Convex geometries are extremal for the generalized Sauer-Shelah bound

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Abstract

The Sauer-Shelah lemma provides an exact upper bound on the size of set families with bounded Vapnik-Chervonekis dimension. When applied to lattices represented as closure systems, this lemma outlines a class of extremal lattices obtaining this bound. Here we show that the Sauer-Shelah bound can be easily generalized to arbitrary antichains, and extremal objects for this generalized bound are exactly convex geometries. We also show that the problem of classification of antichains admitting such extremal objects is NP-complete.

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1 Introduction

The Sauer-Shelah lemma is a renowned result from extremal set theory, establishing an exact bound on the number of sets in a family not shattering any k-set.

Lemma 1 (Sauer-Shelah). *If $\mathcal{F}$ is a family of subsets of U, $|U| = n$, and $|\mathcal{F}| > H(n, k)$, where $H(n, k)$ is a sum of first k binomial coefficients of n, that is,*

$$H(n,k) := \sum_{i=0}^{k-1} \binom{n}{i},$$

then $\mathcal{F}$ shatters some k-set.

This bound is trivially reached by the family of all subsets of the base set of size at most $k-1$. If we restrict ourselves to some specific set families, the upper bound remains intact. However, its sharpness may be lost or difficult to prove. In [1] and its extended version [2] A.Albano and the author, although in lattice-theoretic terms, have proved the sharpness of this bound for families of closed sets, and characterized families reaching this bound.

Another natural generalization of the Sauer-Shelah lemma is to change constraints and seek for families that do not shatter any set from some fixed antichain $\mathcal{A}$, the original bound corresponding to the antichain $\mathcal{A}_k$ of all k-sets. For arbitrary set families such generalization is straightforward. In this paper we make similar generalization for families of closed sets. Extremal objects for this generalized case turn out to be exactly convex geometries.

The structure of the paper is as follows. In Section 2 we introduce basic terminology. In Section 3 we give a generalized version of the Sauer-Shelah lemma and define extremal families of closed sets. In Section 4 we define extremal lattices, argue about their relation to extremal closure systems, and give some examples. In Section 5 we explore some degenerate cases of extremal families and ways to deal with them. In Section 6 we characterize extremal closure families as convex geometries and extremal lattices as meet-distributive lattices. In Section 7 we reformulate circuit characterization of antimatroids in order to obtain characterization of extremal families of closed sets in terms of implications. Finally, in Section 8 we consider a problem of characterization of antichains yielding extremal closure families. As we show, this problem turns out to be NP-complete.

2 Preliminary definitions

If not stated otherwise, all objects in our paper are finite. Typically, we are dealing with subsets of a *base set* U, $|U| = n$. A *k-set* is a set with k elements. At times, we denote by $\mathbf{k}$ some fixed k-set. For a set $X \subseteq U$ and a set family $\mathcal{F}$, a *trace* of $\mathcal{F}$ on X, denoted $\mathrm{T}_{\mathcal{F}}(X)$, is defined as:

$$\mathrm{T}_{\mathcal{F}}(X) = \{F \cap X \mid F \in \mathcal{F}\}.$$

The power set of X is denoted by $\mathrm{P}(X)$. The set family $\mathcal{F}$ *shatters* X if $\mathrm{T}_{\mathcal{F}}(X) = \mathrm{P}(X)$. The Vapnik-Chervonekis (VC) dimension of $\mathcal{F}$ is the maximal size of a set shattered by $\mathcal{F}$.

A family $\mathcal{I}$ is *hereditary* if it contains every subset of A, for all A in $\mathcal{I}$. A family $\mathcal{C}$ is a *closure system* if it contains U and is closed under set intersection. Interchangeably, we call closure systems *families of closed sets*. When dealing with closure systems we denote by $\overline{X}$ the least closed subset containing X.

Closure systems are of special interest to us because of their intimate connection with lattices. Every closure system $\mathcal{C}$, partially ordered by set inclusion, is a lattice, which we denote by $L(\mathcal{C})$, or simply $L_{\mathcal{C}}$. On the other hand, if for a lattice L and an element $x \in L$ we denote by $\mathrm{J}(L)$ the set of join irreducible elements of L, and $\mathrm{J}(x) = (x] \cap \mathrm{J}(L)$, then we can define closure system $\mathcal{C}(L)$, or $\mathcal{C}_L$, over $\mathrm{J}(L)$ as:

$$\mathcal{C}_L = \big\{\mathrm{J}(x) \mid x \in L\big\}.$$

With this notation it holds:

$$L(\mathcal{C}(L)) \cong L$$

and

$$\mathcal{C}(L(\mathcal{C}_L)) \cong \mathcal{C}_L.$$

Notice, however, that for an arbitrary $\mathcal{C}$ in general $\mathcal{C}(L(\mathcal{C})) \ncong \mathcal{C}$, due to the fact that non-isomorphic set families may be isomorphic as posets. We call $\mathcal{C}_L$ the *canonical closure system* for L.

We denote a boolean lattice with k atoms by $B(k)$. We say that a lattice L is $B(k)$*-free* if it does not have an order-embedding of $B(k)$. As we show later in Lemma 8, a lattice L is $B(k)$-free if and only if $\mathcal{C}$ does not shatter a k-set, for any $\mathcal{C}$ such that $L = L(\mathcal{C})$. Notice that we can use here any closure system, not only the canonical one. Using this correspondence, we can define VC-dimension of a lattice L as a smallest k such that L is $B(k+1)$-free, and this definition trivially agrees with the one for set families.

In [1] it was shown that it is possible to construct a closure system $\mathcal{C}$ of size $H(n,k)$ not shattering any k-set, or alternatively, to construct a $B(k)$-free lattice with $H(n,k)$ elements. Lattices obtained that way were called (n,k)-extremal and were completely characterized as lattices obtained by recursive application of a specific doubling construction. Now we proceed to characterize corresponding objects for the generalized Sauer-Shelah bound.

3 Generalized Sauer-Shelah lemma and extremality

In order to pave the road for the generalization of the Sauer-Shelah lemma to arbitrary antichains, we will introduce some simple notation.

Let $\mathcal{A}$ be an antichain of sets over U. We say that $\mathcal{F}$ does not shatter $\mathcal{A}$ if it does not shatter any set A from $\mathcal{A}$. By $\mathcal{I}(\mathcal{A})$ we denote the family of all subsets of U not containing any set from $\mathcal{A}$ as a subset, that is,

$$\mathcal{I}(\mathcal{A}) = \{X \mid X \not\supseteq A, \text{ for all } A \text{ in } \mathcal{A}\},$$

and by $s(\mathcal{A})$ we denote the size of $\mathcal{I}(\mathcal{A})$. Trivially, $\mathcal{I}(\mathcal{A})$ is hereditary and does not shatter $\mathcal{A}$.

As a technical tool, we need the following lemma, used implicitly in [6], and formulated in its present state as [10, Theorem 10.2]. The sketch of the proof follows the latter.

