On two-sided gamma-positivity for simple permutations

Ron M. Adin

Department of Mathematics
Bar-Ilan University
Ramat-Gan 52900, Israel

radin@math.biu.ac.il

Eli Bagno Estrella Eisenberg Shulamit Reches Moriah Sigron

Department of Mathematics
Jerusalem College of Technology
21 Havaad Haleumi St., Jerusalem, Israel

bagnoe@g.jct.ac.il,{estchoc,shulamit.reches}@gmail.com,
moria.sigron@mail.huji.ac.il

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Abstract

Gessel conjectured that the two-sided Eulerian polynomial, recording the common distribution of the descent number of a permutation and that of its inverse, has non-negative integer coefficients when expanded in terms of the gamma basis. This conjecture has been proved recently by Lin.

We conjecture that an analogous statement holds for simple permutations, and use the substitution decomposition tree of a permutation (by repeated inflation) to show that this would imply the Gessel-Lin result. We provide supporting evidence for this stronger conjecture.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

Eulerian numbers enumerate permutations according to their descent numbers. The *two-sided Eulerian numbers*, studied by Carlitz, Roselle, and Scoville [6] constitute a natural generalization. These numbers count permutations according to their number of descents as well as the number of descents of the inverse permutation.

Explicitly, the *descent set* of a permutation $\pi \in S_n$ is defined as:

$$\mathrm{Des}(\pi) = \{i \in [n-1] \mid \pi(i) > \pi(i+1)\}.$$

Denote $des(\pi) = |\operatorname{Des}(\pi)|$ and $\operatorname{ides}(\pi) = des(\pi^{-1})$, the *descent numbers* of π and π^{-1}. For example, if $\pi = 246135$ then $\operatorname{Des}(\pi) = \{3\}$, $des(\pi) = 1$, $\operatorname{Des}(\pi^{-1}) = \{1, 3, 5\}$ and $\operatorname{ides}(\pi) = 3$.

A polynomial $f(q)$ is *palindromic* if its coefficients are the same when read from left to right as from right to left. Explicitly, if $f(q) = a_r q^r + a_{r+1} q^{r+1} + \cdots + a_s q^s$ with $a_r, a_s \neq 0$ and $r \leqslant s$, then we require $a_{r+i} = a_{s-i}$ ($\forall i$); equivalently, $f(q) = q^{r+s} f(1/q)$. Following Zeilberger [17], we define the *darga* of $f(q)$ as above to be $r+s$; the zero polynomial is considered to be palindromic of each nonnegative darga. The set of palindromic polynomials of darga $n-1$ is a vector space of dimension $\lfloor (n+1)/2 \rfloor$, with *gamma basis*

$$\{q^j (1+q)^{n-1-2j} \mid 0 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor\}.$$

The (one-sided) *Eulerian polynomial*

$$A_n(q) = \sum_{\pi \in S_n} q^{des(\pi)}$$

is palindromic of darga $n-1$, and thus there are real numbers $\gamma_{n,j}$ such that

$$A_n(q) = \sum_{0 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor} \gamma_{n,j} q^j (1+q)^{n-1-2j}.$$

See [12, pp. 72, 78] for details. Foata and Schützenberger [7] proved that the coefficients $\gamma_{n,j}$ are actually non-negative integers. The result of Foata and Schützenberger was reproved combinatorially, using an action of the group $\mathbb{Z}_2^n$ on S_n which leads to an interpretation of each coefficient $\gamma_{n,j}$ as the number of orbits of a certain type. This method, called "valley hopping", is described in [8, 4]. A nice exposition appears in [11].

Now let $A_n(s,t)$ be the *two-sided Eulerian polynomial*

$$A_n(s,t) = \sum_{\pi \in S_n} s^{des(\pi)} t^{\operatorname{ides}(\pi)}.$$

It is well known (see, e.g., [11, p. 167]) that the bivariate polynomial $A_n(s,t)$ satisfies

$$A_n(s,t) = (st)^{n-1} A_n(1/s, 1/t) \tag{1}$$

as well as

$$A_n(s,t) = A_n(t,s). \tag{2}$$

In fact, (1) follows from the bijection from S_n onto itself taking a permutation to its reverse, while (2) follows from the bijection taking each permutation to its inverse.

A bivariate polynomial satisfying Equations (1) and (2) will be called (bivariate) *palindromic of darga* $n-1$. Note that if we arrange the coefficients of a bivariate palindromic polynomial in a matrix, then this matrix is symmetric with respect to both diagonals.

