Toward the Schur expansion of Macdonald polynomials

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Abstract

We give an explicit combinatorial formula for the Schur expansion of Macdonald polynomials indexed by partitions with second part at most two. This gives a uniform formula for both hook and two column partitions. The proof comes as a corollary to the result that generalized dual equivalence classes of permutations are in explicit bijection with unions of standard dual equivalence classes of permutations for certain cases, establishing an earlier conjecture of the author, and suggesting that this result might be generalized to arbitrary partitions.

Mathematics Subject Classifications: 05E05, 05A15, 05A19, 05A30, 33D52

1 Introduction

The transformed Macdonald polynomials, $\widetilde{H}_\mu(X;q,t)$, a transformation due to Garsia of the polynomials introduced by Macdonald [20] in 1988, are the simultaneous generalization of Hall–Littlewood and Jack symmetric functions with two parameters, q and t. The *Kostka-Macdonald coefficients*, denoted $\widetilde{K}_{\lambda,\mu}(q,t)$, give the change of basis from Macdonald polynomials to Schur functions, namely,

$$\widetilde{H}_\mu(X;q,t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t) s_\lambda(X).$$

A priori, $\widetilde{K}_{\lambda,\mu}(q,t)$ is a rational function in q and t with rational coefficients.

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The Macdonald Positivity Theorem [14], first conjectured by Macdonald [20], states that $\widetilde{K}_{\lambda,\mu}(q,t)$ is in fact a polynomial in q and t with nonnegative integer coefficients. Garsia and Haiman [11] conjectured that the transformed Macdonald polynomials $\widetilde{H}_\mu(X;q,t)$ could be realized as the bi-graded characters of certain modules for the diagonal action of the symmetric group S_n on two sets of variables. Once resolved, this conjecture gives a representation theoretic interpretation of Kostka-Macdonald coefficients as the graded multiplicity of an irreducible representation in the Garsia-Haiman module, and hence $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. Following an idea outlined by Procesi, Haiman [14] proved this conjecture by analyzing the algebraic geometry of the isospectral Hilbert scheme of n points in the plane, thereby establishing Macdonald Positivity. This proof, however, is purely geometric and does not offer a combinatorial interpretation for $\widetilde{K}_{\lambda,\mu}(q,t)$.

In 2004, Haglund [12] conjectured a combinatorial formula for the monomial expansion of $\widetilde{H}_\mu(X;q,t)$ that was subsequently proved by Haglund, Haiman and Loehr [15]. This formula establishes that $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{Z}[q,t]$ but comes short of proving non-negativity.

Combinatorial formulas for $\widetilde{K}_{\lambda,\mu}(q,t)$ have been found for certain special cases. In 1995, Fishel [7] gave the first combinatorial interpretation for $\widetilde{K}_{\lambda,\mu}(q,t)$ when μ is a partition with 2 columns, and there are now other formulas for two column Macdonald polynomials [22, 17, 12, 3]. In all cases, finding broad extensions for these formulas has proven elusive. Haglund [12] conjectured a formula for three columns Macdonald polynomials, and this was later proved by Blasiak [5] who noted that his methods would not extend beyond this case. Both Roberts [21] and Loehr [18] noticed that Haglund's formula for two column Macdonald polynomials [12] extends to partitions with one additional box by observing that the proof using dual equivalence [4] works for this case as well.

In this paper, we extend the combinatorial formula for two column Macdonald polynomials to partitions with second part at most 2. This case simultaneously contains two column and hook partitions. The proof is purely combinatorial and combines the bijective proofs of the two column and single row Macdonald polynomials in [3] utilizing the structure of dual equivalence [4]. In particular, it proves [4](Conjecture 5.6) for $\mu_2 \leqslant 2$.

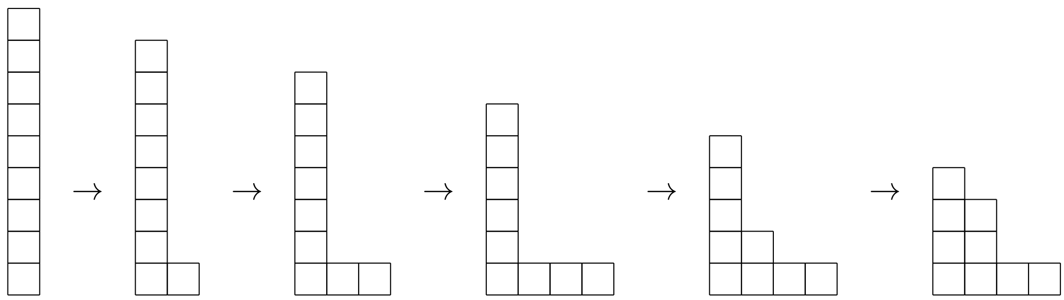

Figure 1: Folding (1^9) to $(4,2,2,1)$.

The main idea of this paper is to relate the standard dual equivalence classes for permutations, which coincide with the generalized dual equivalence classes for single column Macdonald polynomials, with the generalized dual equivalence classes for an arbitrary partition μ. We do this incrementally, as indicated in Figure 1, by folding the leg of the partition to form the rows, from bottom to top. In this paper, we succeed in defining bijections φ_μ for any partition μ with $\mu_2 \leqslant 2$ such that standard dual equivalence classes are combined into generalized μ-dual equivalence classes. Thus we have a characterization of Kostka-Macdonald polynomials as

$$\widetilde{K}_{\lambda,\mu}(q,t) = \sum_{u \in \mathrm{SS}(\lambda)} q^{\mathrm{inv}_\mu(\varphi_\mu(u))} t^{\mathrm{maj}_\mu(\varphi_\mu(u))},$$

where the sum is over certain permutations and inv and maj are Haglund's statistics.

Our proofs are elementary and combinatorial, and this paper is largely self-contained. In Section 2, we recall Haglund's permutation statistics from [12] and use them to define Macdonald polynomials combinatorially as done in [15]. In Section 3, we recall the basic ideas of dual equivalence in [2, 4] and show how this machinery can be used to prove Macdonald positivity. In Section 4, we recall Foata's bijection on permutations [8] and use it to prove our formula for hook partitions. In Section 5, we recall a bijection from [3] and use it to prove our formula for partitions with second part at most 2. Finally, in Section 6, we discuss some obstacles to overcome in generalizing these results.

2 Macdonald polynomials

We begin with terminology, notation and conventions. A *permutation of* n is an ordering on the numbers $12 \cdots n$. Though these are elements of the symmetric group $\mathfrak{S}_n$, we will not make use of the group structure. A *word*, for our purposes, is a subword of a permutation, i.e., has no repeated letters. We regard n as fixed throughout. Finally, let X denote the infinite variable set $x_1, x_2, \ldots$.

Given a permutation w and an index $i < n$, i is an *inverse descent of* w if $i+1$ lies to the left of i in w. Denote the *inverse descent set of* w by $\mathrm{iDes}(w)$. For example,

$$\mathrm{iDes}(583691724) = \{2, 4, 7\}$$

For w a permutation of n, $\mathrm{iDes}(w)$ is a subset of $[n-1] = \{1, 2, \ldots, n-1\}$.

