Binary covering arrays on tournaments

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Abstract

We introduce graph-dependent covering arrays which generalize covering arrays on graphs, introduced by Meagher and Stevens (2005), and graph-dependent partition systems, studied by Gargano, Körner, and Vaccaro (1994). A covering array $\mathrm{CA}(n; 2, G, H)$ (of strength 2) on column graph G and alphabet graph H is an $n \times |V(G)|$ array with symbols $V(H)$ such that for every arc $ij \in E(G)$ and for every arc $ab \in E(H)$, there exists a row $\vec{r} = (r_1, \ldots, r_{|V(G)|})$ such that $(r_i, r_j) = (a, b)$. We prove bounds on n when G is a tournament graph and $E(H)$ consists of the edge $(0, 1)$, which corresponds to a directed version of Sperner's 1928 theorem. For two infinite families of column graphs, transitive and so-called circular tournaments, we give constructions of covering arrays which are optimal infinitely often.

Mathematics Subject Classifications: 05D05, 05B30

1   Introduction

Covering arrays are generalizations of orthogonal arrays that have been largely studied for their combinatorial interest and applications to software and network testing (see [1] and references therein).

Meagher and Stevens [9] extended the definition of a covering array to include a graph structure related to its columns, which we call a *column graph*. While covering arrays optimize the number of tests required to test all pairwise interactions between parameters of a system, the graph structure introduced in [9] is used to specify pairs of parameters that are known not to interact, yielding further reductions in the number of tests required. In

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addition to this increase in efficiency, their paper provides interesting connections between graph homomorphisms and Sperner systems.

Tracing back to the work of Gargano, Körner, and Vaccaro [2, 3] on Sperner capacities, another generalization of covering arrays, different from that introduced in [9], is implicitly considered. A graph (which we call an *alphabet graph* to distinguish it from the column graphs considered in [9]) is used to encode pairs of symbols which must appear in every two distinct columns of the array. Using the terminology given in [3], classical covering arrays of strength 2 correspond to "graph-dependent partition systems", where the alphabet graph being considered is complete with a loop on every vertex. While their work includes the case of classical covering arrays, it also implicitly defines another generalization of covering arrays, where the pairwise coverage requirement is restricted to pairs of symbols specified by the edges of the alphabet graph.

In this paper, we unify both types of graphs associated with covering arrays, generalizing the definition of covering arrays to graph-dependent covering arrays. Like in [9], the binary alphabet case provides interesting connections with extremal set theory. We also consider the more general case of directed alphabet graphs.

For a positive integer n, we write $[n]$ to denote the set of integers $\{1, \ldots, n\}$. Throughout, the symbols n, v, and k are used to represent positive integers. A complete graph on k vertices is denoted K_k, and K_v^{loop} denotes the graph K_v with a loop on each vertex.

Definition 1. (GRAPH-DEPENDENT COVERING ARRAY) Let G be a loopless directed graph with no parallel arcs, and let $V(G) = [k]$. Let H be a directed graph with no parallel arcs, and let $V(H) = [v]$. A *covering array on G and H*, denoted $\text{CA}(n; 2, G, H)$, is an $n \times k$ array with symbols from the alphabet $[v]$ and satisfying the following property: for every arc $ij \in E(G)$ and for every arc $ab \in E(H)$, there exists a row $r_l = (r_{l1}, \ldots, r_{lk})$ such that $(r_{li}, r_{lj}) = (a, b)$. The graphs G and H of a $\text{CA}(n; 2, G, H)$ are called the *column graph* and *alphabet graph*, respectively. The *covering array number for G and H*, denoted $\text{CAN}(2, G, H)$, is the smallest integer n for which a $\text{CA}(n; 2, G, H)$ exists.
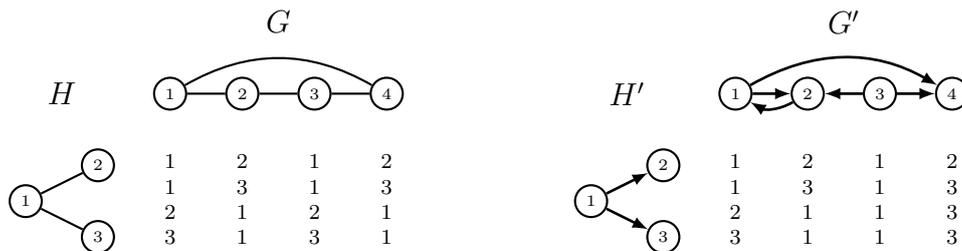

Figure 1: Graph-dependent covering arrays.

In Figure 1, we give two examples of column–alphabet graph pairs with corresponding covering arrays. The left example has undirected column and alphabet graphs, whereas the right example is directed; the underlying graphs of G' and H' are G and H, respectively.

Let $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ be two v-ary n-tuples, that is, $\vec{x}, \vec{y} \in [v]^n$. If, for every arc $ab \in E(H)$, there exists an index $i \in [n]$ such that $x_i = a$ and $y_i = b$, then we say that $\vec{x}$ is H-*dependent* with $\vec{y}$. In a $\mathrm{CA}(n; 2, G, H)$, for all arcs $ij \in E(G)$, the ith column must be H-dependent with the jth column.

Alphabet graphs and column graphs can in fact be undirected; if so, then our convention is to consider them as directed graphs in which each undirected edge corresponds to two oppositely oriented arcs. With our convention, the notion of H-dependence is a symmetric relation if and only if H is an undirected alphabet graph. If one of G or H is undirected, then, in terms of covering arrays on G and H, we can assume that both G and H are undirected. There may be situations in which it makes sense to allow alphabet graphs and column graphs to have parallel arcs, and to allow column graphs to have loops. Parallel arcs could encode the desired multiplicities of coverage (for specified pairs of symbols indicated by parallel arcs of the alphabet graph or entirely between columns specified by parallel arcs of the column graph). This would be analogous to the "index" parameter of classical covering arrays. In the present paper, we do not consider these situations, but mention them as possible lines of future investigation.

Some special cases that have been considered in the literature are the following:

- If the alphabet graph is K_2 and the column graph is K_k, then the columns of a $\mathrm{CA}(n; 2, K_k, K_2)$ correspond to an antichain of subsets of $[n]$. In this case, Sperner's Theorem [10] gives $\mathrm{CAN}(2, K_k, K_2) = \min\left\{ n : k \leqslant \binom{n}{\lfloor n/2 \rfloor} \right\}$.

- If the alphabet graph is complete on v vertices and has a loop on each vertex and the column graph is complete, then a $\mathrm{CA}(n; 2, K_k, K_v^{\mathrm{loop}})$ corresponds to a classical v-ary covering array of strength two, denoted $\mathrm{CA}(n; 2, k, v)$. The columns of a $\mathrm{CA}(n; 2, k, v)$ correspond to what is known as a "qualitatively independent" collection of v-partitions of $[n]$.[1] For a survey on classical covering arrays, see [1].

- If the alphabet graph is K_v^{loop} and the column graph is G, then a $\mathrm{CA}(n; 2, G, K_v^{\mathrm{loop}})$ is what is known as a "covering array on G," which we denote by $\mathrm{CA}(n; 2, G, v)$. Covering arrays on (undirected column) graphs model testing applications where some factors are known not to interact. See [9] for more details.

- If the alphabet graph is H and the column graph is complete, then the columns of a $\mathrm{CA}(n; 2, K_k, H)$ correspond to an "H-dependent collection" of $|V(H)|$-ary n-tuples, that is, a collection $\mathscr{C} = \{\vec{x}_1, \ldots, \vec{x}_k\}$ in which $\vec{x}_i$ is H-dependent with $\vec{x}_j$ for every ordered pair of two distinct elements $\vec{x}_i, \vec{x}_j \in \mathscr{C}$. Asymptotic information-theoretic results related to H-dependent collections have been studied in [2, 3].

In Section 2, we define H-dependence graphs and characterize graph-dependent covering arrays in terms of graph homomorphisms to H-dependence graphs. In the remaining sections, we focus on directed column graphs together with the binary alphabet

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[1]For the particular alphabet graph K_v^{loop}, the notion of being K_v^{loop}-*dependent* corresponds to being *qualitatively independent*. Despite this discord in terminology, we adopted "graph-dependent" since it captures the fact that the symbol coverage of our covering arrays on G and H depends on the edges of both of these accessory graphs. Moreover, the terminology and definition of "graph-dependent partition systems" was already introduced in [3] in order to generalize qualitative independence.

graph T_2, where $V(T_2) = \{0, 1\}$ and $E(T_2) = \{(0, 1)\}$. In Section 3, we give bounds for $\mathrm{CAN}(2, G', T_2)$, where G' is a directed column graph. In Section 4, we establish bounds for $\mathrm{CAN}(2, G', T_2)$, where G' is a tournament, that is, an orientation of a complete graph. In this case, determining $\mathrm{CAN}(2, G', T_2)$ corresponds to an oriented version of Sperner's Theorem. In Section 4.1, we give optimal constructions for covering arrays on transitive tournament column graphs, and we use these in Section 4.2 to obtain tight asymptotic bounds for the covering array number for all tournaments with alphabet graph T_2. In Section 4.3, we consider another particular family of tournament column graphs which we call circular tournaments. We give constructions for covering arrays on circular tournaments and alphabet graph T_2; we show that these constructions are optimal infinitely often. In Section 4.4, we conclude by giving some experimental data and open problems for binary covering arrays on tournaments.