Example 1. The two-sided Eulerian polynomial for S_4 is:

$$A_4(s,t) = 1 + 10st + 10(st)^2 + (st)^3 + st^2 + s^2t.$$

Its matrix of coefficients is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and is clearly symmetric with respect to both diagonals.

It can be proved (see [12, p. 78]) that the set of bivariate palindromic polynomials of darga $n-1$ is a vector space of dimension $\lfloor (n+1)/2 \rfloor \cdot \lfloor (n+2)/2 \rfloor$, with *bivariate gamma basis*

$$\{(st)^i(s+t)^j(1+st)^{n-1-j-2i} \mid i,j \geqslant 0,\, 2i+j \leqslant n-1\}.$$

A bivariate palindromic polynomial is called *gamma-positive* if all the coefficients in its expression in terms of the bivariate gamma basis are nonnegative. Gessel (see [4, Conjecture 10.2]) conjectured that the two-sided Eulerian polynomial $A_n(s,t)$ is gamma-positive. This has recently been proved by Lin [10]. Explicitly:

Theorem 2. (Gessel's conjecture, Lin's theorem) *For each $n \geqslant 1$ there exist nonnegative integers $\gamma_{n,i,j}$ $(i,j \geqslant 0,\, 2i+j \leqslant n-1)$ such that*

$$A_n(s,t) = \sum_{i,j} \gamma_{n,i,j}(st)^i(s+t)^j(1+st)^{n-1-j-2i}.$$

An explicit recurrence for the coefficients $\gamma_{n,i,j}$ was described by Visontai [15]. This recurrence does not directly imply the positivity of the coefficients, but Lin [10] managed to use it to eventually prove Gessel's conjecture. Unlike the univariate case, no combinatorial proof of Gessel's conjecture is known.

Simple permutations (for their definition see Section 2) serve as building blocks of all permutations. We propose here a strengthening of Gessel's conjecture, for the class of simple permutations.

Conjecture 3. For each positive n, the bivariate polynomial

$$\mathrm{simp}_n(s,t) = \sum_{\sigma \in \mathrm{Simp}_n} s^{des(\sigma)} t^{\mathrm{ides}(\sigma)}$$

is gamma-positive, where Simp_n is the set of simple permutations of length n.

Using the substitution decomposition tree of a permutation (by repeated inflation), we show how this cojecture implies the Gessel-Lin result. A combinatorial proof of the conjecture will give a combinatorial proof of the Gessel-Lin result. We also provide supporting evidence for this stronger conjecture.

The rest of the paper is organized as follows. Section 2 contains background material concerning simple permutations, inflation, and the substitution decomposition tree of a permutation. In Section 3 we introduce combinatorial involutions on the tree, and use them to give a combinatorial proof of Gessel's conjecture for a certain class of permutations, $H(5) \cap S_n$. In Section 4 we show how, more generally, Lin's theorem (Gessel's conjecture) follows combinatorially from Conjecture 3. Finally, in Section 5, we give a formula for $\mathrm{simp}_n(s,t)$ which may have independent value.

2 Simple permutations and inflation

We start by presenting some preliminaries concerning simple permutations, inflation and the substitution decomposition tree. Original papers will be mentioned occasionally, but terminology and notation will follow (with a few convenient exceptions) the recent survey [14].

Definition 4. Let $\pi = a_1 \ldots a_n \in S_n$. A *block* (or *interval*) of π is a nonempty contiguous sequence of entries $a_i a_{i+1} \ldots a_{i+k}$ whose values also form a contiguous sequence of integers.

Example 5. If $\pi = 2647513$ then 6475 is a block but 64751 is not.

Each permutation can be decomposed into singleton blocks, and also forms a single block by itself; these are the *trivial blocks* of the permutation. All other blocks are called *proper*.

Definition 6. A permutation is *simple* if it has no proper blocks.

Example 7. The permutation 3517246 is simple.

The simple permutations of length $n \leqslant 2$ are 1, 12 and 21. There are no simple permutations of length $n = 3$. Those of length $n = 4$ are 2413 and its inverse (which is also its reverse). For length $n = 5$ they are 24153, 41352, their reverses and their inverses (altogether 6 permutations).

Definition 8. A *block decomposition* of a permutation is a partition of it into disjoint blocks.

