Defective 3-Paintability of Planar Graphs

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Abstract

A d-defective k-painting game on a graph G is played by two players: Lister and Painter. Initially, each vertex is uncolored and has k tokens. In each round, Lister marks a chosen set M of uncolored vertices and removes one token from each marked vertex. In response, Painter colors vertices in a subset X of M which induce a subgraph $G[X]$ of maximum degree at most d. Lister wins the game if at the end of some round there is an uncolored vertex that has no more tokens left. Otherwise, all vertices eventually get colored and Painter wins the game. We say that G is d-defective k-paintable if Painter has a winning strategy in this game. In this paper we show that every planar graph is 3-defective 3-paintable and give a construction of a planar graph that is not 2-defective 3-paintable.

Mathematics Subject Classifications: 05C15

1 Introduction

All graphs considered in this paper are finite, undirected and contain no loops nor multiple edges. For every $k \geqslant 1$, the set $\{1, \ldots, k\}$ is denoted $[k]$. The size of a graph G, denoted

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$|G|$, is the number of vertices in G. For a vertex v of G, the set of vertices adjacent to v in G is denoted $N(v)$. For a set X of vertices of G, the graph induced by X in G is denoted $G[X]$.

A *d-defective coloring* of a graph G is a coloring of the vertices of G such that each color class induces a subgraph of maximum degree at most d. Thus, a 0-defective coloring of G is simply a proper coloring of G. The famous Four Color Theorem asserts that every planar graph is 0-defective 4-colorable. Defective coloring of graphs was first studied by Cowen, Cowen and Woodall [1]. They proved that every outerplanar graph is 2-defective 2-colorable and that every planar graph is 2-defective 3-colorable. They also showed an outerplanar graph that is not 1-defective 2-colorable, a planar graph that is not 1-defective 3-colorable, and for every d, a planar graph that is not d-defective 2-colorable.

A *k-list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of k permissible colors. A *d-defective L-coloring* of G is a d-defective coloring c of G with $c(v) \in L(v)$ for every vertex v of G. A graph G is *d-defective k-choosable* if for any k-list assignment L of G, there exists a d-defective L-coloring of G. The particular function that assigns the set $[k]$ to each vertex of a graph is a k-list assignment. Therefore, every d-defective k-choosable graph is d-defective k-colorable. The converse is not true. Voigt [7] gave a construction of a planar graph that is not 0-defective 4-choosable. Eaton and Hull [3] and Škrekovski [8] independently proved that every planar graph is 2-defective 3-choosable and every outerplanar graph is 2-defective 2-choosable. They asked the question whether every planar graph is 1-defective 4-choosable. One decade later, Cushing and Kierstead [2] answered this question in the affirmative.

This paper studies the on-line version of list coloring of graphs, defined through a two person game. The study of on-line list coloring was initiated independently by Schauz [5] an Zhu [9].

A *d-defective k-painting game* on a graph G is played by two players: Lister and Painter. Initially, each vertex is uncolored and has k tokens. In each round, Lister marks a chosen set M of uncolored vertices and removes one token from each marked vertex. In response, Painter colors vertices in a subset X of M which induce a subgraph $G[X]$ of maximum degree at most d. Lister wins if at the end of some round there is an uncolored vertex with no more tokens left. Otherwise, after some round, all vertices are colored and Painter wins the game. We say that G is *d-defective k-paintable* if Painter has a winning strategy in this game. For a vertex v of G, let $\theta(v)$ denote the set of neighbors of v that are colored in the same round as v. Thus, in the d-defective painting game we have that for any vertex v, $|\theta(v)| \leqslant d$. We say that vertices in $\theta(v)$ give defect to v.

Let L be a k-list assignment of G with colors in the set $[n]$. Consider the following strategy for Lister. In the i-th round, for $i \in [n]$, Lister marks the set $M_i = \{v : i \in L(v), v \notin X_1, \ldots, X_{i-1}\}$, where X_j is the set of vertices colored by Painter in the j-th round. If Painter wins the game then the constructed coloring is a d-defective L-coloring of G. Therefore, every d-defective k-paintable graph is d-defective k-choosable. The converse is not true. Zhu [9] showed a graph that is 0-defective 2-choosable and is not 0-defective 2-paintable.

Thomassen [6] proved that every planar graph is 0-defective 5-choosable and Schauz [5] observed that every planar graph is also 0-defective 5-paintable. As mentioned above, it is known that every planar graph is 2-defective 3-choosable [3, 8] and 1-defective 4-choosable [2]. Recently, Han and Zhu [4] proved that every planar graph is 2-defective 4-paintable. It remained open questions whether or not every planar graph is 2-defective 3-paintable, or 1-defective 4-paintable.

In this paper, we construct a planar graph that is not 2-defective 3-paintable and prove that every planar graph is 3-defective 3-paintable. The only remaining question is whether or not every planar graph is 1-defective 4-paintable.

In Section 2 we present a strategy for Painter that shows the following.

Theorem 1. *Every planar graph is 3-defective 3-paintable.*

In Section 3 we show that this result is best possible as we construct a graph and a strategy for Lister that shows the following.

Theorem 2. *Some planar graphs are not 2-defective 3-paintable.*

2 Painter's strategy

In this section we prove Theorem 1. The proof provides an explicit, recursive strategy for Painter in a 3-defective 3-painting game on any planar graph. Our proof can be easily transformed into a polynomial-time algorithm that plays the game against Lister.

Let G be a connected non-empty plane graph. By a plane graph we mean a graph with a fixed planar drawing. Let C be the boundary walk of the outer face of G. For a vertex v in C, we define the set of *C-neighbors* of v to be the set of vertices that are consecutive neighbours of v in C. Observe that there may be more than two C-neighbors for a single vertex as C is not necessarily a simple walk. For the purpose of induction, we consider a more general game. We augment the 3-defective 3-painting game and introduce a (G, A, b)-*refined game* in which:

- $A \cup \{b\}$ are *special* vertices – A is a set, possibly empty, of at most two vertices that appear consecutively in C; b is a vertex in C other than the vertices in A. There are additional conditions on marking and coloring special vertices.
- each token has a *value* – when Lister removes a token of value p from a marked vertex v and Painter colors v then at most p neighbors of v are colored in the same round.

We say that a vertex v is an (A, b)-*cut* if $v \notin A, v \neq b$ and there is a vertex a in A such that v is on every path between a and b in G. We call a vertex in C that is neither in A, nor b, nor an (A, b)-cut to be a *regular boundary* vertex.

Let *token function* $f : V(G) \times \{0, \ldots, 3\} \to \mathbb{N}$ be a mapping defined for each vertex v and each value between 0 and 3. Initially, each vertex v has $f(v, p)$ tokens of value p. We denote the vector $(f(v, 0), \ldots, f(v, 3))$ as $f(v)$. We set values of f so that:

- $f(v) = (0, 1, 0, 0)$ if $v \in A$, or $v = b$,

- $f(v) = (0, 0, 1, 0)$ if v is an (A, b)-cut,
- $f(v) = (0, 0, 1, 1)$ if v is a regular boundary vertex,
- $f(v) = (0, 0, 0, 3)$ if $v \notin C$.

See Figure 1 for an example of a graph and a token function.

In each round, Lister marks a chosen set M of uncolored vertices and removes one token from each marked vertex. If $|A| = 2$, then Lister is not allowed to mark simultaneously both vertices in A, i.e. $|M \cap A| \leqslant 1$. Let p_v denote the value of the token removed by Lister from a vertex v in M. In response, Painter colors vertices in a subset X of M such that the degree of any vertex v in the induced subgraph $G[X]$ is at most p_v, i.e. $\forall v \in X : |\theta(v)| \leqslant p_v$. Additionally, if $a \in A$, and $\{a, b\}$ is an edge of C, then no neighbor of a other than b is colored in the same round as a, i.e. $\theta(a) \subseteq \{b\}$. Lister wins if at the end of some round there is an uncolored vertex with no more tokens left. Otherwise, after some round, all vertices are colored and Painter wins.

