqslant \mu(G)$. Particular interest in this parameter stems from the following characterizations:

Theorem 1. *For every graph G:*

1. *$\mu(G) \leqslant 1$ if and only if G is a subgraph of a path.*
2. *$\mu(G) \leqslant 2$ if and only if G is outerplanar.*
3. *$\mu(G) \leqslant 3$ if and only if G is planar.*

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4. $\mu(G) \leqslant 4$ *if and only if* G *is linklessly embeddable.*

Items 1, 2, and 3 were shown by Colin de Verdière in [7]. Robertson, Seymour, and Thomas noted in [25] that $\mu(G) \leqslant 4$ implies that G has a linkless embedding due to their theorem that the Petersen family is the forbidden minor family for linkless embeddings [26]. The other direction for 4 is due to Lovász and Schrijver [18]. See the survey of van der Holst, Lovász, and Schrijver for a thorough introduction to the parameter [13].

There is also a relation between the Colin de Verdière parameter and Hadwiger's conjecture that for every non-negative integer t, every graph with no K_{t+1} minor is t-colorable. Let $\chi(G)$ denote the chromatic number of a graph G and let $h(G)$ denote the Hadwiger number of G. That is, $h(G)$ is the largest integer so that G has the complete graph $K_{h(G)}$ as a minor. Then $\mu(K_{h(G)}) = h(G) - 1$, and so $\mu(G) \geqslant \mu(K_{h(G)}) = h(G) - 1$ [13]. So if Hadwiger's conjecture is true, then for every graph G, $\chi(G) \leqslant \mu(G) + 1$. Colin de Verdière conjectured that every graph satisfies $\chi(G) \leqslant \mu(G) + 1$ in [7]. For graphs with $\mu(G) \leqslant 3$, this statement is exactly the 4-Color Theorem [2, 24].

One way to look for evidence for Hadwiger's conjecture is through considerations of average degree. In particular Mader showed that for every family of graphs $\mathcal{F}$, there is an integer c so that if G is a graph with no graph in $\mathcal{F}$ as a minor, then $|E(G)| \leqslant c|V(G)|$ [19]. It follows by induction on the number of vertices that every graph G with no graph in $\mathcal{F}$ as a minor is $2c+1$-colorable. In fact Mader showed that:

Theorem 2. *[20] For* $t \leqslant 5$, *if* G *is a graph with* $h(G) \leqslant t+1$ *and* $|V(G)| \geqslant t$, *then* $|E(G)| \leqslant t|V(G)| - \binom{t+1}{2}$.

However asymptotically, as noted by Kostochka [16] and Thomason [31], based on Bollobás et al. [5]:

Theorem 3. *[16, 31] There exists a constant* $c \in \mathbb{R}^+$ *such that for every positive integer* t *there exists a graph* G *with* $h(G) \leqslant t+1$ *and* $|E(G)| > ct\sqrt{\log t}|V(G)|$.

Furthermore, Kostochka showed that the lower bound in Theorem 1.3 also serves as an upper bound [16]. This gives the best known bound on Hadwiger's conjecture, that graphs G with no K_t minor have $\chi(G) \leqslant \mathcal{O}(t\sqrt{\log t})$. We will study the following conjecture, that is an analog of Theorem 2 for the Colin de Verdière parameter:

Conjecture 4. For every integer t, if G is a graph with $\mu(G) \leqslant t$ and $|V(G)| \geqslant t$, then $|E(G)| \leqslant t|V(G)| - \binom{t+1}{2}$.

Nevo asked if this is true and showed that his Conjecture 1.5 in [22] implies Conjecture 4. Tait also asked this question as Problem 1 in [29] in relation to studying graphs with maximum spectral radius of their adjacency matrix, subject to having Colin de Verdière parameter at most t. Butler and Young showed the following weakening of Conjecture 4:

Theorem 5. *[6] For every integer* t, *if* G *is a graph on at least* t *vertices with zero forcing number no more than* t, *then* $|E(G)| \leqslant t|V(G)| - \binom{t+1}{2}$. *This bound is tight.*

The zero forcing number is a graph parameter that is always at least the Colin de Verdière parameter of a graph [1, 3]. We observe that since the zero forcing number of a graph is also always at least the pathwidth of the graph [3], grids have unbounded zero forcing number and Colin de Verdière parameter at most three.

There are no known explicit constructions of graphs satisfying Theorem 3. The essential observation is instead that the Hadwiger number of Erdős-Rényi random graphs is too small. So it would be very interesting to know if random graphs are a counterexample to Conjecture 4, and in particular to answer the following.

