Flows in signed graphs with two negative edges

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Abstract

The presented paper studies the flow number $F(G, \sigma)$ of flow-admissible signed graphs (G, σ) with two negative edges. We restrict our study to cubic graphs, because for each non-cubic signed graph (G, σ) there is a set of cubic graphs obtained from (G, σ) such that the flow number of (G, σ) does not exceed the flow number of any of the cubic graphs. We prove that $F(G, \sigma) \leqslant 6$ if (G, σ) contains a bridge, and $F(G, \sigma) \leqslant 7$ in general. We prove better bounds, if there is a cubic graph (H, σ_H) obtained from (G, σ) which satisfies some additional conditions. In particular, if H is bipartite, then $F(G, \sigma) \leqslant 4$ and the bound is tight. If H is 3-edge-colorable or critical or if it has a sufficient cyclic edge-connectivity, then $F(G, \sigma) \leqslant 6$. Furthermore, if Tutte's 5-Flow Conjecture is true, then (G, σ) admits a nowhere-zero 6-flow endowed with some strong properties.

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1 Introduction

In 1954 Tutte stated a conjecture that every bridgeless graph admits a nowhere-zero 5-flow (*5-flow conjecture*, see [17]). Naturally, the concept of nowhere-zero flows has been

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extended in several ways. In this paper we study one generalization of them – nowhere-zero integer flows on signed graphs. Signed graphs are graphs where each edge is either positive or negative. It was conjectured by Bouchet [1] that signed graphs that admit a nowhere-zero flow have a nowhere-zero 6-flow. Recently, it was announced by DeVos [2] that such signed graphs admit a nowhere-zero 12-flow, which is the best current general approach to Bouchet's conjecture.

Bouchet's conjecture has been confirmed for particular classes of graphs [9, 8, 12] and also for signed graphs with restricted edge-connectivity (for example [10]). By Seymour [13] it is also true for signed graphs with all edges positive, because they correspond to the unsigned case. For more details on flows on signed graphs, consult [5].

In this paper we study signed graphs with two negative edges. It is the minimum number of negative edges for which Bouchet's conjecture is open, because signed graphs with one negative edge are not flow-admissible. This class of signed graphs is further interesting for its connection with Tutte's 5-flow conjecture. Suppose there exists k such that every signed graph with k negative edges admits a nowhere-zero 5-flow. Take any bridgeless graph G and identify a vertex of all-positive G with a vertex of a flow-admissible signed graph with k negative edges. The resulting signed graph is flow-admissible with k negative edges. If it admits a nowhere-zero 5-flow, then G admits it as well. Therefore the following holds.

Observation 1. *If there exists k such that every flow-admissible signed graph with k negative edges admits a nowhere-zero 5-flow, then Tutte's conjecture is true.*

Since for every $k \geqslant 3$ there is a flow-admissible signed graph with k negative edges which does not admit a nowhere-zero 5-flow (see [12]), but there is no such example known for $k = 2$, the class of signed graphs with two negative edges is of a great importance. In the opposite direction we will prove that Tutte's conjecture implies Bouchet's conjecture for signed graphs with two negative edges.

In the next section we introduce necessary notions and provide a couple of well-known results on flows. In Section 3 we show how to deal with small edge-cuts, and finally, in Sections 4-6 we prove results on flows for signed graphs with two negative edges.

2 Preliminaries

Graphs in this paper are allowed to have multiple edges and loops. *A signed graph* (G, σ) consists of a graph G and a function $\sigma : E(G) \to \{-1, 1\}$. The function σ is called *a signature*. The set of edges with negative signature is denoted by N_σ. It is called *the set of negative edges*, while $E(G) - N_\sigma$ is called *the set of positive edges*. If all edges of (G, σ) are positive, i. e. when $N_\sigma = \emptyset$, then (G, σ) will be denoted by $(G, \mathtt{1})$ and will be called an *all-positive signed graph*.

Let $e \in E(G)$ be an edge, which is incident with vertices u and v. We divide e into two half-edges h_u^e and h_v^e, one incident with u and one incident with v. The set of the half-edges of G is denoted by $H(G)$. For each half-edge $h \in H(G)$, the corresponding edge in $E(G)$ is denoted by e_h. For a vertex v, $H(v)$ denotes the set of half-edges incident with

v. An *orientation* of (G,σ) is a function $\tau : H(G) \to \{\pm 1\}$ such that $\tau(h_u^e)\tau(h_v^e) = -\sigma(e)$, for each edge $e = uv$. The function τ can be interpreted as an assignment of a direction to each edge in the following way. A positive edge can be directed like $\circ\!\!\longrightarrow\!\!\circ$ or like $\circ\!\!\longleftarrow\!\!\circ$. A negative edge can be directed like $\circ\!\!\longleftrightarrow\!\!\circ$ (so-called *extroverted edge*) or like $\circ\!\!\rightarrow\!\leftarrow\!\!\circ$ (so-called *introverted edge*). In what follows we will often start with a particular orientation, and then redirect some half-edges (change their direction to the opposite one) in order to obtain a new orientation. Note that if we redirect one half-edge of an oriented positive edge, we obtain an oriented negative edge. If we redirect both half-edges of an oriented positive edge, we obtain a positive edge with the opposite orientation. This can be viewed as redirection of a positive edge. It will be used when we study unsigned or all-positive signed graphs. Half-edges incident with v that are oriented towards v (away from v, respectively) will be called *incoming half-edges* (*outgoing half-edges*, respectively) and will be denoted by $\delta^-(v)$ ($\delta^+(v)$, respectively). In case of positive edges we also use $e \in \delta^+(v)$ ($e \in \delta^-(v)$) if e is incident with v, and if there is no danger of confusion.

Let (G,σ) be a signed graph. *A switching at v* defines a signed graph (G,σ') with $\sigma'(e) = -\sigma(e)$ if e is incident with v, and $\sigma'(e) = \sigma(e)$ otherwise. We say that signed graphs (G,σ) and (G,σ^*) are *equivalent* if they can be obtained from each other by a sequence of switchings. We also say that σ and σ^* are *equivalent signatures* of G. If we consider a signed graph with an orientation τ, then switching at v is a change of the orientations of the half-edges that are incident with v, i. e. on half-edges h_v^e for every e incident with v. If τ^* is the resulting orientation, then we say that τ and τ^* are *equivalent orientations*.

Let A be an abelian group. An *A-flow* (τ,ϕ) on (G,σ) consists of an orientation τ and an assignment $\phi : E(G) \to A$ satisfying *Kirchhoff's law*: for every vertex v the sum of values on incoming half-edges equals the sum of values on outgoing half-edges. If $\phi(e) \neq 0$ for every edge e, then we say that the A-flow is *nowhere-zero*. Let k be a positive integer. A nowhere-zero $\mathbb{Z}$-flow such that $-k < \phi(e) < k$ for every $e \in E(G)$ is called a *nowhere-zero k-flow*. A signed graph (G,σ) is *flow-admissible* if it admits a nowhere-zero k-flow for some k. The flow number of a flow-admissible signed graph (G,σ) is

$$F((G,\sigma)) = \min\{k : (G,\sigma) \text{ admits a nowhere-zero } k\text{-flow}\}.$$

This minimum always exists. We will abbreviate $F((G,\sigma))$ to $F(G,\sigma)$.

If (G,σ) admits a nowhere-zero A-flow (τ,ϕ) and (G,σ^*) is equivalent to (G,σ), then there exists an equivalent orientation τ^* to τ such that (τ^*,ϕ) is a nowhere-zero A-flow on (G,σ^*). To find τ^* it is enough to switch at the vertices that are switched in order to obtain σ^* from σ. Thus, it is easy to see that $F(G,\sigma) = F(G,\sigma^*)$.