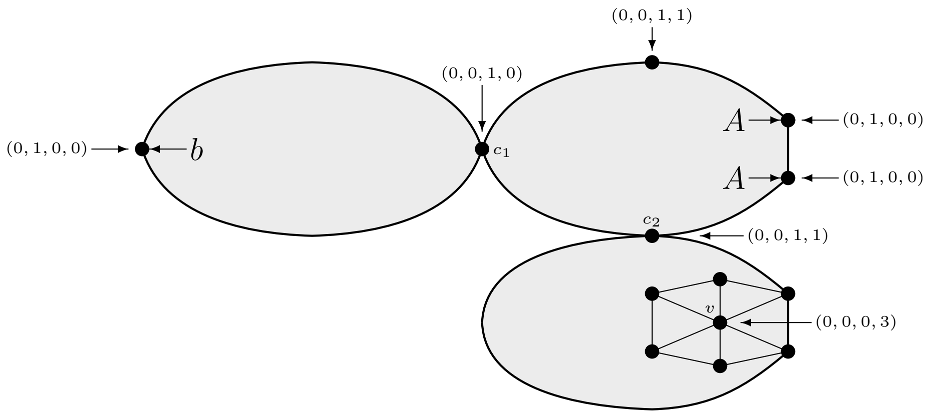

Figure 1: An example of (G, A, b)-refined game. Vertex c_1 is an (A, b)-cut. Vertex c_2 is a regular boundary vertex (c_2 is a cut in G, but not an (A, b)-cut). Since for every $a \in A$, $\{a, b\}$ is not an edge of C, each vertex in A can get one defect from any of its neighbours outside A (vertices of A are not marked simultaneously).

Lemma 3. *Painter has a winning strategy in the* (G, A, b)*-refined game.*

Before the proof, we show how to use Lemma 3 to prove Theorem 1.

Proof of Theorem 1. Suppose to the contrary that a planar graph G is not 3-defective 3-paintable. Adding some edges to G introduce additional constraints for Painter in the 3-defective 3-painting game. Thus, we can assume that G is connected. Choose any plane embedding of G. Choose any b on the boundary of the outer face. By Lemma 3, Painter has a winning strategy S in the $(G, \varnothing, b)$-refined game. The strategy S is a valid winning strategy in the 3-defective 3-painting game on G. □

Before we present the proof of Lemma 3, we briefly introduce some techniques that we frequently use in the proof.

Assume that Painter has a winning strategy S_1 in the (G_1, A_1, b_1)-refined game Γ_1. Now, if we modify the initial state of the game by adding some more tokens, or increasing value of some tokens, then obviously Painter has a winning strategy in the resulting game. Thus, in the proof of Lemma 3 when some vertex has too many tokens, or has tokens of too great value, we can *devalue* the token function and use the winning strategy S_1. We say that a token function g is *sufficient* for Γ_1 if it is equal to or can be devalued to the token function in Γ_1.

In order to find a winning strategy for Painter in the (G, A, b)-refined game Γ, we often divide the graph G into k, possibly overlapping, parts $G_1 = G[V_1], \dots, G_k = G[V_k]$ and consider (G_i, A_i, b_i)-refined games. The division of the graph and choice of special vertices $A_1, b_1, \dots, A_k, b_k$ depends on the structure of G. Then, we can use induction and assume that Painter has a winning strategy S_i in each (G_i, A_i, b_i)-refined game Γ_i. We present the following *composed* strategy S in Γ that uses strategies $S_1, \dots, S_k$ sequentially.

For a vertex v, let i_v be the first index i such that $v \in V_i$. Strategy S will use strategy S_{i_v} to decide whether v gets colored. If $v \in V_j$ for some $j > i_v$ then we will have that $v = b_j$ or $v \in A_j$. This way we get that vertex v has only one token and will be marked only once in game Γ_j – in the round v gets colored in S_{i_v}.

Now, we introduce a very useful technique. For a vertex v, let *slack* of v be the highest number s such that we can remove s most valuable tokens from v and the resulting token function is sufficient for Γ_{i_v}. The slack of any vertex is at most 2. Now, let $U(v)$ be some carefully selected set of neighbors of v in G. We say that v *gives away a token* to each u in $U(v)$ to describe the following behavior. Assume that the size of $U(v)$ does not exceed the slack of v, and, for each u in $U(v)$, either u has only one token in Γ or $i_u < i_v$. In particular, in every round, when we use strategy S_{i_v} to decide whether or not to color v, we already know if any vertex in $U(v)$ will be colored in this round. We say that v is *blocked* in some round by $u \in U(v)$ if u is colored in this round, i.e. either u has only one token and u is marked, or $i_u < i_v$ and S_{i_u} colored u. When vertex v is blocked in some round then we will not mark it in Γ_{i_v}. So, vertex v will be marked in game Γ_{i_v} possibly fewer times than it is marked in game Γ. Each vertex $u \in U(v)$ blocks v at most once during the game, and the number of times vertex v is blocked will not exceed the slack of v.

Let M be a set of vertices marked by Lister in some round. For $i = 1, \dots, k$, Painter constructs the set M_i and uses strategy S_i to find a response X_i for move M_i in the game Γ_i. The set M_i depends on the responses given by strategies $S_1, \dots, S_{i-1}$ and is defined as

$$M_i = \{v \in M \cap V_i : (v \text{ is not blocked and } i = i_v) \text{ or } (i > i_v \text{ and } v \in X_{i_v})\}.$$

Additionally, we need to decide the value of the token removed from each marked vertex. Observe that the regular boundary vertices are the only vertices that have tokens of distinct values, i.e. one token of value 2 and one token of value 3. Usually, this will be a natural and simple decision. In many cases, we will simply use the same value as Lister chose in Γ. Nevertheless, in some scenarios, we will have to be more careful about this choice. Details will be presented when needed.

Strategy S colors the set $X = \{v \in M : v \in X_{i_v}\}$. In order to prove that the composed strategy S is a winning strategy in the game Γ we need to argue that:

- The token function in Γ after removal of tokens that were given away is sufficient for each Γ_i. This is an easy calculation and we will omit it in most of the cases.
- The defects that any single vertex receives in games $\Gamma_1, \dots, \Gamma_k$ do not exceed the value of a token removed by Lister in game Γ. This will usually be the most important argument.
- If $a \in A$ is a C-neighbor of b, then $\theta(a) \subseteq \{b\}$.
- In each round $|M_i \cap A_i| \leqslant 1$. In order to guarantee this, we will have that if some A_i has two elements, then either $A_i = A$, or one of the vertices in A_i gave away a token to the other. Observe that if some vertex v and $u \in U(v)$ are vertices in G_i, then they are never both marked in the same round in the game Γ_i.

In figures that present game divisions we use the following schemas:

- Vertices of $G_1, \dots, G_k$ lie inside or on the boundary of regions filled with different shades of gray.
- We denote A_i, and b_i with A and b inside the region corresponding to G_i.
- We draw an edge directed from v to u to mark that v gives away a token to u.

Proof of Lemma 3. We prove the lemma by induction. Assume, that G is the smallest, in terms of the number of vertices, connected plane graph for which the lemma does not hold. Assume, that all internal faces of G are triangulated, as adding edges that do not change the boundary walk introduce additional constraints only for Painter. Let C be the boundary walk of the outer face of G, and A and b be the special vertices. Any closed walk W in G divides the plane into connected regions. Let $\operatorname{int}[W]$ denote the subgraph of G induced by the vertices that are inside the closure of bounded connected regions of the plane. For a simple path P and two distinct vertices u, v on that path, let $P[u, v]$ denote the subpath of P that traverses P from vertex u to vertex v. Similarly, for a simple cycle D in G and two distinct vertices u, v on that cycle, let $D[u, v]$ denote the subpath of D that traverses D in the clockwise direction from vertex u to vertex v. For a path $Q[u, v]$, we use notation $Q(u, v)$, $Q[u, v)$, and $Q(u, v]$ to denote $Q[u, v] - \{u, v\}$, $Q[u, v] - v$, and $Q[u, v] - u$ respectively.

