On the isomorphism problem for Cayley graphs of abelian groups whose Sylow subgroups are elementary abelian or cyclic

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Abstract

We show that if certain arithmetic conditions hold, then the Cayley isomorphism problem for abelian groups, all of whose Sylow subgroups are elementary abelian or cyclic, reduces to the Cayley isomorphism problem for its Sylow subgroups. This yields a large number of results concerning the Cayley isomorphism problem, perhaps the most interesting of which is the following: if $p_1, \ldots, p_r$ are distinct primes satisfying certain arithmetic conditions, then two Cayley digraphs of $\mathbb{Z}_{p_1}^{a_1} \times \cdots \times \mathbb{Z}_{p_r}^{a_r}$, $a_i \leqslant 5$, are isomorphic if and only if they are isomorphic by a group automorphism of $\mathbb{Z}_{p_1}^{a_1} \times \cdots \times \mathbb{Z}_{p_r}^{a_r}$. That is, that such groups are CI-groups with respect to digraphs.

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1 Introduction

The history of the modern Cayley isomorphism problem begins in 1967 when Ádám [1] conjectured that any two Cayley graphs of the cyclic group $\mathbb{Z}_n$ of order n are isomorphic if and only if they are isomorphic by a group automorphism of $\mathbb{Z}_n$. While Ádám's conjecture was quickly shown to be false [19], the conjecture nonetheless generated much interest in the following obvious generalization: Are two Cayley graphs of a group G isomorphic if and only if they are isomorphic by a group automorphism of G? If so, we say that G is a **CI-group with respect to graphs**. This problem naturally generalizes to any class of combinatorial objects (see [33] for several equivalent formulations of the precise definition of a combinatorial object), and in fact was considered much earlier for designs

[2, 27]. Our question is then: For a group G is it true that any two Cayley objects of G in some class $\mathcal{K}$ of combinatorial objects are isomorphic if and only if they are isomorphic by a group automorphism of G? If the answer to this question is yes, we say that **G is a CI-group with respect to $\mathcal{K}$**. If G is a CI-group with respect to *every* class of combinatorial objects, we say that G is a **CI-group**. In 1987, Pálfy [34] proved the following remarkable result:

Theorem 1. *A group G of order n is a CI-group if and only if* $\gcd(n, \varphi(n)) = 1$ *or* $n = 4$, *where φ is Euler's phi function.*

Definition 2. Given a group G and $g \in G$, define $g_L : G \mapsto G$ by $g_L(h) = gh$, and $G_L = \{g_L : g \in G\}$. Then $G_L \cong G$ is a group, the **left regular representation of G**. A **Cayley object of G** is a combinatorial object X with $G_L \leqslant \mathrm{Aut}(X)$, the automorphism group of X.

We remark that a classical result of Sabidussi [35] gives that the definition of a Cayley object above is consistent with the usual definition of a Cayley digraph when the object is a digraph.

An essential tool in proving Pálfy's Theorem is the following result of Babai, which characterizes the CI-property:

Lemma 3. *For a group G and a class $\mathcal{K}$ of combinatorial objects the following are equivalent:*

1. *G is a CI-group with respect to $\mathcal{K}$,*
2. *whenever X is a Cayley object of G in $\mathcal{K}$ and $\delta \in \mathrm{Sym}(G)$ such that $\delta^{-1}G_L\delta \leqslant \mathrm{Aut}(X)$, then G_L and $\delta^{-1}G_L\delta$ are conjugate in $\mathrm{Aut}(X)$.*

Babai's Lemma has been generalized to give a similar characterization of the solution to the isomorphism problem for Cayley objects X of G in classes $\mathcal{K}$ when G is not a CI-group with respect to $\mathcal{K}$ (see [33, Lemma 1.1], [8, Lemma 13], and [13, Lemma 20]). All such results basically reduce to determining the conjugacy classes of G_L in $\mathrm{Aut}(X)$. In fact, in the positive direction, Pálfy showed that if $\gcd(n, \varphi(n)) = 1$, then there is always one conjugacy class of $(\mathbb{Z}_n)_L$ in $\langle(\mathbb{Z}_n)_L, \delta^{-1}(\mathbb{Z}_n)_L\delta\rangle$ (we remark that every group of order n is cyclic if and only if $\gcd(n, \varphi(n)) = 1$ [36, Theorem 9.2.7]). To place the work in this paper in its proper context, it will be useful to discuss the structure of Pálfy's proof in more detail, and to do this, we will need the following additional definition:

Definition 4. Let G be a transitive group acting on Ω. Let Y be the set of all complete block systems of G. Define a partial order on Y by $\mathcal{B} \preceq \mathcal{C}$ if and only if every block of $\mathcal{C}$ is a union of blocks of $\mathcal{B}$. Let $n = \Pi_{i=1}^r p_i^{a_i}$ be the prime factorization of n and define $\Omega : \mathbb{N} \mapsto \mathbb{N}$ by $\Omega(n) = \Sigma_{i=1}^r a_i$. Let $m = \Omega(n)$. A transitive group G of degree n is **m-step imprimitive** if there exists a sequence of complete block systems $\mathcal{B}_0 \prec \mathcal{B}_1 \prec \cdots \prec \mathcal{B}_m$. A complete block system $\mathcal{B}$ will be said to be **normal** if $\mathcal{B}$ is formed by the orbits of a normal subgroup. We will say that G is **normally m-step imprimitive** if each $\mathcal{B}_i$, $0 \leqslant i \leqslant m$, is formed by the orbits of a normal subgroup of G.

Note that $\mathcal{B}_0$ consists of singletons, while $\mathcal{B}_m = \{\Omega\}$. Also, $\mathcal{B}_1$ consists of blocks of prime size, and in general, $\mathcal{B}_i$ consists of blocks of size a product of i (not necessarily distinct) primes, and of course each block of $\mathcal{B}_i$ is a union of blocks of $\mathcal{B}_{i-1}$.

The proof of the positive direction of Theorem 1 is broken into two parts which use very different techniques. In the first part, Pálfy essentially shows that if $\delta \in \mathrm{Sym}(n)$, then there exists $\gamma \in \langle (\mathbb{Z}_n)_L, \delta^{-1}(\mathbb{Z}_n)_L\delta\rangle$ such that $\langle (\mathbb{Z}_n)_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ is normally m-step imprimitive. This is shown using the fact that all doubly-transitive groups are known [3, Theorem 5.3], a consequence of the Classification of the Finite Simple Groups. He then shows that if $\langle (\mathbb{Z}_n)_L, \delta^{-1}(\mathbb{Z}_n)_L\delta\rangle$ is normally m-step imprimitive, then there exists $\gamma \in \langle (\mathbb{Z}_n)_L, \delta^{-1}(\mathbb{Z}_n)_L\delta\rangle$ such that $\gamma^{-1}\delta^{-1}(\mathbb{Z}_n)_L\delta\gamma = (\mathbb{Z}_n)_L$. The techniques for showing this do not depend upon the Classification of the Finite Simple Groups, and in fact one can simply use the Sylow Theorems to show this [11].

Muzychuk [33, Theorem 1.9] showed that Pálfy's Theorem could be generalized to cyclic groups $\mathbb{Z}_n$, for more values of n. More specifically, let $k = p_1 \cdots p_r$ be such that $\gcd(k, \varphi(k)) = 1$, and $n = p_1^{a_1} \cdots p_r^{a_r}$. Muzychuk showed that the Cayley isomorphism problem for cyclic groups $\mathbb{Z}_n$ can be reduced to the Cayley isomorphism problem of the Sylow subgroups of $\mathbb{Z}_n$. The general structure of his proof is the same as that of Pálfy's proof. He showed that after an appropriate conjugation, $\langle (\mathbb{Z}_n)_L, \delta^{-1}(\mathbb{Z}_n)_L\delta\rangle$ is normally m-step imprimitive using the Classification of the Finite Simple Groups [33, Theorem 4.9] (with no restrictions on n), and then used very different "Sylow type" arguments to show that after an additional conjugation, the group $\langle (\mathbb{Z}_n)_L, \delta^{-1}(\mathbb{Z}_n)_L\delta\rangle$ has very nice properties (these properties are equivalent to being nilpotent), which are then used to deduce his main result.

In 2003, the author [8] showed that the second part of each of the above proofs could be generalized to show that if G is an abelian group of order n (n again satisfies the conditions given in the previous paragraph) and $\langle G_L, \delta^{-1}G_L\delta\rangle$ is normally m-step imprimitive, then there exists $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ is nilpotent, in which case the Cayley isomorphism problem for G reduces to the Cayley isomorphism problem for the Sylow subgroups of G. It turns out that the "first part" of the above proofs does not generalize to arbitrary abelian groups of order n. The author and Spiga [16, Theorem 2.1] give examples of abelian groups of certain orders n and $\delta \in \mathrm{Sym}(n)$ such that there is no $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ is normally m-step imprimitive (this is not explicitly stated in [16, Theorem 2.1], but is established in the proof of that result). Recently, the author [13] extended the results in [8] to nilpotent groups, as well as to a larger class of abelian groups.

In this paper, we will show in Section 4 that for $k = p_1 \cdots p_r$, $n = p_1^{a_1} \cdots p_r^{a_r}$, whenever G is an abelian group of order n with Sylow p_i-subgroup P_i that is elementary abelian or cyclic and the only prime divisor of $|\mathrm{Aut}(P_i)|$ that divides n is p_i, then whenever $\delta \in \mathrm{Sym}(G)$ then there exists $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ is nilpotent, and consequently, normally m-step imprimitive (Theorem 22). Consequences of this result are considered in Section 5, where in particular we show for such G that the Cayley isomorphism problem for G reduces to the Cayley isomorphism problem for P_i, and in the case of color digraphs, we show that if P_i is elementary abelian and $1 \leqslant a_i \leqslant 5$, then

G is a CI-group with respect to color digraphs (Corollary 39). Finally, additional open problems are discussed in Section 6.

An additional comment is in order. Our main motivation in considering the Cayley isomorphism problem in general is to investigate the Cayley isomorphism problem for Cayley digraphs. This is what has guided our restricting Sylow p-subgroups to be either elementary abelian or cyclic, as the Cayley isomorphism problem for digraphs has been has been studied for these groups (many of the corresponding results are mentioned in Section 5), while for other abelian p-groups the isomorphism problem has been solved only for the group $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ [6].

Definition 5. Let G be a transitive group acting on Ω and $\mathcal{B}$ a complete block system of G. By $\mathrm{fix}_G(\mathcal{B})$ we mean the subgroup of G which fixes each block of $\mathcal{B}$ set-wise. That is, $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$. We denote by $\mathrm{Stab}_G(x)$ the stabilizer of $x \in \Omega$, and for $B \in \mathcal{B}$, $\mathrm{Stab}_G(B)$ is the set-wise stabilizer of the block B. That is, $\mathrm{Stab}_G(x) = \{g \in G : g(x) = x\}$ and $\mathrm{Stab}_G(B) = \{g \in G : g(B) = B\}$. Finally, for $g \in G$ we denote by $g/\mathcal{B}$ permutation of $\mathcal{B}$ induced by g, and $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$.

The following result of C. H. Li [28, Theorem 1.1] will be crucial.