For example, the permutation $\sigma = 67183524$ can be decomposed as 67 1 8 3524. In this example, the relative order between the blocks forms the permutation 3142, i.e., if we take for each block one of its digits as a representative then the set of representatives is order-isomorphic to 3142. Moreover, the block 67 is order-isomorphic to 12, and the block 3524 is order-isomorphic to 2413. These are instances of the concept of *inflation*, defined as follows.

Definition 9. Let $n_1, \ldots, n_k$ be positive integers with $n_1 + \ldots + n_k = n$. The *inflation* of a permutation $\pi \in S_k$ by permutations $\alpha_i \in S_{n_i}$ $(1 \leqslant i \leqslant k)$ is the permutation $\pi[\alpha_1, \ldots, \alpha_k] \in S_n$ obtained by replacing the i-th entry of π by a block which is order-isomorphic to the permutation α_i on the numbers $\{s_i{+}1, \ldots, s_i{+}n_i\}$ instead of $\{1, \ldots, n_i\}$, where $s_i = n_1 + \ldots + n_{i-1}$ $(1 \leqslant i \leqslant k)$.

Example 10. The inflation of 2413 by $213, 21, 132$ and 1 is

$$2413[213, 21, 132, 1] = 546\ 98\ 132\ 7.$$

A very important fact is that inflation is additive on both *des* and ides.

Observation 11. *Let* $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$*. Then*

$$des(\sigma) = des(\pi) + \sum_{i=1}^{n} des(\alpha_i)$$

and

$$\mathrm{ides}(\sigma) = \mathrm{ides}(\pi) + \sum_{i=1}^{n} \mathrm{ides}(\alpha_i).$$

Two special cases of inflation, deserving special attention, are the *direct sum* and *skew sum* operations, defined as follows.

Definition 12. Let $\pi \in S_m$ and $\sigma \in S_n$. The *direct sum* of π and σ is the permutation $\pi \oplus \sigma \in S_{m+n}$ defined by

$$(\pi \oplus \sigma)_i = \begin{cases} \pi_i, & \text{if } i \leqslant m; \\ \sigma_{i-m} + m, & \text{if } i > m, \end{cases}$$

and their *skew sum* is the permutation $\pi \ominus \sigma \in S_{m+n}$ defined by

$$(\pi \ominus \sigma)_i = \begin{cases} \pi_i + n, & \text{if } i \leqslant m; \\ \sigma_{i-m}, & \text{if } i > m. \end{cases}$$

Example 13. If $\pi = 132$ and $\sigma = 4231$ then $\pi \oplus \sigma = 1327564$ and $\pi \ominus \sigma = 5764231$

Note that $\pi \oplus \sigma = 12[\pi, \sigma]$ and $\pi \ominus \sigma = 21[\pi, \sigma]$.

Definition 14. A permutation is *sum-indecomposable* (respectively, *skew-indecomposable*) if it cannot be written as a direct (respectively, skew) sum.

The following proposition shows that every permutation has a canonical representation as an inflation of a simple permutation.

Proposition 15. *[2, Theorem 1][14, Proposition 3.10] Let* $\sigma \in S_n$ $(n \geqslant 2)$*. Then there exist a unique integer* $k \geqslant 2$*, a unique simple permutation* $\pi \in S_k$*, and a sequence of permutations* $\alpha_1, \ldots, \alpha_k$ *such that*

$$\sigma = \pi[\alpha_1, \ldots, \alpha_k].$$

If $\pi \notin \{12, 21\}$ *then* $\alpha_1, \ldots, \alpha_k$ *are also unique.*

If $\pi = 12$ $(\pi = 21)$ *then* α_1, α_2 *are unique as long as we require, in addition, that* α_2 *is sum-indecomposable (respectively, skew-indecomposable).*

Example 16. The permutation $\sigma = 452398167$ can be written as an inflation of the simple permutation 2413:

$$\sigma = 2413[3412, 21, 1, 12].$$

Remark 17. The additional requirements for $\pi = 12$ and $\pi = 21$ are needed for uniqueness of the expression. To see that, note that the permutation 123 can be written as $12[12, 1] = 12\ 3$ but also as $12[1, 12] = 1\ 23$. The first expression is the one preferred above (with α_2 sum-indecomposable).