Lemma 2 (Frankl)**.** *For an arbitrary set family $\mathcal{F}$ there is a hereditary family $\mathcal{I}$ of equal size, such that $|\mathrm{T}_S(\mathcal{I})| \leqslant |\mathrm{T}_S(\mathcal{F})|$, for any $S \subseteq U$.*

Proof. For a non-hereditary $\mathcal{F}$, let us pick a "bad coordinate", that is, a point x in U, for which there is $A \in \mathcal{F}$, such that $A - x \notin \mathcal{F}$. With x and $\mathcal{F}$ fixed, let us define an injective transformation $T_x \colon \mathcal{F} \to \mathrm{P}(U)$ in the following way:

$$T_x(A) = \begin{cases} A, & x \notin A; \\ A, & x \in A, A - x \in \mathcal{F}; \\ A - x, & x \in A, A - x \notin \mathcal{F}. \end{cases}$$

It is now easy to check that for $\mathcal{F}_x = T_x(\mathcal{F})$, it holds: $|\mathcal{F}_x| = |\mathcal{F}|$ and $|\mathrm{T}_S(\mathcal{F}_x)| \leqslant |\mathrm{T}_S(\mathcal{F})|$, for any $S \subseteq U$. Moreover, $\mathcal{F}_x$ has same bad coordinates as $\mathcal{F}$, except for x. The statement of the lemma now follows by induction on the number of bad coordinates. □

Theorem 3 (Sauer-Shelah, generalized)**.** *If $\mathcal{A}$ is an antichain of sets in U and $\mathcal{F}$ is a family with $|\mathcal{F}| > s(\mathcal{A})$ then $\mathcal{F}$ shatters $\mathcal{A}$.*

Proof. Obviously, a hereditary family shatters $\mathcal{A}$ if and only if it contains some set from $\mathcal{A}$. Thus, $\mathcal{I}(\mathcal{A})$ is the maximal hereditary family not shattering $\mathcal{A}$. Now, let $\mathcal{F}$ be a family not shattering $\mathcal{A}$. By Lemma 2 there is a hereditary family $\mathcal{F}'$ with $|\mathcal{F}'| = |\mathcal{F}|$, not shattering $\mathcal{A}$. But then $\mathcal{F}' \subseteq \mathcal{I}$, and thus $|\mathcal{F}| \leqslant |\mathcal{I}| = s(\mathcal{A})$. □

We say that a set family $\mathcal{F}$ is *extremal for an antichain* $\mathcal{A}$ if $|\mathcal{F}| = s(\mathcal{A})$ and if $\mathcal{F}$ does not shatter $\mathcal{A}$. We say that $\mathcal{F}$ is *extremal* if it is extremal for some antichain. For any family $\mathcal{F}$ in U, not necessarily extremal, we define a *blocking antichain* $\mathcal{A}_{\mathcal{F}}$ *of* $\mathcal{F}$ as the family of all subsets of U, minimal with respect to being not shattered by $\mathcal{F}$. Trivially, $\mathcal{A}_{\mathcal{F}}$ is an antichain. Note also, that $\mathcal{I}(\mathcal{A}_{\mathcal{F}})$ is the family of all subsets shattered by $\mathcal{F}$.

An easy example of extremal families is the hereditary families.

Lemma 4. *Every hereditary family is extremal.*

Proof. Let $\mathcal{J}$ be a hereditary family. We claim that in this case $\mathcal{J}$ is extremal for $\mathcal{A}_{\mathcal{J}}$. Indeed, $\mathcal{J}$ does not shatter any set from $\mathcal{A}_{\mathcal{J}}$, and suppose that $|\mathcal{J}| < s(\mathcal{A}_{\mathcal{J}}) = |\mathcal{I}|$, where $\mathcal{I} = \mathcal{I}(\mathcal{A}_{\mathcal{J}})$. As $\mathcal{I}$ is maximal of all hereditary sets not shattering $\mathcal{A}_{\mathcal{J}}$, we have $\mathcal{J} \subsetneq \mathcal{I}$, and we can take a minimal set B in $\mathcal{I} - \mathcal{J}$. But from minimality of B it follows that $B - x \in \mathcal{J}$, for all $x \in \mathcal{J}$. Thus, $\mathcal{J}$ does not shatter B, but shatters $B - x$, for any $x \in B$, which means that $B \in \mathcal{A}_{\mathcal{J}}$. But, as $B \in \mathcal{I}$, it follows that $\mathcal{I}$ shatters $\mathcal{A}_{\mathcal{J}}$, a contradiction. □

Our next goal is to show that, given a family $\mathcal{F}$, in order to check its extremality we do not need to go through all possible antichains. In fact, the only antichain for which $\mathcal{F}$ can be extremal is $\mathcal{A}_{\mathcal{F}}$.

For two antichains $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$ *refines* $\mathcal{B}$, denoted $\mathcal{A} \ll \mathcal{B}$, if for each $b \in \mathcal{B}$ there is $a \in \mathcal{A}$ such that $a \leqslant b$. In particular, $A \ll \mathcal{B}$ whenever $\mathcal{B} \subseteq \mathcal{A}$. The refinement relation is a partial order on antichains, see for example [8, Lemma 1.15]. The following proposition is straightforward:

Proposition 5. *If $\mathcal{A} \ll \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$ then $s(\mathcal{A}) < s(\mathcal{B})$.*

Lemma 6. *If $\mathcal{F}$ is extremal for an antichain $\mathcal{B}$ then $\mathcal{B} = \mathcal{A}_{\mathcal{F}}$.*

Proof. From the definition of $\mathcal{A}_{\mathcal{F}}$ it follows that for any $B \in \mathcal{B}$ there is $A \in \mathcal{A}_{\mathcal{F}}$ with $A \subseteq B$, that is, $\mathcal{A}_{\mathcal{F}}$ refines $\mathcal{B}$. From Proposition 5 it follows that $|\mathcal{F}| \leqslant s(\mathcal{A}_{\mathcal{F}}) < s(\mathcal{B})$ whenever $\mathcal{B} \neq \mathcal{A}_{\mathcal{F}}$, which contradicts extremality. □

We say that $\mathcal{F}$ *almost shatters* $A \subseteq U$ if $|\mathrm{P}(A)| - |\mathrm{T}_{\mathcal{F}}(A)| = 1$, that is, if there is a unique X in $\mathrm{P}(A) - \mathrm{T}_{\mathcal{F}}(A)$.