The proof divides into several cases. The analysis of Case 1 is the basis of the induction and shows that G has at least four vertices. The analysis of Cases 2 and 3 shows that G is biconnected. The analysis of Cases 4 and 5 shows that vertex b is not adjacent to vertices in A. Case 7 is the final case of the induction and shows that G does not exist.

Case 1. G has at most three vertices.

Observe that each vertex has a token of value at least 1. If there are at most two vertices in G, then all vertices can be colored simultaneously in the same round.

Now, assume that G has exactly three vertices. Observe that all vertices are in C. If A is empty, choose any vertex x other than b, devalue token function for x and set $A = \{x\}$. The winning strategy in the resulting game is also a winning strategy in the original game.

If A has exactly one element a, let x be the third vertex other than a and b. If x is an (A, b)-cut, then all three vertices can be colored simultaneously in the same round. If x is adjacent to a, but not an (A, b)-cut, then x has two tokens, x gives away a token to a, devalue token function for x and set $A = \{a, x\}$. The winning strategy in the resulting game is also a winning strategy in the original game.

If x is not adjacent to a, and not an (A, b)-cut, then x has two tokens. We add the edge $\{a, x\}$ to the graph. Vertex x gives away a token to a and set $A = \{a, x\}$.

If A has two elements, then both elements of A are not marked in the same round. Thus, Painter can color each vertex v in the first round that v is marked in.

Case 2. G has a bridge.
Let edge $e = \{x, y\}$ be a bridge in G. Let G_1 and G_2 be the two connected components of $G - e$ with x in G_1, and y in G_2. Without loss of generality, assume that the special vertex b is in G_1. We divide this case depending on the position of A relative to e.

Case 2.1. $A \subset G_1$.
In this case, vertex y is not an (A, b)-cut and has two tokens. We divide the game into smaller games:

- $\Gamma_1 = (G_1, A, b)$,
- $\Gamma_2 = (G_2, \varnothing, y)$.

Vertex y gives away a token to x. As a result, we have that x gets defect only in Γ_1, and y gets defect only in Γ_2. If $a \in A$ is a C-neighbor of b then game Γ_1 ensures that $\theta(a) \subseteq \{b\}$.

Case 2.2. $A \cap G_1 \neq \varnothing$, and $A \cap G_2 \neq \varnothing$.
In this case we have that $A = \{x, y\}$. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{x\}, b)$,
- $\Gamma_2 = (G_2, \varnothing, y)$.

As Lister is not allowed to mark both x and y in the same round, vertex x gets defect only in Γ_1, and vertex y gets defect only in Γ_2. Vertex y is not adjacent to b, and if x is a C-neighbor of b then game Γ_1 ensures that $\theta(x) \subseteq \{b\}$.

Case 2.3. $A \subseteq G_2$.
Observe that $A \neq \varnothing$, as otherwise we could apply Case 2.1. Thus, vertex x is either an (A, b)-cut or $x = b$, and similarly vertex y is either an (A, b)-cut or $y \in A$. We divide this case further depending on the size of G_1 and these possibilities.

Case 2.3.1. $|G_1| \geqslant 2$, x is an (A, b)-cut.
In this case, vertex x has a single token of value 2, and vertices in A are not adjacent to b. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{x\}, b)$,
- $\Gamma_2 = (G_2 + x, A, x)$.

As a result, vertex x gets at most one defect in Γ_1, and at most one defect in Γ_2.

Case 2.3.2. $|G_1| \geqslant 2$, $x = b$.

Let z be a C-neighbor of x in G_1. Vertex z is not an (A, b)-cut and has two tokens. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{b\}, z)$,
- $\Gamma_2 = (G_2 + b, A, b)$.

Vertex z gives away a token to b. As a result, vertex b gets at most one defect in Γ_2 and no defect in Γ_1 – rules of the game Γ_1 enforce $\theta(b) \subseteq \{z\}$ and z gave away a token to b. If $y \in A$ then y is a C-neighbor of b and game Γ_2 ensures that $\theta(y) \subseteq \{b\}$. Vertices in A other than y are not adjacent to b.

Case 2.3.3. $|G_1| = 1$, y is an (A, b)-cut.

We observe that $|G_1| = 1$ implies $x = b$ and divide the game into smaller games:

- $\Gamma_1 = (G_1, \varnothing, x)$,
- $\Gamma_2 = (G_2, A, y)$.

Vertex x obviously gets at most one defect. Vertex y gets at most one defect from x and at most one defect in Γ_2.

Case 2.3.4. $|G_1| = 1$, $y \in A$, y has at least two neighbors in G_2.

In this case, vertex y has at least two C-neighbors in G_2. Choose vertex z, a C-neighbor of y in G_2 that is not in A. Vertex z is not an (A, b)-cut and has two tokens. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \varnothing, x)$,
- $\Gamma_2 = (G_2, A, z)$.

Vertex z gives away a token to y. Vertex x obviously gets at most one defect. Vertex y gets at most one defect from x and no defect in Γ_2 – rules of the game Γ_2 enforce $\theta(y) \subseteq \{z\}$ and z gave away a token to y.

Case 2.3.5. $|G_1| = 1$, $y \in A$, y has only one neighbor in G_2.

Let z be the only neighbor of y in G_2. In this case, we apply Case 2.2 if $|A| = 2$, or Case 2.1 if $|A| = 1$ of the induction for the bridge $\{y, z\}$.

In the analysis of the following cases we assume that each vertex has degree at least two. Indeed, a vertex of degree one is incident to a bridge. For each vertex v not in C, the neighbors of v traversed clockwise induce a simple cycle in G. Let $NC(v)$ denote this cycle. Furthermore, we assume that A has exactly two elements, say a_1 and a_2. If $A = \varnothing$, choose any vertex a_1 in C other than b and set $A = \{a_1\}$. If $A = \{a_1\}$, choose a vertex a_2,

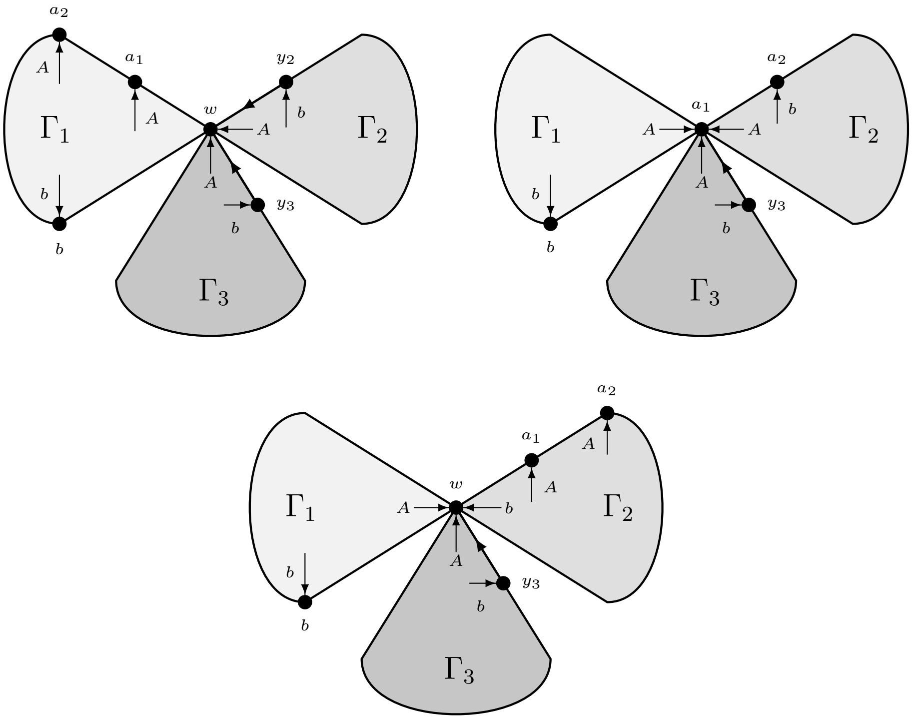

Figure 2: Game division in Cases 3.1, 3.2, and 3.3, respectively.

a C-neighbor of a_1 other than b and not an $(\{a_1\}, b)$-cut. Vertex a_2 gives away a token to a_1 and set $A = \{a_1, a_2\}$.