One can continue the process of decomposition by inflation for the constituent permutations α_i, recursively, until all the resulting permutations have length 1. In the example above, 3412 can be further decomposed as $3412 = 21[12, 12]$, so that

$$\sigma = 2413[21[12, 12], 21, 1, 12]$$

and, eventually,

$$\sigma = 2413[21[12[1, 1], 12[1, 1]], 21[1, 1], 1, 12[1, 1]].$$

This information can be encoded by a tree, as follows.

Definition 18. Represent each permutation σ by a corresponding *substitution decomposition tree* T_σ, recursively, as follows.

- If $\sigma = 1 \in S_1$, represent it by a tree with one node.
- Otherwise, write $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$ as in Proposition 15, and represent σ by a tree with a root node, labeled π, having k ordered children corresponding to $\alpha_1, \ldots, \alpha_k$. Replace each child α_i by the corresponding tree T_{α_i}.

Example 19. Figure 1 depicts the substitution decomposition tree T_σ for $\sigma = 452398167$. For clarity, the leaves are labeled by the corresponding values of the permutation σ, instead of simply 1.

Inflation can be extended to sets of permutations (an operation called *wreath product* in [3]).

Definition 20. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of permutations. Define

$$\mathcal{A}[\mathcal{B}] = \{\alpha[\beta_1, \ldots, \beta_k] \mid \alpha \in \mathcal{A},\ \beta_1, \ldots, \beta_k \in \mathcal{B}\}.$$

Example 21. Let $A = \{12\}$ and $B = \{21, 132\}$. Then

$$A[B] = \{2143, 21354, 13254, 132465\}.$$

Definition 22. A set C of permutations is *substitution-closed* if $C = C[C]$. The *substitution closure* $\langle C \rangle$ of a set C of permutations is the smallest substitution-closed set of permutations which contains C.

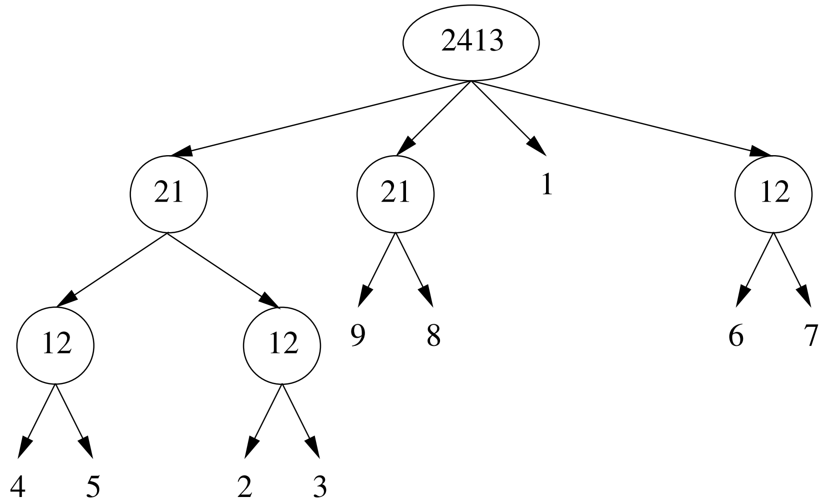

Figure 1: The tree T_σ for $\sigma = 452398167$

The inflation operation is associative. Defining $C_1 = C$ and $C_{n+1} = C[C_n]$, we clearly have

$$\langle C \rangle = \bigcup_{n=1}^{\infty} C_n.$$

Definition 23. For a positive integer n, let Simp_n ($\mathrm{Simp}_{\leqslant n}$) be the set of all simple permutations of length n (respectively, of length at most n). Let $H(n) = \langle \mathrm{Simp}_{\leqslant n} \rangle$, the substitution closure of $\mathrm{Simp}_{\leqslant n}$.

Example 24. $H(2) = \langle \mathrm{Simp}_{\leqslant 2} \rangle = \langle \{1, 12, 21\} \rangle$ is the set of all permutations that can be obtained from the trivial permutation 1 by direct sums and skew sums. These are exactly the *separable permutations*, counted by the *large Schröder numbers;* see [16]. Separable permutations can also be described via pattern avoidance, namely

$$H(2) = \mathrm{Av}(3142, 2413).$$

For more details see [5, following Proposition 3.2].

3 Gamma-positivity for $H(5)$

In this section we present a combinatorial proof of Gessel's conjecture (Lin's theorem) for the subset $H(5) \cap S_n$ of S_n (for any positive n).