Problem 6. What is the Colin de Verdière parameter of the Erdős-Rényi random graph?

Hall et al. studied this problem for some parameters related to the Colin de Verdière parameter [10]. These related parameters are at least the vertex connectivity of a graph [Theorem 4, 12]. So random graphs do not give counterexamples to the analog of Conjecture 4 for these other parameters. Thus it seems that new techniques particular to the Colin de Verdière parameter will be needed to solve this problem.

We also observe that there is a relation between Conjecture 4 and the graph complement conjecture for the Colin de Verdière parameter. Let $\overline{G}$ denote the complement of G. The graph complement conjecture for the Colin de Verdière parameter is as follows:

Conjecture 7. For every graph G, $\mu(G) + \mu(\overline{G}) \geqslant |V(G)| - 2$.

This conjecture was introduced by Kotlov, Lovász, and Vempala, who showed that the conjecture is true if G is planar [17]. Their result is used in this paper and will be stated formally in Section 4. Conjecture 7 is also an instance of a Nordhaus-Gaddum sum problem. See the recent paper by Hogben for a survey of Nordhaus-Gaddum problems for the Colin de Verdière and related parameters, including Conjecture 7 [11]. We observe that:

Observation 8. *If there exists a constant $c \in \mathbb{R}^+$ so that for every graph G, $|E(G)| \leqslant c\mu(G)|V(G)|$, then there exists a constant $p \in \mathbb{R}^+$ so that for every graph G, $\mu(G)+\mu(\overline{G}) \geqslant p|V(G)|$.*

This follows from noting that we would have $c\mu(G)|V(G)| + c\mu(\overline{G})|V(G)| \geqslant |E(G)| + |E(\overline{G})| = \binom{|V(G)|}{2}$. So Conjecture 4 would imply an asymptotic version of the graph complement conjecture for the Colin de Verdière parameter. This weaker version is currently not known. In the other direction we will show in Section 2 that:

Observation 9. *If for every graph G, $\mu(G) + \mu(\overline{G}) \geqslant |V(G)| - 2$, then every graph G has $|E(G)| \leqslant (\mu(G) + 1)|V(G)| - \binom{\mu(G)+2}{2}$.*

Then in particular the graph complement conjecture for Colin de Verdière parameter would imply that all graphs G are $2\mu(G)+2$-colorable. We next comment on the tightness of Conjecture 4. We say that a graph G is the *join* of non-empty graphs H_1 and H_2 if the vertex set of G is the disjoint union of $V(H_1)$ and $V(H_2)$, and for $i = 1, 2$ the induced subgraph of G on vertex set $V(H_i)$ is the graph H_i, and for every pair of vertices $u \in V(H_1)$ and $v \in V(H_2)$, $uv \in E(G)$. We will show in Section 2 that:

Observation 10. *Let H be any edge-maximal planar graph on at least four vertices and let $t \geqslant 3$ be an integer. Let G denote the join of H and K_{t-3}. Then $\mu(G) = t$ and $|E(G)| = t|V(G)| - \binom{t+1}{2}$.*

So for every positive integer t, Conjecture 4 is tight for infinitely many graphs. We say a graph G is *chordal* if for every cycle C of G of length greater than 3, the induced subgraph of G with vertex set $V(C)$ has some edge that is not in $E(C)$. The main result we prove is Theorem 11:

Theorem 11. *Suppose G is a graph such that either:*

- *G is chordal, or*
- *$\overline{G}$ is chordal, or*
- *$\mu(G) \leqslant 7$, or*
- *$\mu(G) \geqslant |V(G)| - 6$.*

Then $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$.

Note that it is equivalent to say that for such graphs, for every integer t with $\mu(G) \leqslant t \leqslant |V(G)|$, $|E(G)| \leqslant t|V(G)| - \binom{t+1}{2}$.

The proof of Theorem 11 for graphs with $\mu(G) \leqslant 7$ relies on difficult results of Mader [20], Jørgensen [15], and Song and Thomas [28] on the extremal function of graphs with Hadwiger number no more than eight. A conjecture of Thomas and Zhu on the extremal function of graphs with no K_{10} minor would similarly imply the theorem for $\mu(G) \leqslant 8$ [Conjecture 1.4, 30].