2 *H*-dependence graphs

In this section, we give a class of graphs, called H-dependence graphs, which characterizes the problem of determining $\mathrm{CAN}(2, G, H)$ in terms of graph homomorphisms. The H-dependence graphs are the natural extension of qualitative independence graphs which were defined by Meagher and Stevens [9] when they characterized covering arrays on (undirected column) graphs via homomorphisms.

Let G and H be graphs. A map $f : V(G) \to V(H)$ is a *homomorphism* of G to H if $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. If there exists a homomorphism of G to H, then we say that G is *homomorphic to* H and we write $G \to H$. If $G \to H$ and $H \to G$, then we say that G and H are *homomorphically equivalent*.

Definition 2. Let H be an alphabet graph with $V(H) = [v]$. The H-*dependence graph*, denoted $\mathrm{QI}(n, H)$, is the graph with $V(\mathrm{QI}(n, H)) = [v]^n$ and $\vec{x}\vec{y} \in E(\mathrm{QI}(n, H))$ if and only if $\vec{x}$ is H-dependent with $\vec{y}$.

If H is undirected, then the H-dependence graph is a symmetric digraph, hence we consider it as an undirected graph.

Remark 3. If the alphabet graph is K_v^{loop}, that is, complete with a loop on every vertex, then our K_v^{loop}-dependence graph $\mathrm{QI}(n, K_v^{\mathrm{loop}})$ is a proper supergraph of the qualitative independence graph with parameters n and v, denoted $\mathrm{QI}(n, v)$; see [9, Definition 10]. In fact, $\mathrm{QI}(n, K_v^{\mathrm{loop}})$ is homomorphically equivalent to the qualitative independence graph $\mathrm{QI}(n, v)$. That $\mathrm{QI}(n, v) \to \mathrm{QI}(n, K_v^{\mathrm{loop}})$ is clear by inclusion. That $\mathrm{QI}(n, K_v^{\mathrm{loop}}) \to \mathrm{QI}(n, v)$ follows from [9, Theorem 1].

The next two results are straightforward extensions of analogous results given in [9].

Proposition 4. *Let G be a column graph and let H be an alphabet graph. Then there exists a $CA(n; 2, G, H)$ if and only if there is a homomorphism $G \to QI(n, H)$. In particular,*

$$CAN(2, G, H) = \min\{n : G \to QI(n, H)\}.$$

Proof. Let $V(G) = [k]$ and let $V(H) = [v]$. If there exists a CA$(n; 2, G, H)$, then its columns are vertices of QI(n, H). Define a map $f : V(G) \to V(\text{QI}(n, H))$ given by $f(i) = C_i$, where C_i denotes the ith column of the CA$(n; 2, G, H)$ and $C_i = (c_{1i}, \ldots, c_{ni}) \in [v]^n$ for each $i \in [k]$. If $ij \in E(G)$, then, for all $ab \in E(H)$, there exists an index $l \in [n]$ such that $c_{li} = a$ and $c_{lj} = b$. By definition of the H-dependence graph QI(n, H), the vertex C_i is adjacent to C_j in QI(n, H). Thus, f defines a homomorphism.

If there exists a homomorphism $f : V(G) \to V(\text{QI}(n, H))$, then we can build an $n \times k$ array with columns $C_1, \ldots, C_k$ given by $C_i = f(i)$ for each $i \in [k]$. By the definition of QI(n, H), it follows that this array is a CA$(n; 2, G, H)$. $\square$

Proposition 5. *Let H be an alphabet graph and let G_1 and G_2 be column graphs. If there is a homomorphism $G_1 \to G_2$, then*

$$CAN(2, G_1, H) \leqslant CAN(2, G_2, H).$$

In particular, for an undirected column graph G, we have

$$CAN(2, K_{\omega(G)}, H) \leqslant CAN(2, G, H) \leqslant CAN(2, K_{\chi(G)}, H),$$

where $\omega(G)$ and $\chi(G)$ denote the clique number and chromatic number of G. $\square$

If H_1 and H_2 are alphabet graphs such that $V(H_1) = V(H_2) = [v]$ and $E(H_1) \subseteq E(H_2)$, then every pair of H_2-dependent v-ary n-tuples $\vec{x}$ and $\vec{y}$ are necessarily H_1-dependent. Thus, we have the following lemma.

Lemma 6. *Let H_1 and H_2 be alphabet graphs such that $V(H_1) = V(H_2)$ and $E(H_1) \subseteq E(H_2)$. Then, for every n, the H_2-dependence graph QI(n, H_2) is a subgraph of the H_1-dependence graph QI(n, H_1).* $\square$

Going forward, the focus of this paper is on one particular directed alphabet graph: the transitive tournament on two vertices, which we denote by T_2. The vertex set of T_2 is $V(T_2) = \{0, 1\}$ and its edge set is $E(T_2) = \{(0, 1)\}$. The alphabet graph T_2 is the natural directed alphabet graph to consider on a binary alphabet.

In Figure 2, we give the T_2-dependence graphs QI$(2, T_2)$ and QI$(3, T_2)$.

A graph that is not homomorphic to any proper subgraph of itself is called a *core*. Every graph G is homomorphic to a unique, up to isomorphism, core, denoted $G^\bullet$, such that $G^\bullet$ is an induced subgraph of G; see [4, Corollary 1.32].

Since every pair of distinct vertices in the T_2-dependence graph QI(n, T_2) is joined by at least one arc, and since QI(n, T_2) has no loops, it follows that QI(n, T_2) is a core.

Lemma 7. *For each $n \geqslant 1$, the T_2-dependence graph QI(n, T_2) is a core.* $\square$

The vertex set of QI(n, T_2) is the set of binary n-tuples, hence $|V(\text{QI}(n, T_2))| = 2^n$. Binary n-tuples are in bijection with the subsets of $[n]$ as follows. If $\vec{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$, then $\vec{x}$ corresponds to the subset $A_{\vec{x}} = \{i \in [n] : x_i = 1\}$.

In terms of subsets of $[n]$, a binary n-tuple $\vec{x}$ is T_2-dependent with another binary n-tuple $\vec{y}$ if and only if the corresponding subsets $A_{\vec{x}}, A_{\vec{y}} \subseteq [n]$ satisfy $A_{\vec{y}} \nsubseteq A_{\vec{x}}$. We often
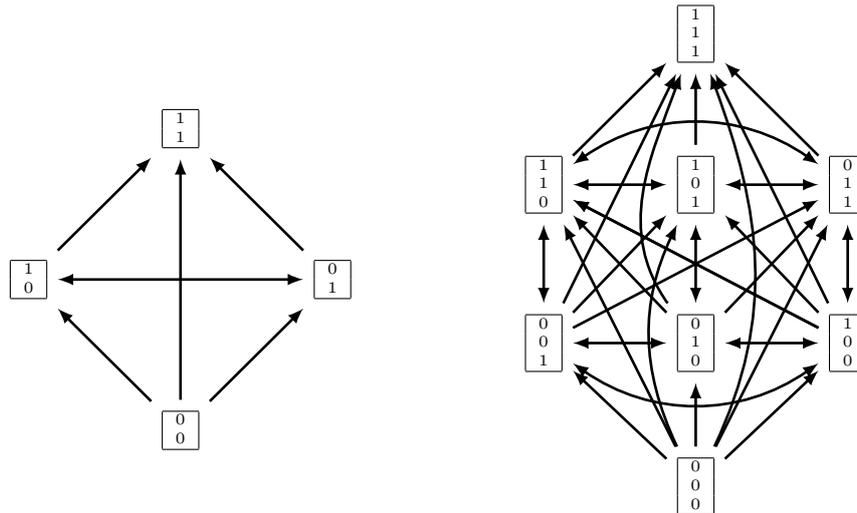
Figure 2: The T_2-dependence graphs $\mathrm{QI}(2, T_2)$ and $\mathrm{QI}(3, T_2)$.

make use of this correspondence and refer to vertices of $\mathrm{QI}(n, T_2)$ as subsets or n-tuples interchangeably. The *rank* of a binary n-tuple $\vec{x}$ is the number of times that '1' is an entry of $\vec{x}$. Equivalently, the rank of $\vec{x}$ is the cardinality $|A_{\vec{x}}|$.

Lemma 8. *The graph $\mathrm{QI}(n, T_2)$ has exactly $2^{2n-1} + 2^{n-1} - 3^n$ pairs of distinct vertices A and B such that both AB and BA are arcs of $\mathrm{QI}(n, T_2)$. In particular, as $n \to \infty$, the proportion $(2^{2n-1} + 2^{n-1} - 3^n)/\binom{2^n}{2} \to 1$, where $\binom{2^n}{2}$ is the total number of pairs of vertices in $\mathrm{QI}(n, T_2)$.*

Proof. First, we show that the number of pairs of distinct vertices $A, B \in V(\mathrm{QI}(n, T_2))$ such that exactly one of AB and BA is an arc of $\mathrm{QI}(n, T_2)$ is $3^n - 2^n$.

