A note on non-4-list colorable planar graphs

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Abstract

The Four Color Theorem states that every planar graph is properly 4-colorable. Moreover, it is well known that there are planar graphs that are non-4-list colorable. In this paper we investigate a problem combining proper colorings and list colorings. We ask whether the vertex set of every planar graph can be partitioned into two subsets where one subset induces a bipartite graph and the other subset induces a 2-list colorable graph. We answer this question in the negative strengthening the result on non-4-list colorable planar graphs.

Mathematics Subject Classifications: 05C10, 05C15

Let $G = (V, E)$ be a simple graph and for every vertex $v \in V$ let $L(v)$ be a set (list) of available colors. A *k-assignment* is a list assignment with $|L(v)| = k$ for all $v \in V(G)$. A graph G is called *L-colorable* if there is a proper coloring c of the vertices with $c(v) \in L(v)$ for all $v \in V(G)$ and $c(v) \neq c(w)$ for all edges $vw \in E(G)$. If G is L-colorable for all possible k-assignments then G is called *k-list colorable.*

In this note we consider simple planar graphs. Since 1993 it is known by Thomassen [6] and Voigt [7] that every planar graph is 5-list colorable but there are planar graphs that are non-4-list colorable.

Recently, Choi and Kwon [2] introduced the concept of a *t-common k-assignment* which is a k-assignment satisfying $|\bigcap_{v\in V(G)} L(v)| \geqslant t$. Using the Four Color Theorem [1, 5], it is easy to see that every planar graph is L-colorable for every 3-common 4-assignment L. Moreover, Choi and Kwon [2] constructed a planar graph G with a 1-common 4-assignment L such that G is not L-colorable and they explicitly asked the following problem.

Problem 1. Is every planar graph L-colorable for every 2-common 4-assignment L?

Since every proper coloring of the vertices gives a partition of the vertex set we may look for the problem from another point of view.

Problem 2. Let G be a planar graph. Is it possible to partition the vertex set of G into two sets in such a way that one partition set induces a bipartite graph and the other one induces a 2-list colorable graph?

If such a partition would always exist for planar graphs, then it would strengthen the Four Color Theorem. Moreover, we have the following relationship to Problem 1.

Claim 3. *If the vertex set V of a planar graph G can be partitioned into V_1 and V_2 such that V_1 induces a bipartite graph and V_2 induces a 2-list colorable graph then G is L-colorable for every 2-common 4-assignment L.*

Proof. Let G be a planar graph and L be a 2-common 4-assignment for the vertices of G with $\{\alpha, \beta\} \subseteq L(v)$ for all $v \in V(G)$. Properly color the subgraph induced by V_1 with α and β and set $L'(v) = L(v) \setminus \{\alpha, \beta\}$ for all $v \in V_2$. Since the subgraph induced by V_2 is 2-list colorable it can be colored from the remaining lists $L'(v)$. $\square$

Since every acyclic graph is 2-list colorable we may put a stronger question for the partition of G.

Problem 4. Let G be a planar graph. Is it possible to partition the vertex set V into V_1 and V_2 such that the subgraph induced by V_1 is a bipartite graph and the subgraph induced by V_2 is a forest?

Unfortunately, this is not possible for every planar graph as shown by Wegner in 1973 [8].

Theorem 5. *There is a planar graph G such that in every proper 4-coloring of G the vertices of every two color classes induce a subgraph containing a cycle.*

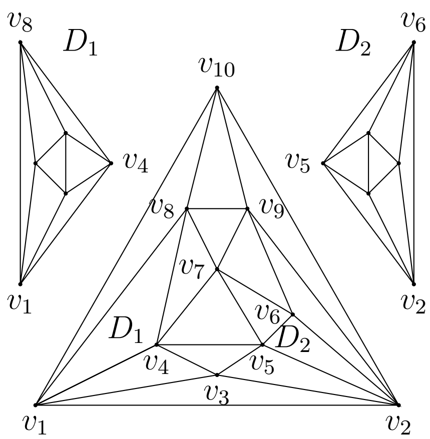

Figure 1: Subgraph G_1 of G

The construction of Wegner does not give an answer to Problem 1 but based on his construction, we were able to find our construction.

Theorem 6. *There is a planar graph G and a 2-common 4-assignment L such that G is not L-colorable.*

Proof. We will construct a planar graph G and a 2-common 4-assignment L in two steps. In the first step we consider the subgraph G_1 of G, which is shown in Figure 1. The structures inside the triangles D_1 and D_2 are depicted separately outside.