The proof of Theorem 11 for graphs with $\mu(G) \geqslant |V(G)| - 6$ relies on work of Kotlov, Lovász, and Vempala [17] on the graph complement conjecture for the Colin de Verdière parameter, and on forbidden topological minor characterizations for graphs with $\mu(G) \leqslant 3$. The main difficulty in extending this result to graphs with $\mu(G) \geqslant |V(G)| - 7$ is that the graph complement conjecture for the Colin de Verdière parameter is not currently known for graphs with $\mu(G) \leqslant 4$.

We also note that the analog of Theorem 11 for the Hadwiger number is false. For $n_1, n_2, \ldots, n_k \in \mathbb{Z}^+$, let $K_{n_1,n_2,\ldots,n_k}$ denote the complete multipartite graph with independent sets of size $n_1, n_2, \ldots, n_k$. The complement of every complete multipartite graph is chordal. Furthermore, as observed in the literature (see [20] and [27]), $K_{2,2,2,2,2,2}$ has $h(K_{2,2,2,2,2,2}) = 7$, yet $|E(K_{2,2,2,2,2,2})| > 6|V(K_{2,2,2,2,2,2})| - \binom{6+1}{2}$.

Another way of generalizing the example of complete multipartite graphs would be to study the join operation. In particular it would be nice to know if Conjecture 4 is tight for the join of two graphs H_1 and H_2 for which the conjecture is tight. This is true if H_1 or H_2 is a clique, as in Observation 3. The join of a path and a clique also appears as an extremal graph in Tait's work on the Colin de Verière parameter and spectral radius [Theorem 1, 29]. Barioli et al study the effect of the join operation on the graph complement conjecture for Colin de Verdière type-parameters [Theorem 3.8, 4]. It would be interesting to see if something similar can be shown for the extremal function:

Problem 12. Show that if G is the join of non-empty graphs H_1 and H_2 so that for $i = 1, 2$, $|E(H_i)| \leqslant \mu(H_i)|V(H_i)| - \binom{\mu(H_i)+1}{2}$, then $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$.

2 Definitions and Preliminaries

In this section we begin by briefly introducing our notation. Then we state the definition and some basic facts on the Colin de Verdière parameter, prove the observations from the introduction, and prove two lemmas that will be used in both of the next sections. In Section 3 we prove our main theorem, Theorem 11, for chordal graphs and the complement of chordal graphs. Finally, in Section 4 we prove Theorem 11 for graphs G with $\mu(G) \leqslant 7$ or $\mu(G) \geqslant |V(G)| - 6$.

Let G be a graph. We will write an edge connecting vertices u and v as uv. We write $\delta(G)$ for the minimum degree, $\Delta(G)$ for the maximum degree, and $\omega(G)$ for the clique number of G. The set of vertices adjacent to a vertex v is denoted $N(v)$. The degree of a vertex v in G is written $d_G(v)$, or simply $d(v)$ if the graph is understood from the context. For $S \subseteq V(G)$, we write $G[S]$ for the induced subgraph of G with vertex set S, and $G - S$ for the induced subgraph of G with vertex set $V(G) - V(S)$. For a vertex v we will write $G - v$ for $G - \{v\}$. If e is an edge of G, we write G/e for the graph obtained from G by contracting e and deleting all parallel edges. We will use $A \coloneqq B$ to mean that A is defined to be B.

Next we give the definition of the Colin de Verdière parameter. Let n be the number of vertices of G. It will be convenient to assume that $V(G) = \{1, 2, \ldots, n\}$ and that G is connected. If G is not connected, then define $\mu(G)$ to be the maximum among all connected components H of G of $\mu(H)$. We denote $I \coloneqq \{ii : i \in \{1, 2, \ldots, n\}\}$.

Definition 13. The Colin de Verdière parameter $\mu(G)$ is the maximum corank of any real, symmetric $n \times n$ matrix M such that:

1. $M_{i,j} = 0$ if $ij \notin E(G) \cup I$, and $M_{ij} < 0$ if $ij \in E(G)$.
2. M has exactly one negative eigenvalue.
3. If X is a symmetric $n \times n$ matrix such that $MX = 0$ and $X_{ij} = 0$ for $ij \in E \cup I$, then $X = 0$.

From the survey of van der Holst, Lovász, and Schrijver, we have:

Theorem 14. *[13] Let G be a graph, let H be a minor of G, and let $v \in V(G)$. Then*

(i) $\mu(H) \leqslant \mu(G)$

(ii) For every positive integer t, $\mu(K_t) = t - 1$.