Case 3. G has a cut-point.

Let vertex w be a cut-point in G. Let $G_1, \ldots, G_k$ be the components of $G - w$. Supplement each graph G_i by adding vertex w back to it, i.e., adding w and the edges between w and G_i to G_i. Without loss of generality, assume that b is in G_1 and that A is contained either in G_1, or in G_2. Let y_i, for $i = 1 \ldots, k$, be any C-neighbor of w in G_i. We divide this case depending on the position of A relative to w. Figure 2 depicts the game divisions that we use in the subcases.

Case 3.1. $A \subset G_1$.

We divide the game into smaller games:

- $\Gamma_1 = (G_1, A, b)$,
- $\Gamma_i = (G_i, \{w\}, y_i)$, for $i = 2, \ldots, k$.

Each vertex y_i, for $i = 2, \ldots, k$ gives away a token to vertex w. As a result, vertex w gets no defect in the games $\Gamma_2, \ldots, \Gamma_k$. If $a \in A$ is a C-neighbor of b, then game Γ_1 ensures that $\theta(a) \subseteq \{b\}$.

Case 3.2. $A \subset G_2$, $w \in A$.

Without loss of generality, $w = a_1$. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{a_1\}, b)$,
- $\Gamma_2 = (G_2, \{a_1\}, a_2)$,
- $\Gamma_i = (G_i, \{w\}, y_i)$, for $i = 3, \ldots, k$.

Each vertex y_i, for $i = 3, \ldots, k$ gives away a token to vertex w. Vertices $w = a_1$ and a_2 are not marked in the same round. As a result, vertex w gets no defect in the games $\Gamma_2, \ldots, \Gamma_k$. If a_1 is a C-neighbor of b, then game Γ_1 ensures that $\theta(a_1) \subseteq \{b\}$. Vertex a_2 is not adjacent to b.

Case 3.3. $A \subset G_2$, $w \notin A$.

Observe that $w \neq b$, as otherwise we could apply Case 3.1. Thus, vertex w is an (A, b)-cut. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{w\}, b)$,
- $\Gamma_2 = (G_2, A, w)$,
- $\Gamma_i = (G_i, \{w\}, y_i)$, for $i = 3, \ldots, k$.

Each vertex y_i, for $i = 3, \ldots, k$ gives away a token to vertex w. Vertex w gets at most one defect in each of the games Γ_1, Γ_2 and no defect in the games $\Gamma_3, \ldots, \Gamma_k$. Vertices in A are not adjacent to b.

In the analysis of the following cases we assume that G is biconnected. Thus, the boundary walk C is a simple cycle. For a vertex v in C we define v^+, and v^- to be respectively the next, and the previous vertex in C when C is traversed clockwise. We define the path $NP(v)$ that traverses neighbors of v clockwise from v^+ to v^-. For any two vertices u and v in C, let $N(u, v)$ denote the set of common neighbors of u and v, i.e. $N(u) \cap N(v)$. Now, assume u and v are C-neighbors. The *minimum common neighbor* of u and v, denoted $\operatorname{minn}(u, v)$, is a vertex w in $N(u, v)$ such that $\operatorname{int}[u, v, w, u]$ contains no other common neighbor of u and v. The *maximum common neighbor* of u and v, denoted $\operatorname{maxn}(u, v)$, is a vertex w in $N(u, v)$ such that $\operatorname{int}[u, v, w, u]$ contains all other common neighbors of u and v. As u and v are C-neighbors, any two common neighbors x_1, x_2 of u and v are on the same side of the edge $\{u, v\}$. Thus, we have that one of the sets $\operatorname{int}[x_1, u, v, x_1]$, $\operatorname{int}[x_2, u, v, x_2]$ is contained in the other and that both $\operatorname{minn}(u, v)$ and $\operatorname{maxn}(u, v)$ exist. Let a_1 and a_2 be the elements of A so that a_1, a_2, b appear in this order when C is traversed clockwise.

Case 4. C is a triangle.

We divide this case depending on the existence of a common neighbor of a_1, a_2, and b.

Case 4.1. Vertex d is adjacent to a_1, a_2, and b.

Figure 3 depicts the game division that we use in this case. Let G_1 be the graph $\operatorname{int}[a_1, d, b, a_1]$. Let P_1 be the path $NP(a_1)(d, b)$. Let G_2 be the graph $\operatorname{int}[a_2, b, d, a_2]$. Let P_2 be the path $NP(a_2)(b, d)$. Let G_3 be the graph $\operatorname{int}[a_1, a_2, d, a_1]$. Vertex d, each vertex in P_1, and each vertex in P_2 has three tokens. We divide the game into smaller games:

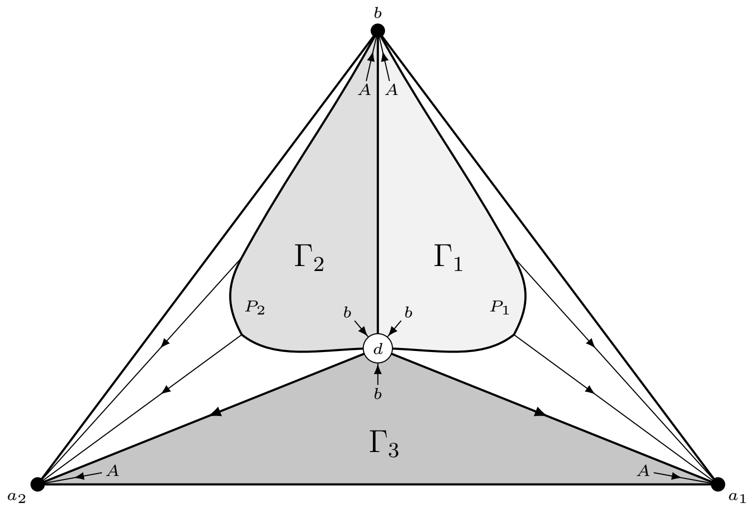

Figure 3: Game division in Case 4.1.

- $\Gamma_1 = (G_1 - a_1, \{b\}, d)$,
- $\Gamma_2 = (G_2 - a_2, \{b\}, d)$,
- $\Gamma_3 = (G_3, \{a_1, a_2\}, d)$.

Vertex d gives away a token to a_1, and one token to a_2. Each vertex in P_1 gives away a token to a_1. Each vertex in P_2 gives away a token to a_2.

As a result, vertices a_1, and a_2 get no defect in Γ_1, Γ_2, Γ_3. Thus, the only vertex that can give defect to a_1, or a_2 is b. Vertex d gets at most one defect in each of the games Γ_1, Γ_2, Γ_3. We have that $\theta(b) \subseteq \{a_1, a_2, d\}$ and each of these vertices is colored in a different round. Thus, vertex b gets at most one defect.

Case 4.2. There is no common neighbor of a_1, a_2, and b.