Lemma 7. *If $\mathcal{F}$ is extremal then $\mathcal{F}$ almost shatters every set in $\mathcal{A}_{\mathcal{F}}$.*

Proof. By Lemma 2 there is a hereditary family $\mathcal{I}$ of size $|\mathcal{F}|$ such that $|Tr_{\mathcal{I}}(A)| \leqslant |Tr_{\mathcal{F}}(A)|$ for all $A \in \mathcal{A}_{\mathcal{F}}$, in particular $\mathcal{I}$ does not shatter $\mathcal{A}_{\mathcal{F}}$. Due to extremality of $\mathcal{F}$, there is exactly one such hereditary family, namely $\mathcal{I}(\mathcal{A}_{\mathcal{F}})$. Thus, $2^{|A|} > |Tr_{\mathcal{F}}(A)| \geqslant |Tr_{\mathcal{I}}(A)| = 2^{|A|} - 1$, implying $|\mathrm{P}(A)| - |\mathrm{T}_{\mathcal{F}}(A)| = 1$. □

The converse of Lemma 7 does not hold, that is, there are families which almost shatter all sets from its blocking antichain, yet are not extremal. For example, let $\mathcal{F}$ be a family of sets in $U = \{1,2,3,4\}$ not covering 123 and intersecting 234, that is,

$$\mathcal{F} = \{2, 3, 4, 12, 13, 14, 23, 24, 34, 124, 134, 234\}.$$

Then $\mathcal{A}_{\mathcal{F}} = \{123, 234\}$, $\mathcal{F}$ almost shatters both sets from $\mathcal{A}_{\mathcal{F}}$, and it is easy to see that $s(\mathcal{A}_{\mathcal{F}}) = 13 > |\mathcal{F}| = 12$.

As a conclusion to this section we also note that extremal set families are also studied under the name *shattering-extremal families*, see [13] for a comprehensive survey on the topic.

4 Extremal lattices

Apart from describing extremal closure systems we also want to describe extremal lattices. We start with proving the result announced in Section 2, which relates shattered sets in closure systems and boolean suborders in lattices. This result implicitly appears in [2] as an easy corollary (although not stated explicitly) of Lemmas 1 and 4. However, this approach requires introducing a large bulk of terminology from Formal Concept Analysis, so instead we will now prove it directly.

Lemma 8. *Let us fix a lattice L and a closure system $\mathcal{C}$, such that $L = L(\mathcal{C})$. Then L has an order-embedding of $B(k)$ if and only if $\mathcal{C}$ shatters some k-set.*

Proof. $(\Rightarrow)$: Let us fix an order-embedding $B(k)$ in L, we denote its elements by e_Z, $Z \subseteq \mathbf{k}$, in a straightforward way. We also associate elements of L with corresponding subsets in U, where U is a base set for $\mathcal{C}$. Thus, $e_Z \subseteq U$, for all $Z \subseteq \mathbf{k}$. Now, for all $x \in \mathbf{k}$ let us fix $a_x \in e_{\{x\}}$ such that $a_x \notin e_{\mathbf{k}-x}$, this could be done as $e_{\{x\}}$ and $e_{\mathbf{k}-x}$ are incomparable as elements in L, and thus as subsets in U.

We argue that $\mathcal{C}$ shatters the k-set $A = \{a_x \mid x \in \mathbf{k}\}$. Indeed, let B be a subset of A, then it can be represented as $B = \{a_x \mid x \in W\}$, for some $W \subseteq \mathbf{k}$. Then $B = A \cap e_W$. Indeed, for $x \in W$ we get $a_x \in e_{\{x\}} \subseteq e_W$, and for $x \notin W$ we get $a_x \notin e_{\mathbf{k}-x} \supseteq e_W$.

$(\Leftarrow)$: Let $\mathcal{C}$ shatter $A \subseteq U$. We argue that all the sets $\{\overline{X} \mid X \subseteq A\}$ are different, from which it is immediate that in $L(\mathcal{C})$ they form an order-embedding of $B(k)$.

Indeed, let X and Y be different subsets of A, and suppose there is $y \in Y - X$ (otherwise switch X and Y). But then, as $\mathcal{C}$ shatters A, there is some $C \in \mathcal{C}$ such that $C \cap A = X$. Then $\overline{X} \subseteq \overline{C} = C$, implying $y \notin \overline{X}$. But as $y \in Y \subseteq \overline{Y}$, then $\overline{Y} \neq \overline{X}$, finishing the proof. □

Now, let us define extremal lattices. We say that a lattice L shatters $A \subseteq \mathrm{J}(L)$ if $\mathcal{F}_L$ shatters $\mathcal{A}$. We say that L is extremal if $\mathcal{F}_L$ is extremal, in which case we denote $\mathcal{A}(\mathcal{F}_L)$ by $\mathcal{A}_L$.

There are two problems with this definition. First, let us consider the closure system $\mathcal{C} = \{\emptyset, 12\}$ over $U = \{1, 2\}$. It is easy to see that $\mathcal{C}$ is not extremal. The corresponding lattice $L_\mathcal{C}$ is a two-element lattice and $\mathcal{C}' = \mathcal{C}(L_\mathcal{C})$ may be represented as $\mathcal{C}' = \{\emptyset, 1\}$ over $U' = \{1\}$. Now, $\mathcal{C}'$ is extremal, and consequently so is $L_\mathcal{C}$. Thus, non-extremal closure systems may give rise to extremal lattices. Of course, this can happen only when $\mathcal{C}$ is not canonical, but still it is unfortunate.

For the second example let us consider $\mathcal{C} = \{1, 12\}$ over $U = \{1, 2\}$, which is extremal. However, $\mathcal{C}$ is not canonical for any lattice, as every canonical closure system contains the empty set. Thus, there are extremal non-canonical closure systems.

Luckily, in the next section we show that both these situations can be easily handled and make no substantial difference. We end this section with several easy examples of extremal lattices.

Example 9. A lattice with a single element is extremal with $\mathcal{F}_L$ being an empty family over an empty set of join irreducible elements, $\mathcal{A}_L = \{\emptyset\}$.

Example 10. Let L be a distributive lattice represented as a family of order-ideals over a poset P. Then L is extremal and $\mathcal{A}_L$ is an antichain over P given by

$$\mathcal{A}_L = \big\{\{x, y\} \mid x < y\big\}$$

Example 11. Every (n, k)-extremal lattice L defined in [1] is extremal over U, $|U| = n$, and $\mathcal{A}_L$ is the antichain of all k-sets.

5 Reduction of antichains and closure systems

In this section we deal with some trivial cases of extremal families and the way to surpass them. This part is rather boring, but we need it for the sake of generality.

We call an antichain $\mathcal{A}$ *redundant* if in $\mathcal{A}$ there is a one-point set, and *irredundant* otherwise. For an antichain $\mathcal{A}$ we define a *reduction* of $\mathcal{A}$, denoted $R(\mathcal{A})$, as an antichain

$$R(\mathcal{A}) = \mathcal{A} - \big\{\{x\} \mid \{x\} \in \mathcal{A}\big\}$$

over a set

$$U_R = U - \{x \mid \{x\} \in \mathcal{A}\}.$$

We say that a set family $\mathcal{F}$ over X is redundant if there is $x \in X$ such that $|Tr_\mathcal{F}(x)| = 1$. We define a reduction of $\mathcal{F}$, denoted $R(\mathcal{F})$ as a family

$$R(\mathcal{F}) = \big\{F \cap X - \{x \mid |Tr_\mathcal{F}(x)| = 1\}\big\}$$

over a set U_R of *irredundant elements* of $\mathcal{C}$

$$U_R = X - \{x \mid |Tr_\mathcal{F}(x)| = 1\}.$$

For closure systems we introduce additional notation. We say that a closure system $\mathcal{C}$ over U is *ambiguous* if $\overline{x} = \overline{X}$ for some $x \in U_R$ and $X \subseteq U_R - x$, otherwise we call $\mathcal{C}$ *unambiguous.* In terms of lattices, if we fix $L = L_{\mathcal{C}}$, then irredundant elements of $\mathcal{C}$ are those mapped to 0_L by closure operator, and $\mathcal{C}$ is ambiguous if either two distinct irredundant elements in U are mapped to the same element in L, or some irredundant element in U is mapped to a join reducible element in L.

