On the poset and asymptotics of Tesler matrices

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Abstract

Tesler matrices are certain integral matrices counted by the Kostant partition function and have appeared recently in Haglund's study of diagonal harmonics. In 2014, Drew Armstrong defined a poset on such matrices and conjectured that the characteristic polynomial of this poset is a power of $q - 1$. We use a method of Hallam and Sagan to prove a stronger version of this conjecture for posets of a certain class of generalized Tesler matrices. We also study bounds for the number of Tesler matrices and how they compare to the number of parking functions, the dimension of the space of diagonal harmonics.

Mathematics Subject Classifications: 05A05, 05A16

1 Introduction

Tesler matrices were introduced by Glenn Tesler to study Macdonald polynomials. They have been recently studied due to their relationship with diagonal harmonics. Haglund proved in [9] that the bigraded Hilbert series for the space of diagonal harmonics, denoted DH_n, is the sum over Tesler matrices of a bivariate weight.

$$\mathrm{Hilb}(DH_n; q, t) = \sum_A \mathrm{wt}_{q,t}(A) \tag{1}$$

where $A = (a_{i,j})$ is a Tesler matrix and the weight $\mathrm{wt}_{q,t}(\cdot)$ is

$$\mathrm{wt}_{q,t}(A) := (-M)^{|\{a_{i,j}>0\}|-n} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t} \text{ with } M = \frac{t-1}{q-1} \text{ and } [b]_{q,t} = \frac{q^b - t^b}{q-t}. \tag{2}$$

In Equation (1), the Hilbert series is over the space DH_n which has dimension $(n+1)^{n-1}$. For more on this space, see [6,8]. Although the enumeration and asymptotics of Tesler matrices are not known, there are some nice product formulas when considering specializations of the alternating weight $\mathrm{wt}_{q,t}(\cdot)$. For instance, it was shown in [3] that

$$q^{\binom{n}{2}} \sum_{A} \mathrm{wt}_{q,q^{-1}}(A) = [n+1]_q^{n-1} \tag{3}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$. Furthermore, it was also shown in [14] that

$$\sum_{A} \mathrm{wt}_{q,0}(A) = [n]_q!. \tag{4}$$

Equations (3) and (4) show product formulas involving alternating sums of Tesler matrices. In this paper, we prove another such result that was initially conjectured by Armstrong in [1] by using a different alternating sum. He defines a *poset* on the set of Tesler matrices which we will denote as $P(1^n)$ and refer to as the *Tesler poset*. Recall that the **characteristic polynomial** on the poset $(P, \preceq)$, denoted $\chi(P; q)$, is a Möbius function weighted rank generating function. That is,

$$\chi(P; q) = \sum_{A \in P} \mu(\hat{0}, A) q^{\rho(P) - \rho(A)}$$

where we use the terminology and notation of [20, Ch.3] for the Möbius function $\mu(\cdot)$, the rank of an element $A \in P$ and of a poset P as $\rho(A), \rho(P)$ respectively, and $\hat{0}$ for the unique least element. We will look at the characteristic polynomial of the Tesler poset, but we first need to give necessary definitions and conventions to discuss Tesler matrices in a precise manner.

Let U_n be the set of $n \times n$ upper-triangular matrices with non-negative integer entries. Given $A \in U_n$, where $A = (a_{i,j})$, we define the **hook sum** h_k for $1 \leqslant k \leqslant n$ to be the sum of all entries weakly right of $a_{k,k}$ minus all entries strictly above it. That is,

$$h_k := (a_{k,k} + a_{k,k+1} + \cdots + a_{k,n}) - (a_{1,k} + a_{2,k} + \cdots + a_{k-1,k}).$$

We define the **hook sum vector** as the n-dimensional vector $(h_1, \ldots, h_n)$. A **Tesler matrix** $A \in U_n$ is such that $h_k = 1$ for all $1 \leqslant k \leqslant n$.

Example 1. The matrix below is a 3×3 Tesler matrix as $h_3 = 2 - 1 - 0 = \mathbf{1}$, $h_2 = 1 + 1 - 1 = \mathbf{1}$, and $h_1 = 0 + 0 + 1 = \mathbf{1}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{pmatrix}$$

We denote the number of matrices in U_n with a hook sum vector of $(\alpha_1, \ldots, \alpha_n)$ as $T(\alpha_1, \ldots, \alpha_n)$ and the set of such matrices as $\mathcal{T}(\alpha_1, \ldots, \alpha_n)$ and refer to these as **generalized Tesler matrices**. We often use short hand of $T(1^n)$ and $\mathcal{T}(1^n)$ for the number of and set of Tesler matrices respectively.

Conjecture 2 (Armstrong [1])**.** Let $P(1^n)$ be the poset on Tesler matrices $\mathcal{T}(1^n)$, then

$$\chi(P(1^n); q) = (q-1)^{\binom{n}{2}}.$$

The method that we use in this paper extends to the larger class of generalized Tesler matrices with binary hook sums and settles Armstrong's conjecture with a simple calculation.

Theorem 3. *Let $\alpha = (\alpha_{n-1}, \ldots, \alpha_0) \in \{0,1\}^n$ and $P(\alpha)$ be the poset on generalized Tesler matrices $\mathcal{T}(\alpha)$. Then, letting $w(\alpha) = \sum_{i=0}^{n-1} i \cdot \alpha_i$, we have that*

$$\chi(P(\alpha); q) = (q-1)^{w(\alpha)}.$$

To see why this theorem settles Armstrong's conjecture, note that $w(1,1,\ldots,1) = \binom{n}{2}$. In addition, this theorem is also consistent with a well known result on the Boolean lattice (see Prop. 18). In order to prove this theorem, we will adapt a method [11] of Joshua Hallam and Bruce Sagan. We also show that certain powers of $(q-1)$ divide the characteristic polynomial of the Tesler poset corresponding to a hook sum vector with either a trailing or a leading binary word. (See Corollary 36.)

Although Tesler matrices have been connected in [8] to diagonal harmonics via a bivariate weight and in [15] were shown to be a solution to the *Kostant partition function*, there are still many enumerative questions on Tesler matrices that have yet to be answered. The best known bound in the literature for $T(1^n)$ is $n! \leqslant T(1^n) \leqslant 2^{\binom{n}{2}}$ [15, §4]. In Section 5, through simple observations of an enumerative tool that we call the **Armstrong polynomial**, we are able to improve the lower bound such that

$$T(1^n) \geqslant (2n-3)!!.$$

In addition, we can similarly get a tighter upper bound. There are also interesting enumerative results when considering generalized Tesler matrices. Let $C_i = \frac{1}{i+1}\binom{2i}{i} \sim 4^i/(\sqrt{\pi} i^{\frac{3}{2}})$ be the ith Catalan number. Zeilberger [23] showed that

$$T(1,2,\ldots,n) = \prod_{i=1}^{n} C_i.$$

Thus $T(1,2,\ldots,n) = e^{\Theta(n^2)}$, which motivated the following question.

Question 4 (Pak)**.** True or False: The number of Tesler matrices have the following asymptotics

$$T(1^n) = e^{\Theta(n^2)}.$$

Remark 5. Note that even the improved lower bound needs to be significantly improved further to give an affirmative answer to Question 4. However, the existing data in the OEIS A008608 suggests that $\log(T(1^n)) = O(n^{1.6})$ as noted in [17].

We denote the hook sum vector $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with k 1's and $(n-k)$ 0's as $(1^k, 0^{n-k})$. This set of generalized Tesler matrices have previously been studied in [12] and we analyze the set $\mathcal{T}(1^k, 0^{n-k})$ in Section 6 to get some insight into Tesler matrices. For fixed k, we will show that

$$T(1^k, 0^{n-k}) \geqslant (k+1)^{n-1} \text{ for sufficiently large } n.$$

This leads us to conjecture that for n large enough, the number of Tesler matrices can be bounded below by the dimension of DH_n, which is $(n+1)^{n-1}$ (also the number of parking functions of size n). We also find generating functions $T_k(x)$ for particular values of k. When $k = 1$, $T(1, 0^{n-1}) = 2^{n-1}$, so this generating function is trivial. However, when $k = 2$ we find the generating function in Proposition 47 [12]. While the case where $k = 3$ is still open, further study could provide insight about a generating function for $T(1^n)$.

Outline: In Section 2, we will highlight some previous results and methods that will be pertinent in this paper. Then, in Section 3, we introduce the Tesler poset, some its properties, and show that a specific hook sum vector yields a poset which is isomorphic to the well-known Boolean lattice that was initially noticed by Alejandro H. Morales in [16]. Using these results, we will then prove Theorem 3 in Section 4 and explore some of its corollaries. Finally, in the last two sections, we will explore asymptotics and other enumerative questions regarding generalized Tesler matrices. We then conclude by exploring the significance of settling Conjecture 2 in respect to the asymptotics of Tesler matrices.

2 Background

2.1 Tesler Generating Algorithm

We will discuss a method for generating generalized Tesler matrices as given by Drew Armstrong [1]. Fix a generalized Tesler matrix $A = (a_{i,j})$ of size n with a hook sum vector $(\alpha_1, \ldots, \alpha_n)$. Then, consider the main-diagonal entries of A as an n-tuple $(d_1, \ldots, d_n)$ with $d_i := a_{i,i}$. We will create a generalized Tesler matrix $A' = (a_{i,j}{}')$ with hook sum vector $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ by first constructing its main-diagonal $(d_1{}', \ldots, d_{n+1}{}')$. For all i such that $1 \leqslant i \leqslant n$, we take d_i and replace it with $d_i{}'$ where $0 \leqslant d_i{}' \leqslant d_i$ and set $a_{n+1,i}{}' = d_i - d_i{}'$ so that the ith hook sum h_i remains unchanged. Then, let $d_{n+1}{}'$ be such that the sum of our newly constructed main-diagonal $(n+1)$-tuple adds up to $\sum\limits_{k=1}^{n+1} \alpha_k$ and let the other entries in the matrix remain unchanged.