Figure 4 depicts the game division that we use in this case. Let G_0 be the subgraph of G induced by the vertices $\{a_1, a_2, b\}$. Let $x_i = \text{maxn}(a_i, b)$, and $y_i = \text{minn}(a_i, b)$, for $i = 1, 2$. Similarly, let $x_3 = \text{maxn}(a_1, a_2)$, and $y_3 = \text{minn}(a_1, a_2)$. As there is no common neighbor of a_1, a_2, and b, vertices x_1, x_2, and x_3 are pairwise different. Let G_i, for $i = 1, 2, 3$, be the connected component of $G - \{a_1, a_2, b, x_1, x_2, x_3\}$ that contains vertex y_i. In particular, if $x_i = y_i$ then G_i is an empty graph. Let G' be the graph obtained from G by removing a_1, a_2, b, $V(G_1)$, $V(G_2)$, and $V(G_3)$. Observe that the choice of x_1, x_2, x_3 guarantees that the boundary walk C' of G' is a simple cycle, and that each vertex in C' except x_1, x_2, x_3 is a neighbor of exactly one of the vertices a_1, a_2, or b. Let z_5 be the first (closest to x_1) neighbor of x_3 on the path $C'[x_1, x_3]$. Such a neighbor exists, and it is possible that $z_5 = x_1$. Similarly, let z_6 be the last (closest to x_2) neighbor of x_3 on the path $C'[x_3, x_2]$. It is possible that $z_6 = x_2$. Let P denote the inverted path $NC(x_3)(z_6, z_5)$. Let G_4 be the graph $\text{int}\,[z_5, P, C'[z_6, z_5]]$. Let G_5 be the graph $\text{int}\,[C'[z_5, x_3], z_5]$. Let G_6 be the graph $\text{int}\,[C'[x_3, z_6], x_3]$. Supplement each G_i, for $i = 1, 2, 3$, by adding vertex x_i to it. We divide the game into smaller games:

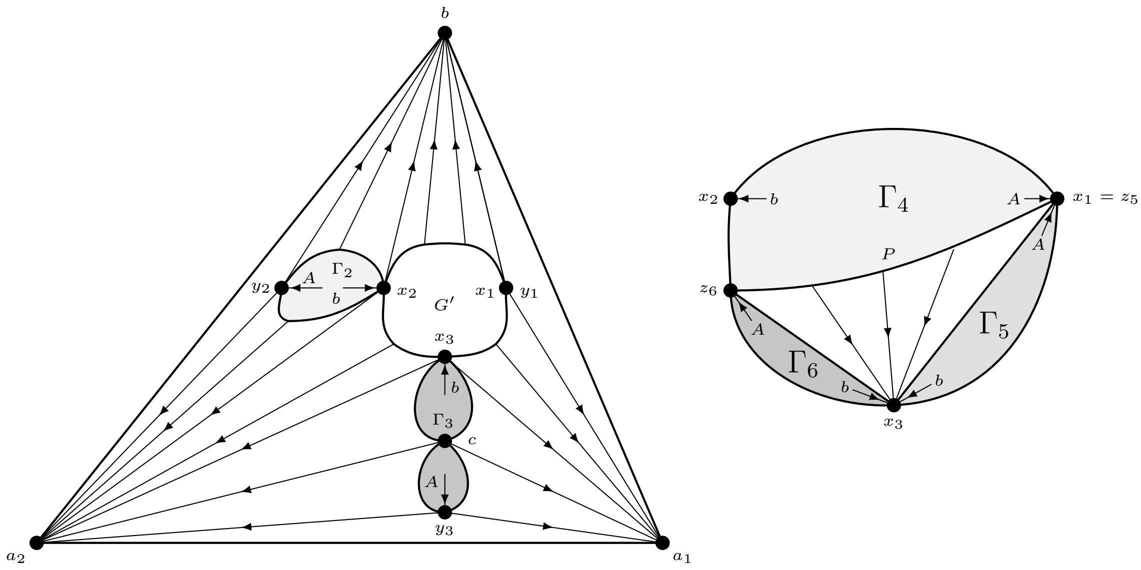

Figure 4: Game division in Case 4.2. On the left: graph G. Since $x_1 = y_1$, G_1 consists only of x_1. Vertex c is an (A, b)-cut in Γ_3. On the right: graph G' with boundary C'.

- $\Gamma_0 = (G_0, A, b)$,
- $\Gamma_1 = (G_1, \{y_1\}, x_1)$ (if $x_1 = y_1$, Γ_1 is not used),
- $\Gamma_2 = (G_2, \{y_2\}, x_2)$ (if $x_2 = y_2$, Γ_2 is not used),
- $\Gamma_3 = (G_3, \{y_3\}, x_3)$ (if $x_3 = y_3$, Γ_3 is not used),
- $\Gamma_4 = (G_4, \{x_1\}, x_2)$,
- $\Gamma_5 = (G_5, \{z_5\}, x_3)$,
- $\Gamma_6 = (G_6, \{z_6\}, x_3)$.

Each vertex adjacent to a_1, a_2, or b gives away a token to each of the adjacent vertices in $\{a_1, a_2, b\}$. This way we get that each vertex a_1, a_2, b receives at most one defect and that $\theta(a_1)$, and $\theta(a_2)$ are contained in $\{b\}$.

There is no common neighbor of a_1, a_2, and b, so each vertex gives away at most two tokens. Vertices that give away exactly two tokens to special vertices are x_i, y_i, and $(\{y_i\}, x_i)$-cuts in G_i, for $i = 1, 2, 3$. Thus, each vertex in G_1, G_2, G_3 has enough tokens for the games Γ_1, Γ_2, Γ_3. Each vertex x_1, x_2, x_3 gets at most one defect in the games Γ_1, Γ_2, Γ_3. Each vertex x_1, x_2, x_3 gets at most two defects in the games Γ_4, Γ_5, Γ_6.

Vertices in C' other than x_1, x_2, x_3 give away only one token to special vertices and have two tokens of value 3 left. When vertex z_5 is different than x_1, then it is marked with a token of value 2 in game Γ_4 in the round when x_3 is colored. This way we get that vertex z_5 different than x_1 gets at most two defects in Γ_4 and one defect in Γ_5 if it gets colored in the same round as x_3. If z_5 is colored in a different round, then it gets at most three defects in Γ_4 and no defect in Γ_5.

Similarly, when vertex z_6 is different than x_2, then it is marked with a token of value 2 in game Γ_4 when x_3 is colored in this round.

Case 5. Special vertex b is adjacent to an element of A.
We assume that b is adjacent to a_1. The other case, that b is adjacent to a_2, is fully symmetric. We divide this case depending on whether $\{a_1, b\}$ is an edge of C or not.

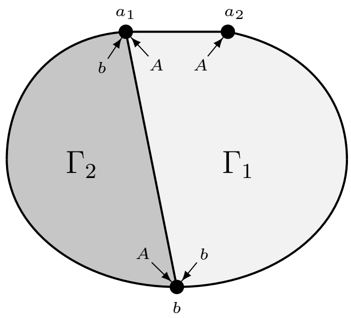

Figure 5: Game division in Case 5.1.

Case 5.1. $\{a_1, b\}$ is a chord of C.
Figure 5 depicts the game division that we use in this case. Observe that in this case a_1 is not a C-neighbor of b and a_1 can get defect from any vertex in G. Let G_1 be int $[C[a_1, b], a_1]$. Let G_2 be int $[C[b, a_1], b]$. We divide the game into smaller games:

- $\Gamma_1 = (G_1, A, b)$,
- $\Gamma_2 = (G_2, \{b\}, a_1)$.

When vertex b gets defect in Γ_2 then b gets the defect from a_1. Similarly, when vertex a_1 gets defect in Γ_1 then a_1 gets the defect from b. Thus, each vertex a_1, b gets at most one defect in both games Γ_1, Γ_2.

Case 5.2. $\{a_1, b\}$ is an edge of C.
Observe that $\{a_2, b\}$ is not an edge of C, as then C would be a triangle and we could apply Case 4. We divide this case further, depending on whether a_1 has a neighbor in C other than a_2 and b. Without the loss of generality, we may assume that b, a_1, a_2 appear in this order on C in a clockwise ordering of the vertices of C. (The assumption also holds in the following case.) Figure 6 depicts the game divisions that we use in the subcases.

Case 5.2.1. a_1 is not incident to a chord of C.
Let d be the vertex $\mathrm{maxn}(a_1, b)$. Assume that d is different than a_2, as otherwise the edge $\{a_2, b\}$ would be a chord of C and we could apply Case 5.1. Observe that d is an internal vertex, as a_1 is not incident to a chord of C.