Theorem 6. *Let X be a primitive permutation group of degree n. Then X contains an abelian regular subgroup A if and only if either*

1. $X \leqslant \mathrm{AGL}(d,p)$, *where p is prime, $d \geqslant 1$, and $n = p^d$, or*

2. $X = (\tilde{T}_1 \times \cdots \times \tilde{T}_\ell).O.Q$, $A = A_1 \times \cdots \times A_\ell$, *and* $n = m^\ell$, *where* $\ell \geqslant 1$, *each* $A_i \leqslant \tilde{T}_i$ *with* $|A_i| = m$, $\tilde{T}_1 \cong \ldots \tilde{T}_\ell$, $O \leqslant \mathrm{Out}(\tilde{T}_1) \times \cdots \times \mathrm{Out}(\tilde{T}_\ell)$, *$Q$ is a transitive permutation group of degree ℓ, and one of the following holds:*

   (a) $(\tilde{T}_i, A_i) = (\mathrm{PSL}(2,11), \mathbb{Z}_{11})$, $(\mathrm{M}_{11}, \mathbb{Z}_{11})$, $(\mathrm{M}_{12}, \mathbb{Z}_2^2 \times \mathbb{Z}_3)$, $(\mathrm{M}_{23}, \mathbb{Z}_{23})$;

   (b) $\tilde{T}_i = \mathrm{PGL}(d,q)$, *and* $A_i = \mathbb{Z}_{(q^d-1)/(q-1)}$ *is a Singer subgroup;*

   (c) $\tilde{T}_i = \mathrm{P\Gamma L}(2,8)$ *and* $\mathbb{Z}_9 = A_i \not\leqslant \mathrm{PSL}(2,8)$;

   (d) $\tilde{T}_i = \mathrm{Sym}(m)$ *or* $\mathrm{Alt}(m)$, *and A_i is abelian of order m.*

Ultimately, we will show by induction on $m = \Omega(n)$ that, under certain arithmetic conditions, if $G \leqslant \mathrm{Sym}(n)$ is transitive, abelian and has Sylow subgroups elementary abelian or cyclic, then for $\delta \in \mathrm{Sym}(n)$ there exists $\gamma \in \langle G, \delta^{-1}G\delta\rangle$ such that $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma\rangle$ is m-step imprimitive. We first show in Section 2 that there exists $\gamma \in \langle G, \delta^{-1}G\delta\rangle$ such that $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma\rangle$ is imprimitive. Now, if $\langle G, \delta^{-1}G\delta\rangle$ is imprimitive with $\mathcal{C}$ a complete block system of $\langle G, \delta^{-1}G\delta\rangle$ with no nontrivial block system $\mathcal{B} \prec \mathcal{C}$, then either $\mathcal{C}$ consists of blocks of prime size, in which case we are finished using the induction hypothesis, or the blocks of $\mathcal{C}$ are of composite size and $\mathrm{Stab}_{\langle G,\delta^{-1}G\delta\rangle}(\mathcal{B})|_B$ is primitive in its action on $B \in \mathcal{B}$ by [4, Exercise 1.5.10]. We next show that we need only consider when the blocks of $\mathcal{C}$ are of prime power size. We then analyze the various cases for $\mathrm{Stab}_{\langle G,\delta^{-1}G\delta\rangle}(\mathcal{B})|_B$ given by Theorem 6 in Section 3.

Finally, one of our intentions in this paper is begin to determine when the isomorphism problem for a direct product $\Pi_{i=1}^r G_i$ of groups reduces to the isomorphism problem for the G_i, $1 \leqslant i \leqslant r$ (as opposed to specifically studying abelian groups whose Sylow subgroups are elementary abelian or cyclic). Induction seems to be the best way to attack this problem, but there is an inherent difficulty. Namely, if $\mathcal{B}$ is a complete block system of $\langle G, \delta^{-1}G\delta\rangle$ where G is a transitive abelian group and $\delta \in S_G$, it need not be the case that $G/\mathcal{B} \cong \delta^{-1}G\delta/\mathcal{B}$ or $\mathrm{fix}_G(\mathcal{B}) \cong \mathrm{fix}_{\delta^{-1}G\delta}(\mathcal{B})$. But of course $G/\mathcal{B}$, $\delta^{-1}G\delta/\mathcal{B}$, $\mathrm{fix}_G(\mathcal{B})$, and $\mathrm{fix}_{\delta^{-1}G\delta}(\mathcal{B})$ are all abelian groups. So if possible we will use G_1 and G_2 instead of G and $\delta^{-1}G\delta$, and only assume that G_1 and G_2 are abelian and not assume that $G_1 \cong G_2$. In fact, we will often only need that $\mathrm{fix}_{G_1}(\mathcal{B}) \cong \mathrm{fix}_{G_2}(\mathcal{B})$, and then certain properties of the quotient.

Throughout this paper, all groups are finite. For group theoretic terms not defined in this paper, see [4].

2 We may assume $\langle G_1, G_2\rangle$ is imprimitive with blocks of prime-power order

We shall have need of the following elementary result whose straightforward proof is left to the reader.

Lemma 7. *Let $G_1, G_2 \leqslant \mathrm{Sym}(n)$ be transitive such that both G_1 and G_2 admit $\mathcal{B}$ as a complete block system. Then $\langle G_1, G_2\rangle$ admits $\mathcal{B}$ as a complete block system.*

The following result is proven in more generality than is needed here.

Lemma 8. *Let $n = kp^a$, where p is an odd prime, and $p^a \geqslant 5$. Let $G_1, G_2 \leqslant \mathrm{Sym}(n)$ be transitive such that $\langle G_1, G_2\rangle$ admits a complete block system $\mathcal{C}$ of n/p^a blocks of size p^a such that $\mathrm{Alt}(p^a) \leqslant \mathrm{Stab}_{\langle G_1,G_2\rangle}(C)|_C$ and $\mathrm{fix}_{G_i}(\mathcal{C})$ is abelian and semiregular while $\mathrm{fix}_{G_i}(\mathcal{C})|_C$ is transitive for every $C \in \mathcal{C}$ and $i = 1, 2$. Then there exists $\gamma \in \langle G_1, G_2\rangle$ such that $\langle G_1, \gamma^{-1}G_2\gamma\rangle$ admits a complete block system $\mathcal{B}$ with n/p blocks of size p.*

Proof. If $a = 1$, the result is trivial, so we assume that $a \geqslant 2$. As $p^a \geqslant 5$, $\mathrm{Alt}(p^a)$ is simple. As $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})|_C \triangleleft \mathrm{Stab}_{\langle G_1,G_2\rangle}(C)|_C$ for every $C \in \mathcal{C}$, we have that $\mathrm{Alt}(p^a) \leqslant \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})|_C$ for every $C \in \mathcal{C}$. Let $J_i \leqslant \mathrm{fix}_{G_i}(\mathcal{C})$ be of order p, $i = 1, 2$, and $C_0 \in \mathcal{C}$. As $\mathrm{Alt}(p^a) \leqslant \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})|_{C_0}$, $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})|_{C_0}$ is at least $(p^a - 2)$-transitive. Choose $p^{a-1} - 1$ orbits $\mathcal{O}_1^i, \ldots, \mathcal{O}_{p^{a-1}-1}^i$ of order p of $J_i|_{C_0}$, $i = 1, 2$. Then there exists $\gamma_0 \in \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ such that $\gamma_0^{-1}(\mathcal{O}_j^2) = \mathcal{O}_j^1$, $1 \leqslant j \leqslant p^{a-1} - 1$. We conclude that the p elements of C_0 not in any $\mathcal{O}_i^2$ are mapped by γ_0^{-1} to the p elements of C_0 not in any $\mathcal{O}_i^1$ as $p > 2$. Hence the orbits of $J_1|_{C_0}$ and $\gamma_0^{-1}J_2\gamma_0|_{C_0}$ are identical. Notice also that if the action of $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ on C_0 is equivalent to the action of $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ on $C \in \mathcal{C}$, then we also have that the orbits of $J_1|_C$ and $\gamma_0^{-1}J_2\gamma_0|_C$ are identical.

Define an equivalence relation $\equiv_0$ on $\mathcal{C}$ by $C \equiv_0 C'$ if and only the action of $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ on C is equivalent to the action of $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ on C'. As $\mathrm{Alt}(p^a)$ has only one representation as $p^a \neq 6$ [3, Theorem 5.3], and two transitive actions of a group are equivalent if and only if the stabilizer of a point in one action is the same as the stabilizer of a point in the other [4, Lemma 1.6B], if $C \not\equiv_0 C'$, then the action of $\mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})|_{C\cup C'}$ on C' cannot be a faithful action. Thus if $C \not\equiv_0 C'$, then there exists $\alpha \in \langle G_1, G_2\rangle$ such that $\alpha|_C = 1$ but $\alpha|_{C'} \neq 1$.

Let E_0 be the equivalence class of $\equiv_0$ that contains C_0. Then the orbits of $J_1|_C$ and $\gamma_0^{-1}J_2\gamma_0|_C$ are identical for $C \in E_0$. Let $L_1 = \{\alpha \in \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C}) : \alpha|_{\cup E_0} = 1\}$. Then $L_1 \triangleleft \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ and $L_1|_C \triangleleft \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})|_C$ for every $C \in \mathcal{C}$. If $C_1 \in \mathcal{C}$, $C_1 \not\equiv_0 C_0$, and $\omega \in \mathrm{fix}_{\langle G_1,G_2\rangle}(\mathcal{C})$ such that $\omega|_{C_0} = 1$ but $\omega|_{C_1} \neq 1$ (note that as $C_1 \not\equiv_0 C_0$, such an ω exists), then for every $C \in E_0$, we have that $\omega|_C = 1$ as well. Then $L_1 \neq 1$. We conclude that $\mathrm{Alt}(p^a) \leqslant L_1|_{C_1}$ for every $C_1 \not\equiv_0 C_0$ as $\mathrm{Alt}(p^a)$ is simple. Also observe that as p^a is odd, $J_1|_{C_1}$ and $\gamma_0^{-1}J_2\gamma_0|_{C_1} \leqslant L_1|_{C_1}$ for every $C_1 \not\equiv_0 C_0$. By arguments analogous to those above, there exists $\gamma_1 \in L_1$ such that the orbits of $J_1|_C$ and $\gamma_1^{-1}\gamma_0^{-1}J_2\gamma_0\gamma_1|_C$ are identical for every $C \equiv_0 C_1$. Furthermore, as $L_1|_C = 1$ for every $C \equiv_0 C_0$, we have that the orbits $J_1|_C$ and $\gamma_1^{-1}\gamma_0^{-1}J_2\gamma_0\gamma_1|_C$ are identical for every $C \in E_0 \cup \{C_1\}$. As before, if the action of L_1 on C_1 is equivalent to the action of L_1 on C, then we also have that the orbits of $J_1|_C$ and $\gamma_1^{-1}\gamma_0^{-1}J_2\gamma_0\gamma_1|_C$ are identical. Continuing inductively, we find $\gamma \in \langle G_1, G_2\rangle$ such that the orbits of J_1 and $\gamma^{-1}J_2\gamma$ are identical. Then G_1 admits a complete block system $\mathcal{B}$ formed by the orbits of $J_1 \triangleleft G_1$ and $\gamma^{-1}G_2\gamma$ admits $\mathcal{B}$ as a complete block system formed by the orbits of $\gamma^{-1}J_2\gamma \triangleleft \gamma^{-1}G_2\gamma$, and so by Lemma 7, $\langle G_1, \gamma^{-1}G_2\gamma\rangle$ admits $\mathcal{B}$ as a complete block system of n/p blocks of size p. □

Corollary 9. *Let $G_1, G_2 \leqslant \mathrm{Sym}(n)$ be transitive abelian groups, with n odd and composite. If $\langle G_1, G_2\rangle$ is primitive, then there exists $\delta \in \langle G_1, G_2\rangle$ such that $\langle G_1, \delta^{-1}G_2\delta\rangle$ is imprimitive.*