Gessel [10] defined an important basis of quasisymmetric functions indexed by sets.

Definition 1 ([10])**.** The *fundamental quasisymmetric function* for $D \subseteq [n-1]$ is

$$F_D(X) = \sum_{\substack{i_1 \leqslant \cdots \leqslant i_n \\ j \in D \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}. \tag{1}$$

For example, $F_{\{2,4,7\}}(X)$ contains the monomial $x_1^2 x_3 x_4 x_5 x_6^2 x_8^2$ but not $x_1^2 x_3 x_4 x_5 x_6^3 x_8$.

Inverse descent sets allow us to associate a quasisymmetric function to each permutation. For example, the permutation 583691724 will have the function $F_{\{2,4,7\}}(X)$ associated to it. An important application of this is the Frobenius character H_n of the regular representation of $\mathfrak{S}_n$. As a function, H_n may be written as the quasisymmetric generating function of permutations,

$$H_n(X) = \sum_{w \in \mathfrak{S}_n} F_{\mathrm{iDes}(w)}(X). \tag{2}$$

A *partition of* n is a weakly decreasing sequence of positive integers that sum to n. Given a permutation w and a partition μ, we fill the Young diagram of μ with w from left to right, top to bottom. For example, see Figure 2.

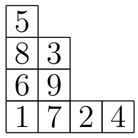

Figure 2: The Young diagram of $(4, 2, 2, 1)$ filled with the permutation 583691724.

Given a partition λ, a *standard Young tableau of shape* λ is a permutation w such that when the letters of w fill the cells of the Young diagram of λ from left to right, top to bottom, the rows increase left to right and the columns increase bottom to top. For example, see Figure 3.

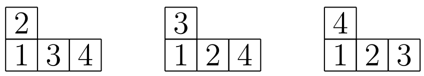

Figure 3: A diagrammatic illustrations of the standard Young tableaux of shape $(3, 1)$.

Abusing notation, we often refer to i as a cell of λ, referring to the cell of the diagram of λ that contains i.

Gessel [10] proved that summing the fundamental quasisymmetric functions indexed by the reading words of all standard Young tableaux of a given shape precisely gives the Schur function indexed by that shape. We take this as our definition.

Definition 2 ([10])**.** The *Schur function* for λ is

$$s_\lambda(X) = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{iDes}(w(T))}(X), \tag{3}$$

where $\mathrm{SYT}(\lambda)$ is the set of standard Young tableaux of shape λ.

Every character for $\mathfrak{S}_n$ has a unique decomposition as a sum of irreducible characters which coincide with Schur functions. The regular representation decomposes as

$$H_n(X) = \sum_{\lambda} f_\lambda s_\lambda(X), \tag{4}$$

where f_λ is the number of standard Young tableaux of shape λ.

Macdonald [20] introduced polynomials that are a two-parameter generalization of H_n. Haglund [12] defined new permutation statistics based on a partition shape that interpolate between the well-known major index and inversion number. Haglund, Haiman, and Loehr [15] proved that the generalized maj_μ and inv_μ statistics precisely give the Macdonald polynomials.

Recall MacMahon's *major index* statistic on permutations, given by

$$\mathrm{maj}(w) = \sum_{w_i > w_{i+1}} i.$$

Haglund generalized this statistic to depend on a partition μ.

Definition 3 ([12])**.** Given a partition μ, define the *μ-descent set of w*, denoted by $\mathrm{Des}_\mu(w)$, to be the set of cells of μ such that the w-entry is strictly greater than the w-entry immediately below in μ. Define the *μ-major index of w*, denoted by $\mathrm{maj}_\mu(w)$, by

$$\mathrm{maj}_\mu(w) = \sum_{c \in \mathrm{Des}_\mu(w)} (\mathrm{leg}(c) + 1)\,, \tag{5}$$

where $\mathrm{leg}(c)$ is the number of cells strictly above c in μ.

For example, from Figure 2, we see that the $(4,2,2,1)$-descent set of 583691724 consists of the cells containing $8, 6, 9$. Therefore the $(4,2,2,1)$-major index of 583691724 is $2+3+2=7$. Notice the (1^n)-major index is the usual major index statistic on permutations.

Inspired by Definition 3, we say that two cells of μ form a *potential μ-descent* if both are in the same column with one immediately above the other. Thus potential μ-descents depend only on the partition μ and not on the particular tableau determined by the permutation w.

Recall the *inversion number* statistic on permutations, given by

$$\mathrm{inv}(w) = \#\{(w_i, w_j) \mid i < j \text{ and } w_i > w_j\}.$$

As with major index, Haglund generalized inversions to depend on a partition μ.

Definition 4 ([12])**.** Given a partition μ, define the *μ-inversion set of w*, denoted by $\mathrm{Inv}_\mu(w)$, to be the set of pairs $(w_i > w_j)$ with $i < j$ and either w_i and w_j are in the same row, or w_j is one row lower and strictly left. Define the *μ-inversion number of w*, denoted by $\mathrm{inv}_\mu(w)$, by

$$\mathrm{inv}_\mu(w) = |\mathrm{Inv}_\mu(w)| - \sum_{c \in \mathrm{Des}_\mu(w)} \mathrm{arm}(c), \tag{6}$$

where $\mathrm{arm}(c)$ is the number of cells in the same row as and strictly right of c in μ.

For the example, 583691724 has $(4,2,2,1)$-inversion set $(8,3), (9,1), (7,2), (7,4)$, and the previously mentioned $(4,2,2,1)$-descent set of $8, 6, 9$. Thus the $(4,2,2,1)$-inversion number is $4-1-1-0=2$. Note the (n)-inversion number is the usual inversion number.

Inspired by Definition 4, we say that two cells i and j of μ form a *potential μ-inversion* if both are in the same row or if one is in the row immediately above the other with the higher cell strictly right of the lower. As with potential μ-descents, potential μ-inversions depend only on the partition μ and not on the particular tableau determined by the permutation w.

As with Schur functions, we take a theorem as definition for Macdonald polynomials.

Definition 5 ([15])**.** The *Macdonald polynomial* for μ is

$$\widetilde{H}_{\mu}(X;q,t) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}_{\mu}(w)} t^{\mathrm{maj}_{\mu}(w)} F_{\mathrm{iDes}(w)}(X). \tag{7}$$

To summarize the running example, the permutation 583691724 will contribute the term $q^2 t^7 F_{\{2,4,7\}}(X)$ to the Macdonald polynomial $\widetilde{H}_{(4,2,2,1)}(X;q,t)$.

It remains an important open problem to prove, combinatorially, that the Schur coefficients of $\widetilde{H}_{\mu}(X;q,t)$ are nonnegative and to give an explicit combinatorial formula for the Schur expansion.