The following easy proposition establishes a correspondence between reduction of antichains and reduction of set families.

Proposition 12. *If a set family $\mathcal{F}$ is extremal for an antichain $\mathcal{A}$, then:*

- *$R(\mathcal{F})$ is extremal for $R(\mathcal{A})$;*
- *$\mathcal{F}$ is redundant if and only if $\mathcal{A}$ is redundant;*
- *if $\mathcal{A} = R(\mathcal{A}')$ for some $\mathcal{A}'$, then there is $\mathcal{F}'$ extremal for $\mathcal{A}'$ such that $\mathcal{F} = R(\mathcal{F}')$. Moreover, $\mathcal{F}'$ can be chosen hereditary (closure system) whenever $\mathcal{F}$ is hereditary (closure system).*

Proposition 13. *An element x is redundant for a closure system $\mathcal{C}$ if and only if $x \in \overline{\emptyset}$.*

Proposition 12, in particular, implies that an extremal closure family can be redundant, as we already saw in the example in Section 4. On the other hand, it cannot be ambiguous.

Lemma 14. *If closure family $\mathcal{C}$ is extremal then $\mathcal{C}$ is unambiguous.*

Proof. Suppose $\mathcal{C}$ is ambiguous and let $\overline{a} = \overline{A}$ for some $a \in U_R$ and $A \subseteq U_R - a$. Let us define an extended closure system $\mathcal{C}_A$ as

$$\mathcal{C}_A = \mathcal{C} \cup \{A' \cap X \mid X \in \mathcal{C}\},$$

where $A' = \overline{A} - a$. Trivially, $\mathcal{C}_A$ is a closure system and it is a proper extension of $\mathcal{C}$, as $A' \in \mathcal{C}_A - \mathcal{C}$.

We claim that $\mathcal{C}$ and $\mathcal{C}_A$ shatter the same sets. Indeed, if X is shattered by $\mathcal{C}$ then it is shattered by $\mathcal{C}_A$ as $\mathcal{C} \subset \mathcal{C}_A$. On the other hand, let $\mathcal{C}_A$ shatter X, and let us take $Y \subseteq X$ and $V \in \mathcal{C}_A$ such that $Y = X \cap V$. If $V \in \mathcal{C}$ then $Y \in Tr_{\mathcal{C}_A}(X)$. Now let $V = W \cap A'$, for $W \in \mathcal{C}$. If $a \notin X$ then

$$Y = X \cap V = X \cap W \cap A' = X \cap W \cap (\overline{A} - a) = X \cap \left(W \cap \overline{A}\right),$$

and again $Y \in Tr_{\mathcal{C}_A}(X)$.

Finally, let $a \in X$ and denote a closure operator in $\mathcal{C}_A$ by $\varphi_{\mathcal{C}_A}$. If $b \in X$ for any $b \in A'$, then $\varphi_{\mathcal{C}_A}(b) \subseteq A' \subset \varphi_{\mathcal{C}_A}(a)$ and $\mathcal{C}_A$ does not shatter X as $X - b \notin Tr_{\mathcal{C}_A}(X)$, which is impossible. Thus, $X \cap A' = \emptyset$, and consequently $Y = X \cap V = X \cap W \cap A' = \emptyset$. However, in this case $Y = X \cap \overline{b}$, for every $b \in A'$, and again $Y \in Tr_{\mathcal{C}_A}(X)$, finishing the proof. □

Lemma 15. *A closure family $\mathcal{C}$ is irredundant and unambiguous if and only if $\mathcal{C} = \mathcal{C}_L$ for a lattice L.*

Proof. $(\Rightarrow)$: Let L be a lattice of closed sets of $\mathcal{C}$ over U and let us denote by $\varphi\colon \mathcal{C} \to L$ the bijection between L and $\mathcal{C}$. As $\mathcal{C}$ is irredundant, $|Tr_{\mathcal{C}}(x)| = 2$, for every $x \in U$. Thus, there is $X_x \in \mathcal{C}$ such that $x \notin X_x$, and consequently $\emptyset = \bigcap\{X_x \mid x \in U\} \in \mathcal{C}$. Thus, $0_L = \varphi(\emptyset) < \varphi(\overline{x})$, for any $x \in U$.

On the other hand, as $\mathcal{C}$ is unambiguous, $\varphi(\overline{x}) \neq \varphi(\overline{Y}) = \bigvee\{\varphi(\overline{y}) \mid y \in Y\}$, for all $x \in U$ and $Y \subseteq U - x$. Thus, $\varphi(\overline{x})$ is join irreducible and $\varphi(\overline{x}) \neq \varphi(\overline{y})$, for all $x, y \in U, x \neq y$. Thus, mapping $x \mapsto \varphi(\overline{x})$ is a bijection between U and $\mathrm{J}(L)$, and $\mathcal{C} \cong \mathrm{J}(L)$.

$(\Leftarrow)$: If $\mathcal{C} = \mathcal{C}_L$ is not irredundant, there is some $x \in \mathrm{J}(L)$ such that $t_x(\mathcal{C}_L) = 1$, that is, x either belongs to $\mathrm{J}(y)$ for every $y \in L$, which is impossible because $x \notin \mathrm{J}(0)$ or x does not belong to any $\mathrm{J}(y)$, which is impossible because $x \in \mathrm{J}(x)$.

Similarly, if $\mathcal{C}$ is ambiguous, then there is $x \in \mathrm{J}(L)$ such that $\overline{x} = \mathrm{J}(x) = \overline{X}$, for $X \subseteq \mathrm{J}(L) - x$. But then $\overline{X} = \mathrm{J}\left(\bigvee(X)\right)$, which is only possible if $\bigvee X = x$, a contradiction. □

6 Characterization of extremal lattices

In our definition of meet-distributive lattices we follow [7], namely, a lattice L is *meet-distributive* if an interval $[x, y]$ is a boolean lattice whenever x is a meet of elements covered by y, for any $y \in L$.

A set $X \in \mathrm{J}(L)$ is called an *irredundant join representation* of $a = \bigvee X$, if an inequality $\bigvee (X - x) < a$ holds, for every $x \in X$. We denote by JIR the family over $\mathrm{J}(L)$ of all irredundant join representations in L.

In a finite lattice every element has at least one (possibly more) irredundant join representations. On the other hand, no two distinct elements share an irredundant join representation. Thus, $|L| \leqslant |\mathrm{JIR}(L)|$.