Proof. Assume $\langle G_1, G_2\rangle$ is primitive. If Theorem 6 (1), (2a) with $(\tilde{T}_i, G_i) \neq (M_{12}, \mathbb{Z}_2^2 \times \mathbb{Z}_3)$ or (2c) occurs, then n is a power of some prime p, and both G_1 and G_2 are p-groups. Then there exists $\delta \in \langle G_1, G_2\rangle$ such that $\langle G_1, \delta^{-1}G_2\delta\rangle$ is a p-group. As a p-group contains a nontrivial center, $\langle G_1, \delta^{-1}G_2\delta\rangle$ contains a normal subgroup of order p, and so has blocks of size p. If Theorem 6 (2) holds and $\ell \geqslant 2$, then $G_1, G_2 \leqslant \tilde{T}_1 \times \tilde{T}_2 \times \cdots \times \tilde{T}_L$, which is imprimitive. Thus $\langle G_1, G_2\rangle$ is imprimitive. Hence $\ell = 1$. If Theorem 6 (2b) holds, then both G_1 and G_2 are cyclic, and the result follows by [32, Theorem 4.9]. Note that Theorem 6 (2a) with $(\tilde{T}_i, G_i) = (M_{12}, \mathbb{Z}_2^2 \times \mathbb{Z}_3)$ cannot hold as n is odd and composite. Finally, if Theorem 6 (2d) holds, then the result follows by Lemma 8 with $\mathcal{C}$ as in Lemma 8 the trivial complete block system consisting of one block of size n. □

Lemma 10. *Let n be a positive integer such that if k is a proper divisor of n and $H_1, H_2 \leqslant \mathrm{Sym}(k)$ are transitive abelian groups, then there exists $\delta \in \langle H_1, H_2\rangle$ such that $\langle H_1, \delta^{-1}H_2\delta\rangle$ is nilpotent. Let $G_1, G_2 \leqslant \mathrm{Sym}(n)$ be transitive abelian groups. If $\langle G_1, G_2\rangle$ is imprimitive, then there exists a prime divisor $p|n$ and $\delta \in \langle G_1, G_2\rangle$ such that $\langle G_1, \delta^{-1}G_2\delta\rangle$ admits a complete block system $\mathcal{B}$ of n/p^a blocks of size p^a, $a \geqslant 1$.*

Proof. Let $\mathcal{C}$ be a nontrivial complete block system of $\langle G_1, G_2\rangle$ such that there exists no nontrivial block system $\mathcal{D}$ with $\mathcal{D} \prec \mathcal{C}$. As both $G_1/\mathcal{C}$ and $G_2/\mathcal{C}$ are transitive abelian groups, and a transitive abelian group is regular, $\mathrm{fix}_{G_i}(\mathcal{C}) \neq 1$, $i = 1, 2$ and acts transitively on $C \in \mathcal{C}$.

Let $C_0 \in \mathcal{C}$. By hypothesis there exists $\delta_0 \in \langle \mathrm{fix}_{G_1}(\mathcal{C}), \mathrm{fix}_{G_2}(\mathcal{C})\rangle$ such that the group $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0\rangle|_{C_0}$ is nilpotent. Note that $\mathrm{fix}_{\delta_0^{-1}G_2\delta_0}(\mathcal{C}) = \delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0$. Similarly, if $C_1 \in \mathcal{C}$ such that $C_1 \neq C_0$, then there exists $\delta_1 \in \langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0\rangle$ such that $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta_1^{-1}\delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0\delta_1\rangle|_{C_1}$ is nilpotent. Further, as $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0\rangle|_{C_0}$ is nilpotent and $\delta_1 \in \langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0\rangle$, we have $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta_1^{-1}\delta_0^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta_0\delta_1\rangle|_{C_0}$ is nilpotent. Continuing this procedure inductively, we find $\delta = \delta_0\delta_1\cdots\delta_{\ell-1}$ such that $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \delta^{-1}\mathrm{fix}_{G_2}(\mathcal{C})\delta\rangle|_C$ is nilpotent for every $C \in \mathcal{C}$, where $\ell = n/|C|$, $C \in \mathcal{C}$. We thus may assume without loss of generality that $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \mathrm{fix}_{G_2}(\mathcal{C})\rangle|_C$ is nilpotent for every $C \in \mathcal{C}$. This then implies that $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \mathrm{fix}_{G_2}(\mathcal{C})\rangle \leqslant \Pi_{C\in\mathcal{C}}\langle \mathrm{fix}_{G_1}(\mathcal{C}), \mathrm{fix}_{G_2}(\mathcal{C})\rangle|_C$ is nilpotent as a direct product of nilpotent groups is nilpotent and a subgroup of a nilpotent group is nilpotent.

Let $p|\ |C|$, be prime, p^a be the largest power of p that divides $|C|$, $C \in \mathcal{C}$, and P_i a Sylow p-subgroup of $\mathrm{fix}_{G_i}(\mathcal{C})$, $i = 1, 2$. Clearly after an appropriate conjugation of G_2, if necessary, we may assume that $P_i \leqslant P$, $i = 1, 2$, where P is a Sylow p-subgroup of $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \mathrm{fix}_{G_2}(\mathcal{C})\rangle$ that contains P_1. Then P_i, $i = 1, 2$, as well as P have orbits of size p^a by [8, Lemma 10] applied to $\langle \mathrm{fix}_{G_1}(\mathcal{C}), \mathrm{fix}_{G_2}(\mathcal{C})\rangle|_C$ for every $C \in \mathcal{C}$. Hence the orbits of P_1, P_2, are the same because $P_1, P_2 \leqslant P$. Then the orbits of P_i form a complete block system $\mathcal{B}$ of n/p^a blocks of size p^a of G_i as $P_i \triangleleft G_i$. Then $\mathcal{B}$ is a complete block system of $\langle G_1, G_2\rangle$ by Lemma 7. □

3 The possibilities for $\mathrm{Stab}_{\langle G_1,G_2\rangle}(C)|_C$

In this section, we will consider the various possibilities for $\mathrm{Stab}_{\langle G_1,G_2\rangle}(C)$ in its action on $C \in \mathcal{C}$. We begin with $T_i = \mathrm{Alt}(p^a)$ and $\ell \geqslant 2$, with $\ell = 1$ in fact considered in the previous section. The results for the case where $T_i = \mathrm{Alt}(p^a)$ will not require any arithmetic conditions on n. In the proofs (but not the statements) of results in this section, we let $G = \langle G_1, G_2\rangle$.

3.1 $T_i = \mathrm{Alt}(p^a)$

We will ultimately reduce this case to the case where $\mathrm{Stab}_{\langle G_1,G_2\rangle}(C)|_C \leqslant \mathrm{AGL}(a, p)$. We begin with a preliminary result.

Lemma 11. *Let p be an odd prime. Then any two regular elementary abelian subgroups of* $\mathrm{Sym}(p^a)$ *are conjugate in* $\mathrm{Alt}(p^a)$, $a \geqslant 1$, *and any two regular elementary abelian subgroups of* $\mathrm{Sym}(p^a)$ *contained in a Sylow p-subgroup P of* $\mathrm{Sym}(p^a)$ *are conjugate in P.*

Proof. First observe that if $H_1, H_2 \leqslant \mathrm{Sym}(p^a)$ are p-groups and p is odd, then $\langle H_1, H_2\rangle \leqslant \mathrm{Alt}(p^a)$. Furthermore, there exists $\delta \in \mathrm{Alt}(p^a)$ such that $\langle H_1, \delta^{-1}H_2\delta\rangle$ is contained in a Sylow p-subgroup P of $\mathrm{Alt}(p^a)$ (which, as p is odd, is also a Sylow p-subgroup of $\mathrm{Sym}(p^a)$). We thus now need only show that any two regular elementary abelian subgroups of P are conjugate in P.

We proceed by induction on a. If $a = 1$, then the result is trivial as a Sylow p-subgroup of $\mathrm{Sym}(p)$ has order p. Assume that the result is true for all $a-1 \geqslant 1$, and let P be a Sylow p-subgroup of $\mathrm{Sym}(p^a)$. Then $P = \mathbb{Z}_p \wr (\mathbb{Z}_p \wr (\cdots \wr \mathbb{Z}_p))$ (a times), which we view as acting canonically on $\mathbb{Z}_p^a$. For $1 \leqslant i \leqslant a$, define $\tau_i : \mathbb{Z}_p^a \to \mathbb{Z}_p^a$ by

$$\tau_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_a) = (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_a).$$

Clearly then $\langle \tau_i : 1 \leqslant i \leqslant a\rangle$ is a regular elementary abelian subgroup of P. Let R be any other regular elementary abelian subgroup of P. Note that P admits a complete block system $\mathcal{B}$ consisting of p blocks of size p^{a-1} formed by the orbits of $1 \wr (\mathbb{Z}_p \wr (\cdots \wr \mathbb{Z}_p))$ ($a-1$ times), and that a Sylow p-subgroup of $\mathrm{fix}_P(\mathcal{B})|_B$ is permutation isomorphic to a Sylow p-subgroup of $\mathrm{Sym}(p^{a-1})$ for every $B \in \mathcal{B}$. As $P/\mathcal{B} \cong \mathbb{Z}_p$, we have that $\mathrm{fix}_R(\mathcal{B})$ is semiregular of order p^{a-1}. Without loss of generality, we assume that $\langle \tau_i : 2 \leqslant i \leqslant a\rangle \leqslant \mathrm{fix}_P(\mathcal{B})$. Let $B_i = \{(i, x_2, \ldots, x_a) : x_j \in \mathbb{Z}_p, 2 \leqslant j \leqslant a\}$ so that $\mathcal{B} = \{B_i : i \in \mathbb{Z}_p\}$. By the induction hypothesis and the fact that $\mathrm{fix}_P(\mathcal{B}) = 1 \wr (\mathbb{Z}_p \wr (\cdots \wr \mathbb{Z}_p))$ ($a-1$ times), for each $i \in \mathbb{Z}_p$ there exists $\gamma_i \in \mathrm{fix}_P(\mathcal{B})$ such that if $j \neq i$, then $\gamma_i|_{B_j} = 1$ and $\gamma_i^{-1}\mathrm{fix}_R(\mathcal{B})|_{B_i}\gamma_i = \langle \tau_j : 2 \leqslant j \leqslant a\rangle|_{B_i}$. Letting $\gamma = \Pi_{i\in\mathbb{Z}_p}\gamma_i$, we have that $\gamma^{-1}\mathrm{fix}_R(\mathcal{B})\gamma|_B = \langle \tau_j : 2 \leqslant j \leqslant a\rangle|_B$ for every $B \in \mathcal{B}$.