We note that flows on signed graphs that are all-positive are equivalent to flows on graphs: a nowhere-zero k-flow (A-flow, respectively) on a graph G can be defined as a nowhere-zero k-flow (A-flow, respectively) on $(G,\mathbb{1})$. This allows us to state known results for flows on graphs in terms of flows on signed graphs, and vice-versa. We will frequently employ this fact and if there is no danger of confusion, we may use the term a nowhere-zero k-flow on a graph G for referring to a nowhere-zero k-flow on the all-positive graph $(G,1)$. While a graph is flow-admissible if and only if it contains no bridge, the definition of flow-admissibility for signed graphs is more complicated – it is closely related to the concept of balanced and unbalanced circuits.

A circuit of (G,σ) is *balanced* if it contains an even number of negative edges; otherwise it is *unbalanced*. Note that a circuit of (G,σ) does not change the parity of negative edges after switching at any vertex of (G,σ). Thus, the set of unbalanced circuits is invariant under switching. The signed graph (G,σ) is *an unbalanced graph* if it contains an unbalanced circuit; otherwise (G,σ) is *a balanced graph*. It is well known (see e.g. [10]) that (G,σ) is balanced if and only if it is equivalent to $(G,\mathtt{1})$. A *barbell* of (G,σ) is the union of two edge-disjoint unbalanced cycles C_1, C_2 and a path P satisfying one of the following properties:

- C_1 and C_2 are vertex-disjoint, P is internally vertex-disjoint from $C_1 \cup C_2$ and shares an endvertex with each C_i, or
- $V(C_1) \cap V(C_2)$ consists of a single vertex w, and P is the trivial path consisting of w.

Balanced circuits and barbells are called *signed circuits*. They are crucial for flow-admissibility of a signed graph.

Lemma 2 (Lemma 2.4 and Lemma 2.5 in [1]). *Let (G,σ) be a signed graph. The following statements are equivalent.*

1. *(G,σ) is not flow-admissible.*
2. *(G,σ) is equivalent to (G,σ') with $|N_{\sigma'}| = 1$ or G has a bridge b such that a component of $G - b$ is balanced.*
3. *(G,σ) has an edge that is contained neither in a balanced circuit nor in a barbell.*

When a signed graph has a single negative edge, it is not flow-admissible by the previous lemma. This can also be seen from the fact that the sum of the flow values over all negative edges is 0 provided that the negative edges have the same direction. Therefore, if a flow-admissible signed graph has two negative edges, which is the case considered in this paper, and the negative edges have opposite orientations, then the flow value on the negative edges is the same for any nowhere-zero k-flow.

Let (τ,ϕ) be a nowhere-zero k-flow on (G,σ). If we reverse the direction of an edge e (or of the two half-edges of e, respectively) and replace $\phi(e)$ by $-\phi(e)$, then we obtain another nowhere-zero k-flow (τ^*,ϕ^*) on (G,σ). Hence, if (G,σ) is flow-admissible, then it has always a nowhere-zero flow with all the flow values positive.

Let $n \geqslant 1$ and let $P = u_0e_1u_1 \cdots e_nu_n$ be a path in G, where $\{u_0,\dots,u_n\} = V(P)$, $\{e_1,\dots,e_n\} = E(P)$ and $e_i = u_{i-1}u_i$ for each $i \in \{1,\dots,n\}$. We say that P is *a v-w-path* if $v = u_0$ and $w = u_n$. Let σ be a signature of G and let (G,σ) be oriented. If P of G does not contain any negative edge and $h_{u_i}^{e_i} \in \delta^-(u_i)$, $h_{u_i}^{e_{i+1}} \in \delta^+(u_i)$ for $i \in \{1,\dots,n-1\}$, and $h_{u_0}^{e_1} \in \delta^+(u_0)$, $h_{u_n}^{e_n} \in \delta^-(u_n)$, then P is *a directed v-w-path*. In case of a positive edge e' we also say that e' is an oriented edge. We will frequently make use of the following well-known lemma and observation.

Lemma 3. *Let G be a graph and (τ, ϕ) be a nowhere-zero $\mathbb{Z}$-flow on $(G, \mathit{1})$. If $\phi(e) > 0$ for every $e \in E(G)$, then for any two vertices u,v of G there exists a directed u-v-path.*

Observation 4. *Suppose that there exists a nowhere-zero k-flow (τ, ϕ) with $\phi(f) = t$ for one direction of f. Let τ_{opp} be the orientation obtained from τ by reversing the direction of each edge of G. Then (τ_{opp}, ϕ) is a nowhere-zero k-flow with $\phi(f) = t$ for the other direction of f.*

Flows on signed graphs were introduced by Bouchet [1], who stated the following conjecture.

Conjecture 5 ([1])**.** Let (G, σ) be a signed graph. If (G, σ) is flow-admissible, then (G, σ) admits a nowhere-zero 6-flow.

The 6-flow theorem of Seymour [13] proves Bouchet's conjecture for all-positive signed graphs.

Theorem 6 ([13])**.** *If $(G, \mathit{1})$ is flow-admissible, then $(G, \mathit{1})$ admits a nowhere-zero 6-flow.*

In this paper, we restrict our study to signed cubic graphs, because for each signed non-cubic graph (G, σ) there is a set $\mathcal{G}(G, \sigma)$ of signed cubic graphs such that $F(G, \sigma) \leqslant \min\{F(H, \sigma_H) : (H, \sigma_H) \in \mathcal{G}(G, \sigma)\}$. The set $\mathcal{G}(G, \sigma)$ is obtained from (G, σ) by suppressing the vertices of degree 2 and by blowing vertices of degree higher than 3 into a circuit. More precisely, if v is a vertex of degree 2 in (G, σ) with u and w being its neighbours, then a new signed graph is obtained by deleting v (together with uv and vw), and by adding a new edge uw whose sign is $\sigma(uv)\cdot\sigma(vw)$. If v is a vertex of degree $d = \deg(v) \geqslant 4$ with neighbours $u_1, \ldots, u_d$, then a new signed graph is obtained by deleting v (together with $vu_1, \ldots vu_d$), adding new vertices $v_1, \ldots, v_d$ that induce an all-positive circuit, and adding new edges $v_1u_1, \ldots, v_du_d$ with $\sigma(v_iu_i) = \sigma(vu_i)$ for $i \in \{1, \ldots, d\}$. The distinct members of $\mathcal{G}(G, \sigma)$ are obtained by repeating the above mentioned methods in distinct order. Sometimes it is useful to apply the methods in such order that an additional property (such as edge-connectivity) is preserved. Note that any member of $\mathcal{G}(G, \sigma)$ is flow-admissible whenever (G, σ) is.

We finish this section by recalling a few standard graph definitions. *A (proper) edge-coloring* of a graph G is an assigment of a color to every edge of G in such a way that any two adjacent edges obtain different colors. We say that G is *k-edge-colorable* if there exists an edge-coloring of G that uses at most k colors. The smallest number of colors needed to edge-color G is the *chromatic index of G*. By Vizing's theorem the chromatic index of a cubic graph is either 3 or 4. Tutte [16, 17] proved that a cubic graph G is 3-edge-colorable if and only if G admits a nowhere-zero 4-flow, and that G is bipartite if and only if G admits a nowhere-zero 3-flow. Bridgeless cubic graphs which do not have a nowhere-zero 4-flow are called *snarks*. We say that a snark G is *critical* if $G - e$ admits a nowhere-zero 4-flow for every edge e. Critical snarks were studied for example in [6, 7, 14].