Example 6. The Tesler matrix in Example 1 has a main-diagonal tuple $(0, 1, 2)$. We will consider the Tesler matrices of size 4 that this matrix generates in Figure 1. In Definition 8, we will define a function that yields the number of Tesler matrices of size $(n+1)$ that a given Tesler matrix of size n generates. By the multiplication principle, the algorithm applied to our initial Tesler matrix with main-diagonal tuple $(0, 1, 2)$ generates six $(1 \cdot 2 \cdot 3)$ main-diagonal 4-tuples. Thus, this initial size 3 Tesler matrix generates six Tesler matrices $(\alpha_4 = 1)$ of size 4.

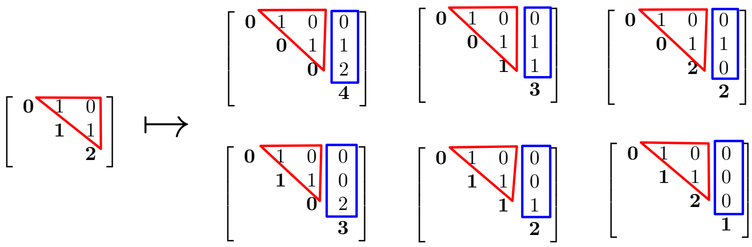

Figure 1: Note that the red triangle is constant and that the blue rectangle corresponds to what was subtracted from the original main-diagonal.

Proposition 7. *Iterating the Tesler Generating Algorithm yields all Tesler matrices.*

Proof. Seeking a contradiction, suppose that there exists a least integer z corresponding to the size of at least one Tesler matrix A that is not generated by this process. By reversing this process, we can then create a Tesler matrix of a smaller size $(z-1)$ that must be generated from this process as it is smaller in size than A. We could then generate A from a matrix that is generated through this process. Hence, this process generates all of the Tesler matrices. $\square$

Fixing $A = (a_{i,j})$ with hook sum vector $(\alpha_1, \ldots, \alpha_n)$, we now consider the number of generalized Tesler matrices of size $(n+1)$ that A generates.

Definition 8. Let $A = (a_{i,j})$ be an $n \times n$ generalized Tesler matrix, then let $d_i := a_{i,i}$ be the ith main-diagonal entry. We define the **diagonal product** of A, denoted $\mathrm{dpro}(A)$, as

$$\mathrm{dpro}(A) = \prod_{i=1}^{n} (d_i + 1).$$

Note that

$$T(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = \sum_{A \in \mathcal{T}(\alpha_1, \ldots, \alpha_n)} \mathrm{dpro}(A). \tag{5}$$

Remark 9. By looking at the right hand side of Equation (5), we note that there is no dependency on the term α_{n+1} and hence for any natural numbers $\alpha_{n+1}, \beta_{n+1}$ we have that

$$T(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = T(\alpha_1, \ldots, \alpha_n, \beta_{n+1}).$$

This follows since if $A = (a_{i,j}) \in \mathcal{T}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then by the definition of the nth hook sum, we have a lower bound on the entry $a_{n,n}$ in that $a_{n,n} \geqslant \alpha_n$.

2.2 Integral Flow Representation

A Tesler matrix of size n can also be represented as an integral flow on the complete directed graph on $(n+1)$ vertices with net flows equal to $(1, 1, \ldots, 1, -n)$ [15]. Given any generalized Tesler matrix with hook sum vector $(\alpha_1, \ldots, \alpha_n)$, we can represent it as an integral flow on the complete directed graph on $(n+1)$ vertices with net flows equal to $(\alpha_1, \ldots, \alpha_n, -\sum_{i=1}^{n} \alpha_i)$.

The bijection in [15] shows that these are equivalent notions. They consider the main-diagonal entry in row i to be the flow sent from the ith vertex to the $(n+1)st$ vertex, which is the rightmost vertex. Then for each entry such that $i < j$, $a_{i,j}$ corresponds to the flow between the ith and jth vertices. See Figure 2 below for an example of this bijection.

$$\begin{pmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{pmatrix} \longleftrightarrow$$

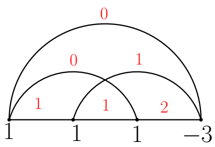

Figure 2: Net flows depicted underneath the complete directed graph.

2.3 Method of Hallam and Sagan

Sagan [18] has previously done work on why the characteristic polynomial of a poset factors. Recently, Sagan and Hallam [11] have introduced a method for showing that the characteristic polynomial of a poset factors. We will apply Hallam and Sagan's method to the Boolean lattice to prove Theorem 3. Their method is to take ranked posets $P_1, \ldots, P_k$ for which the characteristic polynomial is known, and to consider $Q = P_1 \times \cdots \times P_k$. We recall the following facts regarding the characteristic polynomial of posets.

1) If $P \cong P'$, then $\chi(P; q) = \chi(P'; q)$.
2) $\chi(P_1 \times P_2; q) = \chi(P_1; q) \cdot \chi(P_2; q)$

Then, they define an equivalence relation $(\sim)$ to identify elements in Q such that $Q/\sim \;\cong P$. The process of identifying elements leaves the characteristic polynomial unchanged if the equivalence relation satisfies certain conditions. First, an equivalence relation is **homogeneous** if

1) $\hat{0} \in Q$ is in an equivalence class by itself
2) If $X \geqslant Y$ in $Q/\sim$, then for all $x \in X$, there is a $y \in Y$ such that $x \geqslant y$.

Next, we need $\sim$ to preserve rank so that if $x \sim y$, then $\rho(x) = \rho(y)$. Lastly, letting $\mu(\cdot)$ be the Möbius function on Q and considering any nonzero $X \in Q/\sim$ with lower order ideal $L(X) \subseteq Q$,

$$\sum_{y \in L(X)} \mu(\hat{0}, y) = 0. \tag{6}$$

Hallam and Sagan refer to (6) as the summation condition and we adopt this same terminology.

Lemma 10 (Hallam and Sagan [11]). *Let Q be a ranked poset as above and $\sim$ be an equivalence relation on Q which is homogeneous, preserves rank and satisfies the summation condition. Then*

$$\chi({}^{Q}/{\sim}; q) = \chi(Q; q).$$

Remark 11. Hence, given suitable P_i, we see that $\chi(P; q)$ factors. In Hallam and Sagan's paper [11], they use claws CL_n to construct their products. We will use the Boolean lattice to construct our product.

3 The Tesler Poset

We first define the cover relation, introduced by Drew Armstrong [1], and will then use this definition to prove a couple of useful facts which yield some intuition regarding the Tesler poset.

3.1 Definition of Tesler Poset

There are two cases in the example in Figure 3 of the cover relation for the matrix representation depending on the location of the entries.

Definition 12. Fix a hook sum vector α. Then $A = (a_{i,j}) \in \mathcal{T}(\alpha)$ covers $B = (b_{i,j}) \in \mathcal{T}(\alpha)$ and we write $B \preceq A$ if there exists $i < j < k$ such that $a_{i,j} = b_{i,j} + 1$, $a_{j,k} = b_{j,k} + 1$, and $a_{i,k} = b_{i,k} - 1$ or if there exists $i < j$ such that $a_{i,j} = b_{i,j} + 1$, $a_{j,j} = b_{j,j} + 1$, and $a_{i,i} = b_{i,i} - 1$. Note that the notation $B \preceq A$ differs from Stanley's notation in [20].

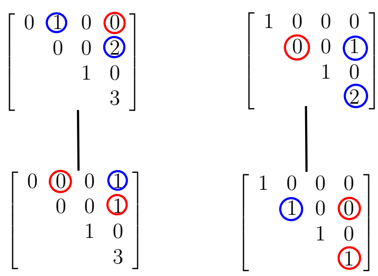

Figure 3: The matrix version of the cover relation

The poset has a least element, $\hat{0}$, with the main-diagonal corresponding to the hook sum vector and all other entries equal to zero. Hence, in the case of a hook sum vector $(1, 1, \ldots, 1)$, the minimal element is the identity matrix of size n.

Remark 13. With the equivalent notion of a Tesler matrix as an integral flow on the complete directed graph, the cover relation for the Tesler poset can also be described in terms of integral flows. Abusing notation, let A, B be the corresponding integral flows to Tesler matrices A and B respectively. Then, integral flow A covers B if there exists vertices $i < j < k$ such that the flow between i and k is 1 more in B than it is in A and the flow from vertices j to k and i to j is 1 more in A than it is in B.

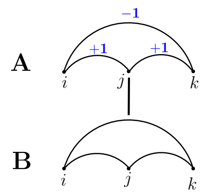

Figure 4: The cover relation for the integral flow representation

Example 14. In the poset below, we see that Armstrong's conjecture is true for the case where $n = 3$. Collecting terms from the bottom-up, we get

$$\chi(P(1^3); q) = q^3 - q^2 - q^2 - q^2 + 2q + q - 1 = (q - 1)^3.$$

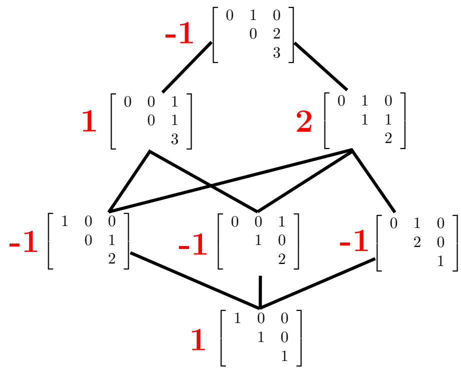

Figure 5: The Tesler poset $P(1^3)$ with the values of the Möbius function in red. See appendix Figure 12 for the Hasse diagram of $P(1^4)$.