Let $\tau_1' \in \gamma^{-1}R\gamma$ such that $\tau_1'/\mathcal{B} = \tau_1/\mathcal{B}$. As $P/\mathcal{B} = \langle \tau_i : i \in \mathbb{Z}_p\rangle/\mathcal{B}$, we have that $\tau_1'(x_1, \ldots, x_a) = (x_1 + 1, \bar{\sigma}_{x_1}(x_2, \ldots, x_a))$, where $\bar{\sigma}_{x_1} \in \mathrm{Sym}(p^{a-1})$. As $\tau_1^{-1}\tau_1' \in \mathrm{fix}_P(\mathcal{B})$, we have that each $\bar{\sigma}_{x_1}$ is in a Sylow p-subgroup of $\mathrm{Sym}(p^{a-1})$. Furthermore, as $|\tau_1'| = p$, we have that $\Pi_{x_1\in\mathbb{Z}_p}\bar{\sigma}_{x_1} = 1$. As $\mathrm{fix}_P(\mathcal{B}) = 1 \wr (\mathbb{Z}_p \wr (\mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p))$ ($a-1$ times), we have that $\sigma_{x_1} \in P$ for every $x_1 \in \mathbb{Z}_p$ where $\sigma_{x_1}|_{B_{x_1}} = \bar{\sigma}_{x_1}$ and $\sigma_{x_1}|_B = 1$ if $B \neq B_{x_1}$. Let $\sigma : \mathbb{Z}_p^a \to \mathbb{Z}_p^a$ by $\sigma(x_1, x_2, \ldots, x_a) = (x_1, \Pi_{j=x_1}^{p-1}\bar{\sigma}_j^{-1}(x_2, \ldots, x_a))$. As each $\sigma_{x_1} \in P$, for $x_1 \in \mathbb{Z}_p$ we have that $\sigma \in P$. Then

$$\begin{aligned}
\sigma^{-1}\tau_1'\sigma(x_1, x_2, \ldots, x_a) &= \sigma^{-1}\tau_1'(x_1, \Pi_{j=x_1}^{p-1}\bar{\sigma}_j^{-1}(x_2, \ldots, x_a)) \\
&= \sigma^{-1}(x_1 + 1, \Pi_{j=x_1+1}^{p-1}\bar{\sigma}_j^{-1}(x_2, \ldots, x_a))) \\
&= (x_1 + 1, x_2, \ldots, x_a),
\end{aligned}$$

so that $\sigma^{-1}\tau_1'\sigma = \tau_1$. Observe that $\gamma^{-1}\mathrm{fix}_R(\mathcal{B})\gamma = \mathrm{fix}_{\gamma^{-1}R\gamma}(\mathcal{B})$ and as $\mathrm{fix}_{\gamma^{-1}R\gamma}(\mathcal{B})|_B \leqslant \langle \tau_j : 2 \leqslant j \leqslant a\rangle|_B$ for every $B \in \mathcal{B}$ and $\mathrm{fix}_{\gamma^{-1}R\gamma}(\mathcal{B})$ is invariant under conjugation by τ_1', we have that each $\bar{\sigma}_{x_i}$ normalizes $(\mathbb{Z}_p^{a-1})_L$. Whence $\mathrm{fix}_{\sigma^{-1}\gamma^{-1}R\gamma\sigma}(\mathcal{B})|_B \leqslant \langle \tau_j : 2 \leqslant j \leqslant a\rangle|_B$ for every $B \in \mathcal{B}$. As every element of $\mathrm{fix}_{\sigma^{-1}\gamma^{-1}R\gamma\sigma}(\mathcal{B})$ commutes with τ_1 (as $\tau_1 \in \sigma^{-1}\gamma^{-1}R\gamma\sigma$) and every element of $\mathrm{fix}_{\sigma^{-1}\gamma^{-1}R\gamma\sigma}(\mathcal{B})$ commutes with every element of $\langle \tau_j : 2 \leqslant j \leqslant a\rangle$ (as $\mathrm{fix}_{\sigma^{-1}\gamma^{-1}R\gamma\sigma}(\mathcal{B})|_B \leqslant \langle \tau_j : 2 \leqslant j \leqslant a\rangle|_B$ for every $B \in \mathcal{B}$), we have that $\mathrm{fix}_{\sigma^{-1}\gamma^{-1}R\gamma\sigma}(\mathcal{B})$ centralizes $\langle \tau_j : 1 \leqslant j \leqslant a\rangle$. As a transitive abelian group is self-centralizing [4, Theorem 4.2A (v)], we conclude that $\sigma^{-1}\gamma^{-1}R\gamma\sigma = \langle \tau_j : 1 \leqslant j \leqslant a\rangle$. The result then follows by induction. □

We remark that it is not the case that the second half of the previous lemma holds for every regular abelian p-group. See [15, Example 6.4].

Definition 12. Let G be a group and $H \leqslant G$. We define the **normal closure of H in G**, denoted H^G, by $H^G = \langle g^{-1}hg : h \in H, g \in G\rangle$. Clearly $H^G \lhd G$.

Lemma 13. *Let p be an odd prime, and $G_1, G_2 \leqslant \mathrm{Sym}(n)$ such that $G = \langle G_1, G_2\rangle$ admits a complete block system $\mathcal{C}$ of n/p^a blocks of size p^a, and both $\mathrm{fix}_{G_1}(\mathcal{C})|_C$ and $\mathrm{fix}_{G_2}(\mathcal{C})|_C$ are regular and elementary abelian for every $C \in \mathcal{C}$. Further, assume that $\mathrm{soc}(\mathrm{Stab}_G(C))|_C = \mathrm{Alt}(m)^\ell$. Then there exists $\gamma \in \mathrm{fix}_G(\mathcal{C})$ such that $\mathrm{Stab}_{\langle G_1,\gamma^{-1}G_2\gamma\rangle}(C)|_C \leqslant \mathrm{AGL}(a,p)$ for every $C \in \mathcal{C}$.*

Proof. If $\mathrm{Alt}(m)$ is not simple, then $\mathrm{Stab}_G(C)|_C \leqslant \mathrm{AGL}(\ell, 3)$ and the result follows. Let $H_i = \mathrm{fix}_{G_i}(\mathcal{C})$, $i = 1, 2$, and $K = \langle H_1, H_2\rangle^G$. Then $K \triangleleft \langle G_1, G_2\rangle$, so $K \triangleleft \mathrm{fix}_G(\mathcal{C})$, and $K \triangleleft \mathrm{Stab}_G(C)$ for every $C \in \mathcal{C}$. As $\mathrm{Stab}_G(C)|_C$ is primitive for some $C \in \mathcal{C}$, $\mathrm{Stab}_G(C)|_C$ is primitive for every $C \in \mathcal{C}$. As a normal subgroup of a primitive group is transitive [41, Theorem 8.8], we have that $K|_C$ is transitive for every $C \in \mathcal{C}$. It is then easy to see that $\mathrm{Alt}(m)^\ell \leqslant K|_C$ for some ℓ and every $C \in \mathcal{C}$ as $\mathrm{Alt}(m)$ is simple. As $H_i|_C$, $i = 1, 2$, are transitive abelian subgroups (acting on C), we have by Theorem 6 that $H_i|_C \leqslant \mathrm{Sym}(m)^\ell$ for every $C \in \mathcal{C}$ and $i = 1, 2$. As p is odd, if $H \leqslant K|_C$ and is regular and elementary abelian, then $H \leqslant \mathrm{Alt}(m)^\ell$. In particular, $K|_C \leqslant \mathrm{Alt}(m)^\ell$ for every $C \in \mathcal{C}$ so that $K|_C = \mathrm{Alt}(m)^\ell$ for every $C \in \mathcal{C}$. Then $K|_C$ admits ℓ complete block systems $\mathcal{B}_{C,i}$ formed by the orbits of one of the ℓ copies of $\mathrm{Alt}(m)$, $1 \leqslant i \leqslant \ell$, each consisting of blocks of size $m = p^{a/\ell}$.

For $C \in \mathcal{C}$ and $1 \leqslant i \leqslant \ell$, let $\pi_{C,i} : K \to \mathrm{Alt}(m)$ be the projection map from K to the i^{th} copy of $\mathrm{Alt}(m)$ in C. Let $\gamma \in K$ such that $\mathrm{fix}_{\langle H_1,\gamma^{-1}H_2\gamma\rangle|_C}(\mathcal{B}_{C,i}) = \mathrm{fix}_{H_1|_C}(\mathcal{B}_{C,i})$ for the maximum number of choices of C and i. Suppose that $C \in \mathcal{C}$ and $1 \leqslant i \leqslant \ell$ is such that $\mathrm{fix}_{\langle H_1,\gamma^{-1}H_2\gamma\rangle|_C}(\mathcal{B}_{C,i}) \neq \mathrm{fix}_{H_1|_C}(\mathcal{B}_{C,i})$. Then there exists $L \leqslant K$ such that $L = \mathrm{Alt}(m)$ and $\pi_{C,i}(L) = \mathrm{Alt}(m)$. Let $1 \leqslant i' \leqslant \ell$ and $C' \in \mathcal{C}$ such that $\pi_{C',i'}(L) = \mathrm{Alt}(m)$. As L is simple, L acts faithfully on both C and C'. By Lemma 11 there exists $\omega \in L$ such that $\pi_{C,i}(\langle H_1, \omega^{-1}\gamma^{-1}H_2\gamma\omega\rangle) = \pi_{C,i}(H_1)$. As $\mathrm{Alt}(m)$ has a unique representation [3, Table] as $m \neq 6$, we have that $\pi_{C',i'}(\omega^{-1}\gamma^{-1}H_2\gamma\omega) = \pi_{C',i'}(H_1)$. Finally, if $1 \leqslant i'' \leqslant \ell$ and $C'' \in \mathcal{C}$ such that $\pi_{C'',i''}(L) = 1$, then $\pi_{C'',i''}(\omega) = 1$, and so $\pi_{C'',i''}(\omega^{-1}\gamma^{-1}H_2\gamma\omega) = \pi_{C'',i''}(\gamma^{-1}H_2\gamma)$. But this contradicts our choice of γ, and hence $\pi_{C,i}(\gamma^{-1}H_2\gamma) = \pi_{C,i}(H_1)$ for every $C \in \mathcal{C}$ and $1 \leqslant i \leqslant \ell$.

As $\pi_{C,i}(\gamma^{-1}H_2\gamma) = \pi_{C,i}(H_1)$ for every $C \in \mathcal{C}$ and $1 \leqslant i \leqslant \ell$, it follows that $\gamma^{-1}H_2\gamma|_C = H_1|_C$ for every $C \in \mathcal{C}$. Now, let $g \in G$. Then g normalizes $\langle H_1|_C : C \in \mathcal{C}\rangle$. Similarly, if $g' \in \gamma^{-1}H_2\gamma$, then g' also normalizes $\langle H_1|_C : C \in \mathcal{C}\rangle$ so that $\langle G_1, \gamma^{-1}G_2\gamma\rangle \leqslant N_{\mathrm{Sym}(n)}(\langle H_1|_C : C \in \mathcal{C}\rangle)$. As $H_1|_C \cong \mathbb{Z}_p^a$ for every $C \in \mathcal{C}$, we have that $\mathrm{Stab}_{\langle G_1,\gamma^{-1}G_2\gamma\rangle}(C)|_C \leqslant \mathrm{AGL}(a,p)$ for every $C \in \mathcal{C}$ as required. □

3.2 A Common Hypothesis

We now consider when $T_i \neq \mathrm{Alt}(p^a)$. All of the results in the rest of this section share some common hypothesis, which we will call **Hypothesis 1**.

Hypothesis 1. Let $n = p_1^{a_1} \dots p_r^{a_r}$ be the prime power factorization of n. Let G_1, G_2 be transitive groups of order n such that $G = \langle G_1, G_2\rangle$ satisfies the following conditions:

1. G admits a complete block system $\mathcal{C}$ of n/p_r^a blocks of size p_r^a, $a \geqslant 1$, and $\mathrm{Stab}_G(C)|_C$ is primitive, $C \in \mathcal{C}$. Additionally, $\mathrm{fix}_{G_i}(\mathcal{C})$ is a p_r-group transitive on each $C \in \mathcal{C}$,

2. no prime divisor of $|N_{\mathrm{Sym}(p_r^a)}(P_1)|$ other than p_r divides n, where P_1 is a Sylow p_r-subgroup of $\mathrm{fix}_{G_1}(\mathcal{C})$ (we denote the Sylow p_r-subgroup of $\mathrm{fix}_{G_2}(\mathcal{C})$ by P_2),

3. Let $\pi = \{p_i : 1 \leqslant i \leqslant r-1\}$. Then $G_i = H_i \times \Pi_i$, where H_i is a π-subgroup and Π_i is a p_r-subgroup, $i = 1, 2$.

3.3 $\mathrm{Stab}_{\langle G_1,G_2\rangle}(C)|_C \leqslant \mathrm{AGL}(a,p)$

We prove the results needed to deal with the case where $\mathrm{Stab}_{\langle G_1,G_2\rangle}(C)|_C \leqslant \mathrm{AGL}(a,p)$.

