Nonempty intersection of longest paths in $2K_2$-free graphs

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Abstract

In 1966, Gallai asked *whether all longest paths in a connected graph share a common vertex*. Counterexamples indicate that this is not true in general. However, Gallai's question is positive for certain well-known classes of connected graphs, such as split graphs, interval graphs, circular arc graphs, outerplanar graphs, and series-parallel graphs. A graph is $2K_2$-free if it does not contain two independent edges as an induced subgraph. In this short note, we show that, in nonempty $2K_2$-free graphs, every vertex of maximum degree is common to all longest paths. Our result implies that all longest paths in a nonempty $2K_2$-free graph have a nonempty intersection. In particular, it strengthens the result on split graphs, as split graphs are $2K_2$-free.

Mathematics Subject Classifications: 05C38

1 Introduction

All graphs considered in this paper are finite and simple. A path P in a graph G is a *longest path* in G if there is no path in G strictly longer than P. In 1966 Gallai asked [5] whether all longest paths in a connected graph have a vertex in common. In 1974, Walther [12] gave a counterexample to the problem. As every hypo-traceable graph (i.e., a graph with no Hamiltonian path where all vertex-deleted subgraphs admit a Hamiltonian path) is clearly a counterexample, there are infinitely many counterexamples to the problem (see Thomassen [11]).

In spite of the negative answer for general graphs, the answer to Gallai's problem when restricted to many classes of graphs is positive. Klavžar and Petkovšek [7] gave an affirmative answer to Gallai's question for connected split graphs and for cacti. An affirmative answer for the class of connected circular-arc graphs was given by Balister

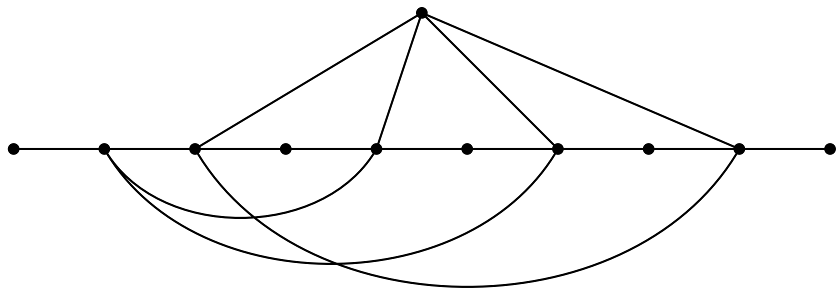

Figure 1: A $3K_2$-free graph with a vertex of maximum degree not belonging to every longest path.

et al. [1] (see also Joos [6]). A positive answer for connected outerplanar graphs and 2-trees was given by de Rezende et al. [4]. Recently, the second author with Chen et al. [2] extended this result, giving a positive solution to Gallai's problem for the class of connected series-parallel graphs. For more information about Gallai's problem and several variations, see [10].

In this paper, we investigate the intersection of all longest paths in connected $2K_2$-free graphs. A graph is *$2K_2$-free* if it contains no two independent edges as an induced subgraph. The class of $2K_2$-free graphs is well studied, for instance, see [3, 8, 9]. It contains the class of *split graphs*, where vertices can be partitioned into a clique and an independent set. One can also easily check that every *cochordal* graph (i.e., a graph that is the complement of a chordal graph) is $2K_2$-free and so the class of $2K_2$-free graphs is at least as rich as the class of chordal graphs. In this note we prove the following.

Theorem 1. *In a nonempty $2K_2$-free graph, every vertex of maximum degree is common to all longest paths.*

In particular, the answer to Gallai's problem is positive for $2K_2$-free graphs. Theorem 1 also strengthens Klavžar and Petkovšek's result for split graphs [7].

Corollary 2. *If G is a nonempty split graph or cochordal graph, then every vertex of maximum degree is common to all longest paths.*

We note that Theorem 1 is best possible in terms of the number of copies of K_2 in the forbidden subgraph. Indeed, in connected $3K_2$-free graphs a vertex of maximum degree does not necessarily belong to the intersection of all longest paths. For example, consider the graph in Figure 1. It is $3K_2$-free; the top vertex is of maximum degree but does not belong to the longest path passing through all the remaining vertices.

For a graph G we will denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. If $uv \in E(G)$, we write $u \sim v$ to denote the adjacency of u and v. For two disjoint subsets $S, T \subseteq V(G)$, we denote by $E_G(S,T)$ the set of edges of G with one end in S and the other in T. If $u \in V(G)$, we denote by $N_G(u)$ the set of neighbors of u in G. If G is clear from the context, we omit the subscript G and write $E(S,T)$ and $N(u)$.

2 Proof of Theorem 1

In this section we prove Theorem 1. We will need the following three lemmas. A path P in a graph G is *dominating* if $G - V(P)$ is edgeless.