Consider the vertices of $\mathrm{QI}(n, T_2)$ as subsets of $[n]$. Each k-set is a proper subset of $\binom{n-k}{i}$ sets of rank $k + i$. For each fixed k-set A, the number of subsets in which A is properly contained is thus equal to

$$\sum_{i=1}^{n-k} \binom{n-k}{i} = 2^{n-k} - 1.$$

Summing over all sets of rank k and summing over all ranks $k \in \{0, 1, \ldots, n\}$, we have

$$\sum_{k=0}^{n} \binom{n}{k}(2^{n-k} - 1) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} - \sum_{k=0}^{n} \binom{n}{k} = 2^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k - 2^n = 3^n - 2^n$$

pairs of distinct vertices $A, B \in V(\mathrm{QI}(n, T_2))$ such that $A \subset B$. These are the only pairs of vertices for which $BA \in E(\mathrm{QI}(n, T_2))$ and $AB \notin E(\mathrm{QI}(n, T_2))$. Thus, there are $3^n - 2^n$ pairs of distinct vertices in $V(\mathrm{QI}(n, T_2))$ such that exactly one of AB and BA is an arc of $\mathrm{QI}(n, T_2)$.

Now, there are $\binom{2^n}{2}$ pairs of distinct vertices in $\mathrm{QI}(n, T_2)$, of which $3^n - 2^n$ pairs are not symmetrically adjacent. Thus, the number of symmetrically adjacent pairs is $\binom{2^n}{2} - [3^n - 2^n] = 2^{n-1}(2^n - 1) - 3^n + 2^n = 2^{2n-1} + 2^{n-1} - 3^n$. Asymptotically, we have

$$\lim_{n \to \infty} \frac{2^{2n-1} + 2^{n-1} - 3^n}{\binom{2^n}{2}} = \lim_{n \to \infty} \frac{4^n + 2^n - 2 \cdot 3^n}{4^n - 2^n} = 1. \qquad \square$$

3   Bounds for binary covering arrays on directed column graphs

In this section, we investigate the problem of finding $\mathrm{CAN}(2, G', T_2)$ for directed column graphs G'. We give bounds which compare the covering array number for a directed column graph G' and alphabet graph T_2 with a binary covering array on the underlying graph G of G'. Throughout this section, we use G to denote the underlying graph of a given directed graph G'.

For a graph G and integers k, v, we denote $\mathrm{CAN}(2, K_k, K_v^{\mathrm{loop}})$ simply as $\mathrm{CAN}(2, k, v)$, and we denote $\mathrm{CAN}(2, G, K_v^{\mathrm{loop}})$ by $\mathrm{CAN}(2, G, v)$.

Theorem 9. *Let G' be a directed graph with at least one arc, and let G be its underlying graph. Then*

$$\lceil \log_2 \chi(G) \rceil \leqslant CAN(2, G', T_2) \leqslant CAN(2, G, 2) - 1.$$

Proof. For the lower bound, let $m = \mathrm{CAN}(2, G', T_2)$, and consider a $\mathrm{CA}(m; 2, G', T_2)$. The columns of this array form a proper vertex-colouring of G since adjacent vertices in G correspond to columns of the $\mathrm{CA}(m; 2, G', T_2)$ that must be T_2-dependent in some direction which makes these columns distinct. Therefore $\chi(G) \leqslant 2^m$, and so $\lceil \log_2 \chi(G) \rceil \leqslant m = \mathrm{CAN}(2, G', T_2)$.

For the upper bound, let $n = \mathrm{CAN}(2, G, 2)$ and consider an optimal $\mathrm{CA}(n; 2, G, 2)$. We may assume without loss of generality that the first row is all zeros. Delete this row and we are left with a $\mathrm{CA}(n - 1; 2, G', K_2)$, for any (directed) subgraph $G' \subseteq G$. Thus, $\mathrm{CAN}(2, G', T_2) \leqslant \mathrm{CAN}(2, G', K_2) \leqslant \mathrm{CAN}(2, G, 2) - 1$. $\qquad \square$

For G' with $\chi(G) = 2$, we have a complete characterization of directed graphs that achieve the lower bound of Theorem 9, based on the following definition and observations.

We call a directed graph G' a *consistently oriented bipartite graph* if the underlying graph of G' is bipartite and there is a bipartition (X, Y) of $V(G')$ such that all arcs of G' are directed from X to Y.

If G' is a directed graph, then $\mathrm{CAN}(2, G', T_2)$ can be equivalently defined as the minimum number of consistently oriented bipartite subgraphs into which one can partition the arc set of G'. Indeed, the set of arcs of G' covered by any particular row of a $\mathrm{CA}(n; 2, G', T_2)$ forms a consistently oriented bipartite subgraph. In particular, the condition $\mathrm{CAN}(2, G', T_2) = 1$ is equivalent to there being a homomorphism $G' \to \mathrm{QI}(1, T_2) \cong T_2$, which is equivalent to G' being a consistently oriented bipartite graph.

For $\chi(G) = 2$, the lower bound of Theorem 9 holds with equality if and only if $\mathrm{CAN}(2, G', T_2) = 1$, which means the lower bound of Theorem 9 is achieved solely by consistently oriented bipartite graphs.

For higher chromatic numbers, we do not have complete characterizations of graphs achieving the lower bound of Theorem 9; however, for $\chi(G) = 3, 4$, we do have a necessary condition, as follows. If G' contains a directed odd cycle, then by [7, Lemma 4.4.11], we have $\mathrm{CAN}(2, G', T_2) \geqslant 3$. For $\chi(G) = 3, 4$, the lower bound of Theorem 9 is given by $\mathrm{CAN}(2, G', T_2) \geqslant 2$. Thus, for these chromatic numbers, a necessary condition to achieve the lower bound of Theorem 9 is for G' to be free of any directed odd cycles.

For all chromatic numbers $\chi \geqslant 2$, there exist directed graphs G' for which the lower bound of Theorem 9 holds with equality. Specifically, if G' is a transitive k-tournament, then the underlying graph of G' is K_k; in this case, $\mathrm{CAN}(2, G', T_2) = \lceil \log_2 k \rceil = \lceil \log_2 \chi(G) \rceil$ (see Section 4.1 for more details).

To see that the upper bound of Theorem 9 holds with equality infinitely often, let G' be a directed graph such that G' contains a directed odd cycle $\vec{C}_{2l+1}$ and the underlying graph G of G' satisfies $\chi(G) = 3$. Since $\chi(G) = 3$, it follows from [9, Theorem 1] that $\mathrm{CAN}(2, G, 2) = 4$. In this case, the upper bound of Theorem 9 is $\mathrm{CAN}(2, G, 2) - 1 = 4 - 1 = 3$. Now, let K_3^* denote K_3 in which each undirected edge is considered as two oppositely oriented arcs. We have that $G' \to K_3^* \to \mathrm{QI}(3, T_2)$ and, since homomorphisms compose, $\mathrm{CAN}(2, G', T_2) \leqslant 3$ by Proposition 4. Finally, since G' contains $\vec{C}_{2l+1}$, it follows from [7, Lemma 4.4.11] that $\mathrm{CAN}(2, G', T_2) \geqslant 3$. Thus $\mathrm{CAN}(2, G', T_2) = 3$, which, in this case, achieves the upper bound of Theorem 9.

More examples of directed column graphs whose covering array numbers achieve the upper bound of Theorem 9 can be found in Appendix A; these examples are tournament column graphs whose underlying graphs have chromatic numbers $\chi = 3, 5, 9$ (see "adjacency vectors" given in Table 2 for $k = 3, 5, 9$). Tournament column graphs are explored in more detail in Section 4.

4 Bounds for binary covering arrays on tournaments

In this section, we give bounds on $\mathrm{CAN}(2, G', T_2)$ where the column graph G' is a k-tournament. We consider two specific families of tournaments as column graphs: transitive tournaments and "circular" tournaments. We give constructions which are optimal infinitely often for these families of column graphs.

A *k-tournament* is an orientation of the complete graph K_k. Our convention for k-tournaments is to label the vertices as $1, 2, \ldots, k$ (except when $k = 2$ in which case we use 0 and 1).

We are interested in k-tournaments as column graphs with T_2 as the alphabet graph. Among all k-tournaments, is there one with the largest covering array number? Do binary covering arrays on k-tournaments generally require as many rows as classical binary covering arrays?

First, since the T_2-dependence graph $\mathrm{QI}(n, T_2)$ has no loops while the underlying graph of a tournament is complete, any homomorphism of a tournament to $\mathrm{QI}(n, T_2)$ must be vertex-injective. Thus, the analogue of Proposition 4 for binary covering arrays on tournaments is the following.

Proposition 10. *Let O_k be any k-tournament. Then there exists a $CA(n; 2, O_k, T_2)$ if and only if O_k is a subgraph of $QI(n, T_2)$. In particular,*

$$CAN(2, O_k, T_2) = \min\{n : O_k \subseteq QI(n, T_2)\}. \qquad \square$$

Since the underlying graph of a k-tournament is the complete graph K_k with $\chi(K_k) = k$, Theorem 9 gives the following bound.