Lemma 14. *In addition to Hypothesis 1, assume that*

1. $\mathrm{Stab}_G(C)|_C$ *normalizes* $P_1|_C$, *for* $C \in \mathcal{C}$, *and*

2. $G/\mathcal{C} = H \times P$ *where H is a solvable π-subgroup, and P is the unique Sylow p_r-subgroup of $G/\mathcal{C}$.*

Then there exists $\gamma \in G$ such that $\langle G_1, \gamma^{-1}G_2\gamma\rangle = \bar{H} \times \bar{\Pi}$, where $\bar{\Pi}$ is a p_r-subgroup and $\bar{H} = \langle H_1, H_2\rangle$.

Proof. Let $|N_{\mathrm{Sym}(p_r^a)}(P_1)| = p_r^v \cdot c$, where $\gcd(p_r, c) = 1$. As $\mathrm{Stab}_G(C)|_C \leqslant N_{\mathrm{Sym}(p_r^a)}(P_1)$ by the Embedding Theorem [30, Theorem 1.2.6] we have $G \leqslant (G/\mathcal{C}) \wr N_{\mathrm{Sym}(p_r^a)}(P_1)$. We conclude that $|G|$ divides $|G/\mathcal{C}|\cdot|N_{\mathrm{Sym}(p_r^a)}(P_1)|^{n/p_r^a}$. As $|G/\mathcal{C}| = |H|\cdot p_r^d$ for some $d \geqslant 0$, and no prime divisor of n other than p_r divides $|N_{\mathrm{Sym}(p_r^a)}(P_1)|$, we have that H is a π-subgroup of G. As $P \triangleleft G/\mathcal{C}$, G admits a (possibly trivial) complete block system $\mathcal{B}'$ consisting of $n/p_r^{a_r}$ blocks of size $p_r^{a_r-a}$. Hence G admits a complete block system $\mathcal{B}$ of $n/p_r^{a_r}$ blocks of size $p_r^{a_r}$ and $\mathcal{C} \preceq \mathcal{B}$. Then $G/\mathcal{B} = H$ is a π-group, and $\mathrm{fix}_G(\mathcal{B})$ is a normal π'-subgroup of G. By the Schur-Zassenhaus Theorem [22, Theorem 6.2.1], G possesses an S_π-subgroup K, which is a complement to $\mathrm{fix}_G(\mathcal{B})$, and any two S_π-subgroups of G are conjugate in G as $G/\mathcal{B} = H$ is solvable.

Let K_i be S_π-subgroups of G that contain H_i, $i = 1, 2$. Then there exists $\gamma_1 \in G$ such that $\gamma_1^{-1}K_2\gamma_1 = K_1$. We may thus assume without loss of generality that $K_1 = K_2$ and H_1, H_2 are contained in the same S_π-subgroup K of G. Assume G acts in the natural fashion on $\mathbb{Z}_{n/p_r^{a_r}} \times \mathbb{Z}_{p_r^{a_r}}$. Then $h_1 \in H_1$ can be written as $h_1(i,j) = (\sigma(i), \omega_i(j))$ and $h_2 \in H_2$ can be written as $h_2(i,j) = (\iota(i), \xi_i(j))$, where $\sigma, \iota \in \mathrm{Sym}(n/p_r^{a_r})$ and each $\omega_i, \xi_i \in \mathrm{Sym}(p_r^{a_r})$. As H_i commutes with Π_i, we see that each ω_i centralizes Π_1 and each ξ_i centralizes Π_2. We now assume that h_1, h_2 are chosen so that for some $b \in \mathbb{Z}_{n/p_r^{a_r}}$ we have that $h_2h_1(b,j) = (b, \xi_{\sigma(b)}\omega_i(j))$. As $h_2h_1 \in K$, we have that p_r does not divide $|\xi_{\sigma(b)}\omega_b|$.

As $G/\mathcal{C} = H \times P$, every element of $G/\mathcal{C}$ of order relatively prime to p_r commutes with every element of $G/\mathcal{C}$ of order a power of p_r. Let Q be the transitive permutation group obtained by the action of P on the blocks of $\mathcal{C}$ whose union is the block $B \in \mathcal{B}$ where $B = \{(b,x) : x \in P\}$, and $\overline{\xi_{\sigma(b)}\omega_b}$ be the permutation obtained by the action of $\xi_{\sigma(b)}\omega_b$ on the blocks of $\mathcal{C}$ whose union is B. Then $\overline{\xi_{\sigma(b)}\omega_b}$ commutes with every element of Q so that $\langle\overline{\xi_{\sigma(b)}\omega_b}, Q\rangle$ admits a complete block system formed by the orbits of $\langle\overline{\xi_{\sigma(b)}\omega_b}\rangle$. As the degree of Q is a power of p_r, we conclude that $\xi_{\sigma(b)}\omega_b$ fixes each block of $\mathcal{C}$ contained in B set-wise. That is, $h_2h_1 \in \mathrm{Stab}_G(C')$ for some $C' \in \mathcal{C}$.

As $P_1|_C \lhd \mathrm{Stab}_G(C)|_C$, there exists a $\hat{P}_1|_{C'}$ permutation isomorphic to $P_1|_C$ such that $\hat{P}_1|_{C'} \lhd \mathrm{Stab}_G(C')|_{C'}$. As $|N_{\mathrm{Sym}(p_r^a)}(P_1)|$ divides $p_r^v \cdot c$, $|N_{\mathrm{Sym}(p_r^a)}(\hat{P}|_{C'})|$ divides $p_r^v \cdot c$. As $\gcd(p_1 \cdots p_{r-1}, c) = 1$, we must have that $|\xi_{\sigma(b)}\omega_b| = 1$. As ω_b centralizes Π_1, we see that $\xi_{\sigma(b)}$ centralizes Π_1. As b was arbitrary, we see that ξ_i centralizes Π_1 for every $i \in \mathbb{Z}_{n/p_r^{a_r}}$. This implies that H_2 centralizes Π_1, and similarly, H_1 centralizes Π_2. Hence $\bar{H} = \langle H_1, H_2\rangle$ centralizes $\bar{\Pi} = \langle \Pi_1, \Pi_2\rangle$. As $\langle H_1, H_2\rangle$ is a π-group, $\langle \Pi_1, \Pi_2\rangle$ is a p_r-group and $G = \langle H_1, H_2, \Pi_1, \Pi_2\rangle$, we see that $G = \langle H_1, H_2\rangle \times \langle \Pi_1, \Pi_2\rangle$. □

Lemma 15. *In addition to Hypothesis 1, assume that*

1. *$G/\mathcal{C}$ is nilpotent,*
2. *for some $C \in \mathcal{C}$, $P_1|_C$ is a Sylow p_r-subgroup of $(\langle P_1, P_2\rangle^G)|_C$. In addition, $\mathrm{fix}_{G_1}(\mathcal{C})|_C$ is isomorphic to $\mathrm{fix}_{G_2}(\mathcal{C})|_C$ for every $C \in \mathcal{C}$.*

Then there exists $\delta \in G$ such that $\langle G_1, \delta^{-1}G_2\delta\rangle$ is nilpotent.

Proof. As $P_1|_C$ is a Sylow p_r-subgroup of $(\langle P_1, P_2\rangle^G)|_C$ for some $C \in \mathcal{C}$, $P_1|_C$ is a Sylow p_r-subgroup of $(\langle P_1, P_2\rangle^G)|_C$ for every $C \in \mathcal{C}$. Let $\hat{P}_i$ be a Sylow p_r-subgroup of $\langle P_1, P_2\rangle^G$ that contains P_i. By a Sylow Theorem, we may assume without loss of generality that $\hat{P}_1 = \hat{P}_2$. Then $\hat{P}_1 \leqslant \langle P_1|_C : C \in \mathcal{C}\rangle$. As $\mathrm{fix}_{G_1}(\mathcal{C})|_C$ is isomorphic to $\mathrm{fix}_{G_2}(\mathcal{C})|_C$ for every $C \in \mathcal{C}$, $\mathrm{fix}_{G_2}(\mathcal{C})|_C = \mathrm{fix}_{G_1}(\mathcal{C})|_C$ for every $C \in \mathcal{C}$. As every element of G_1 normalizes P_1, every element of G_1 normalizes $\langle P_1|_C : C \in \mathcal{C}\rangle$. Similarly, every element of G_2 normalizes $P_2 \leqslant \langle P_1|_C : C \in \mathcal{C}\rangle$, so that every element of G_2 normalizes $\langle P_1|_C : C \in \mathcal{C}\rangle$. We conclude that both G_1 and G_2 normalize $\langle P_1|_C : C \in \mathcal{C}\rangle$. Then $P_1|_C \lhd \mathrm{Stab}_G(C)|_C$, and the result follows by Lemma 14. □

3.4 The case where $\tilde{T}_i \neq \mathrm{Alt}(p^a)$

We will need the following result [10, Lemma 3.3].

Lemma 16. *Let $\mathrm{PGL}(d,q) \leqslant X \leqslant \mathrm{P\Gamma L}(d,q)$ be primitive of degree $n = (q^d-1)/(q-1)$ and contain a regular cyclic subgroup, where q is a prime-power and $d \geqslant 2$. If $n = p^k$ for some odd prime p and $(d,q) \neq (2,8)$, then a Sylow p-subgroup of X is regular and cyclic.*

Definition 17. A group is **homocyclic** if it is the direct product of isomorphic cyclic groups.

In the following result, we will use the terminology of Theorem 6.

Theorem 18. *In addition to Hypothesis 1, set $m = \Omega(n)$, and assume that the following conditions hold:*

1. *p_i does not divide $p_j - 1$ for any distinct primes $p_i, p_j | n$,*

2. *for any $\delta \in \mathrm{Sym}(n)$ and nontrivial complete block system $\mathcal{D}$ of $\langle G_1, \delta^{-1}G_2\delta\rangle$, there exists $\omega \in \langle G_1, \delta^{-1}G_2\delta\rangle/\mathcal{D}$ such that $\langle G_1, \omega^{-1}\delta^{-1}G_2\delta\omega\rangle/\mathcal{D}$ is nilpotent,*

3. *$\mathrm{fix}_{G_1}(\mathcal{C})$ and $\mathrm{fix}_{G_2}(\mathcal{C})$ are semiregular and elementary abelian or cyclic, and*

4. *setting $X = \mathrm{Stab}_G(\mathcal{C})|_C$ if $\ell \geqslant 2$ then $\tilde{T}_i \neq \mathrm{Alt}(z)$, $\mathrm{Sym}(z)$, or $\mathrm{P\Gamma L}(2,8)$ for some z (here we are using the notation of Theorem 6).*

Then there exists $\gamma \in G$ such that $\langle G_2, \gamma^{-1}G_2\gamma\rangle$ is nilpotent.

Proof. By (2) we may, after an appropriate conjugation of G_2, assume that $G/\mathcal{C}$ is nilpotent. As X is primitive and contains a regular abelian subgroup, by Theorem 6 we have that one of the following is true:

1. $X \leqslant \mathrm{AGL}(a,p)$, or

2. $X = (\tilde{T}_1 \times \cdots \times \tilde{T}_\ell).O.Q$, $P_i = A_1 \times \cdots \times A_\ell$, and $p^a = z^\ell$, where $\ell \geqslant 1$, each $A_j \leqslant \tilde{T}_j$ with $|A_j| = z$, $\tilde{T}_1 \cong \ldots \tilde{T}_\ell$, $O \leqslant \mathrm{Out}(\tilde{T}_1) \times \cdots \times \mathrm{Out}(\tilde{T}_\ell)$, Q is a transitive permutation group of degree ℓ, and one of the following holds:

   (a) $\tilde{T}_j = \mathrm{PGL}(d,q)$, and $P_i = \mathbb{Z}_{(q^d-1)/(q-1)}$ is a Singer subgroup;

   (b) $\tilde{T}_j = \mathrm{P\Gamma L}(2,8)$ and $\mathbb{Z}_9 = P_i \not\leqslant \mathrm{PSL}(2,8)$;

   (c) $\tilde{T}_j = \mathrm{Sym}(z)$ or $\mathrm{Alt}(z)$, and P_i is abelian of order z.