Fu, Lin and Zeng [9] proved the following (univariate) gamma-positivity result.

Proposition 25. *For each n there exist nonnegative integers $\gamma_{n,k}$ $(0 \leqslant k \leqslant \lfloor (n-1)/2 \rfloor)$ such that*

$$\sum_{\pi \in H(2) \cap S_n} t^{des(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k}$$

By Observation 11, if $\pi \in H(2)$ then $des(\pi) = \mathrm{ides}(\pi)$. Hence, one can conclude the following restricted version of Gessel's conjecture for the set of separable permutations.

Theorem 26. *For each n there exist non-negative integers $\gamma_{n,k}$ $(0 \leqslant k \leqslant \lfloor (n-1)/2 \rfloor)$ such that:*

$$\sum_{\pi \in H(2) \cap S_n} s^{des(\pi)} t^{\mathrm{ides}(\pi)} = \sum_{\pi \in H(2) \cap S_n} (st)^{des(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} (st)^k (1+st)^{n-1-2k}.$$

In order to extend Theorem 26 further, let us introduce some more definitions.

Definition 27. Let T be a tree with all internal nodes labeled by simple permutations. A *binary right chain* (BRC) is a maximal nonempty chain composed of consecutive right descendants, all of which are from the set $\{12, 21\}$. The *length* of a BRC is the number of nodes in it. Denote by $r_{odd}(T)$ the number of BRC of odd length in T.

Example 28. The tree T_σ in Figure 1 has 4 BRC, and $r_{odd}(T_\sigma) = 3$.

Definition 29. A tree T is called a *G-tree* if it satisfies:

1. Each leaf is labeled by 1.
2. Each internal node is labeled by a simple permutation ($\neq 1$), and the number of its children is equal to the length of the permutation.
3. The labels in each BRC alternate between 12 and 21.

Denote by $\mathcal{GT}_n$ the set of all G-trees with n leaves.

Lemma 30. *The map $f_n : S_n \to \mathcal{GT}_n$ sending each permutation σ to its substitution decomposition tree T_σ, as in Definition 18, is a bijection.*

Proof. Follows immediately from Proposition 15. The last condition in Definition 29 reflects the extra restrictions for the cases $\pi \in \{12, 21\}$ in Proposition 15. $\square$

Let $T = T_\pi$ be a G-tree, and let $\{C_i \mid 1 \leqslant i \leqslant r_{odd}(T)\}$ be the set of all BRC of odd length in T. For each i, let $\phi_i(T)$ be the tree obtained from T by switching 12 and 21 in each of the nodes of C_i. (A similar action was introduced in [9] for univariate polynomials.) Clearly, each operator ϕ_i is an involution, and the various ϕ_i commute. By Observation 11, each ϕ_i changes both $des(\pi)$ and $\mathrm{ides}(\pi)$ by ± 1.

Example 31. Consider $\pi = 6713254$. The corresponding tree $T = T_\pi$ appears on the left side of Figure 2, and has $r_{odd}(T) = 2$. If C_1 is the unique BRC of length 3 in T, then $\phi_1(T)$ is the tree on the right side of the figure. The permutation corresponding to $\phi_1(T)$ is 1257634. Note that ϕ_1 decreased both $des(\pi)$ and $\mathrm{ides}(\pi)$ by 1.

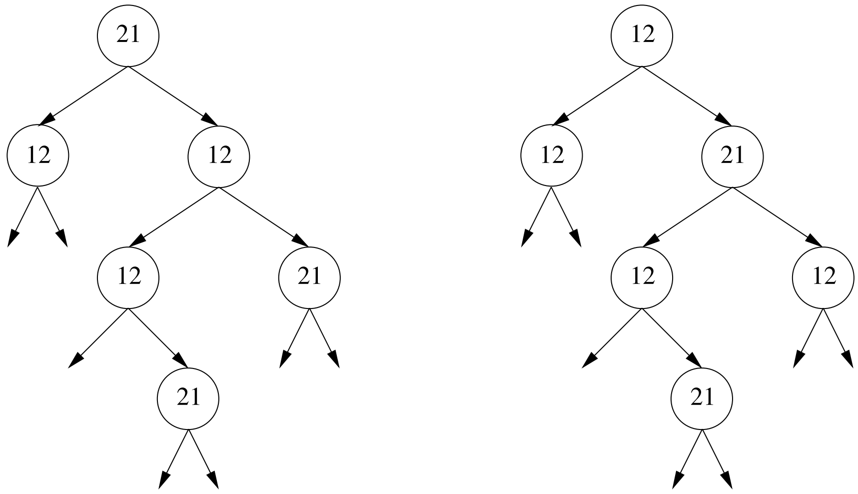

Figure 2: Left: the tree T. Right: The tree $\phi_1(T)$.