For the following fact we refer to [9, Theorem 44], however, as was noted by the anonymous referee, it goes back to the paper of Dilworth [5]:

Lemma 16. *A finite lattice L is meet-distributive if and only if every element of L has a unique irredundant join representation.*

The following two lemmas are straightforward and relate irredundant join representations with extremality.

Lemma 17. *For every lattice L, the family* $\mathrm{JIR}(L)$ *is hereditary.*

Proof. Let X be an irredundant join representation and suppose $X - x \notin \mathrm{JIR}(L)$, for some $x \in X$. This means that $\bigvee(X - x) = \bigvee(X - \{x, y\})$, for some $y \in X$, $y \neq x$. But then

$$\bigvee X = x \vee \bigvee(X - x) = x \vee \bigvee(X - \{x, y\}) = \bigvee(X - y),$$

a contradiction. □

Lemma 18. *A lattice L shatters $A \in \mathrm{J}(L)$ if and only if* $\mathrm{JIR}(L)$ *shatters A.*

Proof. ($\Rightarrow$) : Let $x = \bigvee A$. As L shatters A, for every $a \in A$, let us take $x_a \in L$ such that $\mathrm{J}(x_a) \cap A = A - a$, in particular, $x_a < x$. Thus, for every $a \in A$, $\bigvee(A - a) \subseteq \bigvee \mathrm{J}(x_a) = x_a < x$. Thus, A is an irredundant join representation of x, in particular, $\mathrm{JIR}(L)$ shatters A.

($\Leftarrow$) : If $\mathrm{JIR}(L)$ shatters A then $A \in \mathrm{JIR}(L)$. For any $X \subseteq A$ let us take $\overline{X} \in \mathcal{C}_L$. If $X' = \overline{X} \cap A \supsetneq X$ then $\bigvee X' = \bigvee X$, and thus $X' \notin \mathrm{JIR}(L)$, which is impossible as $X' \subset A \in \mathrm{JIR}(L)$ and $\mathrm{JIR}(L)$ is hereditary. Thus, for every $X \subseteq A$ holds $\overline{X} \cap A = X$ and $\mathcal{C}_L$ shatters A. □

A *convex geometry* is a base set U endowed with a closure operator $\varphi \colon X \mapsto \overline{X}$ satisfying the *anti-exchange property*:

$$x \in \overline{A \cup y} \ \text{ implies } \ y \notin \overline{A \cup x},$$

for all closed A, $x, y \notin A$, $x \neq y$. Trivially, every convex geometry is a closure system. As it was shown by Edelman, convex geometries are closely related to meet-distributive lattices, see [7, Theorem 3.3]. A good modern survey of the state of affairs with convex geometries can be found in [3].

Lemma 19. *A lattice L is a lattice of closed sets of convex geometry if and only if L is meet-distributive.*

Notice that Lemma 19 works only in one direction, that is, if a lattice of a closure system $\mathcal{C}$ is meet-distributive, it does not imply that $\mathcal{C}$ is a convex geometry, for example let $\mathcal{C} = \{\emptyset, 12\}$ over $U = \{1, 2\}$ (we considered this example in Section 4), then $L_{\mathcal{C}} \cong B(1)$, which is meet-distributive. On the other hand $1, 2 \notin \emptyset$, but $2 \in \overline{1}$ and $1 \in \overline{2}$.

We use the notions of reduction and unambiguity, introduced before, to draw a more detailed connection between convex geometries and meet-distributive lattices.

Lemma 20. *A closure system $\mathcal{C}$ is a convex geometry if and only if $R(\mathcal{C})$ is.*

Proof. By Proposition 13, an element x is redundant if and only if $x \in \overline{\emptyset}$. Thus, in anti-exchange property we can consider only irredundant elements x and y, and the statement of the lemma follows. □

Lemma 21. *If $\mathcal{C}$ is a convex geometry then it is unambiguous.*

Proof. Suppose otherwise and take $x \in U_R$ and $X \subseteq (U_R - x)$ such that $\overline{x} = \overline{X}$. Let $Y = \overline{X} - x$, then $\overline{Y} = \overline{X}$, in particular, Y is not closed. Let A be any maximal closed set in Y. As Y is not closed, it follows that $A \subsetneq Y$ and we can chose $y \in Y - A$.

As A is maximal closed in Y, $\overline{A \cup y} \not\subseteq Y$, but $\overline{A \cup y} \subseteq \overline{Y} = Y \cup x$, thus, $x \in \overline{A \cup y}$. On the other hand, $y \in \overline{X} = \overline{x}$, hence $y \in \overline{A \cup x}$, a contradiction. □

As a consequence we might formulate the following characterization of the extremal lattices.

Theorem 22. *For a finite lattice L and a finite closure system $\mathcal{C}$, it holds:*

1. *L is extremal if and only if it is meet-distributive;*

2. *$\mathcal{C}$ is extremal if and only if it is a convex geometry.*

Proof. (1) : By Lemma 18, the family $\mathrm{JIR}(L)$ is hereditary, and so by Lemma 4 it is extremal. Let $\mathcal{A} = \mathcal{A}(\mathrm{JIR}(L))$, implying $\mathrm{JIR}(L) = \mathcal{I}(\mathcal{A})$. By Lemma 18, L shatters same sets as $\mathrm{JIR}(L)$, and thus L is extremal if and only if it is extremal for $\mathcal{A}$. This means that L is extremal if and only if $|L| = |\mathcal{I}(\mathcal{A})| = |\mathrm{JIR}(L)|$, however this holds if and only if every element of L has a unique irredundant join representation. By Lemma 16, this happens if and only if L is meet-distributive.

(2) : By Lemma 20 and Proposition 12 it is sufficient to establish this correspondence only for irredundant $\mathcal{C}$, and by Lemma 21 and Lemma 14 we can also consider only unambiguous $\mathcal{C}$. By Lemma 15, irredundant and unambiguous $\mathcal{C}$ corresponds to $\mathcal{C}_L$, for some lattice L. Lemma 19 and the definition of extremality of lattices show that convex geometries and extremal closure systems correspond to meet-distributivity and extremality of corresponding lattices, and their equivalence was proven in part (1). □

It is an interesting fact that for convex geometries VC-dimension was studied in its own right under the name of Erdős-Szekeres number, see for example Section 3.4 in [12]. To be more accurate, this number is introduced for antimatroids, however this does not play a crucial role as convex geometries and antimatroids are dual notions.

7 Convex geometries and implications

We now aim at a characterization of extremal closure systems in terms of implications. The following notation is from [9].

A set $T \subseteq U$ *respects an implication* $A \to B$ if $A \not\subseteq T$ or $B \subseteq T$. An implication holds in a family $\mathcal{F}$ if every set of $\mathcal{F}$ respects it. Given a family of implications $\mathcal{I}$, we denote by $\mathcal{C}(\mathcal{I})$ a family of sets respecting all implications in $\mathcal{I}$. It is evident that in this case $\mathcal{C}(\mathcal{I})$ is a closure system.