If $X \leqslant \mathrm{AGL}(a,p)$, then X normalizes $\mathbb{Z}_p^a$ and the result follows by Lemma 15.

If $\tilde{T}_i = \mathrm{PGL}(d,q)$, and $A_i = \mathbb{Z}_{(q^d-1)/(q-1)}$ is a Singer subgroup then by Lemma 16, a Sylow p-subgroup of $\tilde{T}_1 \times \cdots \times \tilde{T}_\ell$ is regular and homocyclic. Then $P_i|_C$, $i = 1,2$, are regular homocyclic subgroups and $\langle P_1, P_2\rangle^G \leqslant \tilde{T}_1 \times \cdots \times \tilde{T}_\ell$. Hence $P_1|_C$ is a Sylow p-subgroup of $(\langle P_1, P_2\rangle^{\langle G_1, G_2\rangle})|_C$ for every $C \in \mathcal{C}$. By Lemma 15, there exists $\gamma_1 \in \langle G_1, G_2\rangle$ such that $\langle G_1, \gamma_1^{-1}G_2\gamma_1\rangle$ is nilpotent, and the result follows.

If $\tilde{T}_i = \mathrm{Alt}(z)$, $\mathrm{Sym}(z)$, or $\mathrm{P\Gamma L}(2,8)$, then by hypothesis we must have $\ell = 1$. If $\tilde{T}_i = \mathrm{Alt}(z)$ or $\mathrm{Sym}(z)$, then by Lemma 8, there exists $\gamma_1 \in \langle G_1, G_2\rangle$ such that $\langle G_1, \gamma_1^{-1}G_2\gamma_1\rangle$ admits a complete block system $\mathcal{B}$ of n/p blocks of size p. After a suitable conjugation, we may assume that $\langle G_1, \gamma_1^{-1}G_2\gamma_1\rangle/\mathcal{B}$ is nilpotent, in which case $\langle G_1, \gamma_1^{-1}G_2\gamma_1\rangle$ is normally m-step imprimitive [8, Lemma 9], and the result follows by [8, Theorem 12].

If $\tilde{T}_1 = \mathrm{P\Gamma L}(2,8)$, then $p^a = 9$ and a Sylow p-subgroup of $\tilde{T}_1$ has order 27. Also, a Sylow 3-subgroup of $\mathrm{fix}_{G_1}(\mathcal{C})$ is cyclic of order 9, say $\mathrm{fix}_{G_1}(\mathcal{C}) = \langle\rho\rangle$. As $\mathrm{P\Gamma L}(2,8)$ contains a regular cyclic subgroup of order 9, we have by [17, Theorem 9] and [17, Lemma 9], that a Sylow 3-subgroup of $\mathrm{P\Gamma L}(2,8)$ is the same as a Sylow 3-subgroup $\hat{P}$ of $\{x \to ax+b : a \in \mathbb{Z}_9^*, b \in \mathbb{Z}_9\}$. It is not difficult to see that the center $Z(\hat{P})$ of $\hat{P}$ is $\{x \to x+3b : b \in \mathbb{Z}_3\}$. We may assume, after an appropriate conjugation, that a Sylow 3-subgroup of $\mathrm{fix}_{G_2}(\mathcal{C})$ is contained in the same Sylow 3-subgroup P of $\mathrm{fix}_G(\mathcal{C})$ as $\mathrm{fix}_{G_1}(\mathcal{C})$. As a transitive abelian group is self-centralizing [4, Theorem 4.2A (v)], we must have that $Z(P|_C) \leqslant \langle\rho\rangle|_C$ for every $C \in \mathcal{C}$, and as $Z(\hat{P}) = \{x \to x+3b : b \in \mathbb{Z}_3\}$ we have that $Z(P|_C) = \langle\rho^3\rangle|_C$ for every $C \in \mathcal{C}$. Also, as $\mathrm{fix}_{G_2}(\mathcal{C})|_C$ is a regular cyclic subgroup for every $C \in \mathcal{C}$, we must also have that $\langle \rho^3 \rangle|_C \leqslant \mathrm{fix}_{G_2}(\mathcal{C})|_C$ for every $C \in \mathcal{C}$. We conclude that every element of both G_1 and G_2 normalizes $\langle \rho^3|_C : C \in \mathcal{C}\rangle$, and so $M = G \cap \langle \rho^p|_C : C \in \mathcal{C}\rangle \lhd G$. Then G admits a complete block system with blocks of size 3, and the result follows by arguments above. $\square$

4 The main tool

In this section, we prove the main tool that will be used to prove our main results. We begin with some preliminary results.

Lemma 19. *Let p be a prime, $a \geqslant 1$, and π the set of distinct primes dividing $p(p-1)$. If $P \leqslant \mathrm{Sym}(p^a)$ is a transitive p-group that contains a regular cyclic subgroup R, then $N_{\mathrm{Sym}(p^a)}(P)$ is a π-group.*

Proof. As R is cyclic, by [41, Exercise 1.6.5], R admits unique complete block systems $\mathcal{B}_0, \ldots, \mathcal{B}_a$ consisting of p^{a-i} blocks of size p^i, $0 \leqslant i \leqslant a$. As $R \leqslant P$, P also admits $\mathcal{B}_0, \ldots, \mathcal{B}_a$ as complete block systems as P is a-step imprimitive [8, Lemma 9]. Clearly then P only admits $\mathcal{B}_0, \ldots, \mathcal{B}_a$ as complete block systems. Let $g \in N_{\mathrm{Sym}(p^a)}(P)$. Then $g^{-1}\mathrm{fix}_P(\mathcal{B}_i)g \lhd P$ and has orbits of size p^i, $0 \leqslant i \leqslant a$. We conclude by the uniqueness of complete block systems of P that the orbits of $g^{-1}\mathrm{fix}_P(\mathcal{B}_i)g$ are the same as the orbits of $\mathrm{fix}_P(\mathcal{B}_i)$, and then by order arguments that $g^{-1}\mathrm{fix}_P(\mathcal{B}_i)g = \mathrm{fix}_P(\mathcal{B}_i)$ for $0 \leqslant i \leqslant a$. Hence $N_{\mathrm{Sym}(p^a)}(P)$ admits each $\mathcal{B}_i$, $0 \leqslant i \leqslant a$, as a complete block system, and so is normally a-step imprimitive.

We now show that $N_{\mathrm{Sym}(p^a)}(P)$ is a π-group by induction on a. If $a = 1$, then $P \cong \mathbb{Z}_p$, $N_{\mathrm{Sym}(p)}(P) = \mathrm{AGL}(1,p)$ is or order $p(p-1)$, and so is a π-group. We now assume that $N_{\mathrm{Sym}(p^{a-1})}(P)$ is a π-group for all P satisfying the hypothesis with $a-1 \geqslant 1$, and let $P \leqslant \mathrm{Sym}(p^a)$ satisfy the hypothesis. Then $N_{\mathrm{Sym}(p^a)}(P)$ admits $\mathcal{B}_1$ as a complete block system consisting of p^{a-1} blocks of size p, and $N_{\mathrm{Sym}(p^a)}(P)/\mathcal{B}_1$ is $(a-1)$-step imprimitive as $N_{\mathrm{Sym}(p^a)}(P)$ is a-step imprimitive. By the induction hypothesis, $N_{\mathrm{Sym}(p^a)}(P)/\mathcal{B}_1$ is a π-group. Furthermore, $\mathrm{fix}_{N_{\mathrm{Sym}(p^a)}(P)}(\mathcal{B}_1)|_B$ normalizes a regular cyclic subgroup of degree p for every $B \in \mathcal{B}_1$, so that $\mathrm{fix}_{N_{\mathrm{Sym}(p^a)}(P)}(\mathcal{B}_1)|_B$ is a π-group, $B \in \mathcal{B}_1$. Hence $\mathrm{fix}_{N_{\mathrm{Sym}(p^a)}(P)}(\mathcal{B}_1)$ is a π-group. As $|N_{\mathrm{Sym}(p^a)}(P)| = |N_{\mathrm{Sym}(p^a)}(P)/\mathcal{B}_1| \cdot |\mathrm{fix}_{N_{\mathrm{Sym}(p^a)}(P)}(\mathcal{B}_1)|$, we have that $N_{\mathrm{Sym}(p^a)}(P)$ is a π-group, and the result follows by induction. $\square$

Definition 20. For a prime p, and $r \geqslant 1$, let $f(p^r) = \Pi_{i=1}^r (p^i - 1)$.

Lemma 21. *Let P be a transitive p-subgroup of $\mathrm{Sym}(p^a)$ for some prime p and $a \geqslant 1$ that contains a regular elementary abelian or regular cyclic subgroup R. Let π be the set of all primes dividing $N_{\mathrm{Sym}(p^a)}(P)$ and π_1 be the set of all primes dividing $\mathrm{Aut}(R)$. Then $\pi \subseteq \pi_1$.*

Proof. If R is elementary abelian, then $|\mathrm{Aut}(R)| = \Pi_{i=0}^{a-1}(p^a - p^i) = p^t \cdot f(p^a)$ for some $t \geqslant 1$. The result then follows by [36, 7.3.11]. If R is cyclic, then the result follows by Lemma 19. $\square$

Theorem 22. *Let n be odd with prime-power decomposition $n = p_1^{a_1} \cdots p_r^{a_r}$, and G be a transitive abelian group of order n such that every Sylow p_i-subgroup P_i of G is elementary abelian or cyclic. If no prime divisor of $|\mathrm{Aut}(P_i)|$ other than p_i divides n for every $1 \leqslant i \leqslant r$, then whenever $\delta \in \mathrm{Sym}(n)$ there exists $\gamma \in \langle G, \delta^{-1}G\delta\rangle$ such that $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma\rangle$ is nilpotent. Consequently, $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma\rangle$ is normally m-step imprimitive.*

Proof. The second statement follows from the first and [9, Lemma 9], so it suffices to show the first statement. First, if $r = 1$ then by a Sylow Theorem there exists $\gamma \in \langle G, \delta^{-1}G\delta\rangle$ such that $\langle G, \gamma^{-1}\delta^{-1}G\delta\gamma\rangle$ is a p_1-group, and so nilpotent. We thus assume that $r \geqslant 2$. We now proceed by induction on $m = \Omega(n)$. If $m = 1$, then the result follows by the immediately preceding argument, so we assume the result holds for all n with $1 \leqslant \Omega(n) \leqslant m-1$. Let n be such that $\Omega(n) = m$, $G \leqslant \mathrm{Sym}(n)$ satisfy the hypothesis, and $\delta \in \mathrm{Sym}(n)$. By Corollary 9, we may assume without loss of generality that $\langle G, \delta^{-1}G\delta\rangle$ is imprimitive, and by Lemma 10, that $\langle G, \delta^{-1}G\delta\rangle$ admits a complete block system $\mathcal{B}$ consisting of n/p^a blocks of size p^a, for some prime $p|n$ and $a \geqslant 1$.