Let P be the inverted path $NP(a_1)(a_2, d]$. Let Q be the inverted path $NP(b)(d, b^-]$. Let G_1 be int $[Q, P, C[a_2, b^-]]$. Let G_2 be int $[b, a_1, d, b]$. We divide the game into smaller games:

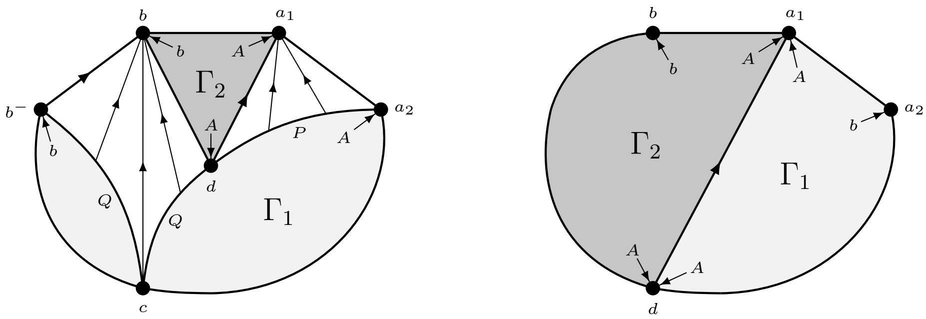

Figure 6: Game division in Cases 5.2.1 and 5.2.2, respectively. On the left: vertex c is an (A, b)-cut in Γ_1.

- $\Gamma_1 = (G_1, \{a_2\}, b^-)$,
- $\Gamma_2 = (G_2, \{a_1, d\}, b)$.

Vertices in P give away a token to a_1. Vertices in Q give away a token to b. Observe that vertices other than b^- that are both in C and Q are $(\{a_2\}, b^-)$-cuts in G_1. Vertex d is marked with the token of value 2 in Γ_1 if b is marked in the same round. If this occurs, then d gets at most one defect in Γ_2 (from b) and at most two defects in Γ_1. Otherwise, should vertex d be colored in this round, it gets no defect in Γ_2.

Case 5.2.2. $\{a_1, d\}$ is a chord of C.

Let G_1 be $\operatorname{int}[C[a_1, d], a_1]$. Let G_2 be $\operatorname{int}[C[d, a_1], d]$. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{a_1, d\}, a_2)$,
- $\Gamma_2 = (G_2, \{a_1, d\}, b)$.

Vertex d gives away a token to a_1. Vertex a_1 gets no defect in Γ_1. Vertex d gets at most one defect in Γ_1 and at most one defect in Γ_2.

Case 6. A chord $\{a_1, d\}$ of C separates a_2 from b.

Observe that the same game division as in Case 5.2.2 (see Figure 6) works also in this case. If a chord $\{a_2, d\}$ of C separates a_1 from b, then the same argument also works.

Case 7. Final case: None of the cases above holds.

Let P denote the path $C[a_2, b]$. Let Q denote the unique longest simple path from a_1 to b^+ in the subgraph induced by $V(G) - V(P)$ that traverses only vertices adjacent to P in G.

Let $p_1 = a_2$, and let $p_2, p_3, \ldots, p_{m-1}$ be the set of all interior vertices of path P that have at least two neighbors in Q, and occur in this order in P, and let $p_m = b$. As G is near-triangulated, for $i = 1, \ldots, m-1$, vertices p_i and p_{i+1} have a unique common neighbor in Q. Let $q_0 = a_1$, $q_m = b^+$, and for $i = 1, \ldots, m-1$, let q_i be the common neighbor of p_i and p_{i+1} in Q. Note that $q_1, \ldots, q_{m-1}$ are pairwise different. Moreover, if $q_0 = q_1$ then $\{a_1, p_2\}$ is a chord of C that separates a_2 from b and we can apply Case 6.

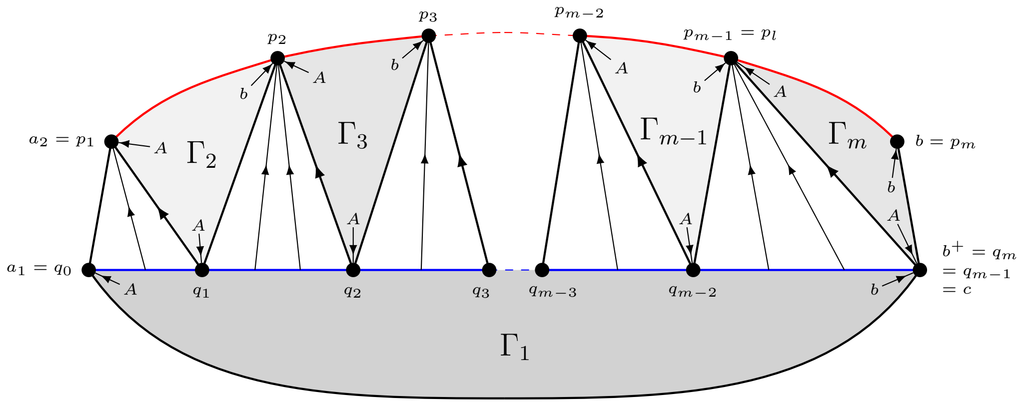

Figure 7: Game division in Case 7. In this figure $q_{m-1} = q_m$, $l = m - 1$, and $c = b^+$. Path P is depicted in red, path Q is depicted in blue.

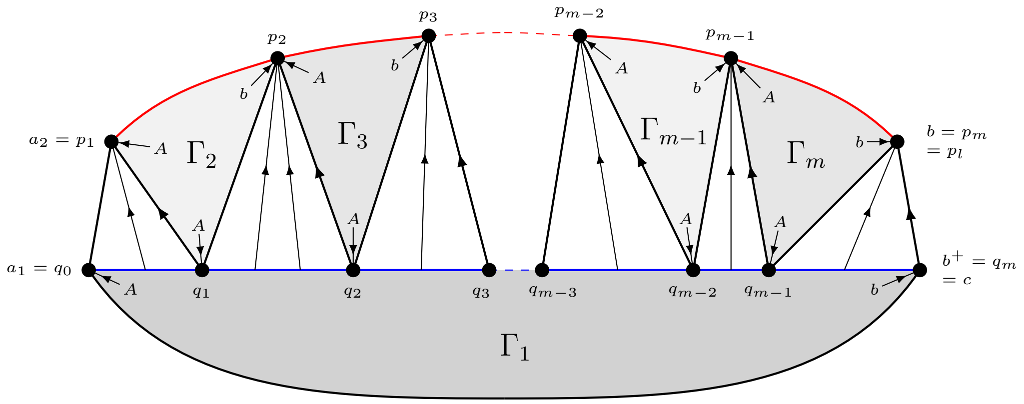

Figure 8: Game division in Case 7. In this figure $q_{m-1} \neq q_m$, $l = m$, and $c = b^+$. Path P is depicted in red, path Q is depicted in blue.

However, it is possible that $q_{m-1} = q_m$. See Figure 7 to see $q_{m-1} = q_m$, and Figure 8 to see $q_{m-1} \neq q_m$. In Figures 7, 8, the path Q does not intersect $C(b^+, a_1)$. See Figure 9 to see a non-empty intersection of Q and $C(b^+, a_1)$.

Observe, that for $i = 1, \ldots, m-1$, vertex q_i is not adjacent to any vertex p_j other than p_i and p_{i+1}. Indeed, an edge connecting q_i with p_j for $j < i$ would have to intersect with edges connecting p_i with Q. Similarly, an edge connecting q_i with p_j for $j > i+1$ would have to intersect with edges connecting p_{i+1} with Q. Each vertex in $Q(q_{i-1}, q_i)$ is not adjacent to any vertex in P other than p_i. Each vertex in $P(p_i, p_{i+1})$ is not adjacent to any vertex in Q other than q_i. Moreover, the definition of Q guarantees that there are no vertices in $\operatorname{int}[Q[q_i, q_{i+1}], p_{i+1}]$ other than the vertices of $Q[q_i, q_{i+1}]$, and p_{i+1}.

Let p be the first (closest to a_2) vertex on path P such that p is adjacent to a vertex in $C[b^+, a_1)$. The vertex p exists since $p_m = b$ is adjacent to b^+. Let c be the first (closest to b^+) neighbor of p on the path $C[b^+, a_1)$ and observe that c is a vertex of Q. Let l be the minimal l such that $c = q_l$ or that c is in $Q(q_{l-1}, q_l)$ and observe that p_l is the first (closest to a_2) vertex on path P that is adjacent to c. Thus, we have $p = p_l$.