MacMahon [19] proved major index and inversion number are *equidistributed*, meaning

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)}.$$

Foata [8] gave a bijective proof of this result by defining an explicit bijection φ on permutations such $\mathrm{inv}(\varphi(w)) = \mathrm{maj}(w)$, and Foata and Schützenberger [9] together proved that this bijection preserves the inverse descent set, i.e. $\mathrm{iDes}(\varphi(w)) = \mathrm{iDes}(w)$. Augmenting the equation above with inverse descent sets, we have an equality between q-graded characters with the grading given by either major index or inversion number,

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} F_{\mathrm{iDes}(w)}(X) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{\mathrm{iDes}(w)}(X).$$

Incorporating Haglund's generalized statistics into this, note that

$$\begin{array}{rclcrcl} \mathrm{maj}_{(1^n)}(w) & = & \mathrm{maj}(w) & \qquad & \mathrm{maj}_{(n)}(w) & = & 0 \\ \mathrm{inv}_{(1^n)}(w) & = & 0 & & \mathrm{inv}_{(n)}(w) & = & \mathrm{inv}(w). \end{array}$$

Therefore we can relate the Macdonald polynomials for single columns and single rows by

$$\widetilde{H}_{(1^n)}(X;q,t) = \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{\mathrm{iDes}(w)}(X) = \sum_{w \in \mathfrak{S}_n} t^{\mathrm{inv}(w)} F_{\mathrm{iDes}(w)}(X) = \widetilde{H}_{(n)}(X;t,q),$$

where we emphasize that the q and t have exchanged places on the right hand side, demonstrating a special case of the q,t-symmetry of Macdonald polynomials. In particular, Haglund's statistics combined with Foata's bijection proves that the Schur positivity of $\widetilde{H}_{(1^n)}(X;q,t)$, which is easily established, implies Schur positivity for $\widetilde{H}_{(n)}(X;q,t)$.

We use this idea, and indeed Foata's bijection itself, to resolve the Schur positivity of Macdonald polynomials for the case when $\mu_2 \leqslant 2$. Precisely, we give a characterization of a set of *super-standard* words together with a simple bijection φ on permutations that preserves the inverse descent set such that

$$\widetilde{H}_\mu(X;q,t) = \sum_\lambda \left(\sum_{u \in \mathrm{SS}(\lambda)} q^{\mathrm{inv}_\mu(\varphi(u))} t^{\mathrm{maj}_\mu(\varphi(u))} \right) s_\lambda(X), \tag{8}$$

where $\mathrm{SS}(\lambda)$ is the set of super-standard words of shape λ.

3 Dual equivalence classes

Dual equivalence involutions on permutations were studied in depth by Haiman [13], though they also appear in earlier work of Edelman and Greene [6]. We recall the basic definition here, along with a variation introduced by Assaf [2].

Definition 6 ([13])**.** Define the *elementary dual equivalence involution* d_i, $1 < i < n$, on permutations w as follows. If i lies between $i-1$ and $i+1$ in w, then $d_i(w) = w$. Otherwise, d_i interchanges i and whichever of $i \pm 1$ is further away from i.

Since elementary dual equivalence involutions exchange consecutive values that are not adjacent, they preserve the descent set of a permutation. In particular, they partition permutations of a given size and major index into equivalence classes. For example, Figure 4 shows the dual equivalence classes for permutations of 4 with major index 2.

$$\{2314 \xleftrightarrow{d_2} 1324 \xleftrightarrow{d_3} 1423\} \qquad \{2413 \underset{d_3}{\overset{d_2}{\Longleftrightarrow}} 3412\}$$

Figure 4: The dual equivalence classes of $w \in \mathfrak{S}_4$ with $\mathrm{maj}(w) = 2$.

Define the *de-standardization of* w, denoted by $\mathrm{dst}(w)$, to be the word obtained by changing $1, \ldots, i_1$ to 1, and changing $i_1+1, \ldots, i_2$ to 2, and so on, where $\mathrm{iDes}(w) = \{i_1 < i_2 < \ldots\}$. For example,

$$\mathrm{dst}(583691724) = 342341312.$$

The *weight* of a de-standardization is the composition whose ith part is the number of occurrences of the letter i. For example, the weight of $\mathrm{dst}(583691724)$ is $(2,2,3,2)$.

A permutation w is *super-standard* if for every $k = n, \ldots, 1$, the $\mathrm{dst}(w)_k \cdots \mathrm{dst}(w)_n$ has at least as many $(i-1)$s as is for every i. That is, w is super-standard if every suffix of $\mathrm{dst}(w)$ has partition weight. In this case, we say that w has weight λ whenever $\mathrm{dst}(w)$ has weight λ. For example, the following satisfies the suffix property,

$$\mathrm{dst}(719852364) = 314321121,$$

and so 719852364 is super-standard of weight $(4,2,2,1)$.

Theorem 7. *Every permutation is dual equivalent to a unique super-standard permutation. Moreover, the quasisymmetric generating function of a dual equivalence class of a super-standard permutation is equal to the Schur function indexed by its weight.*

Proof. The first statement follows from basic properties of the Robinson-Schensted insertion algorithm along with the characterization that u is dual equivalent to v if and only if u and v have the same recording tableau proved in [13]. The super-standard permutations are precisely those whose insertion tableau is filled from bottom to top, left to right with the identity. That each class is a single Schur function now follows from Definition 2, and the characterization of which Schur function from the characterization of dominant elements in [4](Theorem 4.7). □

For example, the quasisymmetric generating function of the dual equivalence classes in Figure 4 are $s_{(3,1)}(X)$ and $s_{(2,2)}(X)$, respectively, with super-standard representatives 1423 of shape $(3,1)$ and 3412 of shape $(2,2)$. In particular, Theorem 7 establishes the Schur positivity of H_n directly from the combinatorial definition and provides an explicit rule for computing the Schur expansion.

Corollary 8. *Let H_n denote the Frobenius character of the regular representation of $\mathfrak{S}_n$. Then*

$$H_n(X) = \sum_{\lambda} (\#\mathrm{SS}(\lambda))\, s_\lambda(X), \tag{9}$$

where $\mathrm{SS}(\lambda)$ *denotes the set of super-standard permutations of weight* λ.

Since the dual equivalence involution d_i will never exchange i and $i \pm 1$ when they are adjacent in a permutation, and since all other letters j compare the same with i and with $i \pm 1$, we have that $w_h > w_{h+1}$ if and only if $d_i(w)_h > d_i(w)_{h+1}$. In particular, the dual equivalence involutions preserve the major index statistic. Therefore we may regard maj as a statistic on dual equivalence classes, thus proving

$$\widetilde{H}_{(1^n)}(X;q,t) = \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{\mathrm{iDes}(w)}(X) = \sum_{\lambda} \left(\sum_{u \in \mathrm{SS}(\lambda)} t^{\mathrm{maj}(u)} \right) s_\lambda(X). \tag{10}$$

In particular, Macdonald polynomials indexed by a single column are Schur positive.

Assaf [2, 4] introduced a variation of the dual equivalence involutions that arise naturally from considering Haglund's permutation statistics for Macdonald polynomials.

Definition 9 ([2, 4])**.** The *elementary twisted dual equivalence involution* $\widetilde{d}_i$, $1 < i < n$, on permutations are defined as follows. If i lies between $i-1$ and $i+1$, then $\widetilde{d}_i(w) = w$. Otherwise, $\widetilde{d}_i$ cyclically rotates $i-1, i, i+1$ so that i lies on the other side of $i \pm 1$.