Remark 15. By looking at the Hasse diagram of the Tesler poset $P(1^3)$ in Figure 5, we see that it is not a lattice.

3.2 Properties

We will show a few properties of the Tesler poset $P(\alpha)$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Proposition 16. *The rank of a matrix in the Tesler poset $P(\alpha)$ exists and is equal to the sum of the non-main-diagonal entries. That is, for $A = (a_{i,j}) \in P(\alpha)$, we have that*

$$\rho(A) = \sum_{i>j} a_{i,j}.$$

Proof. As we see in the definition of the cover relation, for any $A, B \in \mathcal{T}(\alpha)$, if A covers B, then we necessarily have that the sum of the non-main-diagonal entries for A is one more than the sum of the non-main-diagonal entries for B. Note that this yields that the poset $P(\alpha)$ is a ranked poset. The minimal entry has a non-main-diagonal sum of 0 and we get the desired result. $\square$

Corollary 17. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$*, then the rank of the Tesler poset* $P(\alpha)$ *is* $\sum_{i=1}^{n}(n-i)\alpha_i$

Proof. The maximum element, $M \in P(\alpha)$, is such that the entry $M_{i,i+1} = \sum_{k=1}^{i} \alpha_k$ and $M_{n,n} = \sum_{i=1}^{n} \alpha_i$ with all other entries zero. This is easy to see when considering the integral flow representation. The result then follows from the previous proposition. □

3.3 Relation to Boolean Lattice

We now relate the poset formed by generalized Tesler matrices with hook sum vector $a_n = (1, 0, \ldots, 0)$ to the well-known Boolean lattice for subsets of $[n] := \{1, 2, \ldots, n\}$ under the inclusion relation. Throughout this subsection, n is the length of the hook sum vector and hence $(n+1)$ is the length of the vector depicting net flow in an integral flow. In this subsection, we seek to prove the following proposition.

Proposition 18. *Let* $a_n = (1, 0, \ldots, 0)$*, then we have that* $P(a_n) \cong B_{n-1}$ *[16].*

Recall the bijection presented in [15], establishing the equivalent notion of generalized Tesler matrices as integral flows on the complete graph on $(n+1)$ vertices as mentioned in Section 2.2. This bijection, combined with the integral flow cover relation (Figure 4) give that we can consider the poset $P(1, 0, 0, \ldots, 0)$ as the partial order on integral flows. The integral flows would then have net flows $(1, 0, 0, \ldots, 0, -1)$. In order to prove Proposition 18, we need an order-preserving bijection between integral flows on the complete graph of $(n+1)$ vertices and the Boolean lattice B_{n-1} as we will do in Proposition 19 below.

To this end, we will now consider integral flows with net flow $(1, 0, 0, \ldots, 0, -1)$. Note that the outward flow from a vertex in such an integral flow is at most 1. (Suppose there existed a vertex with outward flow of at least 2, then since only the first vertex has positive net flow, we can follow this flow backwards and get that the net flow of the first vertex is at least 2.)

Since the outward flow is either 0 or 1 in integral flows with net flow $(1, 0, 0, \ldots, 0, -1)$, we can equivalently denote such an integral flow by drawing an edge between vertices i and j if there exists an outward flow of 1 from vertex i to vertex j. Otherwise, we draw no such edge.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix} \leftrightarrow$$

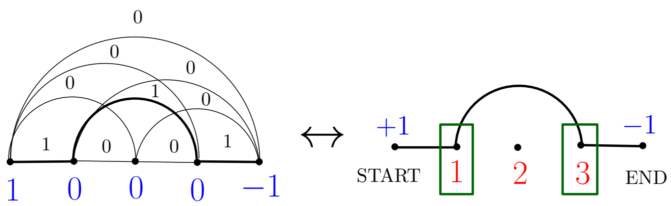

Figure 6: An example with an element from $\mathcal{T}(1, 0, 0, 0)$ with the net flows depicted in blue and the green rectangles in the path hint at the bijection.

Thus, our integral flows with net flow $(1, 0, 0, \ldots, 0, -1)$ can be viewed as a path between vertex 1 and vertex $(n+1)$. Next, for convenience in defining an order-preserving bijection, we label our vertices from 0 to n as opposed to 1 to $(n+1)$ as show in the above Figure 6. We think of vertex 0 as the start of the path and vertex n as the end of the path. Note that the path either hits or does not hit vertex i for all $1 \leqslant i \leqslant (n-1)$. This is how we define our order-preserving bijection.

Proposition 19. *Let the map $\Phi : \mathcal{T}(1, 0, \ldots, 0) \to Pw([n-1])$ be given by, for all $1 \leqslant i \leqslant (n-1)$, we have $i \in \Phi(A)$ if and only if integral flow A, viewed as a path, visits vertex i. Then, Φ is an order-preserving bijection.*

Proof. The map Φ is surjective since given any subset $\{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, (n-1)\}$ we can define a path which visits vertices $i_1, \ldots, i_k$. By filling in the remaining flows with 0's, we get an integral flow A that corresponds with an element from $\mathcal{T}(1, 0, \ldots, 0)$ such that $\Phi(A) = \{i_1, \ldots, i_k\}$. The map Φ is injective since if $\Phi(A) = \Phi(B)$, then the path from vertex 0 to vertex n must be the same. Now, since the outward flows are either 0 or 1, the remaining outward flows must be 0, and hence $A = B$ as integral flows on the complete graph on $(n+1)$ vertices. The order-preserving nature of the map Φ follows immediately from the definition of the integral flow formulation of the cover-relation. (See Figure 4.) $\square$

Proposition 20. *Let $P(1, 0, \ldots, 0, 1)$ and $P(1, 0, \ldots, 0)$ be Tesler posets with hook sum vectors of length n. Then $P(1, 0, \ldots, 0, 1) \cong P(1, 0, \ldots, 0) \cong B_{n-1}$*

Proof. As we mentioned in Remark 9, we see that $\theta_n : \mathcal{T}(1, 0, \ldots, 0) \to \mathcal{T}(1, 0, \ldots, 0, 1)$ given by $M_{n,n} \mapsto (M_{n,n} + 1)$ is a bijection.

It also follows that θ_n is order preserving giving us the first congruence that $P(1, 0, \ldots, 0, 1) \cong P(1, 0, \ldots, 0)$. Thus, by Proposition 18, we have

$$P(1, 0, \ldots, 0, 1) \cong P(1, 0, \ldots, 0) \cong B_{n-1}. \qquad \square$$

Remark 21. The composition of the bijection between the matrix representation and the integral flow representation given in [15] composed with the bijection Φ above, yields a bijection from generalized Tesler matrices with hook sum vector $(1, 0, \ldots, 0)$ to subsets of $\{1, 2, \ldots, (n-1)\}$. This bijection is i is in the corresponding set to a generalized Tesler matrix with hook sum vector $(1, 0, \ldots, 0)$ if there is a non-zero entry in the $(i+1)$st column for $1 \leqslant i \leqslant (n-1)$. Note that it is imperative that the domain is generalized Tesler matrices with hook sum vector $(1, 0, \ldots, 0)$ as opposed to hook sum vector $(1, 0, \ldots, 0, 1)$ since otherwise $(n-1)$ would always be in set for all $A \in \mathcal{T}(1, 0, \ldots, 0, 1)$.

Corollary 22. *The characteristic polynomial of the poset $P(1, 0, \ldots, 0, 1)$ is $(q-1)^{n-1}$.*

Proof. The characteristic polynomial of B_n is known to be $(q-1)^n$, hence the previous proposition result $P(1, 0, \ldots, 0, 1) \cong B_{n-1}$ gives us the desired result. $\square$

4 Application of Hallam-Sagan to the Tesler Poset

4.1 Initial Case

We now can use the Hallam-Sagan method discussed in Section 2.3 for calculating the characteristic polynomial of the Tesler poset. In this subsection, we consider the initial case which serves as a motivating example. Let $\alpha \in \{0,1\}^n$ be such that $\alpha_{n-1} = 0$, and let e_i be the ith elementary vector. In Figure 7, for instance, we have $\alpha = (1,0,1)$ and $\alpha + e_2 = (1,1,1)$. We want to compute the characteristic polynomial for the poset $P(\alpha + e_{n-1})$ using the characteristic polynomials of $P(\alpha)$ and B_1.