Choose $\mathcal{B}$ in such a way that a is as small as possible. By [4, Exercise 1.5.10] $H = \mathrm{Stab}_{\langle G,\delta^{-1}G\delta\rangle}(B)|_B$ is primitive, for $B \in \mathcal{B}$. Note that $p_i - 1$ divides $|\mathrm{Aut}(P_i)|$ for every $1 \leqslant i \leqslant r$ as P_i is abelian and hence a direct product of cyclic groups. By the induction hypothesis, there exists $\gamma_1 \in \langle G, \delta^{-1}G\delta\rangle$ such that $\langle G, \gamma_1^{-1}\delta^{-1}G\delta\gamma_1\rangle/\mathcal{B}$ is nilpotent. We thus assume that $\langle G, \delta^{-1}G\delta\rangle/\mathcal{B}$ is nilpotent.

If $a = 1$, then $\langle G, \delta^{-1}G\delta\rangle$ is m-step imprimitive as $\langle G, \delta^{-1}G\delta\rangle/\mathcal{B}$ is $(m-1)$-step imprimitive [8, Lemma 9]. The result then follows by [8, Theorem 12]. If $a \geqslant 2$, then as H contains a regular abelian subgroup A and is primitive, by Theorem 6 one of the following is true:

1. $H \leqslant \mathrm{AGL}(a, p)$, or
2. $H = (\tilde{T}_1 \times \cdots \times \tilde{T}_\ell).O.Q$, $A = A_1 \times \cdots \times A_\ell$, and $p^a = z^\ell$, where $\ell \geqslant 1$, each $A_i \leqslant \tilde{T}_i$ with $|A_i| = z$, $\tilde{T}_1 \cong \cdots \cong \tilde{T}_\ell$, $O \leqslant \mathrm{Out}(\tilde{T}_1) \times \cdots \times \mathrm{Out}(\tilde{T}_\ell)$, Q is a transitive permutation group of degree ℓ, and one of the following holds:
   (a) $\tilde{T}_i = \mathrm{PGL}(d, q)$, and $A_i = \mathbb{Z}_{(q^d-1)/(q-1)}$ is a Singer subgroup;
   (b) $\tilde{T}_i = \mathrm{P\Gamma L}(2, 8)$ and $\mathbb{Z}_9 = A_i \not\leqslant \mathrm{PSL}(2, 8)$;
   (c) $\tilde{T}_i = \mathrm{Sym}(z)$ or $\mathrm{Alt}(z)$, and A_i is abelian of order z.

Suppose that if $\ell \geqslant 2$ then $\tilde{T}_i \neq \mathrm{Sym}(z)$, $\mathrm{Alt}(z)$, or $\mathrm{P\Gamma L}(2, 8)$. Let P be a Sylow p-subgroup of H. By [36, 3.2.3], $N_{\mathrm{Sym}(p^a)}(P)/Z_{\mathrm{Sym}(p^a)}(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. By [4, Theorem 4.2A (i)], $Z_{\mathrm{Sym}(p^a)}(P)$ is semiregular and consequently a p-group. Thus $|N_{\mathrm{Sym}(p^a)}(P)|$ divides $p^c \cdot |\mathrm{Aut}(P)|$ for some $c \geqslant 0$. By Lemma 21, any prime divisor of $|N_{\mathrm{Sym}(p^a)}(H)|$ divides $|\mathrm{Aut}(R)|$, where $R = \mathrm{Stab}_G(B)|_B$. Then $|\mathrm{Aut}(R)|$ divides $|\mathrm{Aut}(P_j)|$ for some $1 \leqslant j \leqslant r$, and so no prime divisor of $|\mathrm{Aut}(P)|$ other than p divides n. The result then follows by Theorem 18.

If $\tilde{T}_i = \mathrm{Alt}(z)$ or $\mathrm{Sym}(z)$ and $\ell \geqslant 2$, then a regular abelian subgroup of H cannot be cyclic, and so by hypothesis is elementary abelian. By Lemma 13 there exists $\gamma_2 \in$

$\mathrm{fix}_{\langle G,\delta^{-1}G\delta\rangle}(\mathcal{B})$ such that $\mathrm{Stab}_{\langle G,\gamma_2^{-1}\delta^{-1}G\delta\gamma_2\rangle}(B)|_B \leqslant \mathrm{AGL}(a,p)$ for every $B \in \mathcal{B}$, and this case reduces to one considered above. Finally, if $\tilde{T}_i = \mathrm{P\Gamma L}(2,8)$ and $\ell \geqslant 2$, then a regular abelian subgroup of H cannot be cyclic or elementary abelian, a contradiction. □

5 The Main Results

We begin with the basic definitions and results concerning Cayley objects and the Cayley isomorphism problem, some of which we have seen before.

Definition 23. We define a **Cayley object of G** to be a combinatorial object X such that $G_L \leqslant \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the **automorphism group of X**. If X is a Cayley object of G in some class $\mathcal{K}$ of combinatorial objects with the property that whenever Y is another Cayley object of G in $\mathcal{K}$, then X and Y are isomorphic if and only if they are isomorphic by a group automorphism of G, then we say that X is a **CI-object of G in $\mathcal{K}$**. If every Cayley object of G in $\mathcal{K}$ is a CI-object of G in $\mathcal{K}$, then we say that G is a **CI-group with respect to $\mathcal{K}$**. If G is a CI-group with respect to every class of combinatorial objects, then G is a **CI-group**.

We will also have need of the notion of solving sets.

Definition 24. Let G be a finite group. We say that $S \subseteq \mathrm{Sym}(G)$ is a **solving set for a Cayley object X** in a class of Cayley objects $\mathcal{K}$ if for every Cayley object $X' \in \mathcal{K}$ such that $X \cong X'$, there exists $s \in S$ such that $s(X) = X'$. We say that $S \subseteq \mathrm{Sym}(G)$ is a **solving set for a class $\mathcal{K}$ of Cayley objects of G** if whenever $X, X' \in \mathcal{K}$ are Cayley objects of G and $X \cong X'$, then $s(X) = X'$ for some $s \in S$. Finally, a set S is a **solving set for G** if whenever X, X' are isomorphic Cayley objects in any class $\mathcal{K}$ of combinatorial objects, then $s(X) = X'$ for some $s \in S$.

Note that X is a CI-object of G if and only if $\mathrm{Aut}(G)$ is a solving set for X. The following characterization of a solving set for an abelian group is [8, Lemma 15] and generalizes [33, Lemma 1.1].

Lemma 25. *Let G be a finite abelian group, and $S \subseteq \mathrm{Sym}(G)$ a set of permutations. Then the following conditions are equivalent:*

1. *S is a solving set for a Cayley object X in a class $\mathcal{K}$ of Cayley objects of G,*
2. *whenever $\delta \in \mathrm{Sym}(G)$ such that $\delta^{-1}G_L\delta \leqslant \mathrm{Aut}(X)$, there exists $s \in S$ and $v \in \mathrm{Aut}(X)$ such that $v^{-1}\delta^{-1}g_L\delta v = s^{-1}g_L s$ for every $g \in G$.*

The following result [8, Theorems 16] will be needed to apply our main result. We remark that in [8] this result is stated for an arbitrary Cayley object X in an arbitrary class of combinatorial objects).

Theorem 26. *Let k be a positive integer with* $\gcd(k, \varphi(k)) = 1$ *and* $k = p_1 p_2 \cdots p_r$ *be the prime-power decomposition of* k. *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, *where* $a_i \geqslant 1$ *is a positive integer. Let* $G = \Pi_{i=1}^r P_i$ *be an abelian group where* P_i *is a Sylow* p_i*-subgroup of* G, *and let* $S(i)$ *be a solving set for* P_i. *If, whenever* $\delta \in \mathrm{Sym}(G)$ *there exists* $\omega \in \langle G_L, \delta^{-1} G_L \delta\rangle$ *such that* $\langle G_L, \omega^{-1}\delta^{-1} G_L \delta\omega\rangle$ *is normally* m*-step imprimitive, then a solving set for* G *is contained in* $\Pi_{i=1}^r S(i)$.

Let k and n be as above and P_i a Sylow p_i-subgroup of G. Suppose that for $1 \leqslant i \leqslant r$, no prime divisor of n other than p_i divides $|\mathrm{Aut}(P_i)|$. As G is abelian, P_i is abelian, and $p_i - 1$ divides $|\mathrm{Aut}(P_i)|$ as $p_i - 1$ divides $|H|$ for every cyclic group of order p_i^a, $a \geqslant 1$. Thus in this case $\gcd(k, \varphi(k)) = 1$. Combining Theorem 26 with Theorem 22, we have the following result.

Corollary 27. *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ *be the prime-power decomposition of* n. *Let* G *be an abelian group of order* n *such that every Sylow* p_i*-subgroup* P_i *of* G *is elementary abelian or cyclic. Let* $S(i)$ *be the solving set for* P_i. *If no prime divisor of* n *other than* p_i *divides* $|\mathrm{Aut}(P_i)|$, *then a solving set for* G *is* $\Pi_{i=1}^r S(i)$.

As solving sets are known for $\mathbb{Z}_p^2$ [17, Corollary 2] and $\mathbb{Z}_{p^2}$ [24] or [17, Corollary 1], and as all groups of order p^2 are elementary abelian or cyclic, we have the following result.

Corollary 28. *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ *be the prime-power decomposition of* n *and* n *be cube-free. Let* G *be an abelian group of order* n *with Sylow* p_i*-subgroup* P_i, $1 \leqslant i \leqslant r$. *If no prime divisor of* n *other than* p_i *divides* $|\mathrm{Aut}(P_i)|$, *then a solving set for* G *is known.*

Definition 29. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime-power decomposition of n. Define $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $g(n) = \Pi_{i=1}^r f(p_i^{a_i})$ (recall that f was defined in Definition 20).

Let G be an abelian group of order $n = p_1^{a_1} \cdots p_r^{a_r}$ such that every Sylow p_i-subgroup P_i of G is elementary abelian or cyclic. As $|\mathrm{Aut}(\mathbb{Z}_{p^a})| = p^a - p^{a-1}$ and $\mathrm{Aut}(\mathbb{Z}_p^a) = \mathrm{AGL}(a, p)$ so that $|\mathrm{Aut}(\mathbb{Z}_p^a)| = \Pi_{i=0}^{a-1}(p^a - p^i)$, if $\gcd(n, g(n)) = 1$, then no prime divisor of n other than p_i divides $|\mathrm{Aut}(P_i)|$. Hence we have the following results.

Corollary 30. *Let* n *be a positive integer such that* $\gcd(n, g(n)) = 1$, *and* G *an abelian group of order* n *such that every Sylow* p_i*-subgroup* P_i *of* G *is elementary abelian or cyclic. Let* $S(i)$ *be the solving set for* P_i. *Then a solving set for* G *is* $\Pi_{i=1}^r S(i)$.

Corollary 31. *Let* n *be a cube-free positive integer such that* $\gcd(n, g(n)) = 1$, *and* G *be an abelian group of order* n. *Then a solving set for* G *is known.*

It may be interesting to observe that if $\gcd(n, \varphi(n)) = 1$, then every group of order n is cyclic, while if $\gcd(n, g(n)) = 1$ then every group of order n is nilpotent, and if in addition to $\gcd(n, g(n)) = 1$ n is also cube-free, then every group of order n is abelian [36, Theorem 9.2.7].

While the previous results definitely give the flavor of the consequences of the work in this paper, from a strictly computational point of view they can be quite inefficient.

A solving set for P_i will contain the solving sets of every Cayley combinatorial object in any class $\mathcal{K}$ of combinatorial objects, and so in practice can be too large to be useful for efficient isomorphism testing. In practice, for a Cayley object X of G of prime-power order p^a a solving set for X is determined by the Sylow p-subgroup P of $\mathrm{Aut}(X)$ that contains G_L (see also [17, Lemma 15]).