Notice that elementary twisted dual equivalence involutions preserve the number of inversions, though not the set of inversion pairs. In particular, they partition permutations of a given size and inversion number into equivalence classes. For example, Figure 5 shows the unique twisted dual equivalence class for permutations of 4 with 2 inversions.

$$\{2314 \overset{\widetilde{d}_2}{\longleftrightarrow} 3124 \overset{\widetilde{d}_3}{\longleftrightarrow} 2143 \overset{\widetilde{d}_2}{\longleftrightarrow} 1342 \overset{\widetilde{d}_3}{\longleftrightarrow} 1423\}$$

Figure 5: The twisted dual equivalence class of $w \in \mathfrak{S}_4$ with $\mathrm{inv}(w) = 2$.

For permutations u, v of length n, u is twisted dual equivalent to v if and only if $\mathrm{inv}(u) = \mathrm{inv}(v)$ and $u_1 > u_n$ if and only if $v_1 > v_n$ [3]. Using this observation and properties of Foata's bijection [8] described in the following section, we have the following.

Theorem 10 ([3]). *The quasisymmetric generating function of a twisted dual equivalence class of permutations is symmetric and Schur positive.*

For example, the quasisymmetric generating function of the twisted dual equivalence class in Figure 5 is $s_{(3,1)}(X) + s_{(2,2)}(X)$, which is precisely the sum of the generating functions of the dual equivalence classes in Figure 4. While the larger class in Figure 5 is not the union of the elements of the smaller classes in Figure 4, we will show that it is after applying a suitable bijection.

Haglund's statistics interpolate between major index and inversion number. Analogously, the following involutions interpolate between dual equivalence and twisted dual equivalence.

Definition 11 ([2, 4]). Define involutions D_i^μ, $1 < i < n$, on permutations by

$$D_i^\mu(w) = \begin{cases} \widetilde{d}_i(w) & \text{if both of } i-1 \text{ and } i+1 \text{ are potential } \mu\text{-descents} \\ & \text{or potential } \mu\text{-inversions with } i, \\ d_i(w) & \text{otherwise.} \end{cases} \tag{11}$$

For example, both pairs of tableaux in Figure 6 are related by $D_7^{(4,2,2,1)}$. For the left pair $D_7^{(4,2,2,1)}$ acts by d_7, and for the right pair $D_7^{(4,2,2,1)}$ acts by $\widetilde{d}_7$.

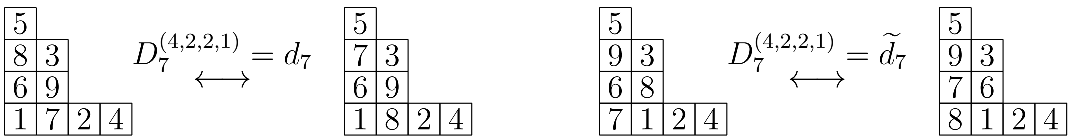

Figure 6: Two $(4, 2, 2, 1)$-dual equivalence moves for $\{6, 7, 8\}$.

Haglund's formula relates Macdonald polynomials to LLT polynomials [16] (see [15] or [2] for details of the equivalence), and so it follows from [4](Proposition 5.2) that these involutions preserve Haglund's statistics.

Proposition 12. *For μ a partition of n, w a permutation, and $1 < i < n$ an integer,*

$$\mathrm{maj}_\mu\left(D_i^\mu(w)\right) = \mathrm{maj}_\mu(w), \tag{12}$$

$$\mathrm{inv}_\mu\left(D_i^\mu(w)\right) = \mathrm{inv}_\mu(w). \tag{13}$$

In particular, these involutions partition permutations of a given inv_μ and maj_μ statistic into equivalence classes. The main motivation for these involutions is the following Conjecture, which is a reformulation of [4](Conjecture 5.6) for Macdonald polynomials.

Conjecture 13 ([4])**.** For μ a partition, the quasisymmetric generating function of each generalized dual equivalence class under D_i^μ is symmetric and Schur positive.

Since generalized dual equivalence classes under D_i^μ have constant inv_μ and maj_μ statistics, Conjecture 13 is enough to establish Macdonald Positivity.

Definition 14. Given a partition μ of n, two permutations u, v of n are *μ-equivalent* if $v = D_{i_k}^\mu \cdots D_{i_1}^\mu(u)$ for some integer sequence $1 < i_1, \ldots, i_k < n$. The set of all permutations that are μ-equivalent to a given permutation is the *μ-equivalence class* of the given permutation.

In [4], Conjecture 13 is shown to hold for μ a two column partition and for μ a single row. In this paper, we merge these two cases to establish Conjecture 13 for partitions μ with $\mu_2 \leqslant 2$. The proof utilizes explicit bijections on words that merge equivalence classes as the shape μ is transformed from a single column, where we know Schur positivity holds.

4 Foata's bijection and hooks

Foata [8] constructed a bijection on words with the property that the major index of a word equals the inversion number of its image, thereby providing a bijective proof of the equi-distribution of these two statistics. Moreover, Foata and Schützenberger [9] proved that his bijection preserves the inverse descent set, and so this proves that the quasisymmetric generating function of permutations with major index k is equal to the quasisymmetric generating function of permutations with k inversions, stated for all k as

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} F_{\mathrm{iDes}(w)}(X) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{\mathrm{iDes}(w)}(X).$$

Foata's bijection makes recursive use of a family of bijections γ_x indexed by a letter x. Given a word w and a letter x not in w, define a partitioning $\Gamma_x(w)$ of w by: if $w_1 < x$, then break before each index i such that $w_i < x$; otherwise break before each index i such that $w_i > x$. For example,

$$\Gamma_5(83691724) = \mid 83 \mid 6 \mid 91 \mid 724.$$

The bijection γ_x is defined by cycling the first letter of each block of Γ_x to the end of the block. Continuing with the example,

$$\gamma_5(83691724) = 38619247.$$

Proposition 15. *For u, v words and x a letter, we have* $\mathrm{iDes}(uxv) = \mathrm{iDes}(ux\gamma_x(v))$.

Proof. We claim that the relative order of consecutive letters is the same in uxv as in $ux\gamma_x(v)$. If they are not in the same block of $\Gamma_x(v)$, either by being in different blocks or by at least one being in ux, then since the relative order of ux and each of the blocks is preserved, their relative order remains the same in $ux\gamma_x(v)$. If they are in the same block of $\Gamma_x(v)$, then they are both larger or both smaller than x, and so their relative order remains unchanged by γ_x. ☐

Define a family of bijections ϕ_k, for $k \in [n]$, on permutations by

$$\phi_k(w) = w_1 \cdots w_k \gamma_{w_k}(w_{k+1} \cdots w_n). \tag{14}$$

For example, $\phi_4(583691724) = 5836\gamma_6(91724) = 583619247$. We use the map ϕ_k to relate $(n-k, 1^k)$-dual equivalence classes with $(n-k+1, 1^{k-1})$-dual equivalence classes as illustrated in Figure 7.

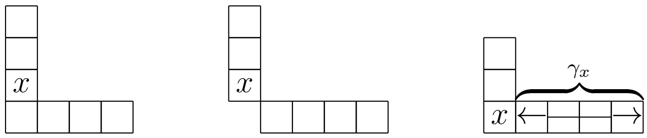

Figure 7: Sliding a hook to lengthen the arm.