We construct our product poset by considering a set of maps between $\mathcal{T}(\alpha)$ and $\mathcal{T}(\alpha + e_{n-1})$. Let $\phi_\emptyset, \phi_{\{1\}} : \mathcal{T}(\alpha) \to \mathcal{T}(\alpha + e_{n-1})$ be such that $\phi_\emptyset : A \mapsto A + \varepsilon_{1,1}$ and $\phi_{\{1\}} : A \mapsto A + \varepsilon_{2,1} + \varepsilon_{2,2}$ where $\varepsilon_{i,j}$ is the elementary matrix of dimension 2. We define how to add these matrices of different dimension in Definition 25. It is then easy to check that these maps are well-defined and that they form a poset isomorphic to B_1 in the sense that $\phi_\emptyset(0) \preceq \phi_{\{1\}}(0)$ where 0 is the $n \times n$ zero matrix. In this motivating example, we define our equivalence relation $\sim$ on the product poset $P(\alpha) \times B_1$ as $(A, \phi_\emptyset) \sim (B, \phi_{\{1\}})$ if and only if $\phi_\emptyset(A) = \phi_{\{1\}}(B)$. As we will show in Section 4.2, $\sim$ satisfies all of the conditions in Lemma 10 so

$$\chi({}^{P(\alpha)\times B_1}/_{\sim}; q) = \chi(P(\alpha) \times B_1; q) = \chi(P(\alpha); q) \cdot \chi(B_1; q) = (q-1)^{n-1} \cdot (q-1) = (q-1)^n.$$

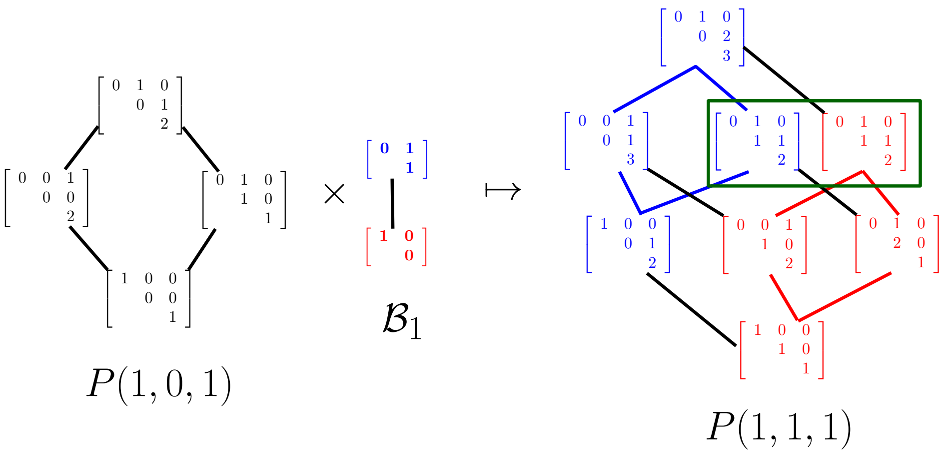

Figure 7: Our method in the case where $n = 3$ with the equivalent elements enclosed in a green rectangle. After identifying the equivalent elements enclosed in the green rectangle, we are left with the poset $P(1,1,1)$ as we see in Figure 5.

4.2 General Case

We will now generalize the idea from the previous section which will lead to our main theorem. For the rest of the subsection, we will fix $n, r \in \mathbb{N}$ such that $r < n$, and also fix

$\alpha \in \{0,1\}^n$ such that $\alpha + e_{n-r+1} \in \{0,1\}^n$. The previous section considers the case where $r = 2$. We seek to show that

$$\chi(P(\alpha + e_{n-r+1}); q) = (q-1)^{r-1}\chi(P(\alpha); q).$$

We will consider a poset of maps from $\mathcal{T}(\alpha)$ to $\mathcal{T}(\alpha + e_{n-r+1})$. While there are certainly other such maps, we will consider a natural, intuitive set of maps which have a nice structure and turn out to be sufficient. In order for $\phi : \mathcal{T}(\alpha) \mapsto \mathcal{T}(\alpha + e_{n-r+1})$ to be well-defined, it must increase the $(n - r + 1)$st hook sum by 1 while not changing the other hook sums. As a result, we consider maps which can be thought of as an $r \times r$ upper triangular matrix with a hook sum vector $(1, 0^{r-1})$, which is exactly an element of $\mathcal{T}(1, 0^{r-1})$. We previously showed that the poset of these matrices are isomorphic to the Boolean lattice, so we often label these maps with their corresponding set.

Example 23. Below is the poset of maps in the case where $r = 3$. This poset is isomorphic to subsets of $\{1, 2\}$ under the inclusion relation.

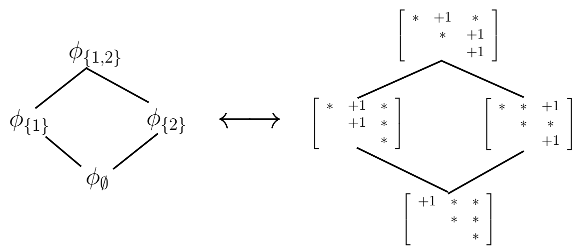

Figure 8: Here we let * indicates no change to the element.

Let Q_A be the subposet of $P(\alpha + e_{n-r+1})$ of the matrices $\phi_A(\mathcal{T}(\alpha))$.

Proposition 24. *We have the following facts:*
(1) Let $A \subseteq [r-1]$*, then* $Q_A \cong P(\alpha)$*.*
(2) $\bigcup_{A\subseteq[r-1]} \phi_A(\mathcal{T}(\alpha)) = \mathcal{T}(\alpha + e_{n-r+1})$

Proof. (1) Clearly ϕ_A is an injective map and is order preserving, so the posets are therefore isomorphic.

(2) By the well-defined nature of all of these maps, we clearly have that $\bigcup_{A\subseteq[r-1]} \phi_A(\mathcal{T}(\alpha)) \subseteq \mathcal{T}(\alpha + e_{n-r+1})$. Now, let us consider the other direction. Let $A \in \mathcal{T}(\alpha + e_{n-r+1})$, then there must be a non-zero element in the $(n - r + 1)th$ row. If this nonzero element is also in the $(n-r+1)th$ column, then one can check that $A \in \phi_\emptyset(\mathcal{T}(\alpha))$. Otherwise, by considering the columns with non-zero entries, we can construct a set $B \subseteq [r-1]$ in the same manner as we noted in Remark 21 such that $A \in \phi_B(\mathcal{T}(\alpha))$. That is, if and only if there is a non-zero entry in the $(n - i + 1)$st column of A for $1 \leqslant i \leqslant (r-1)$, then the element $r - i$ is in the set B so that we get a subset of $[r-1]$ such that $A \in \phi_B(\mathcal{T}(\alpha))$. As a result, we get that $\mathcal{T}(\alpha + e_{n-r+1}) \subseteq \bigcup_{A\subseteq[r-1]} \phi_A(\mathcal{T}(\alpha))$. $\square$

We can form a poset of the maps ϕ_A that is isomorphic to the Boolean lattice as we did in our motivating example and can consider the product poset $P(\alpha) \times B_{r-1}$. Since the maps ϕ_A are additive maps, we often view ϕ_A as a matrix $S_A \in \mathcal{T}(1, 0^{r-1})$.

Definition 25. We define the equivalence relation $\sim$ on $P(\alpha) \times B_{r-1}$ by

$$(A_1, S_1) \sim (A_2, S_2) \text{ if } A_1 + S_1 = A_2 + S_2$$

where the equality is matrix equality. We have to be careful with how we define the addition of these matrices as the dimensions of the square matrices do not match. We extend the matrix S_i such that it is an $n \times n$ matrix in the following manner. The entry $S_{i,j}$ becomes the entry $S_{i+(n-r),j+(n-r)}$ and all other entries of S are zero. Essentially, we are placing our matrix in the lower right corner in order to make addition of matrices defined.

$$\begin{bmatrix} 0 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{bmatrix}$$

Figure 9: The entries circled with the same color are added together to get our resulting $(A + S)$ matrix.

Clearly this is a homogeneous equivalence relation which preserves rank as it satisfies the conditions discussed in Section 2.3. Therefore, we have that $P(\alpha) \times B_{r-1}/\sim$ is a valid poset. We now seek to show that the summation condition (6) holds. In order to do this, we will first need some technical lemmas. The first lemma restricts what elements can be in the same equivalence class.

Lemma 26. *Let A_0 be the minimal element of $P(\alpha)$, and S_0 be the matrix representation of $\phi_\emptyset$ which is the minimal element of B_{r-1}. Then, for non-minimal $A \in P(\alpha)$ and non-minimal $S_d \in B_{r-1}$, we have that (A_0, S_d) and (A, S_0) are in different equivalent classes of the relation $\sim$.*

Proof. We show that $A_0 + S_d \neq A + S_0$ by showing they are not equal in the $(n - r + 1)$st entry along the main-diagonal. That is, the values $(A_0 + S_d)_{(n-r+1,n-r+1)}$ and $(A + S_0)_{(n-r+1,n-r+1)}$ are different. When considering $A_0 + S_d$, we must have that this entry is equal to 0 as it is 0 in both matrices that we are adding. We know that this entry in S_d is 0 since otherwise we would necessarily have that ϕ_d is the minimal element. Considering this same entry for $A + S_0$, we know that since S_0 has a 1 in this particular entry, the non-negativity of elements in A gives that this element in the matrix $A + S_0$ must be greater than or equal to 1. Hence, we do not have matrix equality with the sum and thus the two elements are not equivalent under $\sim$. See Figure 10 below for a visual representative of this argument in a particular case. ☐

$$(A_0, S_d): \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix} + \begin{bmatrix} 0 & * & * \\ & * & * \\ & & * \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & * & * \\ & & & * & * \\ & & & & * \end{bmatrix}$$

$$(A, S_0): \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} = \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & \geq 1 & * & * \\ & & & * & * \\ & & & & * \end{bmatrix}$$

Figure 10: The case when $n = 5$ and $r = 3$. Note that $*$ indicates that we do not know the particular entry.