Lemma 32. *Let X be a Cayley object of the abelian p-group G and P a Sylow p-subgroup of $\mathrm{Aut}(X)$ that contains G_L. Then there exists a Cayley object W in some class of combinatorial objects such that $\mathrm{Aut}(W) = P$ and any solving set for W is a solving set for X.*

Proof. By [40, Theorem 5.12], there exists a Cayley object W of G such that $\mathrm{Aut}(W) = P$. Let Y be a Cayley object of G in the same class of combinatorial objects as X and $\delta \in \mathrm{Sym}(G)$ such that $\delta(X) = Y$. Then $\delta^{-1}G_L\delta \leqslant \mathrm{Aut}(X)$. Hence there exists $v \in \mathrm{Aut}(X)$ such that $v^{-1}\delta^{-1}G_L\delta v \leqslant P$, so we assume without loss of generality that $\delta^{-1}G_L\delta \leqslant P$. As $\delta^{-1}G_L\delta \leqslant \mathrm{Aut}(W)$, $\delta(W)$ is a Cayley combinatorial object of G isomorphic to W. Let S be a solving set for W. By Lemma 25, there exists $s \in S$ and $v \in \mathrm{Aut}(W)$ such that $v^{-1}\delta^{-1}g_L\delta v = s^{-1}g_Ls$ for every $g \in G$. As $\mathrm{Aut}(W) = P \leqslant \mathrm{Aut}(X)$, $v \in \mathrm{Aut}(X)$ and $v^{-1}\delta^{-1}g_L\delta v = s^{-1}g_Ls$ for every $g \in G$. Thus S is a solving set for X by Lemma 25. □

Applying [8, Theorem 16], Theorem 22, and Lemma 32, we have the following result.

Corollary 33. *Let $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ be the prime-power decomposition of n. Let G be an abelian group of order n such that every Sylow p_i-subgroup P_i of G is elementary abelian or cyclic, and X a Cayley object of G. If no prime divisor of n other than p_i divides $|\mathrm{Aut}(P_i)|$, then there exists Cayley objects X_i of P_i such that if $S(i)$ is a solving set for X_i, then a solving set for X is $\Pi_{i=1}^r S(i)$.*

Many of our main results hold for so-called ℓ-closed groups, introduced by Wielandt [40].

Definition 34. Let Ω be a set. An ***ℓ*-ary relational structure** on Ω is an ordered pair (Ω, U), where $U \subseteq \Omega^\ell = \Pi_{i=1}^\ell\Omega$. A group $G \leqslant \mathrm{Sym}(\Omega)$ is called ***ℓ*-closed** if G is the intersection of the automorphism groups of some set of ℓ-ary relational structures. The ***ℓ*-closure of *G***, denoted $G^{(\ell)}$, is the intersection of all ℓ-closed subgroups of $\mathrm{Sym}(\Omega)$ that contain G.

The following result is [8, Theorem 20], and is the analogue of Theorem 26 for ℓ-ary relational structures (again in [8], this result is stated for an arbitrary ℓ-ary relational structure X).

Theorem 35. *Let k be a positive integer with $\gcd(k, \varphi(k)) = 1$ and $k = p_1p_2\cdots p_r$ be the prime-power decomposition of k. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, where $a_i \geqslant 1$ is a positive integer. Let $G = \Pi_{i=1}^r P_i$ be an abelian group where P_i is a Sylow p_i-subgroup of G, and $S_\ell(i)$ be a solving set for P_i in the class $\mathcal{K}$ of ℓ-ary relational structures. If, whenever $\delta \in \mathrm{Sym}(G)$ such that $\delta^{-1}G_L\delta \leqslant \mathrm{Aut}(X)$ there exists $\omega \in \mathrm{Aut}(X)$ such that $\langle G_L, \omega^{-1}\delta^{-1}G_L\delta\omega\rangle$ is normally m-step imprimitive, then a solving set for G in $\mathcal{K}$ is $\Pi_{i=1}^r S_\ell(i)$.*

Combining this result with Theorem 22 we have:

Corollary 36. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime-power decomposition of n. Let G be an abelian group of order n such that every Sylow p_i-subgroup P_i is elementary abelian or cyclic. Let $S_\ell(i)$ be a solving set for P_i in the class of ℓ-ary relational structures. If no prime divisor of n other than p_i divides $|\mathrm{Aut}(P_i)|$, then a solving set for G is $\Pi_{i=1}^r S_\ell(i)$.*

As solving sets for ℓ-ary relational structures of $\mathbb{Z}_p^2$ and $\mathbb{Z}_{p^2}$ are known if $p \leqslant \ell$ [12], we have the following result.

Corollary 37. *Let n be a cube-free positive integer such that $\gcd(n, g(n)) = 1$, and G be an abelian group of order n. Then a solving set for G in the class of ℓ-ary relational structures, $\ell \leqslant p$, is known.*

For binary relational structures (or, if you prefer, Cayley color digraphs), solving sets for $\mathbb{Z}_{p^k}$, $k \geqslant 1$ are known [25], while it is known that $\mathbb{Z}_p^k$, $1 \leqslant k \leqslant 5$ is a CI-group with respect to Cayley color digraphs ($k = 1$ [39], $k = 2$ [21], $k = 3$ [5] or [42], $k = 4$ [23] or [31], $k = 5$ [20] and if $p = 3$ [38]).

Corollary 38. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime-power decomposition of n. Let G be an abelian group of order n such that every Sylow p_i-subgroup P_i is elementary abelian or cyclic. Furthermore, assume that if P_i is elementary abelian then $a_i \leqslant 5$. If no prime divisor of n other than p_i divides $|\mathrm{Aut}(P_i)|$, then a solving set for G in the class of Cayley color digraphs is known.*

Corollary 39. *Let n be an integer such that $\gcd(n, g(n)) = 1$, and G an abelian group of order n such that every Sylow p_i-subgroup P_i is elementary abelian of rank at most 5. Then G is a CI-group with respect to Cayley color digraphs.*

A very special case of the preceding corollary is [7, Theorem 16], while it was shown that $\mathbb{Z}_q \times \mathbb{Z}_p^2$ [26] and $\mathbb{Z}_q \times \mathbb{Z}_p^3$ [37] with $q \geqslant p^3$ are CI-groups with respect to digraphs for any distinct primes p and q.

We finish with an application to the isomorphism problem for codes. The reader interested in the isomorphism problem for codes is referred to [14] for terminology and notation. The following result is [14, Theorem 2.1].

Theorem 40. *Let G be an abelian group of order n such that whenever $\delta \in S_G$ there exists $\gamma \in \langle G_L, \delta^{-1} G_L \delta \rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1} G_L \delta\gamma \rangle$ is normally m-step imprimitive. Let q be a prime-power such that $\gcd(n, q-1) = 1$, and C be a $\hat{G}$-invariant code over $\mathbb{F}_q$. Then every permutation solving set for C is also monomial solving set for C.*

Combining the previous result with Theorem 22, we have the following result.

Theorem 41. *Let n be odd with prime-power decomposition $n = p_1^{a_1} \cdots p_r^{a_r}$, and G be a transitive abelian group of order n such that every Sylow p_i-subgroup P_i of G is elementary abelian or cyclic. Let q be a prime-power such that $\gcd(n, q-1) = 1$, and C be a $\hat{G}$-invariant code over $\mathbb{F}_q$. If no prime divisor of $|\mathrm{Aut}(P_i)|$ other than p_i divides n for every $1 \leqslant i \leqslant r$, then every permutation solving set for C is also monomial solving set for C.*

6 Problems

The isomorphism problem for Cayley digraphs of abelian group seems to be quite difficult in general. There are now at least three obstacles to a group being CI with respect to digraphs that will depend at least to some extent on the structure of the full automorphism group of the digraph. The first is that there may not be an appropriate conjugate such that $\langle G_L, \delta^{-1}G\delta\rangle$ is normally m-step imprimitive [16]. Second, isomorphic regular abelian subgroups of a p-subgroup P of the symmetric group need not be conjugate in P. The third is that in general the direct product of two CI-groups of relatively prime order need not be a CI-group by Theorem 1. All of these obstacles can occur in the automorphism group of a ternary relational structure ([16] in the first instance, and [9] for the latter two). It thus seems wise to begin to break these large difficult problems into more manageable pieces. For example, the following problem seems natural in this context:

Problem 42. Let $\ell \geqslant 1$. Which finite groups G have the property that whenever $\delta \in \mathrm{Sym}(G)$, there exists $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle^{(\ell)}$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ is (normally) m-step imprimitive?

For the previous problem, particular attention should be paid to the groups G which may be CI-groups with respect to digraphs (see [29, Theorem 1.2]) or graphs (see [18, Corollary 21]), and most especially, to groups which may be CI-groups with respect to ternary relational structures (see [16, Theorem 4.2]). Abelian groups also seem worthy of consideration. Of course, p-groups and cyclic groups have the above property, as well as groups given in Theorem 22. Also observe that for any group G, $G^{(1)} = G$, so a special case of the previous problem is to find all groups G such that whenever $\delta \in \mathrm{Sym}(G)$ there exists $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ is (normally) m-step imprimitive.

The problem corresponding to the second obstacle given above is simply to solve the isomorphism problem for abelian p-groups, so we make no more mention of this well-known problem.

Finally, for the third obstacle given above, the following conjecture has been made:

Conjecture 43. Let G and H be CI-groups with respect to digraphs of relatively prime order. Then $G \times H$ is a CI-group with respect to digraphs.

Virtually no progress has been made on this general conjecture, other than showing specific groups are CI-groups with respect to digraphs. We propose a much simpler problem (which still seems quite difficult, and would also be happy for partial solutions for particular G):

Problem 44. Let G be a cube-free abelian group. Assume that whenever $\delta \in \mathrm{Sym}(G)$ such that $\langle G_L, \delta^{-1}G_L\delta\rangle$ is normally m-step imprimitive. Does the isomorphism problem for Cayley digraphs of G reduce to the Cayley isomorphism problem for digraphs of its Sylow subgroups?

Finally, we believe that a much more general result than the positive part of Pálfy's Theorem is true.

Conjecture 45. Let G and H be groups such that

$$\gcd(|\mathrm{Aut}(G)\cdot G|, |H|) = 1 = \gcd(|\mathrm{Aut}(H)\cdot H|, |G|).$$

Then the Cayley isomorphism problem for $G \times H$ reduces to the Cayley isomorphism problem for G and H.

Note that if m and n are positive integers such that $\gcd(m, \varphi(m)) = 1$, $\gcd(n, \varphi(n)) = 1$ and $\gcd(\varphi(n)\cdot n, m) = 1 = \gcd(\varphi(m)\cdot m, n)$, then $\gcd(mn, \varphi(mn)) = 1$, and so Pálfy's Theorem implies the conjecture is true for cyclic groups of order m and n respectively.

Let $k = p_1 \cdots p_r$ be such that $\gcd(k, \varphi(k)) = 1$, and $m = p_1^{a_1} \cdots p_r^{a_r}$, $\ell = q_1 \cdots q_s$ and $\gcd(\ell, \varphi(\ell)) = 1$ and $n = q_1^{a_1} \cdots q_s^{a_s}$, where the p_i's are distinct primes and the q_j's are distinct primes. If in addition, $\gcd(\varphi(n)\cdot n, m) = 1 = \gcd(\varphi(m)\cdot m, n)$, then $p_1, \ldots, p_r, q_1, \ldots, q_s$ are distinct primes, and if $a = p_1 \cdots p_r q_1 \cdots q_s$, then $\gcd(a, \varphi(a)) = 1$. Thus the conjecture holds for cyclic groups of order m and n by Muzychuk's result [33, Theorem 1.9]. More generally, combining [13, Corollary 4.4] and [33, Theorem 4.9], the conjecture is true for all cyclic groups satisfying the hypothesis (this also follows from the results in this paper). Of course, the results in this paper verify the result for abelian groups G and H such that every Sylow subgroup is either elementary abelian or cyclic.

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