Corollary 11. *Let O_k be a k-tournament. Then*

$$\lceil \log_2 k \rceil \leqslant CAN(2, O_k, T_2) \leqslant CAN(2, k, 2) - 1. \qquad \square$$

For k-tournaments, the upper bound of Corollary 11 can be tightened as follows.

Theorem 12. *Let $k \geqslant 2$ and let O_k be any k-tournament. Then*

$$\lceil \log_2 k \rceil \leqslant CAN(2, O_k, T_2) \leqslant \min\left\{ n : k \leqslant 2\binom{n-1}{\lfloor (n-1)/2 \rfloor} \right\}.$$

Proof. The lower bound is given by Corollary 11. To prove the upper bound, we give a construction as follows. Let n be a positive integer such that $2 \leqslant k \leqslant 2\binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Then $\lfloor k/2 \rfloor \leqslant \lceil k/2 \rceil \leqslant \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Take any maximum matching M of K_k (the underlying graph of O_k). Partition $V(O_k)$ into two parts of sizes $\lceil k/2 \rceil$ and $\lfloor k/2 \rfloor$ such that the ends of all edges in M are oriented from the first part to the second in O_k (if k is odd, we add the unmatched vertex to the first part). Relabel the vertices of O_k so that the vertices of the first part of the partition are labelled $1, \ldots, \lceil k/2 \rceil$, and the vertices of the second part are labelled $\lceil k/2 \rceil + 1, \ldots, k$.

To construct a $CA(n; 2, O_k, T_2)$, we first build an $(n-1) \times k$ array with columns $C_1, \ldots, C_k$ corresponding to $\lfloor \frac{n-1}{2} \rfloor$-subsets of $[n-1]$ as follows. Take $\lceil k/2 \rceil$ of the $\lfloor \frac{n-1}{2} \rfloor$-subsets of $[n-1]$ in some order, followed by the same first $\lfloor k/2 \rfloor$ columns repeated in the same order, that is, $C_1, \ldots, C_{\lceil k/2 \rceil}$ are distinct $\lfloor \frac{n-1}{2} \rfloor$-sets and $C_{i+\lceil k/2 \rceil} = C_i$ for $1 \leqslant i \leqslant \lfloor k/2 \rfloor$. Add an nth row to cover the arcs of M. This additional row has zeros as its first $\lceil k/2 \rceil$ entries and ones as its last $\lfloor k/2 \rfloor$ entries.

Since every two distinct $\lfloor \frac{n-1}{2} \rfloor$-sets are K_2-dependent, any two distinct columns C_i and C_j (of length $n-1$) are K_2-dependent except when $|j - i| = \lceil k/2 \rceil$. The extra row ensures that we cover the arcs of M. Note, when $k = 2$, this extra row is in fact the only row of the constructed array. Thus, we have constructed a $CA(n; 2, O_k, T_2)$, and it follows that $CAN(2, O_k, T_2) \leqslant n$ whenever $k \leqslant 2\binom{n-1}{\lfloor (n-1)/2 \rfloor}$. $\qquad \square$

For k-tournament column graphs, the upper bound given in Theorem 12 is an improvement over the upper bound given in Corollary 11 infinitely often. An exact expression for the upper bound of Corollary 11 is known [5, 6], and this can be rewritten as follows:

$$CAN(2, k, 2) - 1 = \min\left\{ n : k \leqslant \binom{n-1}{\lfloor n/2 \rfloor - 1} \right\} - 1 = \min\left\{ n : k \leqslant \binom{n}{\lfloor (n+1)/2 \rfloor - 1} \right\}.$$

For all n, we have $2\binom{n-1}{\lfloor (n-1)/2 \rfloor} > \binom{n}{\lfloor (n+1)/2 \rfloor - 1}$. Consequently, for all $k \geqslant 2$, we have

$$\min\left\{ n : k \leqslant 2\binom{n-1}{\lfloor (n-1)/2 \rfloor} \right\} \leqslant \min\left\{ n : k \leqslant \binom{n}{\lfloor (n+1)/2 \rfloor - 1} \right\} = CAN(2, k, 2) - 1. \quad (1)$$

For each $n \geqslant 1$, for all k lying in the range $\binom{n}{\lfloor(n+1)/2\rfloor-1} < k \leqslant 2\binom{n-1}{\lfloor(n-1)/2\rfloor}$, the inequality in (1) is strict. Thus, for infinitely many k-tournaments, the upper bound given in Theorem 12 is a strict improvement over that given in Corollary 11.

For fixed k, we are interested in the spectrum of covering array numbers which arises when we consider all k-tournament column graphs. Are the bounds of Theorem 12 achieved for all k? Among all k-tournaments, which ones have the largest or smallest covering array numbers?

For the lower bound of Theorem 12, we have a complete answer by applying Proposition 10.

Theorem 13. *A k-tournament O_k achieves the lower bound of Theorem 12 if and only if O_k is a subgraph of the T_2-dependence graph $QI(\lceil \log_2 k \rceil, T_2)$.* $\qquad\square$

4.1 Transitive tournaments

Aside from the general answer given by Theorem 13, we also have a specific infinite family of tournament column graphs achieving the lower bound of Theorem 12, namely, transitive tournaments.

Let $k \geqslant 2$. The *transitive k-tournament*, denoted T_k, is the tournament with $V(T_k) = [k]$ and arcs $ij \in E(T_k)$ if and only if $i < j$.

Theorem 14. *Let $k \geqslant 2$. Then $CAN(2, T_k, T_2) = \lceil \log_2(k) \rceil = \min\{n : k \leqslant 2^n\}$.*

Proof. Let A and B be subsets of $[n]$ (corresponding to vertices of $\mathrm{QI}(n, T_2)$). Notice that $AB \in E(\mathrm{QI}(n, T_2))$ if and only if $B \not\subseteq A$. We can thus order the vertices of $\mathrm{QI}(n, T_2)$ in non-descending order of rank, and this ordering has the property that $AB \in E(\mathrm{QI}(n, T_2))$ whenever A precedes B. With this ordering in place, it is clear that $\mathrm{QI}(n, T_2)$ contains T_{2^n} as an induced subgraph since there is a strict total ordering on the vertices of T_{2^n}. It now follows that for all $k \leqslant 2^n$ there exist homomorphisms $T_k \to T_{2^n} \to \mathrm{QI}(n, T_2)$. By Proposition 4, we have $CAN(2, T_k, T_2) \leqslant \lceil \log_2 k \rceil$. By Theorem 12, we have $CAN(2, T_k, T_2) \geqslant \lceil \log_2 k \rceil$, which completes the proof. $\qquad\square$

The proof of Theorem 14 provides a simple construction for $CA(\lceil \log_2 k \rceil; 2, T_k, T_2)$. If $n = \lceil \log_2 k \rceil$, then k columns corresponding to subsets of $[n]$ in non-descending order of rank form the columns of a $CA(\lceil \log_2 k \rceil; 2, T_k, T_2)$. In the following theorem, we use this construction for transitive tournaments to build binary covering arrays. The construction we provide in Theorem 15 is not optimal in general; however, it is optimal asymptotically, and it produces binary covering arrays built from blocks of rows corresponding to the simple construction for binary covering arrays on transitive tournaments.

Theorem 15. *For each integer $k \geqslant 2$, there exists an integer $m \leqslant \lceil \log_2 k \rceil + 1$ such that*

$$CAN(2, k, 2) \leqslant CAN(2, T_k, T_2) + CAN(2, T_m, T_2) + 2.$$

In particular,

$$CAN(2, k, 2) \leqslant \lceil \log_2 k \rceil + \lceil \log_2(\lceil \log_2 k \rceil + 1) \rceil + 2.$$

Proof. Let $n = \lceil \log_2 k \rceil$, and let m be the minimum number of binomial coefficients needed to write $k \leqslant \binom{n}{a_1} + \binom{n}{a_2} + \cdots + \binom{n}{a_m}$, where $0 \leqslant a_1 < a_2 < \cdots < a_m \leqslant n$. Clearly, $m \leqslant n + 1 = \lceil \log_2 k \rceil + 1$.