Definition 27. We say an element $A \in P(\alpha)$ is *first coordinate isolated relative to X* if $(A, \phi_\emptyset)$ is the only element in $L(X)$ with the first coordinate equal to A. Similarly, we say $\phi_d \in B_{r-1}$ is *second coordinate isolated relative to X* if (A_0, ϕ_d) is the only element in $L(X)$ with the second coordinate equal to ϕ_d.

Now, we fix an element $X \in {}^{P(\alpha)\times B_{r-1}}/\sim$. The next lemma dictates the elements that can be in the lower order ideal $L(X)$. For the rest of the section, we let A_0 be the minimal element of $P(\alpha)$ and ϕ_0 be the minimal element of B_{r-1}.

Lemma 28. *For our fixed $X \in {}^{P(\alpha)\times B_{r-1}}/\sim$, at most one of the conditions hold.*
(1) There exists a non-minimal $A \in P(\alpha)$ that is first coordinate isolated relative to X.
(2) There exists a non-minimal $\phi_d \in B_{r-1}$ that is second coordinate isolated relative to X.

Proof. Since we are keeping X fixed, we omit the "relative to X" when referring to first and second coordinate isolation for the proof. We proceed by contradiction. Suppose that $\phi_d \in B_{r-1}$ is second coordinate isolated and $A \in P(\alpha)$ is first coordinate isolated. Now, since (A, ϕ_0) and (A_0, ϕ_d) are in $L(X)$, there exists a path in $L(X)$ between those elements and a member of the equivalence class X. By a path, we mean a sequence of covers in the poset. Consider such a path $\Gamma_1 : (A, \phi_0) \mapsto (M_l, \phi_l) \sim X$. Note that this path stays in $L(X)$ and is guaranteed to exist by the fact that we are looking at elements in a lower order ideal. (See Figure 11 for a pictorial representation of these paths.)

We start at (A, ϕ_0). In the first cover in our path, we necessarily must change the first coordinate. This is because if we were to change the second coordinate, we would get that $(A, \phi_1) \in L(X)$ which contradicts our hypothesis that $A \in P(\alpha)$ is first coordinate isolated. Therefore, our first cover in Γ_1 must be $(A, \phi_0) \mapsto (A_1, \phi_0)$ for some $A_1 > A \in P(\alpha)$. Now, suppose our second cover resulted in our path Γ_1 going to (A_1, ϕ_1) for some $\phi_1 \in B_{r-1}$. This would imply that $(A_1, \phi_1) \in L(X)$ and hence $(A, \phi_1) \in L(X)$ as this is a lower order ideal. This contradicts our hypothesis that $A \in P(\alpha)$ is first coordinate isolated. Hence, our updated path is $\Gamma_1 : (A, \phi_0) \mapsto (A_1, \phi_0) \mapsto (A_2, \phi_0)$ for some $A_2 > A_1 \in P(\alpha)$. Continuing this argument, we see that our path Γ_1 must have a constant second coordinate ϕ_0. As a result, we have that $X \sim (M_l, \phi_l) \sim (A_m, \phi_0)$ where $A_m > \cdots > A_1 > A \in P(\alpha)$ for some $m \in \mathbb{N}$.

Now, by a similar argument, we have that $\Gamma_2 : (A_0, \phi_d) \mapsto (M_j, \phi_j)$ must have constant first coordinate A_0. As a result, we have that $X \sim (M_j, \phi_j) \sim (A_0, \phi_{d_k})$ where for some $k \in \mathbb{N}$ $\phi_{d_k} > \cdots > \phi_{d_1} > \phi_d \in B_{r-1}$. Note that transitivity implies that $(A_m, \phi_0) \sim (A_0, \phi_{d_k})$. In Lemma 26, we showed that these are necessarily in different equivalence classes, hence we have reached our contradiction. $\square$

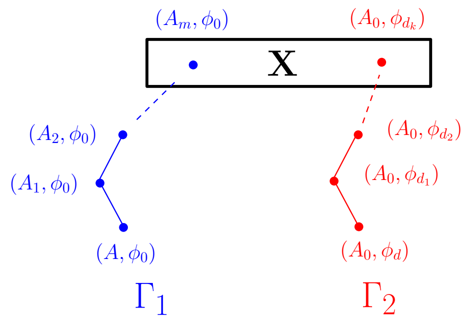

Figure 11: Pictorial representation of paths Γ_1, Γ_2 in proof of Lemma 28.

Proposition 29. *The summation condition* (6) *holds for the poset* ${}^{P(\alpha) \times B_{r-1}}/_{\sim}$.

Proof. Fix a non-minimal equivalence class $X \sim [(A, S)] \in {}^{P(\alpha) \times B_{r-1}}/_{\sim}$. We must show that

$$\sum_{(Y,S) \in L(X)} \mu((\hat{0}, \hat{0}), (Y, S)) = 0.$$

Observe that we can write the LHS in the following two ways:

$$\sum_{(Y,S) \in L(X)} \mu\Big((\hat{0}, \hat{0}), (Y, S)\Big) = \sum_{S_i} \left(\sum_{(Y,S_i) \in L(X)} \mu\Big((\hat{0}, \hat{0}), (Y, S_i)\Big) \right) \tag{7}$$

$$\sum_{(Y,S) \in L(X)} \mu\Big((\hat{0}, \hat{0}), (Y, S)\Big) = \sum_{Y_k} \left(\sum_{(Y_k,S) \in L(X)} \mu\Big((\hat{0}, \hat{0}), (Y_k, S)\Big) \right) \tag{8}$$

Now, since we are considering the lower order ideal of a product, it is easy to evaluate $\sum_{(Y_k,S) \in L(X)} \mu((\hat{0}, \hat{0}), (Y_k, S))$. By the product structure of the lower order ideal $L(X)$, there is a unique maximum $Y_k \in P(\alpha)$ for elements in $L(X)$ with the second coordinate S_i. This follows by supposing that there are at least two incomparable relative maximal elements and using a very similar argument from the previous lemmas. By the recursive nature of the Möbius function, we get that so long as Y_k is not the minimal element in $P(\alpha)$, the inner sum in Equation (8) is always 0 and

$$\sum_{(Y_k,S) \in L(X)} \mu\Big((\hat{0}, \hat{0}), (Y_k, S)\Big) = 0.$$

Similarly, so long as S_i is not the minimal element in B_{r-1} the inner sum in (7) is always 0 so that

$$\sum_{(Y,S_i)\in L(X)} \mu\Big((\hat{0},\hat{0}),(Y,S_i)\Big) = 0.$$

Thus, it suffices to show that we do not have a $A \in P(\alpha)$ which is first coordinate isolated and a $\phi_d \in B_{r-1}$ which is second coordinate isolated. We showed this precise statement in Lemma 28. Hence, (6) holds as we are either adding up all zeroes in (7) or (8). ☐

We are now ready to prove the lemma that we use in our main theorem.

Lemma 30. *Let* $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n$ *where* $\alpha_{n-r+1} = 0$. *Then*

$$\chi(P(\alpha + e_{n-r+1}); q) = \chi(P(\alpha); q) \cdot (q-1)^{r-1}.$$

Proof. It suffices to show that $P(\alpha + e_{n-r+1}) \cong {}^{P(\alpha) \times B_{r-1}}/{\sim}$. By the result from Proposition 18, we consider the elements in B_{r-1} as generalized Tesler matrices with the Tesler cover relation throughout this proof. We have already shown that there is a bijection between the elements of the poset. We now must show that this bijection is order-preserving.

In the forwards direction, if $A = (a_{i,j}), B = (b_{i,j}) \in P(\alpha + e_{n-r+1})$ are such that $B \preceq A$, then we want to show there exists $(A_1, A_2), (B_1, B_2) \in P(\alpha) \times B_{r-1}$ such that $(B_1, B_2) \preceq (A_1, A_2)$ with $B = B_1 + B_2$ and $A = A_1 + A_2$. This is because the cover relation in the quotient poset is defined as $X_B \preceq X_A$ in ${}^{P(\alpha) \times B_{r-1}}/{\sim}$ if we have $(B_1, B_2) \preceq (A_1, A_2)$ in $P(\alpha) \times B_{r-1}$ for some $(B_1, B_2) \in X_B$ and for some $(A_1, A_2) \in X_A$.

Let X_B, X_A be the corresponding equivalence classes in ${}^{P(\alpha) \times B_{r-1}}/{\sim}$ and $(B_1, B_2) \in X_B$ be an element in the equivalent class so that $B = B_1 + B_2$. We will only consider the first type of matrix cover relation from Definition 12; the other type of matrix cover relation has a similar proof. Since $B \preceq A$ in $P(\alpha + e_{n-r+1})$, there exists $i < j < k$ such that $a_{i,j} = b_{i,j} + 1$, $a_{j,k} = b_{j,k} + 1$, and $a_{i,k} = b_{i,k} - 1$. Now since $b_{i,k} > 0$ we know that that either $(B_1)_{i,k} > 0$ or $(B_2)_{i,k} > 0$ where $(B_1)_{i,k}$ corresponds to the matrix entry of B_1 in the ith row and kth column. Without loss of generality, suppose $(B_1)_{i,k} > 0$. Then, we can create $A_1 \in P(\alpha)$ such that $(A_1)_{i,j} = (B_1)_{i,j} + 1$, $(A_1)_{j,k} = (B_1)_{j,k} + 1$, and $(A_1)_{i,k} = (B_1)_{i,k} - 1$ so that $B_1 \preceq A_1$ in $P(\alpha)$. Note that since $A_2 \in B_{r-1}$ then necessarily must be so that $A_2 = B_2$, we have that $(B_1, B_2) \preceq (A_1, A_2)$. Thus, the bijection is order-preserving in the forwards direction.