3 Small edge-cuts

In Section 5 we will show that Bouchet's conjecture holds for signed graphs with two negative edges that contain bridges. In this section we will deal with 2-edge-cuts that do not separate negative edges, and refer to them as non-separating 2-edge-cuts. An idea to reduce non-separating cuts of size less than 3 appeared first in Bouchet's work (see Proposition 4.2. in [1]). However, his reduction uses contraction of a positive edge, which cannot be used in our paper – contraction of an edge of a signed graph from a particular class (e.g. bipartite) may result in a signed graph that does not belong to the same class.

Here, we introduce a reduction of 2-edge-cuts (different from the one introduced by Bouchet [1]), which will be applied several times in the proofs of our main results.

Let X be an edge-cut of (G, σ), and let (W_1, W_2) be a partition of $V(G)$ such that $w_1w_2 \in X$ if and only if $w_1 \in W_1$ and $w_2 \in W_2$. Switching at all vertices of W_1 results in a signed graph (G, σ') such that $\sigma(e) \neq \sigma'(e)$ if and only if $e \in X$. Thus, the following three statements hold.

Lemma 7. *Let (G, σ) be a signed graph and $X \subseteq E(G)$ be an edge-cut of G. If $X = N_\sigma$, then $F(G, \sigma) = F(G, \mathit{1})$.*

Lemma 8. *Let (G, σ) be a signed graph such that all negative edges N_σ belong to an $(|N_\sigma| + 1)$-edge-cut. Then (G, σ) is not flow-admissible.*

Corollary 9. *Let (G, σ) be a signed graph such that $|N_\sigma| = 2$. If (G, σ) is flow-admissible, then the two negative edges of (G, σ) do not belong to any 3-edge-cut.*

2-edge-cuts

When we study (non-separating) 2-edge-cuts, we always assume that the 2-edge-cut is a matching. Let $X = \{e_1, e_2\}$ be a 2-edge-cut of (G, σ) such that $(G - X, \sigma|_{G-X})$ has precisely two components G_1^-, G_2^-, where G_2^- is all-positive. Then X is called a *non-separating 2-edge-cut.* Let $e_1 = u_1u_2$ and $e_2 = v_1v_2$, and $u_i, v_i \in V(G_i^-)$.

If $X = N_\sigma$, then $F(G, \sigma) \leqslant 6$ by Lemma 7 and Theorem 6. For this reason we are interested in non-separating 2-edge-cuts with at least one positive edge, say e_2. A *2-edge-cut reduction of* (G, σ) with respect to the edge-cut X is a disjoint union of two signed graphs (G_1, σ_1) and (G_2, σ_2), where (G_i, σ_i) is obtained from G_i^- by adding an edge f_i between u_i and v_i and setting $\sigma_i(f_i) = \sigma(e_i)$. Note that (G_2, σ_2) is all-positive. We say that (G, σ) is *2-edge-cut reducible* and that (G_1, σ_1) and $(G_2, 1)$ are the *resulting graphs* of the 2-edge-cut reduction of (G, σ) (with respect to the (non-separating) 2-edge-cut X).

In what follows, when we refer to a 2-edge-cut reduction of a signed graph, then we always use the same notation as in the above definition.

Lemma 10. *Let (G, σ) be a flow-admissible signed graph. If (G, σ) is 2-edge-cut reducible with respect to a 2-edge-cut X, then the two resulting graphs are flow-admissible.*

Proof. Let (G_1, σ_1) and $(G_2, 1)$ be the resulting signed graphs of the 2-edge-cut reduction of (G, σ) with respect to $X = \{e_1, e_2\}$. We are going to prove that each edge of the resulting graphs belongs to a signed circuit.

Suppose first that $e \in E(G) \cap E(G_i)$, for $i \in \{1, 2\}$. Since (G, σ) is flow-admissible, there exists a signed circuit C of (G, σ) containing e according to Lemma 2. If $E(C) \subseteq E(G_i)$, then we are done. Otherwise, C contains at least one of $\{e_1, e_2\}$. Since $(G_2, \mathtt{1})$ is all-positive, C must contain both of $\{e_1, e_2\}$. Let P be a path of C that does not belong to G_i (note that P is a path since edges of X are independent, and that $\{e_1, e_2\} \subseteq E(P)$). Then $C - P \cup \{f_i\}$ is a signed circuit of G_i containing e.

Note that any such circuit also contains an edge $f_i \in E(G_i) - E(G)$, so we are done if there exists a signed circuit C of (G, σ) such that $E(C) \nsubseteq E(G_i)$. But if there is no such circuit of (G, σ), then e_i is not contained in any signed circuit, which is a contradiction with flow-admissibility of (G, σ). $\square$

Lemma 11. *Let (G_1, σ_1) and (G_2, σ_2) be the resulting graphs of the 2-edge-cut reduction of (G, σ) with respect to a 2-edge-cut $\{e_1, e_2\}$. Let k be a positive integer, and for $i \in \{1, 2\}$, let (G_i, σ_i) admit a nowhere-zero k-flow (τ_i, ϕ_i). If $\phi_1(f_1) = \phi_2(f_2)$, then $F(G, \sigma) \leqslant k$.*

Proof. Let τ be an orientation of the edges of (G, σ) such that $\tau(e) = \tau_i(e)$ for every edge $e \in E(G_i) \cap E(G)$. By Observation 4, we may assume that $e_1 \in \delta^+(u_2)$ and $e_2 \in \delta^+(v_1)$. Now (τ, ϕ) with $\phi(e) = \phi_i(e)$ for every $e \in E(G_i) \cap E(G)$ and $\phi(e_1) = \phi(e_2) = \phi_1(f_1)$ is a nowhere-zero k-flow on (G, σ). $\square$

For a signed graph (G, σ) with two negative edges we say that an all-positive 2-edge-cut X *separates the negative edges* if the negative edges belong to different components of $G - X$. We note that we will not use an equivalent of a 2-edge-cut reduction for 2-edge-cuts that separate negative edges, because the resulting signed graphs may not be flow-admissible.

4 Nowhere-zero 4-flows

The following lemma is due to Schönberger [11].

Lemma 12 ([11])**.** *If G is a bridgeless cubic graph and e is an edge of G, then G has a 1-factor that contains e.*

Lemma 13. *Let G be a cubic bipartite graph, and let $e, f \in E(G)$. If any 3-edge-cut contains at most one edge of $\{e, f\}$, then there exists a 1-factor of G that contains both e and f.*

Proof. Let U and V be the partite sets of G. Let $e = u_1v_1$ and $f = u_2v_2$ be two edges of G such that $u_1, u_2 \in U$ and $v_1, v_2 \in V$. If e and f are adjacent, they belong to a (trivial) 3-edge-cut of G and there is nothing to prove. Hence, e and f are non-adjacent.

If e and f form a 2-edge-cut, then they must belong to the same color class of a 3-edge-coloring of G and hence, there is a 1-factor that contains e and f.