(iii) $\mu(G) \leqslant \mu(G-v)+1$. *If $N(v) = V(G)-\{v\}$ and $E(G) \neq \emptyset$ then $\mu(G) = \mu(G-v)+1$.*

Then Observation 10, which we restate below, follows from induction on t by (iii) above and noting that for any positive integers $t \geqslant 3$ and n, $(t-1)(n-1)-\binom{t}{2}+n-1 = tn-\binom{t+1}{2}$.

Observation 10. Let H be any edge-maximal planar graph on at least 4 vertices and let $t \geqslant 3$ be an integer. Let G denote the join of H and K_{t-3}. Then $\mu(G) = t$ and $|E(G)| = t|V(G)| - \binom{t+1}{2}$.

To relate the extremal problem to the graph complement conjecture for Colin de Verdière parameter, we will need the following lemma.

Lemma 15. *Let G be a graph on n vertices and let t be an integer with $n \geqslant t$. Then $|E(G)| \leqslant tn - \binom{t+1}{2}$ if and only if $|E(\overline{G})| \geqslant \binom{n-t}{2}$.*

Proof. Observe that $\binom{n-t}{2} + tn - \binom{t+1}{2} = \binom{n}{2} = |E(G)| + |E(\overline{G})|$. $\square$

We will also need the following theorem of Pendavingh.

Theorem 16. [Theorem 5, 23] *If G is a connected graph, then either $|E(G)| \geqslant \binom{\mu(G)+1}{2}$ or $|E(G)| \geqslant \binom{\mu G+1}{2} - 1$ and G is isomorphic to $K_{3,3}$.*

Now we are ready to prove:

Observation 9. If for every graph G, $\mu(G) + \mu(\overline{G}) \geqslant |V(G)| - 2$, then every graph G has $|E(G)| \leqslant (\mu(G)+1)|V(G)| - \binom{\mu(G)+2}{2}$.

Proof. Let G be a graph on n vertices. Since $\mu(\overline{G})$ is the maximum Colin de Verdière parameter of any connected component of G, by Theorem 16 either $\overline{G}$ is isomorphic to the disjoint union of $K_{3,3}$ and an independent set of vertices, or $|E(\overline{G})| \geqslant \binom{\mu(\overline{G})+1}{2}$. In the latter case, $|E(\overline{G})| \geqslant \binom{\mu(\overline{G})+1}{2} \geqslant \binom{n-1-\mu(G)}{2}$. So by Lemma 15, we are done.

If $\overline{G}$ is isomorphic to the disjoint union of $K_{3,3}$ and a set of k independent vertices, then $\mu(\overline{G}) = 3$ and by (iii) of Theorem 14 and since $\mu(\overline{K_{3,3}}) = 2$, $\mu(G) = k + 2$. So then $\mu(G) + \mu(\overline{G}) = n - 1$. So

$$|E(\overline{G})| \geqslant \binom{\mu(\overline{G})+1}{2} - 1 = \binom{n-\mu(G)}{2} - 1 \geqslant \binom{n-1-\mu(G)}{2}$$

and again we are done by Lemma 15. $\square$

We finish this section by proving some basic facts about a counterexample to the main Conjecture 4 such that every induced subgraph on one less vertex satisfies the conjecture. This lemma will be used in Sections 3 and 4 to help prove our main Theorem 11.

Lemma 17. *Let G be an n-vertex graph with $|E(G)| > \mu(G)n - \binom{\mu(G)+1}{2}$. Suppose also that for every $x \in V(G)$, $|E(G-x)| \leqslant \mu(G-x)(n-1) - \binom{\mu(G-x)+1}{2}$. Then $\mu(G) < \delta(G) \leqslant \Delta(G) < n-1$.*

Proof. Suppose v is a vertex of G with $d(v) \leqslant \mu(G)$. Then by Theorem 14, we have $\mu(G-v) \in \{\mu(G), \mu(G)-1\}$ and $\mu(G) \leqslant n-1$. Then

$$\begin{aligned}|E(G)| = |E(G-v)| + d(v) &\leqslant \mu(G-v)(n-1) - \binom{\mu(G-v)+1}{2} + \mu(G) \\ &\leqslant \mu(G)(n-1) - \binom{\mu(G)+1}{2} + \mu(G) = \mu(G)n - \binom{\mu(G)+1}{2}\end{aligned}$$

a contradiction.