In the backwards direction, we have that since $X_B \preceq X_A$ in ${}^{P(\alpha) \times B_{r-1}}/{\sim}$, there exists $(B_1, B_2) \preceq (A_1, A_2)$ in $P(\alpha) \times B_{r-1}$ for some $(B_1, B_2) \in X_B$ and for some $(A_1, A_2) \in X_A$. Then, by the definition of the cover relation in a product poset, we must have a cover in one of the coordinates and the other coordinate fixed. Thus, since the cover relation used in both coordinates is the Tesler cover relation, the corresponding Tesler matrices $A = A_1 + A_2, B = B_1 + B_2 \in P(\alpha + e_{n-r+1})$ are such that $B \preceq A$, so the bijection is order-preserving in the backwards direction. Thus, we have that $P(\alpha + e_{n-r+1}) \cong {}^{P(\alpha) \times B_{r-1}}/{\sim}$, and using Lemma 10 we get

$$\begin{aligned}\chi(P(\alpha + e_{n-r+1}); q) &= \chi({}^{P(\alpha)\times B_{r-1}}/{\sim}; q)\\ &= \chi(P(\alpha)\times B_{r-1}; q)\\ &= \chi(P(\alpha); q)\cdot \chi(B_{r-1}; q) = \chi(P(\alpha); q)\cdot (q-1)^{r-1}. \end{aligned}$$ □

We are now ready to state and prove our main theorem. Note that we have a slight modification in our notation for the hook sum vector α for a more clean result.

Theorem 31. *Let $\alpha = (\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0) \in \{0,1\}^n$ where $\alpha_{n-1} = \alpha_0 = 1$. Then, letting $w(\alpha) = \sum_{i=0}^{n-1} i \cdot \alpha_i$ we have that*

$$\chi(P(\alpha); q) = (q-1)^{w(\alpha)}.$$

Proof. We iterate Lemma 30 for each $\alpha_i = 1$ where $i \in [2, n-1]$. Note that if $\alpha_i = 0$, we are not changing the poset, so the characteristic polynomial is unchanged. One way of representing this using Lemma 30 is to multiply by $(q-1)^{\alpha_i(n-i)}$. This multiplies the characteristic polynomial of the unchanged poset by 1 when $\alpha_i = 0$ and by the desired amount when $\alpha_i = 1$. We start with the hook sum vector $\alpha_{n-1} + \alpha_0$ and then apply the Lemma 30 to get the characteristic polynomial for $\alpha_{n-1} + \alpha_1 + \alpha_0$ as we did in our motivating example. We then do the same thing to get $\alpha_{n-1} + \alpha_2 + \alpha_1 + \alpha_0$, and iterate until we have the characteristic polynomial of the poset corresponding to the hook sum vector α. As a result, we get the following:

$$\begin{aligned}\chi(P(\alpha_{n-1} + \alpha_1 + \alpha_0); q) &= (q-1)^{n-1}\cdot (q-1)^{\alpha_1}\\ \chi(P(\alpha_{n-1} + \alpha_2 + \alpha_1 + \alpha_0); q) &= (q-1)^{n-1}\cdot (q-1)^{\alpha_1}\cdot (q-1)^{2\alpha_2}.\\ &\vdots\\ \chi(P(\alpha); q) &= (q-1)^{n-1}\cdot (q-1)^{\alpha_1}\cdots (q-1)^{(n-2)\alpha_{n-2}}\end{aligned}$$

Collecting powers we obtain $(q-1)^{w(\alpha)}$ as desired. □

Corollary 32. *Let $P(1^n)$ be the Tesler poset and $w(\alpha)$ be as above. Then*

$$\chi(P(1^n); q) = (q-1)^{\binom{n}{2}}.$$

Proof. Since $w(1^n) = \binom{n}{2}$, the result follows by Theorem 31. □

Note that Theorem 31 also is consistent with the well-known result on the Boolean lattice result as the Boolean lattice is isomorphic to the Tesler poset $P(1, 0, \ldots, 0)$. We see this by noting that

$$w(1, 0, \ldots, 0) = w(1, 0, \ldots, 0, 1) = n-1.$$

Remark 33. This result also gives another method of generating Tesler matrices $\mathcal{T}(1^n)$ that is different from the Tesler generating algorithm discussed in Section 2.1. While this method is certainly less efficient that the Tesler generating algorithm, it is possible to construct the set $\mathcal{T}(1^n)$ in this manner without knowledge of the sets $\mathcal{T}(1^1), \ldots, \mathcal{T}(1^{n-1})$ and only using the well known Boolean lattice.

A natural question is to see if this result extends to all generalized Tesler matrices. In the general case, Lemma 26 and Lemma 28 do not hold. For other $\alpha \in \mathbb{N}^n$, we get other factors besides $(q-1)$ as we see in (9). Moreover, in the general case, the characteristic polynomial need not factor over $\mathbb{Z}$ as we see in (10).

Example 34. Let $\alpha = (1,2,3)$ and $\beta = (2,1,1,1)$ and consider the posets $P(\alpha)$ and $P(\beta)$. (For the Hasse diagram of the poset $P(1,2,3)$, see Figure 13 in the Appendix.) Then, one can check:

$$\chi(P(\alpha); q) = q(q-1)^3 \tag{9}$$

$$\chi(P(\beta); q) = (q-1)^4(q^5 - 2q^4 + 4q^3 - 6q^2 + 3q + 1) \tag{10}$$

However, we do have the following divisibility results as corollaries to Theorem 31. The question of $(q-1)$ divisibility in the Tesler poset was initially considered by Drew Armstrong and then communicated in [1].

Corollary 35. *Let $\alpha \in \mathbb{N}^k$ and consider the Tesler poset $P(1,\alpha)$, then*

$$(q-1)^k \textit{ divides } \chi(P(1,\alpha); q).$$

Proof. We start off with the posets $P(\alpha)$ and B_k and consider the product $P(\alpha) \times B_k$ and apply the same equivalence relation from Definition 25 and note that the results from Lemma 26 and Lemma 28 also hold in this case. As a result, we can use Lemma 10 and Proposition 18 to get that

$$\chi(P(1,\alpha); q) = \chi({}^{B_k \times P(\alpha)}/{\sim}; q) = \chi(B_k \times P(\alpha); q) = (q-1)^k \cdot \chi(P(0,\alpha); q). \qquad \square$$

We can now use Corollary 35 and Lemma 30 to get some results about factors of the characteristic polynomial when there are leading and trailing binary words in the hook sum vector.

Corollary 36. *Let $\alpha \in \mathbb{N}^{n-k}$ and $\beta = (\beta_1, \ldots, \beta_k) \in \{0,1\}^k$ and consider the Tesler posets $P(\alpha,\beta)$ and $P(\beta,\alpha)$. Then, letting $w_1(\beta) = \sum_{i=1}^{k}(n-i)\beta_i$ and $w_2(\beta) = \sum_{i=1}^{k}(k-i)\beta_i$, we have*

$$(q-1)^{w_1(\beta)} \textit{ divides } \chi(P(\beta,\alpha); q) \textit{ and} \tag{11}$$

$$(q-1)^{w_2(\beta)} \textit{ divides } \chi(P(\alpha,\beta); q) \tag{12}$$

Proof. First, we look at the statement in (11). We iterate through our binary word β by starting with β_k and ending with β_1. At the ith step, by Corollary 35, we get a factor of $(q-1)^{n-k+i-1}$. Collecting powers of $(q-1)$, and then reordering the sum gives us the desired result of a factor of $(q-1)^{w_1(\beta)}$. Next, we consider the statement in (12). We iterate through our binary word β by starting with β_1 and ending with β_k and use the result from Lemma 30. After collecting powers, we get a factor of $(q-1)^{w_2(\beta)}$. $\square$

5 Armstrong polynomial

In this section, we introduce the Armstrong polynomial to encode the growth of the number of Tesler matrices. Questions regarding asymptotics of the Kostant partition function and hence generalized Tesler matrices have recently appeared in a `Math Overflow` post. [21]

Let $A = (a_{i,j}) \in \mathcal{T}(1^n)$ and $d_i = a_{i,i}$. Recall the diagonal product of A

$$\mathrm{dpro}(A) = \prod_{i=1}^{n} (d_i + 1).$$

Definition 37. We define the Armstrong polynomial $A_n(q)$ to measure the distribution of diagonal products in $\mathcal{T}(1^n)$. That is,

$$A_n(q) := \sum_{A \in \mathcal{T}(1^n)} q^{\mathrm{dpro}(A)}.$$

Example 38. As we see in Figure 5, the diagonals corresponding to the 7 Tesler matrices of size 3 are $(1,1,1), (0,1,2), (0,0,3), (1,0,2), (0,0,3), (0,1,2),$ and $(0,2,1)$ with diagonals products: $8, 6, 4, 6, 4, 6,$ and 6. As a result, we have

$$A_3(q) = 2q^4 + 4q^6 + q^8.$$

5.1 Previously known bounds

By considering the bijection between the integral flow representation and matrix representation of Tesler matrices, we have that the main diagonal of a size n Tesler matrix necessarily sums to n. It then follows that for all $A \in \mathcal{T}(1^n)$, we have
$n + 1 \leqslant \mathrm{dpro}(A) \leqslant 2^n$ where the tightness of the lower and upper bound are obtained with main-diagonals $(0, 0, \ldots, 0, n)$ and $(1, 1, \ldots, 1)$ respectively. The first approximation uses Equation (5) and considers *all* the diagonal products to be $(n + 1)$ to get the lower bound and 2^n to get the upper bound. Through this method we get that

$$n! \leqslant T(1^n) \leqslant 2^{\binom{n}{2}}. \tag{13}$$

Example 39. For $n = 1$ through $n = 5$, we have the following Armstrong polynomials:

$$
\begin{aligned}
A_1(q) &= 1q^2 \\
A_2(q) &= 1q^3 + 1q^4 \\
A_3(q) &= 2q^4 + 4q^6 + 1q^8 \\
A_4(q) &= 7q^5 + 15q^8 + 6q^9 + 11q^{12} + 1q^{16} \\
A_5(q) &= 40q^6 + 93q^{10} + 67q^{12} + 75q^{16} + 55q^{18} + 26q^{24} + 1q^{32}
\end{aligned}
$$

Remark 40. As we will discuss in Section 6, we can also define the Armstrong polynomial $A_n(\alpha, q)$ for certain classes of generalized Tesler matrices with hook sum vector α.