In what follows, we assume that $\{e, f\}$ is not a 2-edge-cut. Let G' be the graph that is constructed from $G - \{e, f\}$ by adding new edges $e' = u_1u_2$ and $f' = v_1v_2$. It follows that G' is cubic and bridgeless (since e and f do not belong to any 3-edge-cut of G). Thus, by Lemma 12, there exists a 1-factor F' of G' containing e'. We claim that F' contains f'. Suppose to the contrary that $f' \notin F'$. Then there exist u_1' and u_2' from U such that v_1u_1' and v_2u_2' are in F'. The graph $G' - \{u_1, u_1', u_2, u_2', v_1, v_2\}$ is bipartite with partite sets of cardinality $|U| - 4$ and $|V| - 2 = |U| - 2$. Note that such a graph does not have any 1-factor, which is a contradiction with the existence of F'. Thus, f' must belong to F'. In that case $F = F' \cup \{e, f\} - \{e', f'\}$ is a 1-factor of G that contains e and f. $\square$

Lemma 14. *Let (G, σ) be a signed cubic graph with $N_\sigma = \{n_1, n_2\}$. If G has a 3-edge-coloring such that n_1 and n_2 belong to the same color class, then (G, σ) admits a nowhere-zero 4-flow (τ, ϕ) such that $\phi(n_1) = \phi(n_2) = 2$.*

Proof. Let $c : E(G) \to \{c_1, c_2, c_3\}$ be a 3-edge-coloring such that $c(n_1) = c(n_2) = c_1$. It is easy to see that $(G, \mathtt{1})$ has a nowhere-zero 4-flow (τ, ϕ) such that $\phi(e) > 0$ for every $e \in E(G)$ and $\phi(f) = 2$ if $f \in c^{-1}(c_1)$. Let $n_1 = u_1u_2$ and $n_2 = v_1v_2$, and let, without loss of generality, $n_1 \in \delta^+(u_1)$ and $n_2 \in \delta^+(v_1)$.

If N_σ is a 2-edge-cut, then the statement follows from Lemma 7.

It remains to consider the case when N_σ is not a 2-edge-cut. Since $c(n_1) = c(n_2)$, the edges n_1 and n_2 do not belong to a 3-edge-cut by parity reasons. Hence, every edge-cut that contains n_1 and n_2 has at least 4 edges. Note, that by Lemma 3, there is a directed v_2-u_1-path P that contains neither n_1 nor n_2. Let τ' be the orientation of (G, σ) which is obtained from τ by reversing the orientation of the half-edges $h_{u_1}^{n_1}$ and $h_{v_2}^{n_2}$ and of the edges of P. Let $\phi'(x) = 4 - \phi(x)$ if $x \in E(P)$, and $\phi'(x) = \phi(x)$ otherwise. It is easy to check that (τ', ϕ') is the required nowhere-zero 4-flow on (G, σ). $\square$

Theorem 15. *Let (G, σ) be a flow-admissible signed cubic graph with $|N_\sigma| = 2$. If G is bipartite, then $F(G, \sigma) \leqslant 4$.*

Proof. Let $N_\sigma = \{n_1, n_2\}$. Since (G, σ) is flow-admissible, n_1 and n_2 do not belong to any 3-edge-cut by Corollary 9. Thus, by Lemma 13, G has a 1-factor containing n_1 and n_2. By Lemma 14, $F(G, \sigma) \leqslant 4$. $\square$

The bound given in Theorem 15 is tight. It is achieved for example on $(K_{3,3}, \sigma)$, where the two negative edges form a matching (see [8]). It is not possible to extend the result of Theorem 15 to cubic bipartite graphs with any number of negative edges. For example, a circuit of length 6, where every second edge is doubled and one of the parallel edges is negative for every pair of parallel edges while all the other edges are positive, has flow number 6 (see [12]).

We would like to note that the choice of the flow value on negative edges is important. The signed graph in Figure 1 (where the depicted values represent the signature) is an example of a signed graph that does not admit a nowhere-zero 4-flow that assigns 1 to negative edges even though it admits a nowhere-zero 4-flow according to Theorem 15. For the proof, suppose to the contrary that (G, σ) admits a nowhere-zero 4-flow that assigns 1 to negative edges. Let all positive edges of (G, σ) be oriented from left to right and from top to bottom with respect to the embedding depicted in Figure 1. Furthermore, let the top negative edge be extroverted, and let the bottom one be introverted, both carrying the flow value 1. If the horizontal positive edges carry the flow values a, $(-a-b)$ and b (assigned from top to bottom), then due to the vertical edges, we have the following constraints on values a and b: $a \neq \pm 1$ and $b \neq \pm 1$ (vertical edges would carry value 0), $a \neq \pm 3$ and $b \neq \pm 3$ (vertical edges would carry value 4 or -4). Thus, $|a| = |b| = 2$, resulting in the flow value 0 or ± 4 on the middle horizontal positive edge, which carries the flow value $(-a-b)$ by Kirchhoff's law. This is a contradiction.

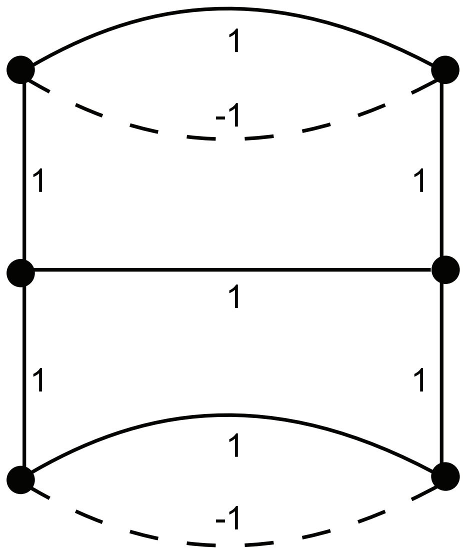

Figure 1: A signed graph for which the choice of the flow value on negative edges is important.

5 Nowhere-zero 6-flows

In this section we prove that Bouchet's conjecture is true for signed graphs with two negative edges where the underlying graph has additional properties. Our first result is on signed graphs with bridges, for which we need some lemmas. Tutte [17] proved that a graph has a nowhere-zero k-flow if and only if it has a nowhere-zero $\mathbb{Z}_k$-flow. This is not true for general signed graphs, but in our paper we will apply the following theorem, which is a straightforward corollary of Theorem 3.2 in [4]. Note that the following theorem as well as the subsequent lemma deal with unsigned graphs, but they can be applied to all-positive signed graphs too.

Theorem 16 ([4])**.** *Let G be a 3-edge-connected graph, and let $v \in V(G)$ be a vertex of degree 3 incident with edges e_1, e_2, e_3. Suppose that τ is an orientation of G such that $\delta^+(v) = \{e_1, e_2\}$, and $\delta^-(v) = \{e_3\}$. If $a_1, a_2, a_3 \in \mathbb{Z}_6 - \{0\}$ are such that $a_1 + a_2 = a_3$, then G admits a nowhere-zero $\mathbb{Z}_6$-flow (τ, ϕ) such that $\phi(e_i) = a_i$, for $i \in \{1, 2, 3\}$.*

Let τ be an orientation of a graph G and $\phi:\ E(G) \to \mathbb{Z}_k$ $(k \geqslant 2)$ be a flow on G. We now calculate in $\mathbb{Z}$. Let $\zeta(v) = \sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e)$ be the (integer) *outflow* of v and let $\Theta(G,(\tau,\phi)) = \sum_{v\in V(G)} |\zeta(v)|$. This notation is used in proof of the following lemma, which is an extension of a classical result by Tutte [17].

Lemma 17. *Let G be a graph, and let w be a vertex of G of degree 3 incident with $e_1, e_2, e_3 \in E(G)$. Suppose that G admits a nowhere-zero $\mathbb{Z}_k$-flow (τ,ϕ) such that*

(i) $\delta^+(w) = \{e_1, e_2\}$ *and* $\delta^-(w) = \{e_3\}$,

(ii) $\phi(e_1) = 1$,

(iii) $\phi(e_2) = a$, *for some* $a \in \{1,\dots,k-2\}$, *and*

(iv) $\phi(e_3) = 1 + a$.