To begin to relate dual equivalence classes with μ-equivalence classes, we consider first the cases when the bijection ϕ_k commutes with the generalized dual equivalence involutions.

Lemma 16. *Let $\mu = (n-k, 1^k)$ for some $k > 0$, and let $\nu = (n-k+1, 1^{k-1})$. If $D_i^\mu(u) = u$, $u_k \notin \{i-1, i, i+1\}$, or $u_j \in \{i-1, i, i+1\}$ for some $j < k$, then $\phi_k(D_i^\mu(u)) = D_i^\nu(\phi_k(u))$.*

Proof. By definition, $D_i^\lambda(w) = w$ if and only if both or neither of $i-1, i$ are in iDes(w). By Proposition 15, ϕ_k preserves iDes, and so $D_i^\mu(u) = u$ if and only if $D_i^\nu(\phi_k(u)) = \phi_k(u)$. Thus we may assume neither is the case.

If $u_k \notin \{i-1, i, i+1\}$, then γ_{u_k} does not differentiate between $i-1, i, i+1$, and so their relative order is the same in u and in $\phi_k(u)$. Moreover, since the number of $i-1, i, i+1$ in the leg is the same for u and in $\phi_k(u)$, D_i^μ and D_i^ν act by the same involution, so $\phi_k(D_i^\mu(u)) = D_i^\nu(\phi_k(u))$.

Finally, if $u_j \in \{i-1, i, i+1\}$ for some $j < k$, then $D_i^\mu(u) = d_i(u)$, and since ϕ_k leaves the leg unchanged, we also have $D_i^\mu(\phi_k(u)) = d_i(\phi_k(u))$. Since d_i exchanges consecutive values, the middle letter of $i-1, i, i+1$ occurring in u compares the same with both values being exchanged. Therefore Γ_{u_k} and $\Gamma_{d_i(u)_k}$ partition the arm in the same way, and so $\phi_k(d_i(u)) = d_i(\phi_k(u))$ as desired. ☐

Generalized dual equivalence involutions do not always commute with ϕ_k. For example, take $\mu = (3, 1, 1)$ and consider $u = 42513$. Then $\phi_2(u) = 42\gamma_2(513) = 42153$ and $\phi_2(D_2^\mu(u)) = 43\gamma_3(512) = 43125 \neq D_2^{(4,1)}(42153)$. They are, however, $(4, 1)$-dual equivalent, as illustrated in Figure 8.

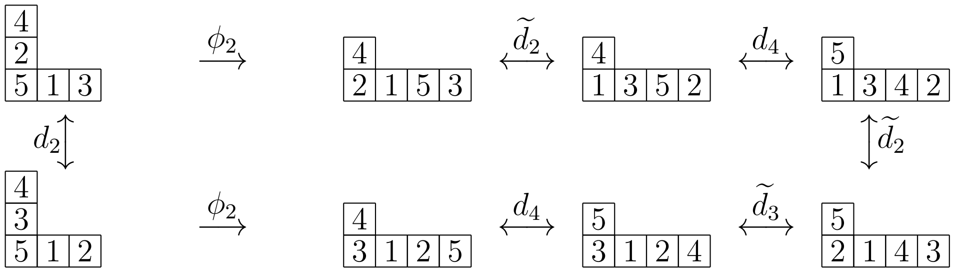

Figure 8: An example showing when two $(3,1,1)$-equivalent tableaux (42513 and 43512) map under ϕ_2 to two $(4,1)$-equivalent permutations (42153 and 43125), but $\phi_2(D_2^{(3,1,1)}(u)) \neq D_2^{(4,1)}(\phi_2(u))$.

Lemma 17. *Let u be a permutation with $u_{k+1} = 3$, $u_j = 1$, and $u_i = 2$ for some $1 < k+1 < j < i$, and let $x = u_{i+1}$. Let v be the permutation with $v_{k+1} = 1$, $v_j = 3$, $v_i = x$, $v_{i+1} = 2$, and $v_h = u_h$ for all other indices h. Then u and v are $(n-k, 1^k)$-equivalent.*

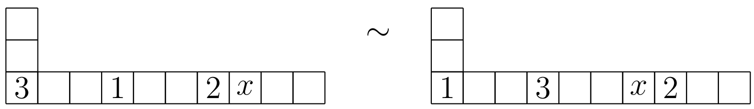

Figure 9: Illustration of the situation described in Lemma 17.

Proof. If the result holds for $x = n$, then it continues to hold whether other letters with higher value are interspersed or not. Therefore assume $x = n$. If $x = 4$, then $\widetilde{d}_3(3124) = 2143 = \widetilde{d}_2(1342)$, so 3124 is (4)-equivalent to 1342. Therefore we may proceed by induction on x assuming $x > 4$.

If only $1, 2, 3, x$ occur in the arm, then we claim that we may apply d_{x-1} to exchange x with $x-1$. If so, then we may use induction since $x-1$ is now in position $i+1$ to move $1, 2, 3, x-1$ to the desired positions, and then apply d_{x-1} again to restore x to the arm, now in position i as desired. To establish the claim, if $x = 5$, the smallest possibly value, then 4 must be in the leg above 3 and so $D_4^{(n-k,1^k)}(u) = d_4(u)$ exchanges 4 and 5 as desired. If $x > 5$, then we may strict our attention to the letters $3, 4, \ldots, x-1$, of which there are at least three, and apply the dual equivalences $d_4, \ldots, d_{x-2}$, at least one of which must act nontrivially, until $x-1$ sits above $x-2$ in the leg. This follows from the basic fact that the relative positions of any two consecutive letters is constant on a dual equivalence class if and only if the class is a single term. At this point, $D_{x-1}^{(n-k,1^k)}(u) = d_{x-1}(u)$ which acts by exchanging $x-1$ with x as claimed. Therefore we may now assume there is some entry $3 < a < x$ also in the arm.

If such an a has position greater than $i+1$ (i.e. lies right of x), then consider the left-most such letter right of x. By induction and ignoring letters larger than a (in particular, ignoring x), the pattern $312xa$ may be exchanged for $13ax2$. Ignoring the 1 which lies precisely in the corner, using that the twisted Knuth move (defined by $w \sim (d_i(w^{-1}))^{-1}$) taking $ax1$ is generalized dual equivalent to $x1a$, we obtain $13x2a$, as desired. Similarly, if some a lies between positions j and i, then by induction and ignoring letters larger than a (in particular, ignoring x), the pattern $13ax2$ may be exchanged for $312xa$. Applying $\widetilde{d}_2$ gives the pattern $231xa$. It is an easy exercise to show that the twisted Knuth move taking $1xa$ is generalized dual equivalent to $a1x$, and so applying $\widetilde{d}_2$ again yields the pattern $31a2x$, proving the result. Therefore the only remaining case is for all entries in the arm other than $1, 2, 3, x$ to lie between positions k and j. If $x-1$ lies in the leg, or can be moved to the leg by some sequence of dual equivalences, then the result will again follow by induction. Otherwise, we are forced to have both $x-2, x-1$ in the arm. If $x = 5$, then the resolution is

$$34125 \stackrel{\widetilde{d}_2}{\longleftrightarrow} 24315 \stackrel{\widetilde{d}_4}{\longleftrightarrow} 23514 \stackrel{\widetilde{d}_2}{\longleftrightarrow} 31524 \stackrel{\text{base case}}{\longleftrightarrow} 13542 \stackrel{\widetilde{d}_4}{\longleftrightarrow} 14352.$$