If u is a vertex with $d(u) = n-1$, first note that $E(G) \neq \emptyset$. Then by (iii) of Theorem 14, $\mu(G-u) = \mu(G) - 1$, and so

$$|E(G)| = |E(G-u)| + n - 1 \leqslant (\mu(G)-1)(n-1) - \binom{\mu(G)}{2} + n - 1 = \mu(G)n - \binom{\mu(G)+1}{2}$$

a contradiction. □

3 Chordal Graphs and Complements of Chordal Graphs

In this section we will show that if G is a graph such that G is chordal or $\overline{G}$ is chordal, then $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$. Define a *simplicial* vertex of a graph G to be a vertex v such that $G[N(v)]$ is a complete graph. We will use the fact that every chordal graph has a simplicial vertex.

Lemma 18. *If G is a chordal graph then $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$.*

Proof. Let G be a vertex-minimal counterexample. Let u be a simplicial vertex of G. Then $d(u) \leqslant \omega(G) - 1 \leqslant \mu(G)$. This is a contradiction to Lemma 17 since every induced subgraph of a chordal graph is chordal and G is a vertex-minimal counterexample. □

For graphs G so that the complement of G is chordal, we need to introduce the following two theorems. Mitchell and Yengulalp showed that:

Theorem 19. *[21] If G is a chordal graph, then $\mu(G) + \mu(\overline{G}) \geqslant |V(G)| - 2$.*

For an integer $t \geqslant 3$, let $K_t - \Delta$ denote the graph obtained from K_t by deleting the edges of a triangle. Fallat and Mitchell proved that:

Theorem 20. *[9] Let G be a chordal graph. Then $\mu(G) = \omega(G)$ if and only if G has $K_{\omega(G)+2} - \Delta$ as an induced subgraph. Otherwise $\mu(G) = \omega(G) - 1$.*

This theorem is related to a result of van der Holst, Lovász, and Schrijver on the behavior of the Colin de Verdiére parameter under clique sums [14]. Indeed, a direct combinatorial proof of Theorem 20 is possible using the result of van der Holst, Lovász, and Schrijver, but we omit it for the sake of concision. We are now ready to prove the final lemma of this section.

Lemma 21. *If G is a graph so that $\overline{G}$ is chordal, then $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$.*

Proof. Let G be a vertex-minimal counterexample, and set $n \coloneqq |V(G)|$. First we show two claims:

Claim 22. $\omega(\overline{G}) \geqslant 2$

Proof. Otherwise G is a complete graph and by (ii) of Theorem 14, $\mu(G) = n - 1$. Then $|E(G)| = \binom{n}{2} = \mu(G)n - \binom{\mu(G)+1}{2}$, a contradiction. □

Claim 23. $\Delta(\overline{G}) < \mu(\overline{G}) + 1$

Proof. Otherwise by Theorem 19, $\Delta(\overline{G}) \geqslant \mu(\overline{G}) + 1 \geqslant n - 1 - \mu(G)$. Then $\delta(G) \leqslant \mu(G)$, a contradiction to Lemma 17 since G is a vertex-minimal counterexample and the complement of every induced subgraph of G is chordal. □

Now, suppose $\mu(\overline{G}) = \omega(\overline{G})$. Then by Theorem 20, $\overline{G}$ has an induced subgraph that is isomorphic to $K_{\omega(\overline{G})+2} - \Delta$. Since $\omega(\overline{G}) \geqslant 2$, we have $\Delta(\overline{G}) \geqslant \Delta(K_{\omega(\overline{G})+2} - \Delta) = \omega(\overline{G}) + 1 = \mu(\overline{G}) + 1$, a contradiction to Claim 2.

So $\mu(\overline{G}) = \omega(G) - 1$. Let $S \subseteq V(\overline{G})$ be the set of vertices of a maximum clique of $\overline{G}$. Write $T := V(\overline{G}) - S$. First we will show that if $x \in S$ and $y \in T$, then $xy \notin E(\overline{G})$. If $xy \in E(\overline{G})$, then $d_{\overline{G}}(x) \geqslant \omega(\overline{G}) = \mu(\overline{G}) + 1$, a contradiction to Claim 2.

If $T = \emptyset$, then $E(G) = \emptyset$ and G would satisfy the lemma. So $T \neq \emptyset$. Then let $u \in S$ and $v \in T$. We have $uv \in E(G)$. Let uv also denote the new vertex of G/uv. Since in $\overline{G}$ the vertex u is adjacent to no vertices in T and v is adjacent to no vertices in S, the vertex uv is adjacent to every other vertex in G/uv. Also, since $|S| \geqslant 2$, G/uv contains an edge. So by (iii) of Theorem 14, $\mu(G/uv) = \mu(G - \{u, v\}) + 1$. Then $|E(G - \{u, v\})| \leqslant (\mu(G) - 1)(n - 2) - \binom{\mu(G)}{2}$.