Then G admits a nowhere-zero k-flow (τ,ψ) such that $\psi(e_i) = \phi(e_i)$, for $i \in \{1,2,3\}$.

Proof. Let G, w, e_1, e_2, e_3, and (τ,ϕ) be as described in the lemma. Since (τ,ϕ) is a nowhere-zero $\mathbb{Z}_k$-flow, $\zeta(v)$ is a multiple of k (including 0) for every $v \in V(G)$. Furthermore, $\sum_{v\in V(G)} \zeta(v) = 0$, because for every edge e of G, $\phi(e)$ contributes to the sum twice (once for each end-vertex of e), but with different signs. If $\Theta(G,(\tau,\phi)) = \sum_{v\in V(G)} |\zeta(v)| = 0$, then (τ,ϕ) is a nowhere-zero k-flow, and we are done. Thus, we may assume that $\Theta(G,(\tau,\phi)) \neq 0$. This and Kirchhoff's law extended to a set of vertices imply that there exist vertices x, y connected by a directed path P such that $\zeta(x) > 0$ and $\zeta(y) < 0$. Let τ' be the orientation of G obtained from τ by reversing the direction of the edges of P. Let $\phi'(e) = k - \phi(e)$ if $e \in E(P)$, and let $\phi'(e) = \phi(e)$ otherwise. Then (τ',ϕ') is a nowhere-zero $\mathbb{Z}_k$-flow on G. Furthermore, $\Theta(G,(\tau',\phi')) < \Theta(G,(\tau,\phi))$. Repeating this procedure eventually gives a nowhere-zero k-flow (τ_1,ϕ_1) on G, and we may assume that $\phi_1(e) > 0$ for all $e \in E(G)$.

From the procedure follows that at least one of e_1, e_2 belongs to $\delta^+(w)$. If $\delta^+(w) = \{e_1, e_2\}$, then $(\tau_2,\phi_2) := (\tau_1,\phi_1)$ is a nowhere-zero k-flow on G which coincides with (τ,ϕ) on $\{e_1, e_2, e_3\}$. It remains to consider that exactly one of e_1, e_2, say e_j, belongs to $\delta^-(w)$. Note that in this case, $e_3 \in \delta^+(w)$, $\phi_1(e_j) = k - \phi(e_j)$, and $\phi_1(e_3) = k - \phi(e_3)$. Now, e_3 is contained in a directed circuit C which also contains e_j. Change the direction of the edges of C to obtain τ_2 and for $e \in E(C)$ replace $\phi_1(e)$ by $k - \phi_1(e)$ to obtain ϕ_2. Then (τ_2,ϕ_2) is a nowhere-zero k-flow on G which coincides with (τ,ϕ) on $\{e_1, e_2, e_3\}$.

As the last step we need to modify edges for which $\tau_2 \neq \tau$. Let $\psi(e) = -\phi_2(e)$ (in $\mathbb{Z}$) for $e \in E(G)$ with $\tau_2(e) \neq \tau(e)$, and let $\psi(e) = \phi_2(e)$ otherwise. Then (τ,ψ) is the required nowhere-zero k-flow of G. □

Corollary 18. *Let G be a cubic graph, $f \in E(G)$ and $t \in \{1,\dots,5\}$. If G is bridgeless, then $(G, \mathbb{1})$ has a nowhere-zero 6-flow (τ,ϕ) such that $\phi(f) = t$, for each possible direction of f, and $\phi(e) > 0$, for each $e \in E(G)$.*

Proof. By Theorem 6, $(G, \mathbb{1})$ admits a nowhere-zero 6-flow (or, equivalently, a nowhere-zero $\mathbb{Z}_6$-flow). We need to show that we can choose the flow value on f. By Observation 4, we only need to prove the statement for different values of t irrespective of the direction of

f. Suppose the contrary, and let G be a counterexample with minimum number of edges. If G is 3-edge-connected, we obtain a contradiction with Theorem 16 and Lemma 17.

Thus, we may assume that G has a 2-edge-cut $X = \{e_1, e_2\}$. Viewing G as $(G, \mathtt{1})$, we may use the 2-edge-cut reduction with respect to X defined in Section 3. By Lemma 10, the resulting graphs G_1 and G_2 are flow-admissible. Moreover, they are both smaller than G, and therefore they admit a nowhere-zero 6-flow such that we can choose the flow value on one edge.

If $f \notin X$, then we first choose the requisite flow for G_i which contains f, and then we choose the flow for G_{3-i} in such a way that the flow values on f_1 and f_2 coincide (recall that $f_i \in E(G_i) - E(G)$, for $i \in \{1, 2\}$). If $f \in X$, then we choose the flow on G_i in such a way that f_i receives the flow value t, for each $i \in \{1, 2\}$. By Lemma 11, G has the required nowhere-zero 6-flow, which is a contradiction. □

Theorem 19. *Let (G, σ) be a flow-admissible signed cubic graph with two negative edges. If (G, σ) contains a bridge, then (G, σ) admits a nowhere-zero 6-flow (τ, ϕ) with the flow value 1 on the negative edges.*

Proof. Let n_1 and n_2 be the two negative edges of (G, σ). Since (G, σ) is flow-admissible, it follows that all bridges are positive edges according to Lemma 2.

We will prove the statement by induction on the order of the graph. If $|V(G)| = 2$, then (G, σ) is the graph with one positive edge and a negative loop on each vertex. We will call this graph the *dumbbell graph*. Clearly, (G, σ) has the desired nowhere-zero 6-flow; indeed it has a nowhere-zero 3-flow.

Let $B = \{b_1, \ldots, b_l\}$ be the set of bridges of (G, σ), where $l \geqslant 1$, and let (G_i, σ_i) be the $l + 1$ bridgeless components of $(G - B, \sigma|_{G-B})$, for $i \in \{1, \ldots, l + 1\}$. Since (G, σ) is flow-admissible, each bridge must belong to a barbell with the property that each of the negative edges belongs to one of the unbalanced circuits of the barbell. It follows that $b_1, \ldots, b_l$ lie on a path, and that the two negative edges are contained in different end-components, say $n_1 \in E(G_1)$ and $n_2 \in E(G_{l+1})$.

We claim that $l = 1$. Suppose to the contrary that $l \geqslant 2$. Let $b_1 = x_1x_2$ and $b_l = x_lx_{l+1}$, where $x_i \in E(G_i)$. Reduce (G, σ) to two smaller graphs (H_1, σ_{H_1}) and (H_2, σ_{H_2}), where (H_1, σ_{H_1}) is obtained from (G_1, σ_1) and (G_l, σ_l) by adding a positive edge x_1x_{l+1}, and (H_2, σ_{H_2}) is the all-positive graph obtained from (G, σ) by removing $V(G_1)$ and $V(G_{l+1})$ and adding a positive edge x_2x_l. By induction hypothesis, (H_1, σ_{H_1}) has a nowhere-zero 6-flow with the flow value 1 on the negative edges. Hence, x_1x_{l+1} has the flow value 2. By Theorem 6 and Corollary 18, (H_2, σ_{H_2}) has a nowhere-zero 6-flow with the flow value 2 on x_2x_l. According to Observation 4, the directions of x_1x_{l+1} and x_2x_l can be chosen appropriately so that these two nowhere-zero 6-flows combine to the desired nowhere-zero 6-flow on (G, σ).

Now let $b = x_1x_2$ be the only bridge of (G, σ) and let $y_i, z_i \in V(G_i)$ be the neighbors of x_i in (G_i, σ_i) for $i \in \{1, 2\}$. It follows that either $y_i \neq z_i$, or $y_i = z_i = x_i$. In the latter case (G_i, σ_i) consists of one vertex with a negative loop. We say that (G_i, σ_i) is a *negative loop*.