Lemma 3. *Let G be a $2K_2$-free graph. Then every longest path in G is dominating.*

Proof. Let $P = v_0v_1 \cdots v_\ell$ be a longest path in G. Assume by contradiction that P is not dominating. Then there exists an edge $uv \in E(G)$ such that $u, v \notin V(P)$. Since G is $2K_2$-free, there must be an edge e' in G which connects the edge uv to the edge v_0v_1. Without loss of generality, we can assume that e' connects v to either v_0 or v_1. If $e' = vv_0$ then $uvv_0v_1 \cdots v_\ell$ is a path in G longer than P. If $e' = vv_1$ then $uvv_1 \cdots v_\ell$ is a path in G longer than P. □

The proof of the following lemma follows from standard arguments.

Lemma 4. *Let G be a graph. Let $P = v_0v_1 \cdots v_\ell$ be a longest path in G and let x be a vertex of G which does not belong to P. Then the following assertions hold.*

(1) The vertex x is not adjacent to the endpoints v_0 and v_ℓ of P.

(2) The vertex x does not have two neighbors which are consecutive vertices v_i, v_{i+1} on P.

(3) If v_a is a neighbor of x then v_0 is not adjacent to v_{a+1}.

(4) If v_a and v_b are distinct neighbors of x then v_{a+1} is not adjacent to v_{b+1}.

The following lemma was proved in [3, Theorem 1].

Lemma 5. *Let G be a nonempty $2K_2$-free graph and let $S \subseteq V(G)$ be an independent set. Let $T \subseteq V(G) - S$. Then there exists $y \in T$ such that $N(y)$ meets all edges in $E(S,T)$.*

Proof of Theorem 1. Let G be a nonempty $2K_2$-free graph and let $P = v_0v_1 \cdots v_\ell$ be a longest path in G. Assume that $x \in V(G)$ is a vertex of maximum degree in G which does not belong to P. Let $k = d(x) = \Delta(G)$. By Lemma 3, $N(x) \subseteq V(P)$. Let

$$S = \{v_0, v_{a+1} \mid v_a \in N(x)\} \subseteq V(P).$$

By (3) and (4) of Lemma 4, S is an independent set. Let $T = V(P) - S$. By (1) and (2) of Lemma 4, $V(P)$ contains at least $2k+1$ vertices. The set $\{v_av_{a+1} \mid v_a \in N(x)\}$ is a set of k independent edges in $E(S,T)$ (i.e., k edges which pairwise do not share an endpoint).

We claim that if $|V(P)| \geqslant 2k+2$ then there are $k+1$ independent edges in $E(S,T)$. Indeed, the k neighbors of x separate P into $k+1$ non-trivial subpaths (see Lemma 4(1)). By the pigeonhole principle one of these subpaths contains at least two vertices in $V(P) - N(x)$. If v_0 is an endpoint of this subpath, then $v_0, v_1 \notin N(x)$ and

$$\{v_0v_1, v_av_{a+1} \mid v_a \in N(x)\} \subseteq E(S,T)$$

is an independent subset of $k+1$ edges. If v_ℓ is an endpoint of this subpath then

$$\{v_0v_1, v_{a+1}v_{a+2} \mid v_a \in N(x)\} \subseteq E(S,T)$$

is an independent subset of size $k+1$. Thus, we can assume that the endpoints of this subpath are $v_p, v_q \in N(x)$ for some $p < q$ in $\{1, \ldots, \ell-1\}$. Then

$$\{v_0v_1\} \cup \{v_{a+1}v_{a+2} \mid a \leqslant p,\ v_a \in N(x)\} \cup \{v_bv_{b+1} \mid b \geqslant q,\ v_b \in N(x)\} \subseteq E(S,T)$$

is a set of $k+1$ independent edges.

Now, by Lemma 5, there is a vertex $y \in T$ such that $N(y)$ meets all edges in $E(S,T)$. If $|V(P)| \geqslant 2k+2$, then y has at least $k+1$ neighbors in $V(P) = S \cup T$ since $E(S,T)$ contains an independent set of $k+1$ edges. Then $d(y) \geqslant k+1 > k = \Delta(G)$, a contradiction. If $|V(P)| = 2k+1$ then $T = N(x)$. Indeed, the disjoint union $S \cup N(x) \subseteq V(P)$ and $|S| = k+1$, $|N(x)| = k$. Hence $N(x) = V(P) - S = T$. In particular, in that case, $y \in N(x)$. Since $N(y)$ meets all edges in $E(S,T)$ and $E(S,T)$ contains an independent set of k edges, y has at least k neighbors in $V(P)$. Since x is also a neighbor of y we have $d(y) \geqslant k+1 > k = \Delta(G)$, a contradiction. □

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