If $x > 5$, then $x-2 > 3$ and by applying other dual equivalence moves as needed (analogous to the leg case), we may assume $x-1$ is left of $x-2$. Then the resolution is

$$3(x–1)(x–2)12x \stackrel{\widetilde{d}_{x-1}}{\longleftrightarrow} 3(x–2)x12(x–1) \stackrel{\substack{\text{by inductive}\\ \text{hypothesis}}}{\longleftrightarrow} 1(x–2)x3(x–1)2 \stackrel{\widetilde{d}_{x-1}}{\longleftrightarrow} 1(x–1)(x–2)3x2.$$

□

For example, we see from Figure 8 that 43125 and 41352 are $(4, 1)$-equivalent.

Theorem 18. *Let $\mu = (m, 1^k)$ be a hook partition for some $m, k > 0$, and let $\nu = (m + 1, 1^{k-1})$. For permutations u, v, if u and v are μ-equivalent, then $\phi_k(u)$ and $\phi_k(v)$ are ν-equivalent.*

Proof. It is enough to prove the result for $v = D^{\mu}_i(u)$. By Lemma 16, we may assume that $D^{\mu}_i(u) \neq u$, $u_k \in \{i-1, i, i+1\}$, and the remaining two of $i-1, i, i+1$ are among $u_{k+1} \cdots u_{k+m}$. In particular, this implies $D^{\mu}_i(u) = d_i(u)$ and $D^{\nu}_i(\phi_k(u)) = \widetilde{d}_i(\phi_k(u))$. By symmetry between u and v, we may assume $u_k = i$. We treat in detail the case where $i-1$ is left of $i+1$, noting that the alternative case is completely analogous.

We compare $\Gamma_i(u_{k+1} \cdots u_n)$ with $\Gamma_{i+1}(v_{k+1} \cdots v_n)$. Since $v = d_i(u)$, we must have $u_{k+1} = v_{k+1}$ and neither is i or $i+1$. Suppose first that $u_{k+1} < i$. If $i-1$ and $i+1$ occur within the same Γ_i-block of u, then we have

$$\begin{aligned} \Gamma_i(u_{k+1} \cdots u_n) &= (\cdots \mid (i-1)A(i+1)B \mid \cdots) \stackrel{\gamma_i}{\longmapsto} i \cdots A(i+1)B(i-1) \cdots \\ \Gamma_{i+1}(v_{k+1} \cdots v_n) &= (\cdots \mid (i-1)A \mid (\ i\ )B \mid \cdots) \stackrel{\gamma_{i+1}}{\longmapsto} i+1 \cdots A(i-1)B(\ i\ ) \cdots \end{aligned}$$

where A, B are words containing letters larger than $i+1$. Therefore $D^{\nu}_i(\phi_k(u)) = \widetilde{d}_i(\phi_k(u)) = \phi_k(v) = \phi_k(d_i(u)) = \phi_k(D^{\mu}_i(u))$. If $i-1$ and $i+1$ do not occur within the same Γ_i-block of u, then computing the Γ-partitioning and applying γ, we have

$$\begin{aligned} \Gamma_i &: (\cdots \mid (i-1)A \mid \cdots \mid xB(i+1)C \mid \cdots) \stackrel{\gamma_i}{\longmapsto} i \cdots A(i-1) \cdots B(i+1)Cx \cdots \\ \Gamma_{i+1} &: (\cdots \mid (i-1)A \mid \cdots \mid xB \mid (\ i\ )C \mid \cdots) \stackrel{\gamma_{i+1}}{\longmapsto} i+1 \cdots A(i-1) \cdots BxC(\ i\ ) \cdots \end{aligned}$$

where A, B, C are words containing letters larger than $i+1$, and $x < i-1$ is a letter. If we consider $\phi_k(v)$ and $\widetilde{d}_i(\phi_k(u))$ where we delete all letters larger than $i+1$ and change smaller letters a to $(i+1)-a+1$, we are precisely in the situation of Lemma 17. Reversing this change and replacing larger letters does not affect the equivalence, and so $\phi_k(v)$ and $\widetilde{d}_i(\phi_k(u))$ are ν-dual equivalent. Since $i-1, i, i+1$ all lie in the arm of ν, we have $\phi_k(u)$ and $\widetilde{d}_i(\phi_k(u))$ are ν-dual equivalent as well. Therefore, by transitivity, $\phi_k(u)$ and $\phi_k(v)$ are ν-dual equivalent.

For the case $u_{k+1} > i+1$, we apply the same analysis. If $i-1$ and i occur within the same Γ_{i+1}-block of v, then we see that $D_i^{\nu}(\phi_k(u)) = \widetilde{d}_i(\phi_k(u)) = \phi_k(v) = \phi_k(d_i(u)) = \phi_k(D_i^{\mu}(u))$, giving the desired ν-dual equivalence. If $i-1$ and i do not occur within the same Γ_{i+1}-block of v, then deleting letters smaller than $i-1$ and changing larger letters a to $a-(i-1)+1$ puts $\phi_k(v)$ and $\widetilde{d}_i(\phi_k(u))$ precisely in the situation of Lemma 17, and so $\phi_k(v)$ and $\widetilde{d}_i(\phi_k(u))$ are ν-dual equivalent. Again, since $i-1, i, i+1$ all lie in the arm of ν, by transitivity $\phi_k(u)$ and $\phi_k(v)$ are ν-dual equivalent. □

Since, by Proposition 15, ϕ_k preserves the inverse descent set, Theorem 18 establishes Conjecture 13 for hooks and gives the following explicit formula.

Corollary 19. *For $\mu = (n-k, 1^k)$ a hook partition, set $\varphi_\mu = \phi_{k+1} \cdots \phi_{n-1}$. Then*

$$\widetilde{H}_\mu(X; q, t) = \sum_{\lambda} \left(\sum_{u \in \mathrm{SS}(\lambda)} q^{\mathrm{inv}_\mu(\varphi_\mu(u))} t^{\mathrm{maj}_\mu(\varphi_\mu(u))} \right) s_\lambda(X). \tag{15}$$

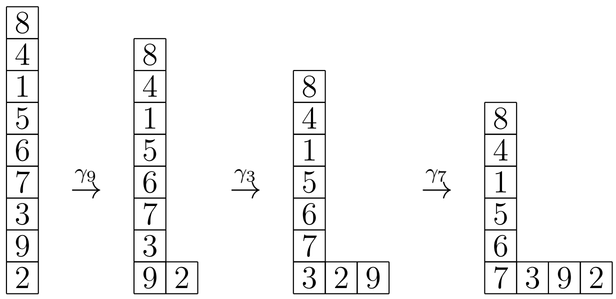

Figure 10: Folding (1^9) to $(4, 1^5)$.