Suppose first that neither (G_1, σ_1) nor (G_2, σ_2) is a negative loop. Reduce (G, σ) to two graphs (H_1, σ_{H_1}) and (H_2, σ_{H_2}), where (H_i, σ_{H_i}) is obtained from (G, σ) by replacing (G_i, σ_i) by a negative loop, for $i \in \{1, 2\}$. Now the result follows easily by induction and a suitable combination of flows on (H_i, σ_{H_i}).

Now assume that one component is a negative loop, say (G_2, σ_2). The case when (G_1, σ_1) is also a negative loop, is discussed above, hence we may assume that (G_1, σ_1) is not a negative loop. We are going to define a nowhere-zero 6-flow on (G, σ) directly.

Let $n_1 = u_1v_1$, and let G_1^* be the underlying graph obtained from the signed graph (G_1, σ_1) by removing n_1 and connecting u_1, v_1 and x_1 to a new vertex w. We claim that G_1^* is 3-edge-connected. It is easy to see that G_1^* is cubic, connected and that it does not have a bridge, because it is obtained from a 2-edge-connected graph (G_1, σ_1) where the deleted edge n_1 is replaced by a path u_1wv_1. Suppose to the contrary that $X \subseteq E(G_1^*)$ is a 2-edge-cut of G_1^*. If u_1, v_1 and x_1 belong to one component of $G_1^* - X$, then X is a non-separating 2-edge-cut of (G, σ). We may apply the 2-edge-cut reduction on (G, σ) with respect to X, and use induction hypothesis and Corollary 18 to obtain a contradiction. Therefore, there is one component of $G_1^* - X$ containing exactly one of u_1, v_1 and x_1. But then X must contain exactly one edge incident to w, otherwise G_1^* contains a bridge. Thus, $G_1^* - w = G_1 - n_1$ contains a bridge. This is possible if and only if n_1 belongs to a 2-edge-cut of (G_1, σ_1), which is a non-separating 2-edge-cut of (G, σ), because it contains n_1. Similarly as above, we may use the 2-edge-cut reduction to obtain a contradiction. We conclude that G_1^* is indeed 3-edge-connected.

By Theorem 16 and by Lemma 17, G_1^* admits a nowhere-zero 6-flow (τ_1^*, ϕ_1^*) such that $\delta^+(w) = \{u_1w, v_1w\}$, $\delta^-(w) = \{x_1w\}$, and $\phi_1^*(u_1w) = \phi_1^*(v_1w) = 1$, and $\phi_1^*(x_1w) = 2$. Let τ be an orientation of (G, σ) defined as follows: n_1 is extroverted, n_2 is introverted, $b_1 \in \delta^+(x_1)$, and $\tau(e) = \tau_1^*(e)$ for every edge $e \in E(G) \cap E(G_1^*)$. Let ϕ be an assignment of integer values to the oriented edges of (G, σ) defined as follows: $\phi(n_1) = \phi(n_2) = 1$, $\phi(b_1) = 2$, and $\phi(e) = \phi_1^*(e)$, for every edge $e \in E(G) \cap E(G_1^*)$. Then (τ, ϕ) is the required nowhere-zero 6-flow of (G, σ). $\square$

Using the previous theorem, we are able to prove Theorem 6 as follows. Consider a 2-edge-connected graph G, and an arbitrary edge $e = uv \in E(G)$. To obtain G' from G, remove e and add new edges uu', vv', $l_{u'}$ and $l_{v'}$, where u' and v' are new vertices, and $l_{u'}$ and $l_{v'}$ are loops incident with u' and v', respectively. Let σ' be a signature on G' such that $\sigma'(l_{u'}) = \sigma'(l_{v'}) = -1$ and $\sigma'(e) = 1$, for every other edge e of $E(G')$. By Theorem 19, (G', σ') admits an all-positive 6-flow with the flow value 1 on $l_{u'}$ and $l_{v'}$, and therefore, the flow value 2 on uu' and vv'. It is easy to see that G admits a nowhere-zero 6-flow with the flow value 2 on e.

Note that Theorem 19 and Theorem 6 are not equivalent, because a stronger statement, namely Theorem 16, is used in the proof of the former one.

In the following we focus on (G, σ) where G is 3-edge-colorable or a critical snark. Recall that a snark G is *critical* if $G - e$ admits a nowhere-zero 4-flow for every edge e.

Lemma 20. *Let G be a cubic graph and $e_1, e_2 \in E(G)$. If G is 3-edge-colorable, then $(G, 1)$ has a nowhere-zero 4-flow (τ, ϕ) such that $\phi(e) > 0$ for every $e \in E(G)$, and*

$\phi(e_1) = \phi(e_2) = 1$.

Proof. Let $c: E(G) \to \{c_1, c_2, c_3\}$ be a 3-edge-coloring, and let $c(e_1) = c_1$ and $c(e_2) \in \{c_1, c_2\}$. Let τ be an orientation of G, and let (τ, ϕ_1) be a nowhere-zero 2-flow on $c^{-1}(c_1) \cup c^{-1}(c_2)$ such that $\phi_1(e_2) = 1$ and (τ, ϕ_2) be a nowhere-zero 2-flow on $c^{-1}(c_2) \cup c^{-1}(c_3)$ such that $\phi_2(e_2) = 1$ if $c(e_2) = c_2$.

In both cases for $c(e_2)$, ϕ is defined as $2 \cdot \phi_2 - \phi_1$. The desired flow on (G, σ) is obtained from (τ, ϕ) by reversing the direction and the value of each edge with negative value. $\square$

Theorem 21. *Let (G, σ) be a flow-admissible signed cubic graph with $N_\sigma = \{n_1, n_2\}$. If G is 3-edge-colorable or a critical snark, then (G, σ) has a nowhere-zero 6-flow (τ, ϕ) such that $\phi(n_1) = \phi(n_2) = 1$.*

Proof. Suppose to the contrary that the statement is not true, and let (G, σ) be a minimal counterexample. By Lemma 2, (G, σ) has no negative bridge. By Lemma 7 and Theorem 6, (G, σ) has no 2-edge-cut containing both negative edges. If (G, σ) has a 2-edge-cut containing exactly one negative edge, then deduce a contradiction with application of the 2-edge-cut reduction and Corollary 18. In what follows we may assume that both edges of any 2-edge-cut are positive. Let $n_1 = u_1 v_1$ and $n_2 = u_2 v_2$.

Case 1: G is 3-edge-colorable. By Lemma 20, there is a nowhere-zero 4-flow (τ', ϕ') on $(G, \mathtt{1})$ such that $\phi'(n_1) = \phi'(n_2) = 1$, and $\phi'(e) > 0$, for every $e \in E(G)$.

Suppose, without loss of generality, that $n_1 \in \delta^+(u_1)$ and $n_2 \in \delta^+(u_2)$ in τ'. Since $\phi'(n_1) = 1$, there is another edge $f = u_1 v \in \delta^+(u_1)$. It follows from Lemma 3, that there is a directed v-v_2-path of $(G, \mathtt{1})$, which together with the edge f forms a directed u_1-v_2-path P of $(G, \mathtt{1})$. We claim that $n_1 \notin E(P)$. Otherwise either n_1 is a bridge, or n_1 belongs to a 2-edge-cut, a contradiction.

If $n_2 \notin E(P)$, then to obtain τ from τ' reverse the direction of $h_{u_1}^{n_1}$ and $h_{v_2}^{n_2}$. Let $\phi(e) = \phi'(e) + 2$ if $e \in E(P)$, and $\phi(e) = \phi'(e)$ otherwise. Then (τ, ϕ) is the desired nowhere-zero 6-flow on (G, σ).