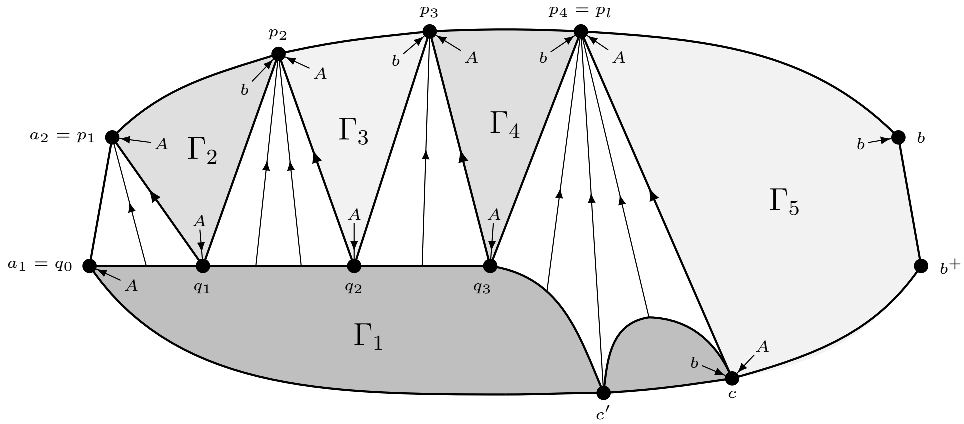

Figure 9: Game division in Case 7. In this figure $c \neq b^+$.

If $p_l = p_1$ then $\{a_2, c\}$ is a chord of C that separates a_1 from b and we can apply Case 6. Thus, we can assume that $2 \leqslant l \leqslant m$. In case $l = m$ we have that $c = b^+$. See Figure 8. For $l = m - 1$ and $q_{m-1} = q_m = b^+$, we also have $c = b^+$. See Figure 7. Otherwise, we have $c \neq b^+$. See Figure 9. Observe, that in any case we have that $Q(a_1, q_{l-1}]$ does not intersect C. On the other hand, $Q(q_{l-1}, c)$ might intersect C if p_l has more than one neighbor in C.

Let G_1 be $\operatorname{int}[C[c, a_1], Q(a_1, c]]$. Let G_i, for $i = 2, 3, \ldots, l$, be $\operatorname{int}[p_{i-1}, p_i, q_{i-1}, p_{i-1}]$. Additionally, if $l < m$, let G_{l+1} be $\operatorname{int}[C[p_l, c], p_l]$. If $l = m$, then G_{l+1} is not defined. Observe that any neighbor of p_l in C other than c is an $(\{a_1\}, c)$-cut in G_1.

For each vertex p_i, for $i = 2, \ldots, \min(l, m-1)$, we devalue the token function by removing the token of value 2. This way we obtain that all vertices p_i for $i = 1, \ldots, l$ have one token. We divide the game into smaller games:

- $\Gamma_1 = (G_1, \{a_1\}, c)$,
- $\Gamma_i = (G_i, \{p_{i-1}, q_{i-1}\}, p_i)$, for $i = 2, 3, \ldots, l$,
- $\Gamma_{l+1} = (G_{l+1}, \{p_l, c\}, b)$ (if $l = m$, Γ_{l+1} is not defined).

For $i = 1, \ldots, l-1$, each vertex in $Q(q_{i-1}, q_i]$ gives away a token to p_i. Each vertex in $Q(q_{l-1}, c]$ gives away a token to p_l. For $i = 1, \ldots, l-1$, vertex q_i different than c has one token of value 2 and one token of value 3 in the game Γ_1. We mark such a vertex with a token of value 2 when p_{i+1} is colored in the same round. Otherwise it is marked with a token of value 3 in Γ_1.

Vertex a_1 gets at most one defect in game Γ_1. Vertex a_2 gets at most one defect in game Γ_2. Vertex b gets at most one defect in game Γ_{l+1} when $l < m$, and at most one defect in game Γ_m when $l = m$. For $i = 2, \ldots, m-1$, vertex p_i gets at most one defect in Γ_i, and at most one defect in Γ_{i+1}. The strategy that chooses the value of a token removed from vertex q_i in Γ_1 guarantees that each vertex q_i receives at most three defects in total. □

3 Lister's strategy

In this section we show a planar graph which is not 2-defective 3-paintable. We begin with a definition of a family of outerplanar graphs that play a crucial role in the construction.

An *l-layered, k-petal daisy* $D(l, k)$ is an outerplanar graph with the vertex set partitioned into l layers, $L_1, \ldots, L_l$, defined inductively as follows:

- 1-layered, k-petal daisy $D(1, k)$ is a single edge $\{u, v\}$, and $L_1 = \{u, v\}$.
- l-layered, k-petal daisy $D(l, k)$ for $l > 1$ extends $D(l-1, k)$ in the following way: for every edge $\{u, v\}$ of $D(l-1, k)$ with $u, v \in L_{l-1}$ we add a path $P(u, v)$ on $2k-1$ new vertices and join the first k vertices of $P(u, v)$ to u and join the last k vertices of $P(u, v)$ to v. The *inner* vertices of path $P(u, v)$ are all the vertices of $P(u, v)$ except the two end-points. We set

$$L_l = \bigcup \{P(u, v) : \{u, v\} \text{ is an edge with both endpoints in } L_{l-1}\}.$$

We draw $D(l, k)$ in an outerplanar way, i.e., such that all vertices are adjacent to the outerface. In particular, in such a drawing all inner faces of $D(l, k)$ are triangles – see Figure 10 for an example.

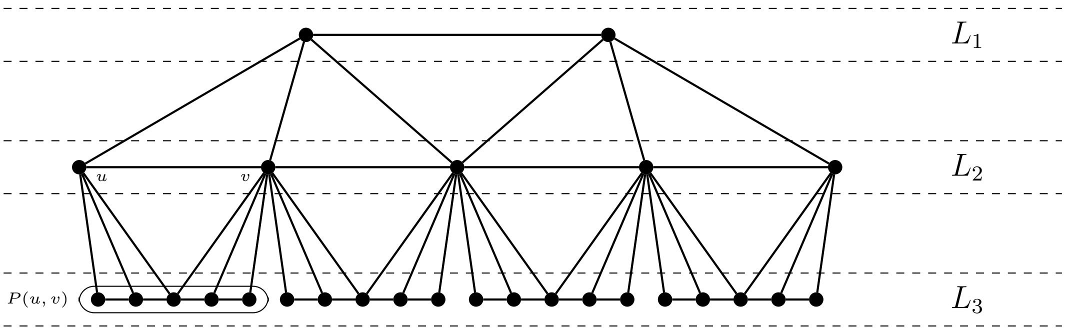

Figure 10: A 3-layered 3-petal daisy $D(3, 3)$.

A planar graph G is an *edge extension of* $D(l, k)$ if G extends $D(l, k)$ in the following way: for every inner face F of $D(l, k)$ we add a vertex $v(F)$ that is adjacent to some two vertices on the boundary of F. A planar graph G is the *face extension of* $D(l, k)$ if G extends $D(l, k)$ in the following way: for every inner face F of $D(l, k)$ we add a set $u(F)$ of four vertices such that one vertex in $u(F)$, say u, is adjacent to all vertices on the boundary of F, and for every edge e on the boundary of F, one vertex in $u(F)$ is adjacent to u and to the endpoints of e. If G is an edge/face extension of $D(l, k)$, the copy of $D(l, k)$ in G is called *the skeleton* of G and is denoted skel(G). Let ext(G) denote the vertices in G that are not in skel(G). See Figure 11 for examples of an edge extension and of a face extension.