Proposition 41. *Let $[q^a]A_n(q)$ be the coefficient of the term of degree a in $A_n(q)$, then:*

- $[q^{2^n}]A_n(q) = 1$
- $[q^{n+1}]A_n(q) = T(1^{n-1})$
- $[q^{3\cdot 2^{n-2}}]A_n(q) = 2^n - n - 1$

Proof. The first statement follows as there is only one diagonal, namely $(1, 1, \dots, 1)$, which results in a diagonal product of 2^n and the identity matrix is the only such matrix with this diagonal. By considering the incoming flow to the $(n+1)$st vertex of the integral flow representation of a Tesler matrix, we get that the sum of the main-diagonal entries must add up to n. Also, since $\alpha_n = 1$, we necessarily must have that $a_{n,n} \geqslant 1$ for $A = (a_{i,j}) \in \mathcal{T}(1^n)$. Thus, when considering the second statement, the only possible main-diagonal with diagonal product $(n+1)$ is $(0, \dots, 0, n)$. Using the Tesler generating algorithm discussed in Section 2.1, the only way to get such a diagonal is to start out with any main-diagonal of size $(n-1)$ and then taking everything away from all elements of the original diagonal. As a result, for each Tesler matrix of size $(n-1)$, we have a unique Tesler matrix of size n with diagonal $(0, \dots, 0, n)$, thus proving the second statement.

Finally, considering the last part of our proposition, let the coefficient of the term with degree $3\cdot 2^{n-2}$ in $A_n(q)$ be a_n. We simply need to show that a_n satisfies the same recurrence relation as the sequence $\{2^n - n - 1\}$. Namely, we need to show that $a_n = (n-1) + 2a_{n-1}$. One can check that the terms with degree $3 \cdot 2^{n-2}$ in $A_n(q)$ come from the diagonal $(2, 1, \dots 1, 0)$ and valid rearrangements of those terms. Starting with diagonals in the form $(2, 1, \dots 1, 0)$ of the previous size, we can either do nothing, or subtract 2 from the 2 term in the diagonal $(2, 1, \dots 1, 0)$. This accounts for the $2a_{n-1}$. We get the $(n-1)$ from noting that we can also generate the diagonal $(2, 1, \dots 1, 0)$ by starting from the unique Tesler matrix with main-diagonal $(1, \dots, 1)$ and subtracting any one of the $(n-1)$ main-diagonal entries that are 1. $\square$

Note that given $k \in \mathbb{N}$ and the Armstrong polynomial, $A_k(q)$, it is possible to read off $T(1^{k-1})$, $T(1^k)$, and $T(1^{k+1})$ from this polynomial as we show in the following proposition.

Proposition 42. *The Armstrong polynomial is so that*

$$T(1^{n+1}) = \frac{d}{dq} A_n(q)|_{q=1}$$

and $A_n(1) = T(1^n)$.

Proof. First, we note that dpro(A) $\geqslant 2$ for all n. Then,

$$\begin{aligned}
\frac{d}{dq} A_n(q)\Big|_{q=1} &= \frac{d}{dq} \sum_{A \in \mathcal{T}(1^n)} q^{\mathrm{dpro}(A)}\Big|_{q=1} = \sum_{A \in \mathcal{T}(1^n)} \frac{d}{dq} q^{\mathrm{dpro}(A)}\Big|_{q=1} \\
&= \sum_{A \in \mathcal{T}(1^n)} \mathrm{dpro}(A) \cdot q^{\mathrm{dpro}(A)-1}\Big|_{q=1} \\
&= \sum_{A \in \mathcal{T}(1^n)} \mathrm{dpro}(A) = T(1^{n+1}).
\end{aligned}$$

The second statement is immediate. $\square$

We can now use the observations in Proposition 41 regarding the Armstrong polynomial to get the following bounds on the number of Tesler matrices.

Theorem 43.

$$\prod_{i=1}^{n-1} (2i-1) \leqslant T(1^n) \leqslant 2^{\binom{n-2}{2}-1} \cdot 3^n. \tag{14}$$

Proof. We use a similar method as we did in our first approximation. This time, however, we know that we have exactly $T(1^{n-1})$ of our terms to have a diagonal product of $(n+1)$ by Proposition 41. We now assume that the remaining Tesler matrices have a diagonal product of $2n$, the second lowest diagonal product. Using this, we note that

$$\begin{aligned}
T(1^{n+1}) &= \sum_{A \in \mathcal{T}(1^n)} \mathrm{dpro}(A) \\
&\geqslant T(1^{n-1})(n+1) + \left[T(1^n) - T(1^{n-1})\right](2n).
\end{aligned}$$

We now use the previously known bounds in (13) that $T(1^n) \geqslant nT(1^{n-1})$ to get that

$$\begin{aligned}
T(1^{n+1}) &\geqslant T(1^n) \left[\frac{T(1^{n-1})}{T(1^n)}(n+1) + \frac{T(1^n) - T(1^{n-1})}{T(1^n)}(2n)\right] \\
&\geqslant T(1^n) \left[\frac{1}{n}(n+1) + \frac{n-1}{n}(2n)\right] \geqslant T(1^n)(2n-1).
\end{aligned}$$

Iterating this, we get our desired lower bound that

$$T(1^{n+1}) \geqslant \prod_{i=1}^{n}(2i-1) = (2n-1)!!.$$

We get the upper bound by the same method and further reductions. $\square$

Remark 44. We note that the lower bound in (14) is better than $n!$ since $\prod_{i=1}^{n-1}(2i-1) \geqslant 2^{n-2} \cdot (n-2)!$ and is $O((\frac{2n}{e})^{n-1})$. Note that this still does not give an affirmative answer to Question 4 and that the upper bound in (14) is still $e^{\Theta(n^2)}$, but is slightly tighter.

6 Understanding Different Hook Sum Vectors

Recall that $\mathcal{T}(1^k, 0^{n-k})$ denotes the set of generalized Tesler matrices with hook sum vector equal to $(1, \ldots, 1, 0, \ldots, 0)$ where there are k 1's and $(n-k)$ 0's and that $T(1^k, 0^{n-k})$ denotes the number of such matrices. In this section, we will refer to Armstrong polynomials for generalized Tesler matrices with hook sum vector α as $A_n(\alpha, q)$ where the Armstrong polynomial from the previous section is such that $A_n(q) := A_n(1, 1, \ldots, 1, q)$.

First, we consider $\mathcal{T}(1, 0^{n-1})$. Taking $\alpha_{k+1} = 0$ from the method discussed in Section 2.1, it follows that $\mathcal{T}(1, 0^{n-1})$ can be produced by the Tesler generating process. Now, we note that there is only one possible diagonal up to reordering of $(1, 0, \ldots, 0)$, so all elements have the same diagonal product and as a result the Armstrong polynomial is always in the form $A_n(1, 0, \ldots, 0, q) = (T(1, 0^{n-2}))q^2$ which yields that $T(1, 0^{n-1}) = 2^{n-1}$ by the first part in Proposition 42. Hence, letting $T_1(x)$ be the generating function for the number of generalized Tesler matrices with hook sum vector $(1, 0^{n-1})$ we get that

$$T_1(x) := \sum_{n \geqslant 0} T(1, 0^n)x^n = \frac{1}{1-2x}.$$

Now, let us consider $\mathcal{T}(1^2, 0^{n-2})$. These matrices have been recently studied in [5, 12]. For the same reason as above, we can consider the corresponding Armstrong polynomial. There are only two possible diagonals up to reordering of $(2, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$ with diagonal products 3 and 4 respectively. We now consider the corresponding Armstrong polynomial $A_n(1^2, 0^{n-2}, q)$.