Now, build a $\mathrm{CA}(n; 2, T_k, T_2)$ as described in the proof of Theorem 14. Without loss of generality, we may assume that the columns of the $\mathrm{CA}(n; 2, T_k, T_2)$ correspond to the subsets of $[n]$ of ranks $a_1, a_2, \ldots, a_{m-1}$ and (as many as needed of) the a_m-subsets of $[n]$, sorted in non-decreasing order of rank. Note, for each $i \in [m]$, the columns corresponding to the a_i-subsets form an antichain and are thus already K_2-dependent. We extend this $\mathrm{CA}(n; 2, T_k, T_2)$ into a $\mathrm{CA}(n + \lceil \log_2 m \rceil; 2, K_k, K_2)$ by appending to it at most $\lceil \log_2(\lceil \log_2 k \rceil + 1) \rceil$ additional rows. These additional rows correspond to the rows of a $\mathrm{CA}(\lceil \log_2 m \rceil; 2, T_m, T_2)$ whose columns we denote by $C_1, C_2, \ldots, C_m$. Under each column of the $\mathrm{CA}(n; 2, T_k, T_2)$ that corresponds to an a_i-subset of $[n]$, we put a copy of C_{m-i+1}, as depicted in Figure 3.
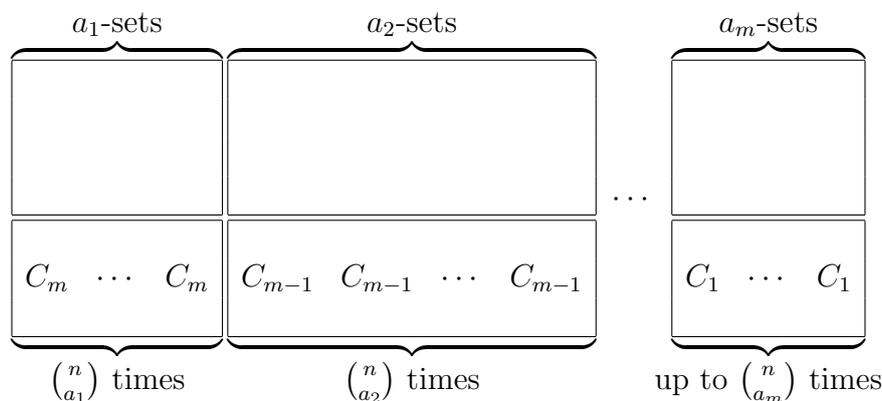

Figure 3: Construction of $\mathrm{CA}(\lceil \log_2 k \rceil + \lceil \log_2 m \rceil; 2, K_k, K_2)$, where $m \leqslant \lceil \log_2 k \rceil + 1$.

The array given in Figure 3 is indeed a $\mathrm{CA}(n + \lceil \log_2 m \rceil; 2, K_k, K_2)$. Thus, we have

$$
\begin{aligned}
\mathrm{CAN}(2, K_k, K_2) &\leqslant \mathrm{CAN}(2, T_k, T_2) + \mathrm{CAN}(2, T_m, T_2) \\
&\leqslant \mathrm{CAN}(2, T_k, T_2) + \mathrm{CAN}(2, T_{\lceil \log_2 k \rceil + 1}, T_2) \\
&= \lceil \log_2 k \rceil + \lceil \log_2(\lceil \log_2 k \rceil + 1) \rceil
\end{aligned}
\tag{2}
$$

By adding two constant rows to the $\mathrm{CA}(n + \lceil \log_2 m \rceil; 2, K_k, K_2)$, namely a row of all zeros and another row of all ones, we get a $\mathrm{CA}(n + \lceil \log_2 m \rceil + 2; 2, k, 2)$. $\qquad\square$

4.2 Asymptotic bounds

For directed column graphs G' and alphabet graph T_2, we now show that, asymptotically, $\mathrm{CAN}(2, G', T_2)$ grows logarithmically with respect to the chromatic number of the underlying graph of G'.

Theorem 16. *For each $c \geqslant 2$, let G'_c be some directed graph with underlying graph G_c satisfying $\chi(G_c) = c$. Then*

$$
\lim_{c \to \infty} CAN(2, G'_c, T_2) / \log_2 \chi(G_c) = 1.
$$

Proof. Since $\mathrm{CAN}(2, G_c', T_2) \leqslant \mathrm{CAN}(2, \chi(G_c), 2)$, it follows from Theorems 9 and 15 that, for each $c \geqslant 2$, we have

$$\lceil \log_2 \chi(G_c) \rceil \leqslant \mathrm{CAN}(2, G_c', T_2) \leqslant \lceil \log_2 \chi(G_c) \rceil + \lceil \log_2(\lceil \log_2 \chi(G_c) \rceil + 1) \rceil + 1.$$

Clearly, the above bounds are asymptotically equal to $\log_2 \chi(G_c)$, as $c \to \infty$. □

Corollary 17. *For each $k \geqslant 2$, let O_k denote some k-tournament. Then*

$$\lim_{k \to \infty} CAN(2, O_k, T_2)/\log_2 k = 1.$$ □

4.3 Circular tournaments

In addition to transitive tournaments, we consider one other infinite family of tournament column graphs, which we call circular tournaments. We prove that the covering array number for circular k-tournaments and alphabet graph T_2 is always $\lceil \log_2 k \rceil$ or $\lceil \log_2 k \rceil + 1$.

Definition 18. The *circular k-tournament*, denoted Ω_k, has vertex set $V(\Omega_k) = [k]$ and arcs as follows. If k is odd, then $ij \in E(\Omega_k)$ if and only if $j = i + t$ for some t such that $1 \leqslant t \leqslant \lfloor k/2 \rfloor$ (addition is done modulo k). If k is even, then $ij \in E(\Omega_k)$ whenever $j = i + t$ for some t such that $1 \leqslant t < k/2$ (addition is done modulo k); moreover, $(i, i + k/2) \in E(\Omega_k)$ for all i such that $1 \leqslant i \leqslant k/2$.

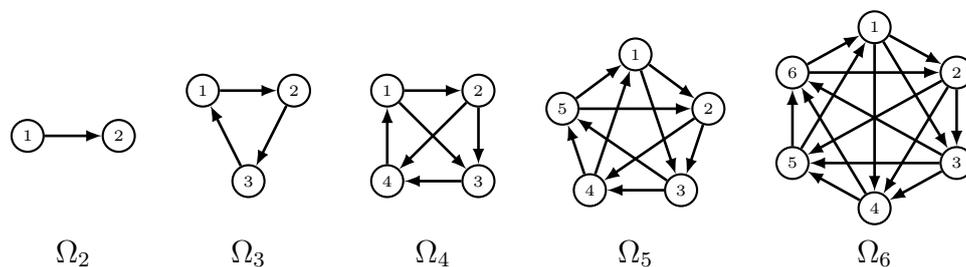In Figure 4, we depict several circular k-tournaments.

Figure 4: Circular tournaments.

Determining $\mathrm{CAN}(2, \Omega_k, T_2)$ for all k is an interesting extremal problem that corresponds to an oriented version of Sperner's Theorem [10]. In terms of covering arrays, Sperner's Theorem determines $\mathrm{CAN}(2, K_k, K_2)$ for all k.

Here, we give some important properties of circular tournaments.

Proposition 19. *If k is odd, then the circular tournament Ω_k is vertex-transitive. If k is even, then the automorphism group of Ω_k is trivial.*

Proof. Let k be odd, and let $i, j \in V(\Omega_k)$ be two distinct vertices. We can write $j = i + x \pmod{k}$ for a unique $x \in \{1, \ldots, k-1\}$. The map $f : V(\Omega_k) \to V(\Omega_k)$ given by

$f(u) = u + x \pmod{k}$ is an automorphism such that $f(i) = j$. Thus, when k is odd, Ω_k is vertex-transitive, as claimed.

Now, let k be even and write $k = 2l$. The vertices of Ω_k are of two types: the vertices labelled $1, 2, \ldots, l$ which have outdegree l and indegree $l - 1$, and the vertices labelled $l+1, l+2, \ldots, 2l$ which have outdegree $l-1$ and indegree l. Suppose $f : V(\Omega_k) \to V(\Omega_k)$ is an automorphism such that $f(i) = j$ for some vertices $i, j \in V(\Omega_k)$. If $i \in \{1, \ldots, l\}$, then in order for its in- and outdegree to match, j must also be from among the vertices in the set $\{1, \ldots, l\}$. In order for the neighbours of i and j to have the correct in- and outdegrees, both i and j must have the same number of neighbours from the set $\{l + 1, \ldots, 2l\}$. This can happen only if $i = j$. The argument when $i \in \{l + 1, \ldots, 2l\}$ is similar. Thus, when k is even, the automorphism group of Ω_k is trivial. $\qquad\square$

Proposition 20. *For all $k \geqslant 3$, the circular tournament Ω_k contains an induced subgraph isomorphic to Ω_{k-1}. In particular, when $k = 2l + 1$, we can delete any vertex to obtain a copy of Ω_{2l}. When $k = 2l$, we can delete either vertex l or $l + 1$ from Ω_{2l} in order to obtain a copy of Ω_{2l-1}.*

Proof. Let $k = 2l + 1$, and let Y denote the subgraph of Ω_{2l+1} in which the vertex $2l + 1$ is deleted. Define a bijection $g : V(\Omega_{2l}) \to V(Y)$ by $g(i) = i$ for $1 \leqslant i \leqslant 2l$. By definition of Ω_{2l}, we have $ij \in E(\Omega_{2l})$ whenever either $i < j$ and $j = i+t$ for some $t \in \{1, \ldots, l-1\}$, or $i > j$ and $j + 2l = i + t$ for some $t \in \{1, \ldots, l - 1\}$, and for all $j = i + l$ such that $i \in \{1, \ldots, l\}$. By definition of Ω_{2l+1}, we have $ij \in E(\Omega_{2l+1})$ whenever either $i < j$ and $j = i + t$ for some $t \in \{1, \ldots, l\}$, or $i > j$ and $j + 2l + 1 = i + t$ for some $t \in \{1, \ldots, l\}$. Thus, whenever $ij \in E(\Omega_{2l})$, it follows that $g(i)g(j) = ij \in E(Y)$. Therefore, g is a vertex-injective homomorphism of Ω_{2l} to Y. Since Ω_{2l} is a tournament, it follows that g is an isomorphism.