Also, $d_{\overline{G}}(u) = \omega(\overline{G}) - 1$, so $d_G(u) = n - \omega(\overline{G}) = n - 1 - \mu(\overline{G}) \leqslant \mu(G) + 1$ by Theorem 19. By Lemma 17, $d_G(y) < n - 1$. Then

$$\begin{aligned} |E(G)| = |E(G - \{u, v\})| + d_G(u) + d_G(v) - 1 &\leqslant (\mu(G) - 1)(n - 2) - \binom{\mu(G)}{2} + \mu(G) + n - 2 \\ &= \mu(G)n - \binom{\mu(G) + 1}{2} \end{aligned}$$

a contradiction. □

4 Graphs with Small or Large Parameter

In this section we will show that graphs G such that either $\mu(G) \leqslant 7$ or $\mu(G) \geqslant |V(G)| - 6$ have $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$. First we give some definitions related to clique sums.

Let k be a non-negative integer and let G_1 and G_2 be two vertex-disjoint graphs. For $i = 1, 2$ let $C_i \subseteq V(G_i)$ be a clique of size k of G_i. Then let G denote the graph obtained from G_1 and G_2 by identifying the vertices in cliques C_1 and C_2 by some bijection. We say G is a *pure* k-clique sum of G_1 and G_2.

Let H be some fixed graph and let k be a non-negative integer. We say a graph G is *built* by pure k-sums of H if either G is isomorphic to H, or if G is a pure k-clique sum of graphs H_1 and H_2, where H_1 and H_2 are built by pure k-sums of H. The following generalization of Theorem 2 is due to Jørgensen.

Theorem 24. *[15] Let G be a graph with $h(G) \leqslant 7$, $|V(G)| \geqslant 6$, and $|E(G)| > 6|V(G)| - 21$. Then $|E(G)| = 6|V(G)| - 20$, and G can be built by pure 5-sums of $K_{2,2,2,2,2}$.*

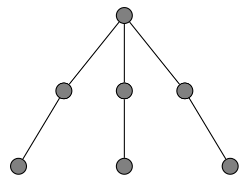

Figure 1: The graph $P_{3,2}$

For graphs with no K_9 minor, Song and Thomas proved:

Theorem 25. *[28] Let G be a graph with $h(G) \leqslant 8$, $|V(G)| \geqslant 7$, and $|E(G)| > 7|V(G)| - 28$. Then $|E(G)| = 7|V(G)| - 27$, and either G is isomorphic to $K_{2,2,2,3,3}$, or G can be built by pure 6-sums of $K_{1,2,2,2,2,2,2}$.*

We will also make use of the following theorem due to Kotlov, Lovász, and Vempala.

Theorem 26. *[17] If G is a graph with $\mu(G) \leqslant 3$, then $\mu(G) + \mu(\overline{G}) \geqslant |V(G)| - 2$.*

Kotlov, Lovász, and Vempala also characterized exactly which graphs G have $\mu(G) \geqslant |V(G)| - 3$ [Theorems 3.3 and 5.2, 17]. Let $P_{3,2}$ denote the graph formed from three disjoint paths of length two by identifying one end from each path. That is, $P_{3,2}$ is the graph in Figure 1. We will make use of the following corollary of these theorems:

Corollary 27. *[17] If G is a graph such that $\overline{G}$ contains no $P_{3,2}$ subgraph and no cycle, then $\mu(G) \geqslant |V(G)| - 3$.*

Now we are ready to prove the following lemma.

Lemma 28. *Let G be a graph with $\mu(G) \leqslant 7$. Then $|E(G)| \leqslant \mu(G)|V(G)| - \binom{\mu(G)+1}{2}$.*

Proof. First note that $\mu(K_{2,2,2,2,2,2}) \geqslant 7$, $\mu(K_{1,2,2,2,2,2,2}) \geqslant 8$, and $\mu(K_{2,2,2,3,3}) \geqslant 8$ by Theorem 26, since $\mu(\overline{K_{2,2,2,2,2,2}}) = 1$, $\mu(\overline{K_{1,2,2,2,2,2,2}}) = 1$, and $\mu(\overline{K_{2,2,2,3,3}}) = 2$.