Proposition 45. *Let* $A_{n-1}(1^2, 0^{n-3}, q) = a_{n-1}q^3 + b_{n-1}q^4$*. Then, we have*

$$A_n(1^2, 0^{n-2}, q) = (2a_{n-1} + b_{n-1})q^3 + (a_{n-1} + 3b_{n-1})q^4.$$

Proof. We only need to that prove the value of the coefficient of q^3 is as stated as the other coefficient is determined by the fact we know the total number of matrices that are in this set from the $A_{n-1}(1^2, 0^{n-2}, q)$ term. Thus, we consider the ways to get the diagonal $(2, 0, \ldots, 0)$ from the previous set. First, we can do nothing in the diagonal part of the

Tesler generating process and add a zero to each of the $(2, 0, \dots, 0)$ of the previous case. Second, for all of the previous size matrices, we subtract everything from the diagonal and then add 2 yielding the diagonal $(0, \dots, 0, 2)$. As a result, we generate $2a_{n-1} + b_{n-1}$ distinct terms with diagonal $(2, 0, \dots, 0)$. □

Proposition 46. *Let $t_n := T(1^2, 0^n)$. Then $t_n \geqslant 3^{n+1}$ for $n \geqslant 5$.*

Proof. Generating these matrices through a computer program, we note that $t_3 = 90$. Thus, since 3 is the smallest possible diagonal product we have
$t_n \geqslant 3t_{n-1} \cdots \geqslant 3^{n-3}t_3 = 90(3^{n-3}) \geqslant 3^{n+1}$. □

Proposition 47 (See also [12])**.** *The ordinary generating function for t_n is*

$$T_2(x) := \sum_{n\geqslant 0} T(1^2, 0^n)x^n = \frac{2 - 3x}{1 - 5x + 5x^2}.$$

Proof. Using the results from Proposition 42 and applying it to Proposition 45 with the Armstrong polynomial $A_{n-1}(1^2, 0^{n-3}, q) = a_{n-1}q^3 + b_{n-1}q^4$, we can explicitly compute the values for t_{n-1}, t_n, t_{n+1} in terms of a_{n-1} and b_{n-1}. From this, we get that t_n satisfies the following recurrence relation $t_{n+1} = 5t_n - 5t_{n-1}$. From this difference equation, and the initial conditions $t_0 = 2$ and $t_1 = 7$, we can find the generating function for t_n via standard methods. □

Proposition 48. *For all k, there exists some $N_k \in \mathbb{N}$ such that for all $n \geqslant N_k$ we have*

$$T(1^k, 0^{n-k}) \geqslant (k + 1)^{n-1}.$$

Proof. For a given k, the smallest possible diagonal product in $\mathcal{T}(1^k, 0^{n-k})$ is $(k + 1)$. Using similar methods of generating the diagonals of the form $(0, 0, \dots, 0, k)$, we can see that less than half of the terms in the set $\mathcal{T}(1^k, 0^{n-k})$ have a diagonal product of $(k+1)$. Hence, noting that the next lowest diagonal product is $2k$, the expected value of the diagonal product is *at least* ${}^{(3k+1)}/_{2}$. Since ${}^{(3k+1)}/_{(2k+2)} > 1$ for $k \geqslant 2$, we will eventually have an N_k such that $T(1^k, 0^{N_k - k}) \geqslant (k + 1)^{N_k - 1}$. □

6.1 Conjectures and Future Work

The sequence $\{T(1^n)\}$ appears in the OEIS A008608. Based on the 25 entries in this sequence, and the insight from Proposition 48, we make the following conjecture.

Conjecture 49. Let $n, k \in \mathbb{Z}$ be such that $n \geqslant k \geqslant 11$. Then, we have

$$T(1^k, 0^{n-k}) \geqslant (k + 1)^{n-1}.$$

Remark 50. This conjecture would prove that for $n \geqslant 11$, we have $T(1^n) \geqslant (n + 1)^{n-1}$ which is a significant because $(n + 1)^{n-1}$ is the value of (1) with $t = 1$ and $q = 1$ (i.e. the dimension of DH_n). We note that for $k = 11$, this conjecture is true as
$T(1^{11}) = 515, 564, 231, 770$ which is bigger than 12^{10}. Thus if we can show that
$T(1^{n+1}) \geqslant e \cdot (n + 2)T(1^n)$ for $k \geqslant 11$, then we have proven the conjecture. Here the number e comes from looking at the fraction of the next term over the previous term which gives us $(\frac{n+2}{n+1})^{n-1} \cdot (n + 2)$ where the first term is bounded below by e.

The statistics dinv and area, which are mentioned in more detail in [8], are used in the now settled Haglund-Loehr conjecture [10]. Carlsson and Mellit show in [4] that

$$\mathrm{Hilb}(DH_n; q, t) = \sum_{\pi} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)} \tag{15}$$

where the sum is over *parking functions* π of size n.
Haglund's Tesler matrix approach to showing (15) reduces to proving that

$$\sum_{\pi} q^{dinv(\pi)} t^{area(\pi)} = \sum_{A=(a_{i,j})\in\mathcal{T}(1^n)} wt_{q,t}(A). \tag{16}$$

where $\mathrm{wt}_{q,t}(\cdot)$ is as in (2).

It was shown in [3] that by plugging in $t = 1$ and $q = 1$ we get

$$(n+1)^{n-1} = \sum_{A=(a_{i,j})\in\mathcal{PT}(1^n)} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t}. \tag{17}$$

We note here that the only terms that survive on the RHS of (16) after plugging in $t = 1$ and $q = 1$ are Tesler matrices with exactly one nonzero element in each row. These are called *Permutation Tesler matrices*. This relationship between parking functions and Tesler matrices adds intrigue to having the number of parking functions eventually be a lower bound for Tesler matrices since this would imply there is a lot of cancellation in the alternating sum on the RHS of (16). We will now explore a way to affirmatively answer Question 4 using $\chi(P(1^n); q)$.

Proposition 51. *Let $\mu(\cdot)$ be the Möbius function for the Tesler poset $P(1^n)$. If for all $A \in \mathcal{T}(1^n)$ we have that $|\mu(\hat{0}, A)| \leqslant f(n)$, then we have that:*

$$T(1^n) \geqslant \frac{2^{\binom{n}{2}}}{f(n)}.$$

Proof. We note that by Corollary 32, we have

$$\sum_{A} |\mu(\hat{0}, A)| \geqslant 2^{\binom{n}{2}}.$$

Hence, if for all $A \in \mathcal{T}(1^n)$ we have that $|\mu(\hat{0}, A)| \leqslant f(n)$, then we would have that $T(1^n) \cdot f(n) \geqslant 2^{\binom{n}{2}}$ which gives the desired result. □

Remark 52. We would find such a bound on the Möbius function for the Tesler poset $P(1^n)$ by analyzing the size of the equivalence classes that we get when we use Hallam-Sagan's method from Section 2.3. In their Lemma 10, they show that the Möbius function of the equivalence class $[X]$ is equal to the sum of the Möbius function evaluated at the elements in the equivalence class $[X]$.

Conjecture 53. Let $\alpha = (1, 1, \ldots, 1)$ and $P(\alpha)$ be the Tesler poset with Möbius function $\mu(\cdot)$. Then we can have the following lower bound on the Möbius function

$$|\mu(\hat{0}, A)| \leqslant n!.$$

We have been able to computationally able to verify Conjecture 53 in the Tesler poset corresponding to hook sum vectors $(1, 1, \ldots, 1)$ up to size 5. A proof of this conjecture would give an affirmative answer to Question 4 since

$$T(1^n) \geqslant \frac{2^{\binom{n}{2}}}{n!} = e^{\Theta(n^2)}.$$

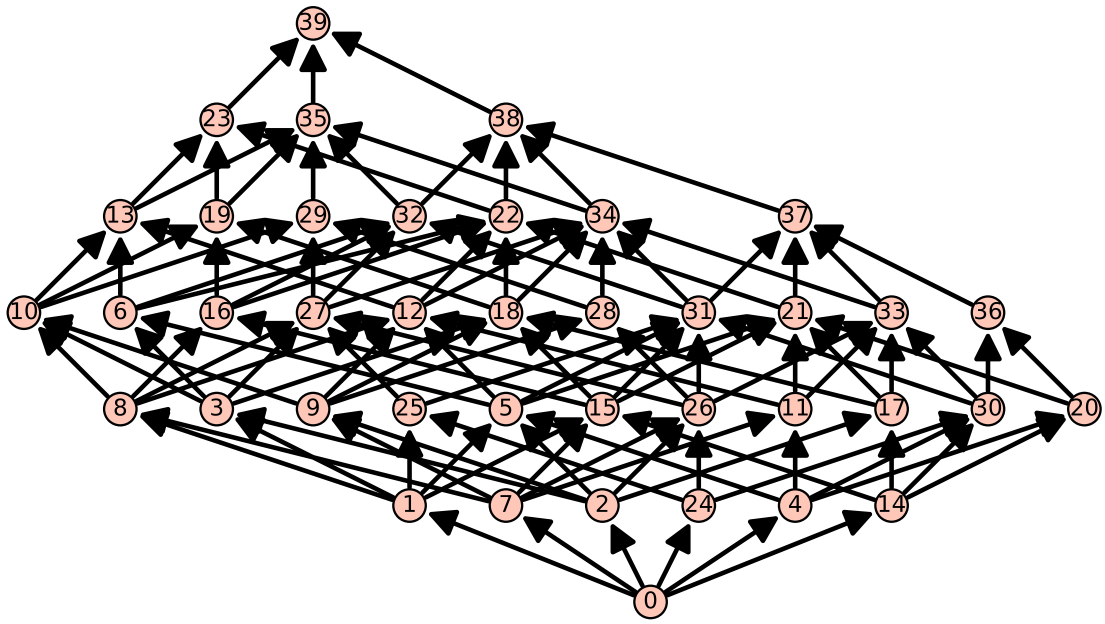

Figure 12: The Tesler poset corresponding to hook sum vector $(1, 1, 1, 1)$. Note that the numbers in the image signify the order in which the matrices were generated in SAGE.

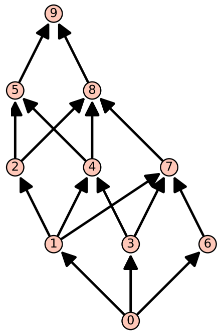

Figure 13: The Tesler poset corresponding to hook sum vector $(1, 2, 3)$.

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