Let $T = T_\pi$ be a G-tree, and let $l_1, \ldots, l_k$ be the labels of nodes in T that belong to the set $\mathrm{Simp}_4 = \{2413, 3142\}$. Define $\psi_j(T)$ $(1 \leqslant j \leqslant k)$ to be the tree obtained from T by switching the label l_j from 2413 to 3142, or vice versa. Again, it is easy to see that the ψ_j are commuting involutions, and each ψ_j commutes with each ϕ_i. Switching from 2413 to 3142 increases $des(\pi)$ by 1 while decreasing $\mathrm{ides}(\pi)$ by 1.

Remark 32. Each of the 6 simple permutations $\pi \in \mathrm{Simp}_5$ has $des(\pi) = \mathrm{ides}(\pi) = 2$, and we don't need to define involutions for them.

Definition 33. For any two G-trees T_1 and T_2, write $T_1 \sim T_2$ if T_2 can be obtained from T_1 by a sequence of applications of the involutions ϕ_i and ψ_j.

Clearly $\sim$ is an equivalence relation, partitioning the set $\mathcal{GT}_n$ (equivalently, the group S_n) into equivalence classes.

Definition 34. For each equivalence class in $\mathcal{GT}_n$, let T_0 be the unique tree in this class in which each odd BRC begins with 12 and each node representing a simple permutation of length 4 is labeled 2413. The corresponding permutation π_0 has the minimal number of descents in its class. The tree T_0 and the permutation π_0 are called the *minimal representatives* of their equivalence class.

Lemma 35. *Let A be an equivalence class of permutations in $H(5) \cap S_n$. There exist nonnegative integers i and j such that*

$$\sum_{\sigma \in A} s^{des(\sigma)} t^{\mathrm{ides}(\sigma)} = (st)^i (s+t)^j (1+st)^{n-1-2i-j}.$$

Proof. Let π_0 be the minimal representative of A, and let $T_0 = T_{\pi_0}$. For $i \in \{2, 4, 5\}$, let v_i be the number of nodes of T_0 having labels of length i. Since T_0 has exactly n leaves, its total number of nodes (including leaves) is

$$v = n + v_2 + v_4 + v_5.$$

On the other hand, counting the children of each node gives

$$v - 1 = \sum_{i \in \{2,4,5\}} i v_i.$$

It follows that

$$n - 1 = v_2 + 3v_4 + 4v_5.$$

Let $r = r_{odd}(T_0)$ be the number of odd BRC in T_0, and let d_2 be the number of nodes labeled 21. By definition, each BRC alternates between 12 and 21 and each odd BRC in T_0 starts with 12. It follows that

$$v_2 = 2d_2 + r,$$

so that

$$n - 1 = r + 2d_2 + 3v_4 + 4v_5. \tag{3}$$

Now recall that $des(12) = \mathrm{ides}(21) = 0$, $des(21) = \mathrm{ides}(21) = 1$, $des(2413) = 1$, $\mathrm{ides}(2413) = 2$, $des(3142) = 2$, $\mathrm{ides}(3142) = 1$, and for each simple permutation σ of length 5 we have $des(\sigma) = \mathrm{ides}(\sigma) = 2$. it follows from Observation 11 that

$$des(\pi_0) = d_2 + v_4 + 2v_5$$

and

$$\mathrm{ides}(\pi_0) = d_2 + 2v_4 + 2v_5,$$

so that the bivariate monomial corresponding to π_0 is

$$s^{des(\pi_0)} t^{\mathrm{ides}(\pi_0)} = (st)^{d_2 + v_4 + 2v_5} t^{v_4}.$$

Consider now the whole equivalence class A, whose elements are obtained from π_0 by applications of the commuting involutions $\phi_1, \ldots, \phi_r$ and $\psi_1, \ldots, \psi_k$, where $r = r_{odd}(T_0)$ is the number of BRC in T_0 and $k = v_4$. Each application of ϕ_i multiplies the monomial by st, and each application of ψ_j multiplies it by st^{-1}. It follows that

$$\begin{aligned}\sum_{\sigma \in A} s^{des(\sigma)} t^{\mathrm{ides}(\sigma)} &= (st)^{d_2 + v_4 + 2v_5} t^{v_4} (1 + st)^r (1 + st^{-1})^{v_4} \\ &= (st)^{d_2 + v_4 + 2v_5} (s + t)^{v_4} (1 + st)^r.\end{aligned}$$