By Proposition 19, Ω_{2l+1} is vertex-transitive. Therefore, we can delete any vertex of Ω_{2l+1} in order to obtain a copy of Ω_{2l}.

Now, let $k = 2l$ and let X denote Ω_k with the vertex l deleted. We claim that $\Omega_{k-1} \cong X$. Define a bijection $f : V(\Omega_{k-1}) \to V(X)$ by $f(i) = i$ for $1 \leqslant i \leqslant l - 1$ and $f(i) = i + 1$ for $l \leqslant i \leqslant k - 1$. Suppose $ij \in E(\Omega_{2l-1})$. Consider the following cases:

1. If $1 \leqslant i < j \leqslant l - 1$, then $f(i)f(j) = ij$. By definition of Ω_{2l-1}, we must have $j = i+t$ for some $t \in \{1, \ldots, l-2\}$. Thus, $j = i+t$ for some $t < l$, and by definition of Ω_{2l}, we have $f(i)f(j) \in E(X)$.

2. If $1 \leqslant i \leqslant l - 1$ and $l \leqslant j \leqslant 2l - 1$, then $f(i)f(j) = i(j + 1)$. By definition of Ω_{2l-1}, we must have $j = i + t$ for some $t \in \{1, \ldots, l - 1\}$. Thus, $t < l$ and so $f(i)f(j) \in E(X)$.

3. If $l \leqslant i < j \leqslant 2l - 1$, then $f(i)f(j) = (i+1)(j+1)$. By definition of Ω_{2l-1}, we must have $j = i + t$ for some $t \in \{1, \ldots, l-1\}$. Thus, $j + 1 = i + 1 + t$ for some $t < l$ and so $f(i)f(j) \in E(X)$.

4. If $l \leqslant i \leqslant 2l-1$ and $1 \leqslant j \leqslant l-1$, then $f(i)f(j) = (i+1)j$. By definition of Ω_{2l-1}, we must have $j+2l-1 = i+t$ for some $t \in \{1, \ldots, l-1\}$. Thus, $j+2l = i+1+t$ and so, by definition of Ω_{2l}, we have $f(i)f(j) \in E(X)$.

Since the above cases cover all arcs of the tournament Ω_{2l-1}, it follows that f is an isomorphism. Similarly, we can show that Ω_{2l-1} is isomorphic to the subgraph of Ω_{2l} in which the vertex $l-1$ is deleted. $\qquad\square$

Next, we give a recursive construction for covering arrays with alphabet graph T_2 in which we make use of a supergraph of the circular tournament, which we denote Ω_k^+, defined for even values of k as follows. Let k be even. The vertex set of Ω_k^+ is $V(\Omega_k^+) = [k]$ and $ij \in E(\Omega_k)$ if and only if $j = i+t$ for some t such that $1 \leqslant t \leqslant k/2$ (addition is done modulo k). The difference between Ω_k and Ω_k^+ is that Ω_k^+ has a pair of oppositely oriented arcs joining the vertices $i, j \in [k]$ with $|j-i| = k/2$. In particular, Ω_k^+ is not a tournament, but does contain subgraphs isomorphic to Ω_m, for all $m \leqslant k$. Consequently, we have $\mathrm{CAN}(2, \Omega_m, T_2) \leqslant \mathrm{CAN}(2, \Omega_k^+, T_2)$ for all circular tournaments Ω_m with $m \leqslant k$.

Proposition 21. *Let k be even. Then $CAN(2, \Omega_{2k}^+, T_2) \leqslant CAN(2, \Omega_k^+, T_2) + 1$.*

Proof. Let $n = \mathrm{CAN}(2, \Omega_k^+, T_2)$. For $1 \leqslant j \leqslant k$, let C_j denote the jth column of a $\mathrm{CA}(n; 2, \Omega_k^+, T_2)$. Take two copies of this array and interleave their columns as shown in Figure 5. Add a row of alternating zeros and ones. Let us call the array we obtain A.
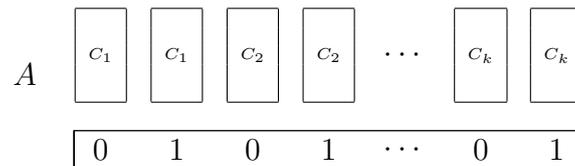
$$A \quad \boxed{C_1} \; \boxed{C_1} \; \boxed{C_2} \; \boxed{C_2} \; \cdots \; \boxed{C_k} \; \boxed{C_k}$$
$$\boxed{0 \quad 1 \quad 0 \quad 1 \quad \cdots \quad 0 \quad 1}$$

Figure 5: Recursive construction for a graph-dependent covering array on Ω_{2k}^+ and T_2.

For each $j \in [2k]$, the jth column of A corresponds to $C_{\lceil j/2 \rceil}$ (the $\lceil j/2 \rceil$th column of the $\mathrm{CA}(n; 2, \Omega_k^+, T_2)$). Let i and j be the indices of two distinct columns of A such that $j = i+t \pmod{2k}$ for some t such that $1 \leqslant t \leqslant k$. We must show that some row of A has a zero in the ith column and a one in the jth column; if there exists such a row, then for short, we say that A *covers* $\{(i, 0), (j, 1)\}$.

Suppose $j > i$. Then $1 \leqslant j-i \leqslant k$. In this case, we consider all possibilities for the parity of i and j. If i and j have the same parity, then $1 \leqslant \lceil j/2 \rceil - \lceil i/2 \rceil = j/2 - i/2 \leqslant k/2$; in these cases, the ith column of A ($C_{\lceil i/2 \rceil}$) and the jth column of A ($C_{\lceil j/2 \rceil}$) cover $\{(i, 0), (j, 1)\}$ since the $\mathrm{CA}(n; 2, \Omega_k^+, T_2)$ must cover $\{(\lceil i/2 \rceil, 0), (\lceil j/2 \rceil, 1)\}$. If i and j have opposite parity, then $1 \leqslant j-i \leqslant k-1$, since k is even. It now follows that $0 \leqslant \lceil j/2 \rceil - \lceil i/2 \rceil \leqslant j/2 + 1/2 - i/2 \leqslant k/2$. If $\lceil j/2 \rceil - \lceil i/2 \rceil = 0$, then i and j correspond to the same column $C_{j/2}$; in this case, $j = i+1$ and the extra row of alternating zeros and ones covers $\{(i, 0), (j, 1)\}$. Otherwise, $1 \leqslant \lceil j/2 \rceil - \lceil i/2 \rceil \leqslant j/2 + 1/2 - i/2 \leqslant k/2$; in these

cases, the ith column of A ($C_{\lceil i/2\rceil}$) and the jth column of A ($C_{\lceil j/2\rceil}$) cover $\{(i,0),(j,1)\}$ since the $\mathrm{CA}(n;2,\Omega_k^+,T_2)$ must cover $\{(\lceil i/2\rceil,0),(\lceil j/2\rceil,1)\}$.

Now, suppose $j < i$. Then $1 \leqslant 2k + j - i \leqslant k$, and the case-analysis is similar to show that the array A covers each required interaction of the form $\{(i,0),(j,1)\}$.

Thus, the array A is a $\mathrm{CA}(n+1;2,\Omega_{2k}^+,T_2)$, and it follows that

$$\mathrm{CAN}(2,\Omega_{2k}^+,T_2) \leqslant \mathrm{CAN}(2,\Omega_k^+,T_2) + 1. \qquad \square$$

In the following theorem, we use Proposition 21 to give tight bounds on the binary covering array number for circular tournaments.

Theorem 22. *For all $k \geqslant 2$,*

$$\lceil \log_2 k\rceil \leqslant CAN(2,\Omega_k,T_2) \leqslant CAN(2,\Omega_{2^{\lceil \log k\rceil}}^+,T_2) \leqslant \lceil \log_2 k\rceil + 1.$$

Proof. It is easy to see that $\mathrm{CAN}(2,\Omega_2^+,T_2) = 2$ since $E(\Omega_2^+) = \{(0,1),(1,0)\}$ and these two arcs cannot be covered in one row whereas two rows suffice. Now, by applying Proposition 21 iteratively, for all $L > 0$, we have

$$\begin{aligned}
\mathrm{CAN}(2,\Omega_{2^{L+1}}^+,T_2) &\leqslant \mathrm{CAN}(2,\Omega_{2^L}^+,T_2) + 1 \\
&\leqslant \mathrm{CAN}(2,\Omega_2^+,T_2) + L \\
&= 2 + L.
\end{aligned} \qquad (3)$$

Let $k \leqslant 2^{L+1}$. Then $\mathrm{CAN}(2,\Omega_k,T_2) \leqslant \mathrm{CAN}(2,\Omega_{2^{L+1}}^+,T_2)$ since $\Omega_k \subseteq \Omega_{2^{L+1}} \subset \Omega_{2^{L+1}}^+$. Using this fact, the bound given in (3), and the lower bound of Theorem 12, for all k such that $2^L < k \leqslant 2^{L+1}$, we have

$$\lceil \log_2 k\rceil \leqslant \mathrm{CAN}(2,\Omega_k,T_2) \leqslant \mathrm{CAN}(2,\Omega_{2^{L+1}}^+,T_2) \leqslant L + 2 = \lceil \log_2 k\rceil + 1. \qquad \square$$

Similarly, we use Proposition 21 to show that there are infinitely many values of k for which $\mathrm{CAN}(2,\Omega_k,T_2) = \lceil \log_2 k\rceil$.