Lemma 4. *Any edge extension of $D(l, k)$ for $l = 4$ and $k = 362$ is not 2-defective 2-choosable.*

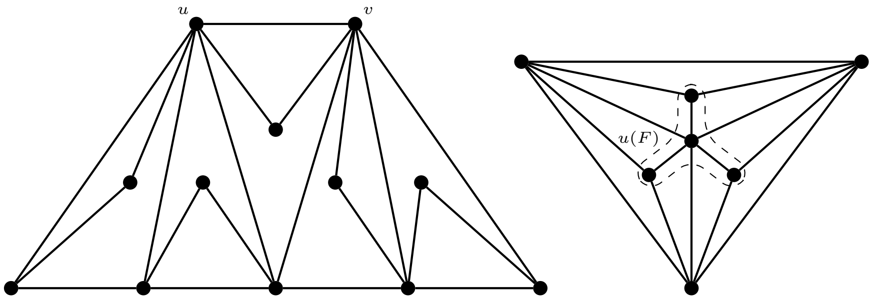

Figure 11: On the left: an edge extension of $D(2,3)$. On the right: the face extension of $D(2,1)$ (which has only one inner face F).

Before the proof of Lemma 4 we show how to use it to construct a planar graph that is not 2-defective 3-paintable.

Proof of Theorem 2. Fix $l = 4$ and $k = 362$. Let $G_1, \ldots, G_9$ be nine copies of the face extension of $D(l,k)$. Let G be a planar graph that is formed of $G_1, \ldots, G_9$ and a vertex v joined to every vertex in $\text{skel}(G_1), \ldots, \text{skel}(G_9)$. We show a winning strategy for Lister in a 2-defective 3-painting game on G.

In the first four rounds Lister plays the following strategy:

- in the i-th round, for $i = 1, 2, 3$, Lister marks v, if it is still uncolored, and the vertices in $\text{skel}(G_{3i-2})$, $\text{skel}(G_{3i-1})$, and $\text{skel}(G_{3i})$.
- in the 4-th round, Lister marks all the vertices in $\text{ext}(G_1), \ldots, \text{ext}(G_9)$.

Clearly, Painter needs to color vertex v in one of the first three rounds. Say, he colors v in the i-th round. All vertices from the skeletons of G_{3i-2}, G_{3i-1}, and G_{3i} are adjacent to v and at most two of them are colored in the i-th round. Let H be a graph, one among G_{3i-2}, G_{3i-1}, G_{3i}, such that no vertex of H is colored in the i-th round. Observe that after three rounds, all vertices from $skel(H)$ are uncolored and have only two tokens left.

In the fourth round, for any inner face F of $skel(H)$, Painter colors at most three vertices in $u(F)$. Thus, for every inner face F of $skel(H)$, at least one vertex in $u(F)$ is still uncolored and has only two tokens left.

Let H' be the graph induced by the set of uncolored vertices in H after the 4-th round. Clearly, H' is a supergraph of some edge extension of $D(l,k)$ and each vertex in H' has two tokens left. The state of the game on H' is the same as the initial state of a 2-defective 2-painting game on H'.

By Lemma 4, graph H' is not 2-defective 2-choosable and hence Lister has a winning strategy in the 2-defective 2-painting game on H'. ☐

Proof of Lemma 4. Fix $l = 4$ and $k = 362$. Let G be an edge extension of $D(l,k)$. For notational convenience, let $D(l,k)$ denote the skeleton of G, and use the notation introduced in the definition of $D(l,k)$.

We split all vertices of the first three layers of $D(l,k)$ into two categories. A vertex $x \in L_i$ for $i < l$ is *bad* if there exist: a vertex $y \in L_i$ adjacent to x; an inner vertex z of the path $P(x,y)$; and a vertex in $\text{ext}(G)$ that is adjacent both to x and z. Otherwise, x is *good*. For example, in Figure 11, vertex u is good, while vertex v is bad.

Let z be a vertex in L_i, $i \in [2]$. Note that the neighborhood of z in L_{i+1} induces one or two paths of size k in G: we denote them $P_1(z)$, and $P_2(z)$. If the neighborhood of z in L_{i+1} induces only one path then $P_2(z)$ is undefined.

We claim that if some vertex z in L_1 or L_2 has at least 15 bad neighbors in $P_j(z)$ for some $j \in [2]$ then G is not 2-defective 2-colorable. Suppose to the contrary that z has 15 bad neighbors in $P_j(z)$ for some $j \in [2]$ and that there is a 2-defective coloring of G with colors α and β. Without loss of generality, z is colored α. Among the neighbors of z in $P_j(z)$ at most two are colored α. Vertices colored α in $P_j(z)$ split $P_j(z)$ into at most three subpaths that consist only of vertices colored β. As there are at least 15 bad vertices in $P_j(z)$ there is a subpath P of $P_j(z)$ that consists of 5 vertices colored β such that the middle vertex of P is bad. Let x be the the middle vertex of P and y, y' be the two neighbors of x in P. As each of the vertices y, x, y' has two neighbors colored β in P, all vertices in $P(x,y)$, and all vertices in $P(x,y')$ are colored α. Since x is bad, there is a vertex w in $\text{ext}(G)$ that is adjacent to x and to some inner vertex t in $P(x,y)$ or in $P(x,y')$. If w is colored α then t has three neighbors colored α. If w is colored β then x has three neighbors colored β. So, the considered coloring is not 2-defective, a contradiction.

For the rest of the proof we assume that every vertex x in layers L_1, L_2 of $D(l,k)$ has at most 14 bad neighbors in $P_j(x)$, $j \in [2]$. Let x be a good vertex in L_2 (such a vertex x exists as $k > 14$) and let y be any neighbor of x in L_2. Let W be a path of 24 good neighbors of x that are inner vertices of $P(x,y)$. Such a path W exists as bad neighbors of x split $P(x,y)$ into at most 15 subpaths of good vertices. For $k = 362$, one of those subpaths has at least 24 inner vertices. We number the consecutive elements of W by $w_1, \ldots, w_{24}$ according to the order they appear on the path $P(x,y)$. Now, for $i \in [23]$, we denote the following vertices:

- c_i – a common neighbor of w_i and w_{i+1} in the path $P(w_i, w_{i+1})$,
- a_i – a vertex $v(F)$ of the face F with boundary x, w_i, w_{i+1},
- b_i – a vertex $v(F)$ of the face F with boundary c_i, w_i, w_{i+1}.

Note that a_i is adjacent to w_i and w_{i+1} as x is good, and that b_i is adjacent to w_i and w_{i+1} as both w_i and w_{i+1} are good. We claim that the graph G' induced by the vertex set

$$\{x\} \cup W \cup \{a_i, b_i, c_i : i \in [23]\}$$

is not 2-defective 2-choosable, which completes the proof of the lemma.

Consider the following 2-list assignment L of G' with colors $\{\alpha, \beta, 1, \ldots, 24\}$:

- $L(x) = \{\alpha, \beta\}$,

- $L(w_i) = \{i, \alpha\}$ for $i \in \{1, \dots, 12\}$,
- $L(w_i) = \{i, \beta\}$ for $i \in \{13, \dots, 24\}$,
- $L(a_i) = L(b_i) = L(c_i) = \{i, i+1\}$.

Now, suppose that c is a 2-defective L-coloring of G'. Without loss of generality we assume that $c(x) = \alpha$. It follows that among the vertices $w_1, \dots, w_{12}$ at most 2 are colored α. Thus, there are four consecutive vertices in $w_1, \dots, w_{12}$, say $w_j, w_{j+1}, w_{j+2}, w_{j+3}$ for some $j \in [9]$, that are not colored α. We have $c(w_l) = l$ for $l \in \{j, \dots, j+3\}$. Since $c(w_j) = j$, at most two vertices in the set $\{a_j, b_j, c_j\}$ are colored j. Thus, at least one vertex in this set is colored $j+1$. Since $c(w_{j+1}) = j+1$, at most one vertex in $\{a_{j+1}, b_{j+1}, c_{j+1}\}$ is colored $j+1$. Thus, at least two vertices in this set are colored $j+2$. Eventually, since $c(w_{j+2}) = j+2$, all vertices in the set $\{a_{j+2}, b_{j+2}, c_{j+2}\}$ are colored $j+3$. However, w_{j+3} is also colored $j+3$ and c is not a 2-defective L-coloring. $\square$

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