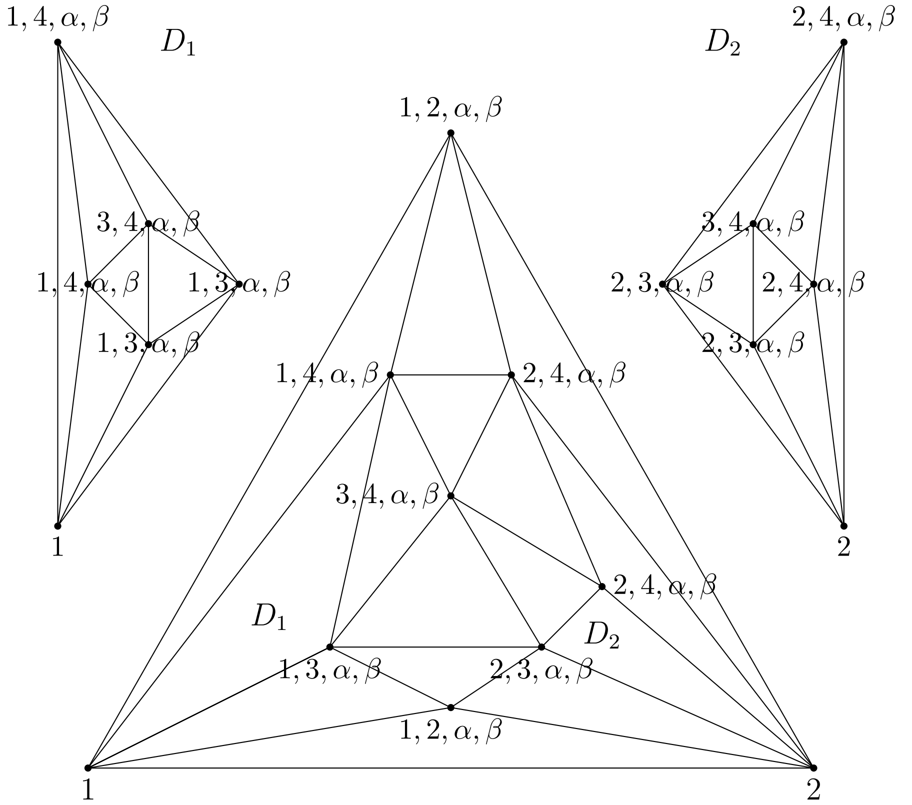

Figure 2: Subgraph G_1 with a 2-common 4-list assignment and two precolored vertices

Let v_1 be precolored by 1 and v_2 be precolored by 2, and consider the list assignment for the other vertices of G_1 given in Figure 2. Assume that there is a proper coloring c that assigns every vertex v a color $c(v) \in L(v)$ such that adjacent vertices get different colors.

At first, let v_{10} be colored by α. Clearly, one of the vertices v_4 and v_5 must be colored by 3 since otherwise v_3 would not be colorable.

- Case 1: $c(v_4) = 3$

  Since $c(v_5) \in \{\alpha, \beta\}$ and $\{c(v_6), c(v_7)\} \subset \{4, \alpha, \beta\}$ for the vertices of the triangle $v_5v_6v_7$ it follows that $c(v_6) = 4$ or $c(v_7) = 4$, which implies $c(v_9) = \beta$ and then $c(v_8) = 4$. Hence, the triangle completely in the interior of D_1 must be colored with colors α and β, a contradiction.

- Case 2: $c(v_5) = 3$

  Clearly $\{c(v_8), c(v_9)\} = \{4, \beta\}$, which implies successively that $c(v_7) = \alpha$, $c(v_4) = \beta$, $c(v_8) = 4$, $c(v_9) = \beta$, and finally $c(v_6) = 4$. Consequently, the triangle in the interior of D_2 must be colored with the two colors α and β, again a contradiction.

Secondly, let v_{10} be colored by β. This can be handled using analogous arguments by interchanging the roles of α and β.

Therefore, the subgraph G_1 of G with given precoloring and list assignment as in Figure 2 is not L-list colorable.

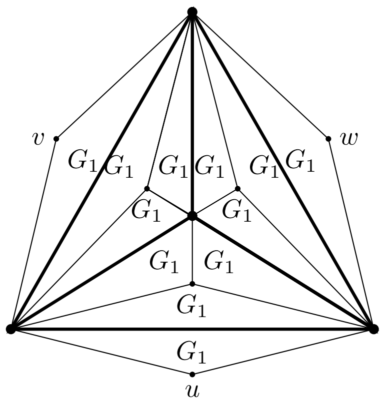

Figure 3: K_4 with twelve inserted G_1s

Next, consider the complete graph K_4 where the list of all vertices is $\{1, 2, \alpha, \beta\}$. Construct the graph G as follows. For every edge xy in K_4 add two copies of the graph G_1 identifying its edge v_1v_2 with the edge xy, once with $x = v_1$ and $y = v_2$ and the other time with $y = v_1$ and $x = v_2$ (see Figure 3) and then identify the vertices u, v, and w.

Clearly, two vertices of of K_4 must be colored by 1 and 2 giving exactly the above precoloring for one of the corresponding subgraphs G_1. Hence, G is planar and not L-list colorable, where all lists of the list assignment L have length 4 and contain the elements α and β. ☐

Since the vertices u, v, and w in Figure 3 are identified, the graph G constructed above has $8 + 12 \cdot 13 = 164$ vertices.

Considering Claim 3 and Theorem 6 we obtain the answer to Problem 2, which also improves the above mentioned result of Wegner. Moreover, in some sense this conclusion is a sharpness result for the Four Color Theorem.

Corollary 7. *There is a planar graph G such that in every proper 4-coloring of G the vertices of every two color classes induce a subgraph that is non-2-list colorable.*

Finally, let us mention a related concept introduced by Kratochvíl et al. in [4]. A list assignment L for a graph $G = (V, E)$ is called a (k, c)-*assignment* if $L(v) = k$ for all $v \in V(G)$ and $|L(v) \cap L(w)| \leqslant c$ for all edges $vw \in E(G)$. In [3] it is mentioned that every planar graph is L-list colorable for every $(4,1)$-assignment L. Moreover, there exist planar graphs G and corresponding $(4,3)$-assignments L such that G is not L-list colorable. So far, there is no result for $(4,2)$-assignments and the following problem remains open.

Problem 8. Is every planar graph L-list colorable for every $(4,2)$-assignment L?

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