Lemma 23. *For an extremal closure system $\mathcal{C}$ and any $A \in \mathcal{A}_{\mathcal{C}}$ there is unique element $a \in A$ such that implication $(A - a) \to a$ holds in $\mathcal{C}$.*

Proof. By Lemma 7, there is exactly one set X in $\mathrm{P}(A) - \mathrm{T}_{\mathcal{C}}(A)$. Notice also that $\mathrm{T}_{\mathcal{C}}(A)$ is a closure system, and thus $|X| = |A - 1|$, otherwise X can be represented as an intersection of two subsets of size $|X| + 1$. Thus, $X = A - a$ for some $a \in A$.

Now, from the definition of trace it follows that for any $C \in \mathcal{C}$, $a \in C$ whenever $(A - a) \subseteq C$, that is, $\mathcal{C}$ respects implication $(A - a) \to a$. □

Following [11], we define a *rooted set* as a pair (X, x), $x \in X$, where x is called the *root*. We also define a *rooted antichain* as a family of rooted sets $\mathcal{A}^R = \{(A_i, a_i)\}$ such that family $\mathcal{A} = \{A_i\}$ is an antichain. In this case we call $\mathcal{A}$ and $\mathcal{A}^R$ *corresponding.* For simplicity, we distinguish corresponding antichain and rooted antichain by upper index

R. In what follows we identify a rooted set (A, a) with an implication $(A - a) \to a$. We say that a rooted antichain $\mathcal{A}^R$ is extremal if $\mathcal{C}(\mathcal{A}^R)$ is.

Using Lemma 23, for a given extremal closure system $\mathcal{C}$ we may uniquely construct a rooted antichain $\mathcal{A}^R_{\mathcal{C}}$ such that $\mathcal{A}_{\mathcal{C}}$ is the extremal set of $\mathcal{C}$ and such that $\mathcal{C}$ respects $\mathcal{A}^R_{\mathcal{C}}$. We call $\mathcal{A}^R_{\mathcal{C}}$ a *rooted blocking antichain of* $\mathcal{C}$.

Lemma 24. *For an extremal closure family* $\mathcal{C}$*, it holds* $\mathcal{C} = \mathcal{C}(\mathcal{A}^R_{\mathcal{C}})$.

Proof. Obviously, $\mathcal{C}$ respects $\mathcal{A}^R_{\mathcal{C}}$, thus $\mathcal{C} \subseteq \mathcal{C}(\mathcal{A}^R_{\mathcal{C}})$. On the other hand, $\mathcal{C}(\mathcal{A}^R_{\mathcal{C}})$ does not shatter any $A \in \mathcal{A}$, thus $|\mathcal{C}| \geqslant |\mathcal{C}(\mathcal{A}^R_{\mathcal{C}})|$. □

It turns out that rooted antichains which give rise to extremal closure families may be easily described using characterization of circuits of antimatroids from [4].

Theorem 25. *A rooted antichain* $\mathcal{A}^R$ *is extremal if and only if the following condition holds*

$$\text{if } a \in B - b \text{ then there is } (C, b) \in \mathcal{A}^R \text{ such that } C \subseteq A \cup B - a,$$

for any rooted sets (A, a) *and* (B, b) *in* $\mathcal{A}^R$.

Proof. Theorem 22 reveals extremal closure systems to be convex geometries, which are turned into antimatroids by complementation. In the same time, rooted sets from the rooted blocking antichain of an extremal closure family $\mathcal{C}$ correspond to rooted circuits of complemented antimatroid $\mathcal{C}^*$, see [11, Lemma 4.4].

Now, the statement of the theorem follows directly from the characterization of rooted families yielding antimatroids given in [4, Theorem 7]. □

8 Deciding the extremality of an antichain is NP-complete

We say that an antichain $\mathcal{A}$ is *extremal*, if it is extremal for some closure family $\mathcal{C}$, or, equivalently, if it is a corresponding antichain for some extremal rooted antichain. Notice that the family $\mathcal{I}(\mathcal{A})$ is extremal for any $\mathcal{A}$, thus the condition for $\mathcal{C}$ to be a closure family is essential.

As we mentioned before, in [1] it was shown that it is possible to construct an extremal closure system for the antichain $\mathcal{A}_k$ of all k-sets of U, for any k, thus all antichains $\mathcal{A}_k$ are extremal. In general, however, it is not easy to say whether a given antichain is extremal. In fact, below we show that this problem is NP-complete.

Lemma 26. *For a given antichain* $\mathcal{A}$ *there is an NP algorithm deciding whether* $\mathcal{A}$ *is extremal.*

Proof. Given an antichain $\mathcal{A}$, the described algorithm first nondeterministically guesses a rooted antichain $\mathcal{A}^R$, corresponding to $\mathcal{A}$, and then checks whether $\mathcal{A}^R$ is extremal using characterization from Theorem 25, which demands time cubic with respect to the number of sets in $\mathcal{A}$. □

Thus, providing nondeterministic polynomial algorithm for checking extremality of the chain is an easy part. In order to prove NP-completeness we now have to design specific terminology.

We call a (rooted) antichain $\mathcal{A}$ *intransitive* if for any distinct A and B and for any point $x \in A \cap B$ there is no $C \subseteq A \cup B - x$. We say that a rooted antichain $\mathcal{A}^R$ is *accordant* if every point from its base set is either the root for all sets from $\mathcal{A}^R$ containing it, or for none of them.

Lemma 27. *An intransitive rooted antichain is extremal if and only if it is accordant.*

Proof. If a rooted antichain is accordant, then $a \notin B - b$, for any rooted sets (A, a) and (B, b) and by Theorem 25 it is rooted, proving one side of the statement.

Now, let $\mathcal{A}^R$ be extremal intransitive rooted antichain. Suppose $\mathcal{A}^R$ is not accordant, that is, there is a point a and two rooted sets (A, a) and (B, b) in $\mathcal{A}^R$, $a \in B - b$. Then by Theorem 25 there is $(C, b) \in \mathcal{A}^R$ such that $C \in A \cup B - a$, but this contradicts the intransitivity of $\mathcal{A}^R$. $\square$

We denote by $R(\mathcal{A}^R)$ the set of roots of $\mathcal{A}^R$, that is, $R(\mathcal{A}^R) = \{a \mid (a, A) \in \mathcal{A}^R\}$. For an antichain $\mathcal{A}$ over the base set U, we say that $S \subseteq U$ is a *root set*, if $|S \cap A| = 1$ for every $A \in \mathcal{A}$, and such that every $x \in S$ lies in some set of $\mathcal{A}$. For such $\mathcal{A}$ and S we can define an accordant corresponding rooted antichain $\mathcal{A}^R(S)$ by setting the root of A to $a = A \cap S$, for every $A \in \mathcal{A}$. Then $S = R(\mathcal{A}^R(S))$. On the other hand, $R(\mathcal{A}^R)$ is a root set for every accordant rooted antichain $\mathcal{A}^R$.