See Figure 10 for details of the map $\varphi_{(4,1^5)}$ on the permutation 841567392. Notice that the inverse descent set for all four fillings is $\{2, 3, 7\}$. From left to right, the Macdonald weights are t^{17}, qt^9, qt^9, q^4t^3. We emphasize that in Corollary 19, the weights change with the bijection, but the set of permutations that determines the Schur expansion of the Macdonald polynomial is the same for every partition.

5 Folding the legs

Refining Foata's bijection, Assaf [3] constructed a family of bijections on words that preserve an interpolating statistic between major index and inversion number. We focus on the first in this family of bijections, which makes use of a bijection β_x indexed by a letter x. The idea of β_x is to swap adjacent pairs of letters, percolating through the word based on a straddling condition.

Given a word w and a letter x not in w, let B_x be the set of indices defined recursively as follows: if x has value between w_1, w_2, then $1 \in B_x$; if $i \in B_x$, then if exactly one of w_i, w_{i+1} has value between w_{i+2}, w_{i+3}, then $i + 2 \in B_x$. For example,

$$B_5(83691724) = \{1, 3, 5\}.$$

The bijection β_x is defined by swapping w_i and w_{i+1} for every $i \in B_x$. Continuing with the example,

$$\beta_5(83691724) = 38967124.$$

Proposition 20. *For u, v, y words and x a letter, we have* $\mathrm{iDes}(uxvy) = \mathrm{iDes}(ux\beta_x(v)y)$.

Proof. Since β_x exchanges pairs of adjacent letters that are necessarily not consecutive, the relative order of any pair of consecutive letters remains unchanged. □

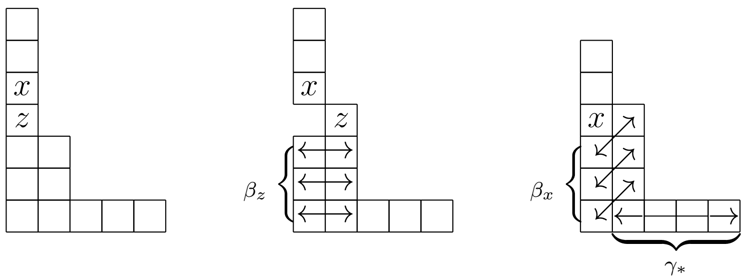

Figure 11: An illustration of sliding the leg and dropping the foot, where β_* cycles pairs of entries and γ_* cycles a row of entries, as indicated by the arrowed ranges.

Figure 11 illustrates the manner in which we will use β_x along with γ_x to relate generalized dual equivalence classes for partitions with second part at most 2. Define a family of bijections $\sigma_{(k,m)}$ by

$$\sigma_{(k,m)}(w) = w_1 \cdots w_k \beta_{w_k}(w_{k+1} \cdots w_{k+m}) w_{k+m+1} \cdots w_n. \tag{16}$$

If we extend Definition 5 to arbitrary diagrams in the first quadrant of the plane, then we may consider the quasisymmetric generating function associated with the middle diagram in Figure 11 and its corresponding generalized dual equivalence classes. Extending [3](Theorem 5.3), we have the following commutativity.

Below we consider partitions of the form $(n-2b-a, 2^b, 1^a)$, where we assume throughout $a \geqslant 2$ and either $n - 2b - a > 2$ or $n - 2b - a = 0$ and $b > 0$ (in which case we have a partition with largest part 2 and we remove the leading 0).

Lemma 21. *Let $\mu = (n-2b-a, 2^b, 1^a)$, and let δ be the diagram obtained by moving the ath cell from the top to the second column. Then $\sigma_{(a,2b+2)}(D_i^\mu(u)) = D_i^\delta(\sigma_{(a,2b+2)}(u))$.*

Proof. For any diagram λ and any permutation w, $D_i^\lambda(w) = w$ if and only if both or neither of $i-1, i$ are inverse descents of w. By Proposition 20, $\sigma_{(k,m)}$ preserves the inverse descent set, and so $D_i^\mu(u) = u$ if and only if $D_i^\delta(\sigma_{(k,m)}(u)) = \sigma_{(k,m)}(u)$. Therefore assume that D_i^μ acts non-trivially on u.

Let $z = u_a$, and consider the action of β_z on $u_{a+1}\cdots u_{a+2b+2}$. From the definition, β_z cannot exchange i and $i \pm 1$ since no letter has value between these. Similarly, β_z exchanges $i-1$ and $i+1$ only if they are the leftmost pair in the same row and i is in the row above. When this is not the case, both D_i^μ and D_i^δ act by the same operator, either d_i or $\widetilde{d}_i$, and the actions are easily seen to commute with $\sigma_{(a,2b+2)}$ since letters $j > i+1$ or $h < i-1$ compare the same with each of $i-1, i, i+1$. When this is the case, one of D_i^μ and D_i^δ acts by d_i and the other by $\widetilde{d}_i$, since in one case the i will be above the left letter and so the action is d_i and in the other it will be above the right forcing the action by $\widetilde{d}_i$. Since the difference between these two actions is precisely exchanging $i-1$ and $i+1$, we again have $\sigma_{(a,2b+2)}(D_i^\mu(u)) = D_i^\delta(\sigma_{(a,2b+2)}(u))$. □

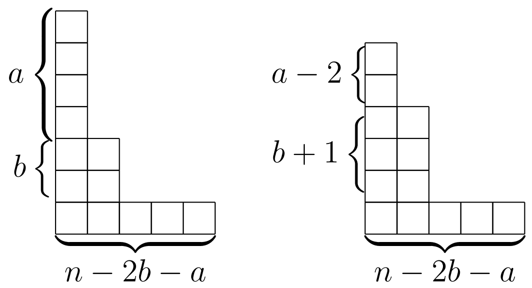

Figure 12: An illustration of $\mu = (n-2b-a, 2^b, 1^a)$ and $\nu = (n-2b-a, 2^{b+1}, 1^{a-2})$.

To compare generalized dual equivalence classes for $(n-2b-a, 2^b, 1^a)$ with those for $(n-2b-a, 2^{b+1}, 1^{a-2})$ (see Figure 12 for an illustration), we use the composite map

$$\phi_{(a,b)}(w) = \begin{cases} \phi_{a+2b+1}\sigma_{(a-1,2b+2)}\sigma_{(a,2b+2)}(w) & \text{if } \sigma_{(a-1,2b+2)} \text{ changes } u_{a+2b+1} \text{ and some } u_i, \\ & \text{for } i \geqslant a+2b+2, \text{ has value between} \\ & u_{a+2b} \text{ and } u_{a+2b+1}, \text{ where } u = \sigma_{(a,2b+2)}(w), \\ \sigma_{(a-1,2b+2)}\sigma_{(a,2b+2)}(w) & \text{otherwise.} \end{cases} \tag{17}$$

The three components of this map are illustrated in Figure 11.

Theorem 22. *Let $\mu = (n-2b-a, 2^b, 1^a)$, and let $\nu = (n-2b-a, 2^{b+1}, 1^{a-2})$. For permutations u, v, if u and v are μ-equivalent, then $\phi_{(a,b)}(u)$ and $\phi_{(a,b)}(v)$ are ν-equivalent.*

Proof. It is enough to prove the result for $v = D_i^\mu(u)$, and we consider two cases.