Theorem 23. *Let m be an even integer. If there exists a $CA(\lceil \log_2 m\rceil; 2,\Omega_m^+,T_2)$, then, for all $L \geqslant 0$ and for all k such that $2^{L+\lceil \log_2 m\rceil-1} < k \leqslant m \cdot 2^L$, we have*

$$CAN(2,\Omega_k,T_2) = \lceil \log_2 k\rceil.$$

Proof. Suppose a $\mathrm{CA}(\lceil \log_2 m\rceil; 2,\Omega_m^+,T_2)$ exists for some even integer m. By applying Proposition 21 iteratively, for all $L > 0$, we have

$$\begin{aligned}
\mathrm{CAN}(2,\Omega_{m\cdot 2^L}^+,T_2) &\leqslant \mathrm{CAN}(2,\Omega_{m\cdot 2^{L-1}}^+,T_2) + 1 \\
&\leqslant \mathrm{CAN}(2,\Omega_m^+,T_2) + L \\
&\leqslant \lceil \log_2 m\rceil + L
\end{aligned} \qquad (4)$$

Now, fix $L \geqslant 0$ and let k be an integer such that $2^{L+\lceil \log_2 m\rceil-1} < k \leqslant m \cdot 2^L$. Then $\lceil \log_2 k\rceil = \lceil \log_2 m\rceil + L$. Since we have the inclusions $\Omega_k \subseteq \Omega_{m\cdot 2^L} \subset \Omega_{m\cdot 2^L}^+$, it follows from Proposition 5 that $\mathrm{CAN}(2,\Omega_k,T_2) \leqslant \mathrm{CAN}(2,\Omega_{m\cdot 2^L}^+,T_2)$. Now, by Theorem 12 and (4), for all k such that $2^{L+\lceil \log_2 m\rceil-1} < k \leqslant m \cdot 2^L$, we have

$$\lceil \log_2 k\rceil \leqslant \mathrm{CAN}(2,\Omega_k,T_2) \leqslant \mathrm{CAN}(2,\Omega_{m\cdot 2^L}^+,T_2) \leqslant \lceil \log_2 m\rceil + L = \lceil \log_2 k\rceil. \qquad \square$$

Corollary 24. *For all integers $L \geqslant 0$ and for any integer k such that $2^{L+4} < k \leqslant 20 \cdot 2^L$, the covering array number for covering arrays on circular k-tournament column graphs and the alphabet graph T_2 is given by*

$$CAN(2, \Omega_k, T_2) = \lceil \log_2 k \rceil.$$

Proof. For $m = 20$, we have $\lceil \log_2 m \rceil = 5$. The following is a CA($\lceil \log_2 20 \rceil; 2, \Omega_{20}^+, T_2$):

```
 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20
 1  1  0  1  0  1  0  0  1  1  0  0  0  0  1  1  1  1  0  0
 1  0  1  1  0  0  1  0  0  0  1  1  0  0  1  1  0  0  1  1
 1  0  0  0  1  1  1  0  0  0  0  0  1  1  0  0  1  1  1  1
 0  1  1  1  1  1  1  1  0  1  0  1  0  1  0  0  0  0  0  0
 0  0  0  0  0  0  0  1  1  1  1  1  1  1  0  1  0  1  0  1
```

By applying Theorem 23 with $m = 20$, it follows that for all $L \geqslant 0$, for all k such that $2^{L+\lceil \log_2 20 \rceil - 1} < k \leqslant 20 \cdot 2^L$, we have $CAN(2, \Omega_k, T_2) = \lceil \log_2 k \rceil$. $\qquad\square$

Remark 25. For any even m such that a CA($\lceil \log_2 m \rceil; 2, \Omega_m^+, T_2$) exists, we can apply Theorem 23 to obtain a result like Corollary 24. For each $L \geqslant 0$, the value of m that maximizes the range $2^{L+\lceil \log_2 m \rceil - 1} < k \leqslant m \cdot 2^L$ corresponds to an even value of m for which $m/2^{\lceil \log_2 m \rceil - 1}$ is as close to 2 as possible.

By computer search, we determined $CAN(2, \Omega_k, T_2)$ for all $k \leqslant 37$. We also determined $CAN(2, \Omega_k^+, T_2)$ for all even values of $k \leqslant 36$. The only values of $k \leqslant 37$ for which $CAN(2, \Omega_k, T_2) = \lceil \log_2 k \rceil$ are $k \in \{2, 9, 17, 18, 19, 20, 33, 34, 35, 36, 37\}$. The only even values of $k \leqslant 36$ for which $CAN(2, \Omega_k^+, T_2) = \lceil \log_2 k \rceil$ are $k \in \{18, 20, 34, 36\}$. Among these values, $m = 20$ was our best candidate for obtaining the strongest version of Corollary 24 with respect to Remark 25.

For $k = 2$ and $k = 9$, examples of the CA($\lceil \log_2 \rceil; 2, \Omega_k, T_2$) obtained from our computer search are given here:

CA($1; 2, \Omega_2, T_2$)

```
 1  2
 0  1
```

CA($4; 2, \Omega_9, T_2$)

```
 1  2  3  4  5  6  7  8  9
 1  1  1  0  1  0  0  0  0
 0  1  0  0  0  0  1  1  1
 0  0  1  0  0  1  0  0  1
 0  0  0  1  1  1  0  1  0
```

One example of a CA($5; 2, \Omega_{20}^+, T_2$) obtained in our search is given in Corollary 24. From this array, we obtain a CA($6; 2, \Omega_{40}^+, T_2$) using Proposition 21. By removing appropriate columns from the CA($5; 2, \Omega_{40}^+, T_2$) which correspond to the way in which $\Omega_{k-1} \subset \Omega_k$, we obtain optimal CA($\lceil \log_2 k \rceil, \Omega_k, T_2$) for $k = 38, 39$. Hence, the next value of k for which the covering array number is unknown is 41.

4.4 Data and questions for binary covering arrays on tournaments

Transitive tournaments and circular tournaments meet the lower bound of Theorem 12 for infinitely many values of k. We are interested in knowing how often the upper bound of Theorem 12 is met, or whether there is a tighter bound for tournament column graphs.

Using a computer search and the database of tournaments given in [8], for each $k \leqslant 9$, we determined $\mathrm{CAN}(2, O_k, T_2)$ for every k-tournament O_k (up to isomorphism). Our findings are summarized in Table 1 and more details are given in Appendix A. In Table 1, we use L.B. and U.B. to denote the lower and upper bounds of Theorem 12, respectively. In Table 1, we write CAN to abbreviate the covering array number on a given tournament column graph with alphabet graph T_2.

k	L.B.	U.B.	k-tournaments achieving U.B.
2	1	1	T_2 (the unique 2-tournament)
3	2	3	only Ω_3
4	2	3	exactly those 4-tournaments that contain Ω_3 as a subgraph
5	3	4	only Ω_5
6	3	4	Ω_6 and 12 others (not all contain Ω_5 as a subgraph)
7	3	5	no 7-tournaments achieve U.B. Ω_7 and 380 others with CAN $= 4$
8	3	5	no 8-tournaments achieve U.B. Ω_8 and 6836 others with CAN $= 4$
9	4	5	only two 9-tournaments achieve U.B. (neither is isomorphic to Ω_9)

Table 1: Summary of analysis for small tournaments.

Our analysis of small tournaments shows that for $k \in \{2, 3, 4, 5, 6, 9\}$, the upper bound of Theorem 12 is attained.

Problem 26. Determine whether the upper bound of Theorem 12 is met infinitely often. If not, determine a tight numerical upper bound on $\mathrm{CAN}(2, O_k, T_2)$ for any k-tournament O_k.

Based on our data in Table 1, we ask whether the bounds of Theorem 22 hold for all k-tournaments.

Problem 27. For any k-tournament O_k, is $\mathrm{CAN}(2, O_k, T_2) \in \{\lceil \log_2 k \rceil, \lceil \log_2 k \rceil + 1\}$?

For circular tournaments in particular, Theorem 22 determines $\mathrm{CAN}(2, \Omega_k, T_2)$ to within 1 row. The following problem is also of interest.

Problem 28. For each $L \geqslant 2$, for all k in the range $2^{L-1} < k \leqslant 2^L$, the covering array number $\mathrm{CAN}(2, \Omega_k, T_2)$ ranges from $\lceil \log_2 k \rceil$ to $\lceil \log_2 k \rceil + 1$, non-decreasingly as k increases. For all $L \geqslant 2$, for each range $2^{L-1} < k \leqslant 2^L$, determine the threshold value of k for which $\mathrm{CAN}(2, \Omega_k, T_2) = \lceil \log_2 k \rceil$ and $\mathrm{CAN}(2, \Omega_{k+1}, T_2) = \lceil \log_2 k \rceil + 1$.