Suppose now that $n_2 \in E(P)$, for every directed u_1-v_2-path P of $(G, \mathtt{1})$ with $n_1 \notin P$. Consider any edge-cut X that contains n_2 and separates u_1 and v_2. Let X divide $V(G)$ into two subsets U and W, where $u_1 \in U$ and $v_2 \in W$. By Kirchhoff's law, the total outflow from U is 0. Since $\phi'(n_2) = 1$ and n_2 does not belong to any 2-edge-cut, there must be another edge of X oriented from U to W under τ'. This is possible if and only if $n_1 \in X$, since every directed u_1-v_2-path P of $(G, \mathtt{1})$ with $n_1 \notin P$ contains n_2. Thus, there are two edges oriented from U to V and they both carry the flow value 1. Therefore, there are at most two edges oriented from V to U, and $3 \leqslant |X| \leqslant 4$. By Corollary 9, $|X| = 4$, and thus, the two edges oriented from V to U carry the flow value 1, according to Kirchhoff's law. Let $f' = u_3 v_3 \in X - N_\sigma$ be one of them and suppose that $f' \in \delta^+(v_3)$ in τ'. Let $P' = P_1 \cup f' \cup P_2$, where P_1 is a directed u_1-u_3-path such that $E(P_1) \cap N_\sigma = \emptyset$ and P_2 is a directed v_3-v_2-path such that $E(P_2) \cap N_\sigma = \emptyset$. Note that P_1 and P_2 may be trivial, but they always exist due to Kirchhoff's law. Similarly as in the case above, we define a nowhere-zero 6-flow (τ, ϕ) on (G, σ). Note that $f' \in \delta^+(u_3)$ in τ and $\phi(f') = 1$.

Case 2: G is a critical snark. Hence, $(G, \mathtt{1}) - n_1$ admits a nowhere-zero 4-flow (τ', ϕ'), and by Lemma 20, we may assume that $\phi'(n_2) = 1$, $n_2 \in \delta^+(u_2)$, and $\phi'(e) > 0$ for every $e \in E(G)$. Consider a directed u_1-v_2-path P_1 and a directed v_1-v_2-path P_2 in $(G - n_1, 1)$. Since $\phi'(n_2) = 1$ and n_2 does not belong to any 2-edge-cut, we may assume that $n_2 \notin E(P_1) \cup E(P_2)$. Note that P_1 and P_2 are not edge-disjoint, because they share an edge whose end-vertex is v_2. Obtain an orientation τ of (G, σ) by letting n_1 be extroverted, reversing the direction of $h_{v_2}^{n_2}$, and $\tau(h) = \tau'(h)$ for every other half-edge h of (G, σ). Let $\phi''(e) = \phi'(e) + 1$ if $e \in E(P_1)$, $\phi''(n_1) = 1$, and $\phi''(e) = \phi'(e)$ if $e \notin E(P_1) \cup \{n_1\}$. The desired nowhere-zero 6-flow on (G, σ) is (τ, ϕ) with $\phi(e) = \phi''(e) + 1$ if $e \in E(P_2)$, and $\phi(e) = \phi''(e)$ otherwise. □

6 General case

In this section we prove the bound 7 for all flow-admissible signed graphs with two negative edges, and the bound 6 if the Tutte's conjecture is true.

Theorem 22. *Let (G, σ) be a flow-admissible signed cubic graph with two negative edges $n_1 = u_1v_1$ and $n_2 = u_2v_2$. Let $G^* = (V(G), E(G) \cup \{n\} - \{n_1, n_2\})$ be an unsigned graph, where $n = u_1u_2 \notin E(G)$. If G^* admits a nowhere-zero k-flow for some integer $k \geqslant 2$ such that n receives the flow value 1, then (G, σ) admits a nowhere-zero $(k+1)$-flow (τ, ϕ) with the following properties:*

1. *$\phi(e) > 0$, for every $e \in E(G)$,*
2. *$\phi(n_1) = \phi(n_2) = 1$, and*
3. *there exists a v_1-v_2-path P such that $\phi^{-1}(k) \subseteq E(P)$ and $\phi^{-1}(1) \cap E(P) = \emptyset$.*

Proof. Let (τ^*, ϕ^*) be a nowhere-zero k-flow of G^* with $\phi^*(e) > 0$ for every $e \in E(G^*)$, $\phi^*(n) = 1$ and $n \in \delta^+(u_1)$. By Lemma 3, there is a directed v_2-v_1-path in G^*. We claim that there is a directed v_2-v_1-path P in $G^* - \{n\}$. If not, then all directed paths from v_2 contain n. Since $\phi^*(n) = 1$, it follows that there is an edge f such that $\{n, f\}$ is a 2-edge-cut of G^* which separates the two sets $\{v_1, u_2\}$ and $\{u_1, v_2\}$. Note that none of the negative edges is a bridge, otherwise the signed graph (G, σ) would not be flow-admissible. Moreover, the negative edges do not belong to any 2-edge-cut by Lemma 7. Hence, $\{n_1, n_2, f\}$ is a 3-edge-cut of (G, σ) that contains two negative edges, contradicting Lemma 8, since (G, σ) is flow-admissible. Thus, there is a directed v_2-v_1-path P in $G^* - \{n\}$.

We define (τ, ϕ) on (G, σ) as follows. For $e \in E(G) \cap E(G^*)$ we set $\tau(e) = \tau^*(e)$. Let n_2 be extroverted and n_1 be introverted, and let $\phi(n_2) = \phi(n_1) = 1$. If $e \notin P$, then $\phi(e) = \phi^*(e)$, and if $e \in P$, then $\phi(e) = \phi^*(e) + 1$. It is easy to see that (τ, ϕ) is the required nowhere-zero $(k+1)$-flow. □

The previous theorem combined with the following observation provides several interesting corollaries.

Observation 23. *Let (G, σ) be a flow-admissible signed cubic graph with two negative edges $n_1 = u_1v_1$ and $n_2 = u_2v_2$. Let $G^* = (V(G), E(G) \cup \{n\} - \{n_1, n_2\})$ be an unsigned graph, where $n = u_1u_2 \notin E(G)$. If no 2-edge-cut of (G, σ) contains a negative edge, then G^* is flow-admissible.*

Proof. Suppose to the contrary that G^* is not flow-admissible. Then G^* contains a bridge b. If $b = n$, then either (G, σ) has two components, each containing a negative edge, or $\{n_1, n_2\}$ is a 2-edge-cut of (G, σ). In the first case (G, σ) is not flow-admissible, and in the second case there is a 2-edge-cut of (G, σ) containing a negative edge, a contradiction. If $b \neq n$, then u_1 and u_2 belong to the same component H of $G^* - b$. If v_1 and v_2 both belong to H, then b is a bridge of (G, σ) with an all-positive signed graph on one side, contradicting the flow-admissibility of (G, σ) due to Lemma 2. If neither v_1 nor v_2 belongs to H, then $\{n_1, n_2, b\}$ is a 3-edge-cut containing two negative edges, contradicting the flow-admissibility of (G, σ) due to Corollary 9. Suppose, finally, that one of v_1 and v_2, say v_1, belongs to H. Then $\{n_2, b\}$ is a 2-edge-cut of (G, σ) containing a negative edge, a contradiction. □

Theorem 24. *If (G, σ) is a flow-admissible signed cubic graph with $N_\sigma = \{n_1, n_2\}$, then (G, σ) has a nowhere-zero 7-flow (τ, ϕ) such that $\phi(n_1) = \phi(n_2) = 1$, and all edges with the flow value 6 lie on a single path.*