Let G be a graph with $\mu(G) \leqslant 7$, and write $n := |V(G)|$. If $\mu(G) \leqslant 5$, then since $h(G) \leqslant \mu(G) + 1$, the lemma follows from Theorem 2. If $\mu(G) = 6$, then G does not contain $K_{2,2,2,2,2,2}$ as a subgraph. So we are done by Theorem 24. If $\mu(G) = 7$, then G does not contain $K_{1,2,2,2,2,2,2}$ or $K_{2,2,2,3,3}$ as a subgraph, and we are done by Theorem 25. $\square$

For the next lemma we need to give some definitions related to subdivisions. Fix a graph H'. We say a graph H is a *subdivision* of H' if H can be formed from H' by replacing edges of H' with internally-disjoint paths with the same ends. Then we say $v \in V(H)$ is a *branch vertex* of H if also $v \in V(H')$. Suppose H' is a bipartite graph with bipartition (A, B). That is, (A, B) is a partition of the vertex set of H' such that every edge of H' has one end in A and one end in B. Then if H is a subdivision of H', we will say that branch vertices u and v of H are *in the same part* of H if either $u, v \in A$ or $u, v \in B$. Now we are ready to prove the final lemma:

Lemma 29. *Let G be an n-vertex graph with $\mu(G) \geqslant n - 6$. Then $|E(G)| \leqslant \mu(G)n - \binom{\mu(G)+1}{2}$.*

Proof. Let G be a vertex-minimal counterexample. Write $n := |V(G)|$ and $c := n - \mu(G)$. First we will show that $\delta(\overline{G}) \geqslant 1$. Let $v \in V(G)$. Then by part (iii) of Theorem 14, $\mu(G - v) \geqslant \mu(G) - 1 \geqslant |V(G - v)| - 6$. So by Lemma 17, $\Delta(G) < n - 1$. So $\delta(\overline{G}) \geqslant 1$.

Next we find upper and lower bounds for n. By Lemma 28, we may assume $\mu(G) \geqslant 8$, so $n = \mu(G) + c \geqslant 8 + c$. By Lemma 15, $|E(\overline{G})| \leqslant \binom{n-\mu(G)}{2} - 1 = \binom{c}{2} - 1$. Then since $\delta(\overline{G}) \geqslant 1$, we have $n \leqslant 2|E(\overline{G})| \leqslant 2(\binom{c}{2} - 1)$. In total, we have $8 + c \leqslant n \leqslant 2(\binom{c}{2} - 1)$. This implies that $c \geqslant 5$.

Now we will show that $\mu(\overline{G}) \geqslant c - 2$. Otherwise, $\mu(\overline{G}) \leqslant c - 3 \leqslant 3$. Then by Theorem 26, $n - 2 \leqslant \mu(G) + \mu(\overline{G}) \leqslant n - 3$, a contradiction. Now we proceed by cases.

Case 1: $c = 5$.

Then since $\mu(\overline{G}) \geqslant c - 2 = 3$, $\overline{G}$ is not outerplanar. So $\overline{G}$ has a subgraph H that is either a subdivision of K_4 or a subdivision of $K_{2,3}$. Let $D \subseteq V(H)$ be the set of branch vertices of H. Remember that by Lemma 15, we have $|E(\overline{G})| \leqslant \binom{c}{2} - 1$. Then since $\delta(\overline{G}) \geqslant 1$ and $n \geqslant 8 + c = 13$,

$$\binom{5}{2} - 1 \geqslant |E(\overline{G})| \geqslant \frac{1}{2}\left(\sum_{x \in D} d_H(x) + \sum_{y \in V(G)-D} d_{\overline{G}}(y)\right) \geqslant \frac{1}{2}\left(\sum_{x \in D} d_H(x) + 13 - |D|\right)$$

In either case we get a contradiction.

Case 2: $c = 6$.

Then $\mu(\overline{G}) \geqslant 4$ and so $\overline{G}$ is not planar. So $\overline{G}$ has a subgraph H that is either a subdivision of K_5 or a subdivision of $K_{3,3}$. If H is a subdivision of K_5 then similarly to before, since $\delta(\overline{G}) \geqslant 1$ and $n \geqslant 8 + c = 14$, we have $\binom{6}{2} - 1 \geqslant |E(\overline{G})| \geqslant \frac{1}{2}(5 * 4 + 9) = \frac{29}{2}$, a contradiction.