Aside from a tight numerical upper bound on $\mathrm{CAN}(2, O_k, T_2)$, we are interested in finding an extremal family of k-tournaments. In particular, we wish to have a structural characterization of some infinite family of k-tournaments, say $\{X_k\}_{k=2}^{\infty}$, for which $\mathrm{CAN}(2, O_k, T_2) \leqslant \mathrm{CAN}(2, X_k, T_2)$ for every k-tournament O_k.

Problem 29. Find and characterize an infinite family $\{X_k\}_{k=2}^{\infty}$ of k-tournaments for which $\mathrm{CAN}(2, O_k, T_2) \leqslant \mathrm{CAN}(2, X_k, T_2)$ for every k-tournament O_k.

It was conjectured in [7, Conjecture 4.5.1] that circular tournaments constitute an extremal family of tournaments as described in Problem 29; however, our data for 9-tournaments shows that this is not the case. Adjacency matrices and other properties for the two exceptional 9-tournaments referred to in Table 1 are given in Appendix A.1.

Acknowledgements

References

[1] C. J. Colbourn. Combinatorial aspects of covering arrays. *Le Matematiche (Catania)*, 59(1-2):125–172, 2004.

[2] L. Gargano, J. Körner, and U. Vaccaro. Sperner capacities. *Graphs and Combinatorics*, 9(1):31–46, 1993.

[3] L. Gargano, J. Körner, and U. Vaccaro. Capacities: From information theory to extremal set theory. *J. Combin. Theory Ser. A*, 68(2):296–316, 1994.

[4] P. Hell and J. Nešetřil. *Graphs and Homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[5] G. O. H. Katona. Two applications (for search theory and truth functions) of Sperner type theorems. *Period. Math. Hungar.*, 3:19–26, 1973.

[6] D. J. Kleitman and J. Spencer. Families of k-independent sets. *Discrete Math.*, 6:255–262, 1973.

[7] E. Maltais. *Graph-dependent Covering Arrays and LYM Inequalities*. Ph.D. thesis. University of Ottawa, 2016.

[8] B. McKay. Combinatorial Data: Digraphs, avaliable at: http://users.cecs.anu.edu.au/~bdm/data/digraphs.html.

[9] K. Meagher and B. Stevens. Covering arrays on graphs. *J. Combin. Theory Ser. B*, 95(1):134–151, 2005.

[10] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27(1):544–548, 1928.

A Extremal k-tournaments for $2 \leqslant k \leqslant 9$

Up to $k = 9$, we determined $\text{CAN}(2, O_k, T_2)$ for all pairwise non-isomorphic k-tournaments O_k by using a computer search and the lists of all k-tournaments given in [8]. For each k-tournament O_k with $k \leqslant 9$, we found that $\text{CAN}(2, O_k, T_2) \in \{\lceil \log_2 k \rceil, \lceil \log_2 k \rceil + 1\}$. In the following table, for each k such that $2 \leqslant k \leqslant 9$, we give the lower and upper bounds of Theorem 12, denoted L.B. and U.B., respectively. We write CAN to abbreviate the covering array number for a covering array on a given tournament column graph with alphabet graph T_2. For $2 \leqslant k \leqslant 9$, we give the number of k-tournaments with CAN $= n$ for each possible value of n in the range L.B. $\leqslant n \leqslant$ U.B.. For some parameters, the actual tournaments are given as the upper triangle of the adjacency matrix in row order, on one line without spaces. We refer to this representation as the *adjacency vector*. For example, the 4-tournament represented by the adjacency vector 000111 has the following adjacency matrix and graph:

$$\begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{1} & \mathbf{1} \\ 1 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

By permuting the rows and columns of its adjacency matrix, any given tournament has several distinct adjacency vector representations. In Table 2, each given adjacency vector is represented exactly as given in the lists in [8], with the exception of the 9-tournaments; the adjacency vectors of the given 9-tournaments have been reconfigured in order to emphasize their structure which we discuss in more detail in Appendix A.1.

Table 2: Data for small tournaments.

$k = 3$:	number of pairwise non-isomorphic 3-tournaments:	2
	number with CAN = L.B. = 2:	1
	number with CAN = U.B. = 3:	1
	adjacency vectors of 3-tournaments with CAN = 3:	101
$k = 4$:	number of pairwise non-isomorphic 4-tournaments:	4
	number with CAN = L.B. = 2:	1
	number with CAN = U.B. = 3:	3
	adjacency vectors of 4-tournaments with CAN = 3:	000101
		100010
		100100

$k = 5$:	number of pairwise non-isomorphic 5-tournaments:	12
	number with CAN = L.B. = 3:	11
	number with CAN = U.B. = 4:	1
	adjacency vectors of 5-tournaments with CAN = 4:	1100101110

$k = 6$:	number of pairwise non-isomorphic 6-tournaments:	56
	number with CAN = L.B. = 3:	43
	number with CAN = U.B. = 4:	13
	adjacency vectors of 6-tournaments with CAN = 4:	000001100101110
		000101000010001
		010001000101100
		010001000110101
		010001000111101
		100001010010001
		100001100101110
		100101001001000
		101001100010101
		101001100100110
		110001001011000
		110001001110101
		110001010110000

$k = 7$:	number of pairwise non-isomorphic 7-tournaments:	456
	number with CAN = L.B. = 3:	75
	number with CAN = 4:	381
	number with CAN = U.B. = 5:	0
	(adjacency vectors omitted)	

$k = 8$:	number of pairwise non-isomorphic 8-tournaments:	6880
	number with CAN = L.B. = 3:	43
	number with CAN = 4:	6837
	number with CAN = U.B. = 5:	0
	(adjacency vectors omitted)	

$k = 9$:	number of pairwise non-isomorphic 9-tournaments:	191536
	number with CAN = L.B. = 4:	191534
	number with CAN = U.B. = 5:	2

adjacency vectors of 9-tournaments with CAN = 5:	101110001000111110001001111101110101
	101110001000111110001001111101110010

A.1 Properties of exceptional 9-tournaments

Since there are only two exceptional 9-tournaments, we give more details about their properties here in the hope that this information may help to solve Problem 29.

Let X_{9a} and X_{9b} denote the 9-tournaments with the following adjacency vectors and adjacency matrices, respectively.

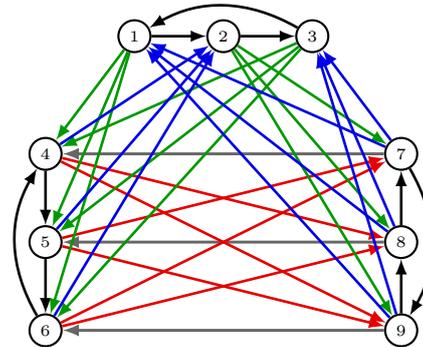**a.** 101110001000111111000100111101110101

b. 101110001000111111000100111101110010

a.	1	2	3	4	5	6	7	8	9
1	0	1	0	1	1	1	0	0	0
2	0	0	1	0	0	0	1	1	1
3	1	0	0	1	1	1	0	0	0
4	0	1	0	0	1	0	0	1	1
5	0	1	0	0	0	1	1	0	1
6	0	1	0	1	0	0	1	1	0
7	1	0	1	1	0	0	0	1	0
8	1	0	1	0	1	0	0	0	1
9	1	0	1	0	0	1	1	0	0

b.	1	2	3	4	5	6	7	8	9
1	0	1	0	1	1	1	0	0	0
2	0	0	1	0	0	0	1	1	1
3	1	0	0	1	1	1	0	0	0
4	0	1	0	0	1	0	0	1	1
5	0	1	0	0	0	1	1	0	1
6	0	1	0	1	0	0	1	1	0
7	1	0	1	1	0	0	0	0	1
8	1	0	1	0	1	0	1	0	0
9	1	0	1	0	0	1	0	1	0
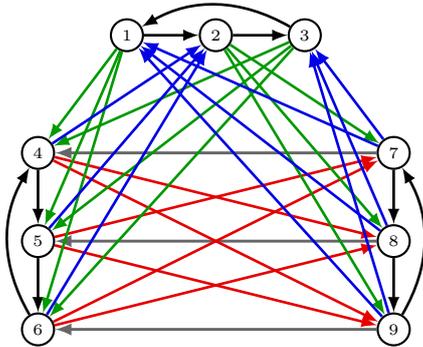
X_{9a}

X_{9b}

The exceptional 9-tournaments X_{9a} and X_{9b} are both regular with in- and outdegree 4. Both are isomorphic to their own reversal. Neither is isomorphic to Ω_9. Both have exactly three automorphisms which correspond to the rotations of the directed 3-cycle $(4, 5, 6)$ (and simultaneous rotations of the directed 3-cycles on vertices $7, 8, 9$) while the vertices $1, 2, 3$ remain fixed. In X_{9a} and X_{9b}, the directed 3-cycles on vertices $7, 8, 9$ are oppositely oriented. From our computer search, we know that $\mathrm{CAN}(2, X_{9a}, T_2) > 4$. On the other hand, since $9 < \binom{5}{2}$, finding a $\mathrm{CA}(5; 2, X_{9a}, T_2)$ is easy; simply take columns corresponding to the 2-subsets of $\{1, \dots, 5\}$ which form an antichain. The same works for X_{9b}.