If $\sigma_{(a-1,2b+2)}$ does not alter $\sigma_{(a,2b+2)}(u)_{a+2b+1}$, then $\phi_{(a,b)}(u) = \sigma_{(a-1,2b+2)}\sigma_{(a,2b+2)}(u)$ acts as in the lower case of (17). By Lemma 21, we have

$$\sigma_{(a-1,2b+2)}\sigma_{(a,2b+2)}(v) = \sigma_{(a-1,2b+2)}(D_i^{\delta}(\sigma_{(a,2b+2)}(u))) = D_i^{\nu}(\sigma_{(a-1,2b+2)}\sigma_{(a,2b+2)}(u)),$$

where the second equality holds precisely because $\sigma_{(a,2b+2)}(u)_{a+2b+1}$ is unchanged, and so there is no change in the entries appearing in the bottom row nor in the rightmost entry in the second row from the bottom. Therefore $\phi_{(a,b)}(u)$ and $\phi_{(a,b)}(v)$ differ by an elementary ν-equivalence.

On the other hand, if $\sigma_{(a-1,2b+2)}$ does alter $\sigma_{(a,2b+2)}(u)_{a+2b+1}$, then $\phi_{(a,b)}$ acts as in the upper case of (17). By the same argument as in the proof of Lemma 16, if $\sigma_{(a,2b+2)}(u)_{a+2b+1}$ is not one of $i-1, i, i+1$ or if one of $i-1, i, i+1$ is not in the bottom row, then again we will have $\phi_{(a,b)}(D_i^{\mu}(u)) = D_i^{\nu}(\phi_{(a,b)}(u))$, and so again $\phi_{(a,b)}(u)$ and $\phi_{(a,b)}(v)$ differ by an elementary ν-equivalence. Finally, the analysis in the proof of Theorem 18 resolves the case when $\sigma_{(a,2b+2)}(u)_{a+2b+1}$ is one of $i-1, i, i+1$ and the others lie in the bottom row. In this final case, $\phi_{(a,b)}(u)$ and $\phi_{(a,b)}(v)$ are ν-equivalent though not necessarily by a single elementary ν-equivalence. □

By Propositions 15 and 20, ϕ_k and $\phi_{(a,b)}$ preserve the inverse descent set. Therefore Theorems 18 and 22 together establish Conjecture 13 for partitions with second part at most 2. Moreover, we have the following explicit formula for the Macdonald polynomial.

Corollary 23. *For $\mu = (n-2b-a, 2^b, 1^a)$, set $\varphi_\mu = \phi_{(a+2,b-1)} \cdots \phi_{(a+2b,0)}\phi_{a+2b+1}\cdots\phi_{n-1}$. Then*

$$\widetilde{H}_\mu(X;q,t) = \sum_\lambda \left(\sum_{u \in \mathrm{SS}(\lambda)} q^{\mathrm{inv}_\mu(\varphi_\mu(u))} t^{\mathrm{maj}_\mu(\varphi_\mu(u))} \right) s_\lambda(X). \tag{18}$$

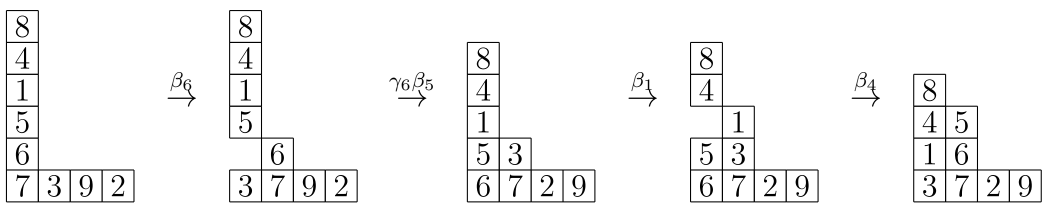

Figure 13: Folding $(4,1^5)$ to $(4,2,2,1)$.

Continuing the example in Figure 10, see Figure 13 for details of the map $\varphi_{(4,2,2,1)}$ on the permutation 841567392. Again, the inverse descent set remains $\{2,3,7\}$ for each filling. The Macdonald weights for the left, middle and right partition shapes are q^4t^3, q^3t^3, and q^2t^5, respectively. Once again, the weights change with the bijection, but the set of permutations that determines the Macdonald polynomial is the same.

6 Toward the general case

The inspiration for this paper comes from the generalized dual equivalence structures imposed on permutations for a given partition presented in [2, 4]. The maps presented herein arise from following through the explicit transformations on dual equivalence graphs presented in [1] and simplifying the result for the specific case of the graph on permutations with edges given by D_i^{μ}.

The first example for which the model presented in this paper does not apply is the partition $(3,3)$. Figure 14 illustrates how one might generalize the techniques of this paper to relate the dual equivalence classes for $(3,3)$ with those for $(3,2,1)$, the latter of which is understood by Corollary 23.

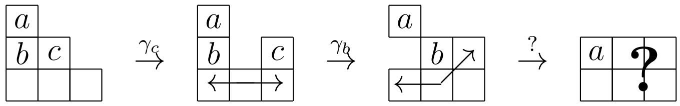

Figure 14: An illustration of how one might try folding $(3,2,1)$ to $(3,3)$.

Moving from the leftmost diagram to the second and then to the third in Figure 14, one can define explicit maps (essentially γ_c then γ_b restricted to the indicated cells) that preserve the inverse descent set and merge generalized dual equivalence classes. However, going from the third to the fourth diagram is problematic. In fact, both of the two middle diagrams have a dual equivalence class with generating polynomial $s_{(4,1,1)}+2s_{(3,2,1)}+s_{(2,2,2)}$, which implies these classes are strictly larger than any dual equivalence class for the partition $(3,3)$. Nevertheless, classes for $(3,3)$ can be realized as unions of classes for $(3,2,1)$. Therefore the lesson here is that the incremental steps of the bijections cannot be too small. Thus one can hope to resolve the following strengthening of Conjecture 13, which has been tested on computer for partitions up to size 11.

Conjecture 24. Let $\mu = (\mu_1, \ldots, \mu_k, 1^m)$ and let $\nu = (\mu_1, \ldots, \mu_{k-1}, \mu_k + 1, 1^{m-1})$, where $m > 0$ and $\mu_{k-1} > \mu_k \geqslant 1$. Then there exists a bijection φ on permutations that preserves the inverse descent set such that if u and v are μ-equivalent, then $\varphi(u)$ and $\varphi(v)$ are ν-equivalent. In particular, for any partition μ, we have

$$\widetilde{H}_{\mu}(X;q,t) = \sum_{\lambda} \left(\sum_{u \in \mathrm{SS}(\lambda)} q^{\mathrm{inv}_{\mu}(\varphi_{\mu}(u))} t^{\mathrm{maj}_{\mu}(\varphi_{\mu}(u))} \right) s_{\lambda}(X), \tag{19}$$

where φ_{μ} is the (unique) sequence of bijections taking (1^n) to μ.

Acknowledgements

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