Lemma 28. *An intransitive antichain is extremal if and only it has a root set.*

Proof. By Lemma 27, an intransitive antichain is extremal if and only if it has corresponding accordant rooted antichain. The statement of the lemma now follows by observing that accordant rooted antichains trivially correspond to root sets. $\square$

We will prove NP-hardness of antichain extremality by polynomial reduction of 3-SAT to this problem. Let us recall that 3-SAT is a problem of determining whether a given boolean formula F in conjunctive normal form (CNF) with each clause containing exactly three literals is satisfiable. Also, a boolean formula in CNF is a conjunction of clauses, each clause is a disjunction of literals and each literal is either a variable or its negation.

We utilize simplified notation for partial valuations on the set of variables, for example we write $a\bar{b}c$ for a partial valuation that assigns True to a and c, and False to b. For a clause cl (partial valuation α) we denote by $V(cl)$ (by $V(\alpha)$) the set of variables of cl (of α). Thus, for a 3-SAT formula F, $|V(cl)| = 3$ for each clause cl of F.

For a partial valuation α and a variable $x \in V(\alpha)$ we denote by $\alpha(x)$ the value of α on x, and we denote by $x : \alpha(x)$ the partial valuation that assigns value $\alpha(x)$ to variable x. Thus, $a\bar{b}c(b) = False$ and $b : a\bar{b}c(b) = \bar{b}$. We say that a partial valuation α is *compliant* with a complete valuation φ if $\alpha(x) = \varphi(x)$ for every $x \in V(\alpha)$.

Theorem 29. *The problem of determining whether a given antichain is extremal is NP-hard.*

Proof. Let F be a 3-CNF formula, V the set of its variables and C the set of its clauses, each consisting of three literals. We are going to construct an antichain which is extremal if and only if F is satisfiable.

For a clause cl let $T(cl)$ be all $2^{|cl|}-1$ partial valuations of variables $V(cl)$ that satisfy cl. For example,

$$T(a \vee \neg b \vee c) = \{abc, ab\overline{c}, a\overline{b}c, a\overline{b}\overline{c}, \overline{a}bc, \overline{a}\overline{b}c, \overline{a}\overline{b}\overline{c}\}.$$

Let us construct the antichain $\mathcal{A}_F$. We use partial valuations, sometimes with indexes, as elements of the base set U of $\mathcal{A}_F$, $U = U_{var} \cup U_{cl} \cup U_{link}$, where

$$\begin{aligned} U_{var} &= \{x, \overline{x} \mid x \in L\}; \\ U_{cl} &= \{\alpha \mid \alpha \in T(c), c \in C\}; \\ U_{link} &= \{\alpha^{v,i} \mid i \in \{1,2\}; v \in V(c), \alpha \in T(c), c \in C\}. \end{aligned}$$

Intuitively, points in U_{var} describe valuations of variables of F, and points in U_{cl} describe values of clauses under given valuations.

The antichain $\mathcal{A}_F$ consists of four parts, $\mathcal{A}_F = \mathcal{A}_{var} \cup \mathcal{A}_{cl} \cup \mathcal{A}^1_{link} \cup \mathcal{A}^2_{link}$ where

$$\begin{aligned} \mathcal{A}_{var} &= \big\{\{x, \overline{x}\} \mid x \in L\big\}; \\ \mathcal{A}_{cl} &= \big\{T(c) \mid c \in C\big\}; \\ \mathcal{A}^1_{link} &= \big\{\{\alpha, \alpha^{v,1}, \alpha^{v,2}\} \mid \alpha \in T(c), v \in V(c)\} \mid c \in C\big\}; \\ \mathcal{A}^2_{link} &= \big\{\{\alpha^{v,2}, v : \alpha(v)\} \mid c \in C, \alpha \in T(c), v \in V(c)\big\}. \end{aligned}$$

For example, for the formula $F^* = x \vee \overline{y}$ with two variables and a single clause, corresponding base set U^* and antichain $\mathcal{A}^*_F$ look as follows:

$$\begin{aligned} U^* = \{&x, \overline{x}, y, \overline{y}, xy, xy^{x,1}, xy^{x,2}, xy^{y,1}, xy^{y,2}, x\overline{y}, x\overline{y}^{x,1}, x\overline{y}^{x,2}, x\overline{y}^{y,1}, x\overline{y}^{y,2}, \\ &\overline{xy}, \overline{xy}^{x,1}, \overline{xy}^{x,2}, \overline{xy}^{y,1}, \overline{xy}^{y,2}\}; \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}^*_{var} &= \big\{\{x, \overline{x}\}, \{y, \overline{y}\}\big\}; \\ \mathcal{A}^*_{cl} &= \big\{\{xy, x\overline{y}, \overline{xy}\}\big\}; \\ \mathcal{A}^{*,1}_{link} &= \big\{\{xy, xy^{x,1}, xy^{x,2}\}, \{xy, xy^{y,1}, xy^{y,2}\}, \\ &\qquad \{x\overline{y}, x\overline{y}^{x,1}, x\overline{y}^{x,2}\}, \{x\overline{y}, x\overline{y}^{y,1}, x\overline{y}^{y,2}\}, \\ &\qquad \{\overline{xy}, \overline{xy}^{x,1}, \overline{xy}^{x,2}\}, \{\overline{xy}, \overline{xy}^{y,1}, \overline{xy}^{y,2}\}\big\}; \\ \mathcal{A}^{*,2}_{link} &= \big\{\{xy^{x,2}, x\}, \{xy^{y,2}, y\}, \{x\overline{y}^{x,2}, x\}, \{x\overline{y}^{y,2}, \overline{y}\}, \{\overline{xy}^{x,2}, \overline{x}\}, \{\overline{xy}^{y,2}, \overline{y}\}\big\}. \end{aligned}$$

Figure 1 below shows this exemplary antichain.

Thus constructed, antichain $\mathcal{A}_F$ is polynomial with respect to the size of F, also, $\mathcal{A}_F$ is intransitive. We claim that $\mathcal{A}_F$ is extremal if and only if F is satisfiable. By Lemma 28, the latter is true if and only if there is a root set for $\mathcal{A}$.

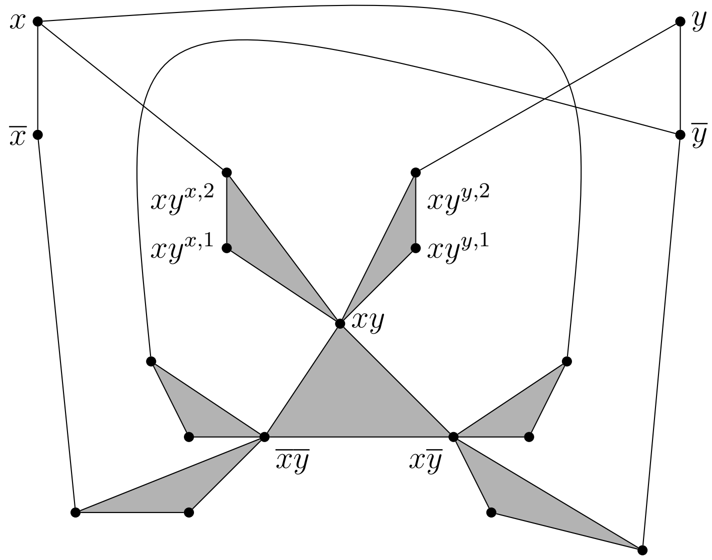

Figure 1: Example of the antichain $\mathcal{A}^*_F$ for $F^* = x \vee \neg y$.