Denoting $i = d_2 + v_4 + 2v_5$ and $j = v_4$ will complete the proof, once we show that

$$2(d_2 + v_4 + 2v_5) + v_4 + r = n - 1;$$

but this follows immediately from equation (3) above. □

Lemma 35 immediately implies one of the main results of this paper:

Theorem 36. *For each $n \geqslant 1$, the polynomial*

$$\sum_{\sigma \in H(5) \cap S_n} s^{des(\sigma)} t^{ides(\sigma)}$$

is gamma-positive.

4 Gamma-positivity for simple permutations

For each positive integer n, the set Simp_n of simple permutations of length n is invariant under taking inverses and reverses. It follows that the bivariate polynomial

$$\mathrm{simp}_n(s,t) = \sum_{\sigma \in \mathrm{Simp}_n} s^{des(\sigma)} t^{\mathrm{ides}(\sigma)}$$

is palindromic, and can be expanded in the gamma basis. Conjecture 3, presented in the Introduction, states that this polynomial is, in fact, gamma-positive. The main result of this section is the following.

Theorem 37. *Conjecture 3 implies Gessel's conjecture (Lin's theorem), Theorem 2.*

Proof. For each permutation σ of length $n \geqslant 2$, consider its substitution decomposition tree T_σ. Each internal node of T_π is labeled by some simple permutation π of length $\ell = \ell(\pi) \geqslant 2$. Replace π by ℓ, to obtain a *simplified tree* T'_σ (with internal nodes labeled by numbers). For permutations $\sigma_1, \sigma_2 \in S_n$ define $\sigma_1 \sim \sigma_2$ if $T'_{\sigma_1} = T'_{\sigma_2}$. Clearly $\sim$ is an equivalence relation on S_n, with each equivalence class corresponding to a unique simplified tree T'. Denote such a class by $A(T')$.

Define a BRC of T' (in analogy to Definition 27) to be a maximal nonempty chain of consecutive right descendants, all labeled 2. How can we recover a permutation $\sigma \in A(T')$ from the tree T'? Each internal node, labeled by a number ℓ, can be relabeled by any simple permutation of length ℓ, with the single restriction that the labels in each BRC must alternate between 12 and 21, starting with either of them. It thus follows, by Observation 11, that for each simplified tree T', the polynomial

$$\sum_{\sigma \in A(T')} s^{des(\sigma)} t^{\mathrm{ides}(\sigma)}$$

is a product of factors, as follows:

- Each internal node with label $\ell \geqslant 4$ contributes a factor $\mathrm{simp}_\ell(s,t)$.
- Each BRC of even length $2k$ contributes a factor $2(st)^k$.
- Each BRC of odd length $2k+1$ contributes a factor $(st)^k(1+st)$.

By Conjecture 3, all those factors are gamma-positive, and so is their product. Summing over all equivalence classes in S_n completes the proof. □

It is clear from the arguments above that a *combinatorial* proof of Conjecture 3 will immediately yield a combinatorial proof of Theorem 2. In fact, the preceding section contains such a combinatorial proof assuming there are only labels $\ell \leqslant 5$, using $\mathrm{simp}_4(s,t) = st(s+t)$ and $\mathrm{simp}_5(s,t) = 6(st)^2$. We were unable to extend the combinatorial arguments to length 6, although the corresponding polynomial is indeed gamma-positive:

$$\mathrm{simp}_6(s,t) = st(s+t)^2(1+st) + 5(st)^2(1+st) + 14(st)^2(s+t).$$

In fact, Conjecture 3 has been verified by computer for all $n \leqslant 12$.

5 The bi-Eulerian polynomial for simple permutations

In [2], the ordinary generating function for the *number* of simple permutations was shown to be very close to the functional inverse of the corresponding generating function for all permutations. In this section we refine this result by considering also the parameters *des* and ides, thus obtaining a formula for $\mathrm{simp}_n(s,t)$.

Recall from Definition 14 the notions of sum-indecomposable and skew-indecomposable permutations.