So H is a subdivision of $K_{3,3}$. Let $u, v \in V(H)$ be distinct branch vertices of H that are in the same part of H such that $d_{\overline{G}}(u) + d_{\overline{G}}(v)$ is maximum. We will show that $\overline{G - \{u, v\}}$ contains no $P_{3,2}$ subgraph and no cycle. Write $k := d_{\overline{G}}(u) + d_{\overline{G}}(v) - 6$. Then $k \geqslant 0$. Since u and v are not adjacent in H and vertices adjacent to u or v in H have degree at least 1 in $\overline{G - \{u, v\}}$, the graph $\overline{G - \{u, v\}}$ has at most k vertices of degree 0.

Suppose $\overline{G - \{u, v\}}$ has a $P_{3,2}$ subgraph. If u and v are not adjacent in $\overline{G - \{u, v\}}$, then

$$\binom{6}{2} - 1 \geqslant |E(\overline{G})| = k + 6 + |E(\overline{G - \{u, v\}})| \geqslant k + 6 + |E(P_{3,2})| + \frac{1}{2}(12 - |V(P_{3,2})| - k) \geqslant \frac{29}{2}$$

a contradiction. If u and v are adjacent in $\overline{G - \{u, v\}}$, then since they are not adjacent in H, we have $k \geqslant 2$. So similarly we have

$$\binom{6}{2} - 1 \geqslant |E(\overline{G})| = k + 5 + |E(\overline{G - \{u, v\}})| \geqslant k + 5 + |E(P_{3,2})| + \frac{1}{2}(12 - |V(P_{3,2})| - k) \geqslant \frac{29}{2}$$

again a contradiction. So G contains no $P_{3,2}$ subgraph.

Now we will show that $\overline{G - \{u,v\}}$ has no cycle. Write $S := V(G) - V(H)$. Let S_1 be the set of vertices in S with degree strictly greater than 1 in $\overline{G}$. Write $d := \sum_{z \in V(H)} d_{\overline{G}}(z) - d_H(z)$. Then since $\delta(\overline{G}) \geqslant 1$ and $n \geqslant 14$, we have:

$$\begin{aligned}\binom{6}{2} - 1 \geqslant |E(\overline{G})| &= \frac{1}{2}\left(\sum_{x \in V(H)} d_H(x) + d + \sum_{y \in S} d_{\overline{G}}(y)\right) \\ &\geqslant \frac{1}{2}\left(\sum_{x \in V(H)} d_H(x) + d + 14 - |V(H)| + |S_1|\right) = \frac{1}{2}\left(|V(H)| + d + |S_1| + 20\right)\end{aligned}$$

So $|V(H)| + d + |S_1| \leqslant 8$. Since $|V(H)| \geqslant 6$, we have that $d + |S_1| \leqslant 2$.

Suppose $\overline{G - \{u,v\}}$ contains a cycle C. If $|V(C) \cap S| \geqslant 3$, then $|S_1| \geqslant |V(C) \cap S| \geqslant 3$, a contradiction. If $|V(C) \cap S| \in \{1,2\}$, then $\overline{G - \{u,v\}}$ has at least two edges with one end in $V(H)$ and the other in S. Then $d \geqslant 2$, and $|S_1| \geqslant |V(C) \cap S| \geqslant 1$, a contradiction.

Finally, suppose $|V(C) \cap S| = 0$. Since u and v are in the same part of H, the graph $H - \{u,v\}$ contains no cycle. So there exist distinct vertices $a, b \in V(H) - \{u,v\}$ so that a and b are adjacent in $\overline{G - \{u,v\}}$ but not in H. Then we have $d_{\overline{G}}(a) - d_H(a), d_{\overline{G}}(b) - d_H(b) > 0$, so $d \geqslant 2$. Then since $|V(H)| + d \leqslant 8$, we have $|V(H)| = 6$ and H is isomorphic to $K_{3,3}$. Then since a and b are not adjacent in H, they are in the same part of H. So by the choice of u and v, we have $d_{\overline{G}}(u) + d_{\overline{G}}(v) \geqslant d_{\overline{G}}(a) + d_{\overline{G}}(b) \geqslant 8$. Then $d \geqslant d_{\overline{G}}(u) + d_{\overline{G}}(v) - 6 + d_{\overline{G}}(a) + d_{\overline{G}}(b) - 6 \geqslant 4$, a contradiction.

We have shown that $\overline{G - \{u,v\}}$ has no cycle and no $P_{3,2}$ subgraph. Then by Corollary 27, we have $n - 6 = \mu(G) \geqslant \mu(G - \{u,v\}) \geqslant |V(G - \{u,v\})| - 3 = n - 5$, a contradiction. This completes the proof. □

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