Proof. Suppose the contrary, and let (G, σ) be a minimal counterexample. By Theorem 19, we may assume that (G, σ) is bridgeless. By Lemma 7 and Theorem 6, N_σ does not form a 2-edge-cut. Suppose that there is a 2-edge-cut X containing one positive and one negative edge. Let (G_1, σ_1) and $(G_2, \mathtt{1})$ be the resulting graphs of the 2-edge-cut reduction of (G, σ) with respect to X (see Section 3 for notation). By Lemma 10, (G_1, σ_1) and $(G_2, \mathtt{1})$ are flow-admissible. Furthermore, (G_1, σ_1) has two negative edges and is smaller than (G, σ). Therefore, (G_1, σ_1) admits a nowhere-zero 7-flow (τ_1, ϕ_1) with the required properties. We may assume that $\phi_1(e) > 0$, for every $e \in E(G_1)$. Note, that the added edge f_1 of (G_1, σ_1) is negative and therefore, $\phi_1(f_1) = 1$. By Corollary 18, there is nowhere-zero 6-flow (τ_2, ϕ_2) on $(G_2, \mathtt{1})$ with $\phi_2(f_2) = \phi_1(f_1)$. By Lemma 11, we can combine (τ_1, ϕ_1) and (τ_2, ϕ_2) to define the desired nowhere-zero 7-flow on (G, σ), a contradiction.

Finally, we may assume that every 2-edge-cut of (G, σ) contains only positive edges. Let $n_1 = u_1v_1$, $n_2 = u_2v_2$ and let $G^* = (V(G), E(G) \cup \{n\} - \{n_1, n_2\})$ be an unsigned graph obtained from (G, σ), where $n = u_1u_2 \notin E(G)$. By Observation 23, G^* is flow-admissible, and by Theorem 6 and Corollary 18, G^* admits a nowhere-zero 6-flow with the flow value 1 on n. We obtain a contradiction by applying Theorem 22. □

We will relate Tutte's 5-flow conjecture and Bouchet's 6-flow conjecture for signed graphs with two negative edges. For this we will need the following lemma.

Lemma 25. *Let G be a cubic graph, $f \in E(G)$ and $t \in \{1, \dots, 4\}$. If G has a nowhere-zero $\mathbb{Z}_5$-flow, then for every possible direction of f, G has a nowhere-zero 5-flow (τ, ϕ) with $\phi(f) = t$, and $\phi(e) > 0$ for each $e \in E(G)$.*

Proof. By Observation 4, we only need to prove the statement for different values of t irrespective of the direction of f. Let (τ, ϕ) be a nowhere-zero $\mathbb{Z}_5$-flow on G. If $t \in \{1, 4\}$, then we may assume that $\phi(f) = 1$, otherwise we will consider $(\tau, c \cdot \phi)$, for $c \cdot \phi(f) = 1 \pmod 5$. If $t \in \{2, 3\}$, then we may assume that $\phi(f) = 2$, otherwise we will consider $(\tau, c \cdot \phi)$, for $c \cdot \phi(f) = 2 \pmod 5$.

Let f_1 and f_2 be edges of G adjacent to f and incident with a common vertex v. We may assume that $\delta^+(v) = \{f, f_1\}$ and $\delta^-(v) = f_2$, since otherwise we revert the edge and the flow value on it. If none of $\phi(f), \phi(f_1), \phi(f_2)$ is 1, then $\phi(f) = \phi(f_1) = 2$ and $\phi(f_2) = 4$. Let C be a directed circuit containing edges f_1 and f_2, which exists by Lemma 3. For the edges of C revert their orientation and replace their value $\phi(e)$ by $5 - \phi(e)$ to obtain a new flow. Now, the flow value of f_2 equals 1. Apply Lemma 17 to obtain a nowhere-zero 5-flow (τ_1, ϕ_1) with $\phi_1(f) = \phi(f)$. If $\phi_1(f) \neq t$, then repeat the trick with a directed circuit C for f to obtain the correct value on f. Finally, the required flow is obtained by reversing the orientations and values of edges with the negative flow value. $\square$

Theorem 26. *If Tutte's* 5*-flow conjecture holds true, then Bouchet's conjecture holds true for all signed graphs with two negative edges. Moreover, for any bridgeless signed graph* (G, σ) *with* $N_\sigma = \{n_1, n_2\}$*, there is a nowhere-zero* 6*-flow* (τ, ϕ) *with* $\phi(e) > 0$ *for every* $e \in E(G)$ *such that* $\phi(n_1) = \phi(n_2) = 1$*, and there is a path* P *such that* $\phi^{-1}(5) \subseteq E(P)$ *and* $\phi^{-1}(1) \cap E(P) = \emptyset$*.*

Proof. Suppose the contrary, and let (G, σ) be a minimal signed graph with two negative edges, for which the theorem does not hold. By Theorem 19, (G, σ) is bridgeless. Let X be a 2-edge-cut of (G, σ). If $X = N_\sigma$, then (G, σ) has a nowhere-zero 5-flow by Lemma 7 and by the assumption, a contradiction.

Suppose that $|X \cap N_\sigma| = 1$. Let (G_1, σ_1) and $(G_2, \mathtt{1})$ be the resulting graphs of the 2-edge-cut reduction of (G, σ) with respect to X (see Section 3 for notation). By Lemma 10, (G_1, σ_1) and $(G_2, \mathtt{1})$ are flow-admissible. Since (G_1, σ_1) is smaller than (G, σ), it admits a nowhere-zero 6-flow (τ_1, ϕ_1) with the required properties. In particular, $\phi_1(f_1) = 1$. By the assumption and by Lemma 25, there is a nowhere-zero 5-flow (τ_2, ϕ_2) on (G_2, σ_2) with $\phi_2(f_2) = 1$. By Observation 4 and by Lemma 11, we obtain a contradiction.

Finally, we may assume that $|X \cap N_\sigma| = 0$ or that G is 3-edge-connected. Let $n_i = u_i v_i$, for $i \in \{1, 2\}$, and let $G^* = (V(G), E(G) \cup \{n\} - \{n_1, n_2\})$ be an unsigned graph such that $n = u_1 u_2 \notin E(G)$. By Observation 23, G^* is bridgeless and therefore, it has a nowhere-zero 5-flow (τ, ϕ). By Lemma 25, we may assume that $\phi(n) = 1$. Now, the result follows from Theorem 22. $\square$

A graph G is *cyclically k-edge-connected* if there exists no edge-cut X with less than k edges such that $G - X$ has two components that contain a circuit. The *oddness* $\omega(G)$ of a cubic graph G is the minimum number of odd circuits of any 2-factor of G. In [15] it is proved that if the cyclic connectivity of a cubic graph G is at least $\frac{5}{2}\omega(G) - 3$, then $F(G, \mathtt{1}) \leqslant 5$. Clearly, if G' is obtained from G by subdividing an edge, then G is cyclically k-edge-connected if and only if G' cyclically k-edge-connected. Hence, the following corollary follows from Lemma 25 and Theorem 22.

Corollary 27. *Let (G,σ) be a flow-admissible signed cubic graph with with two negative edges $n_1 = u_1v_1$ and $n_2 = u_2v_2$. Let $G^* = (V(G), E(G) \cup \{n\} - \{n_1, n_2\})$, where $n = u_1v_1 \notin E(G)$. If G^* is cyclically k-edge-connected and $k \geqslant \frac{5}{2}\omega(G') - 3$, then $F(G,\sigma) \leqslant 6$.*

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