Definition 38. For each positive integer n, denote by I_n^+ (respectively, I_n^-) the set of all sum-indecomposable (respectively, skew-indecomposable) permutations in S_n.

Definition 39. Let

$$F(x,s,t) := \sum_{n=1}^{\infty}\left(\sum_{\pi\in S_n} s^{des(\pi)}t^{\mathrm{ides}(\pi)}\right)x^n,$$

$$I^+(x,s,t) := \sum_{n=1}^{\infty}\left(\sum_{\pi\in I_n^+} s^{des(\pi)}t^{\mathrm{ides}(\pi)}\right)x^n,$$

$$I^-(x,s,t) := \sum_{n=1}^{\infty}\left(\sum_{\pi\in I_n^-} s^{des(\pi)}t^{\mathrm{ides}(\pi)}\right)x^n,$$

$$S(x,s,t) := \sum_{n=4}^{\infty}\left(\sum_{\pi\in \mathrm{Simp}_n} s^{des(\pi)}t^{\mathrm{ides}(\pi)}\right)x^n.$$

Note that the summation in the definition of $S(x,s,t)$ is only over $n \geqslant 4$. We want to find relations between these generating functions.

From now on, we consider $F(x,s,t)$ etc. as formal power series in x, with coefficients in the field of rational functions $\mathbb{Q}(s,t)$. We therefore use the short notation $F(x)$, or even F. For example, the composition $S\circ F$ means that F is substituted as the x variable of $S(x,s,t)$. By Proposition 15 and Observation 11,

$$\begin{aligned} F &= x + I^+F + stI^-F + \sum_{n=4}^{\infty}\mathrm{simp}_n F^n \\ &= x + I^+F + stI^-F + S\circ F \end{aligned}$$

and similarly

$$I^+ = x + stI^-F + S\circ F$$

and

$$I^- = x + I^+F + S\circ F.$$

Rearranging, we have

$$\begin{aligned} FI^+ + stFI^- + (S \circ F + x) &= F \\ -I^+ + stFI^- + (S \circ F + x) &= 0 \\ FI^+ - I^- + (S \circ F + x) &= 0 \end{aligned}$$

This is a system of linear equations in I^+, I^- and $S \circ F + x$. Its unique solution is

$$\begin{aligned} I^+ &= \frac{F}{1+F} \\ I^- &= \frac{F}{1+stF} \\ S \circ F + x &= \frac{F(1-stF^2)}{(1+F)(1+stF)} \end{aligned} \tag{4}$$

Note that the reversal map $\pi \mapsto \pi'$, defined by $\pi'(i) = n - 1 - \pi(i)$ $(1 \leqslant i \leqslant n)$, is a bijection from S_n onto itself (and also from I_n^+ onto I_n^-), satisfying $des(\pi') = n-1-des(\pi)$ and $\mathrm{ides}(\pi') = n - 1 - \mathrm{ides}(\pi)$. Therefore:

$$\begin{aligned} F(x,s,t) &= \frac{1}{st} F(xst, 1/s, 1/t) \\ I^-(x,s,t) &= \frac{1}{st} I^+(xst, 1/s, 1/t). \end{aligned}$$

This agrees with the first two equations in (4). Denoting $u = F(x)$, the third equation in (4) gives an explicit expression for $S(u,s,t)$:

$$S(u,s,t) = -F^{\langle -1 \rangle}(u) + \frac{u(1-stu^2)}{(1+u)(1+stu)}\,,$$

where $x = F^{\langle -1 \rangle}(u)$ is the functional inverse of $u = F(x)$. Further manipulations with partial fractions give the following.

Proposition 40.

$$S(u,s,t) = -F^{\langle -1 \rangle}(u) + \frac{u}{1+stu} + \frac{u}{1+u} - u. \tag{5}$$

Using the expansions

$$S(u,s,t) = \sum_{n \geqslant 4} \mathrm{simp}_n(s,t) u^n$$

and

$$F^{\langle -1 \rangle}(u) = \sum_{n \geqslant 1} f_n^{\langle -1 \rangle}(s,t) u^n,$$

we finally obtain a formula for $\mathrm{simp}_n(s,t)$.

Corollary 41.

$$\mathrm{simp}_n(s,t) = -f_n^{\langle -1 \rangle}(s,t) + (-1)^{n-1} + (-st)^{n-1} \qquad (n \geqslant 4).$$

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