Suppose F is satisfiable, then there is some valuation φ, satisfying F. Let us define set $S \subseteq U$ as:

$$\begin{aligned} S = &\{\alpha \mid \alpha \in U_{var} \cup U_{cl}, \alpha \text{ is compliant with } \varphi\} \cup \\ &\{\alpha^{v,2} \mid \alpha \in U_{link}, \alpha(v) \neq \varphi(x)\} \cup \\ &\{\alpha^{v,1} \mid \alpha \in U_{link}, \alpha(v) = \varphi(x), \alpha \text{ is not compliant with } \varphi\ \}. \end{aligned}$$

We claim that S is a root set of $\mathcal{A}_F$. Obviously, for any $\{x, \overline{x}\} \in \mathcal{A}_{var}$, φ is compliant with exactly one of x and $\overline{x}$, thus $S \cap \{x, \overline{x}\} = 1$, for any variable x. For any clause c of F, φ satisfies c, thus there is exactly one $\alpha \in T(c)$, compliant with φ and a unique intersection of S with $A = T(c)$, for any $A \in \mathcal{A}_{cl}$.

For any $A = \{\alpha, \alpha^{v,1}, \alpha^{v,2}\} \in \mathcal{A}^1_{link}$, either α is compliant with φ and $\alpha \in S$, or α is not compliant with φ but $\alpha(v) = \varphi(x)$, in which case $\alpha^{v,1} \in S$, or $\alpha(v) \neq \varphi(x)$, in which case α is automatically not compliant with φ, and $\alpha^{v,2} \in S$. Thus, there is a unique intersection of S with A.

Finally, for any $A = \{\alpha^{v,2}, v : \alpha(v)\} \in \mathcal{A}^2_{link}$, either $\varphi(v) = \alpha(v)$ and $v : \alpha(v) \in S$, or $\varphi(v) \neq \alpha(v)$, in which case $\alpha^{v,2} \in S$.

Figure 2 below shows the rooted extremal antichain corresponding to our exemplary formula $F^* = x \vee \neg y$ with root set S^* constructed for valuation $\varphi^* = xy$.

Conversely, let F be unsatisfiable, and let S be a root set for $\mathcal{A}_F$. Then for every $x \in V$ exactly one of x and $\overline{x}$ lies in S. Let us now define a complete valuation φ by $\varphi(x) = True$ if $x \in S$ and $\varphi(x) = False$ otherwise. As F is unsatisfiable, there is a clause $c \in C$ such that $\varphi(c) = False$.

Let α be a complete valuation of $V(c)$, which is the root of $T(c) \in \mathcal{A}^1_{link}$. Then α satisfies c and, consequently, α is not compatible with φ. Then there is a variable $x \in V(c)$

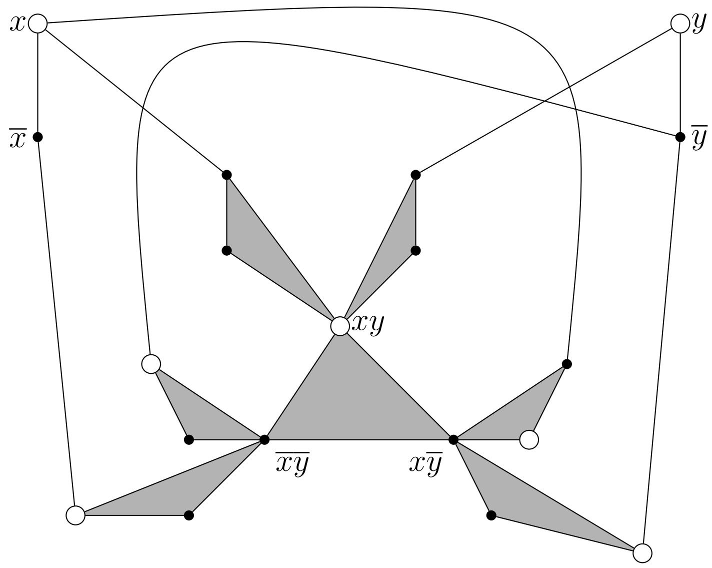

Figure 2: Example of the rooted extremal antichain, corresponding to a valuation $\varphi^* = xy$ for the formula $F^* = x \vee \neg y$.

such that $x : \alpha(x) \neq \varphi(x)$, by the definition of φ this means that $x : \alpha(x) \notin S$. But as S has a common point with $\{\alpha^{x,2}, x : \alpha(x)\} \in \mathcal{A}^2_{link}$, it follows $\alpha^{x,2} \in S$. But then both $\alpha^{x,2}$ and α lie in $S \cap A$ for $A = \{\alpha, \alpha^{x,2}, \alpha^{x,2}\} \in \mathcal{A}^1_{link}$, a contradiction.

Figure 3 below shows that a root set cannot be chosen for a valuation that falsifies the only clause of F^*. □

Theorem 29 together with Lemma 26 prove NP-completeness of determining extremality of an antichain.

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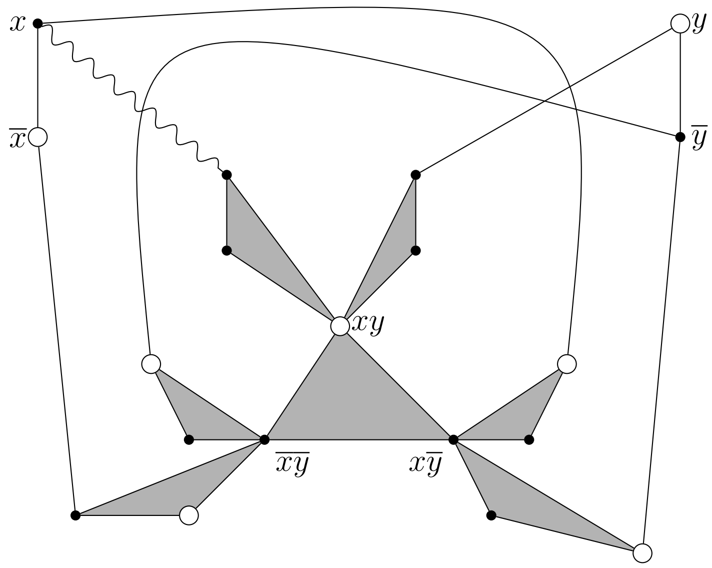

Figure 3: Example of a tentative rooted antichain corresponding to a valuation $\varphi^* = \overline{x}y$, which does not satisfy $F^* = x \vee \neg y$.

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