Hamilton circles in Cayley graphs

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Abstract

For locally finite infinite graphs the notion of Hamilton cycles can be extended to Hamilton circles, homeomorphic images of S^1 in the Freudenthal compactification. In this paper we prove a sufficient condition for the existence of Hamilton circles in locally finite Cayley graphs.

Mathematics Subject Classifications: 05C25, 05C45, 05C63, 20E06, 20F05, 37F20

1 Introduction

In 1969, Lovász, see [1], conjectured that every finite connected vertex-transitive graph contains a Hamilton cycle except five known counterexamples. As the Lovász conjecture is still open, one might instead try to solve the, possibly easier, Lovász conjecture for finite Cayley graphs which states: Every finite Cayley graph with at least three vertices contains a Hamilton cycle. Doing so enables the use of group theoretic tools. Moreover one can ask for what generating sets a particular group contains a Hamilton cycle. There are a vast number of papers regarding the study of Hamilton cycles in finite Cayley graphs, see [8, 9, 16, 24, 25], for a survey of the field see [23].

In this paper we focus on Hamilton cycles in infinite Cayley graphs. As cycles are always finite, we need a generalization of Hamilton cycles for infinite graphs. We follow the topological approach of [4, 5, 7], which extends Hamilton cycles in a sensible way by using the circles in the Freudenthal compactification $|\Gamma|$ of a Γ graph as infinite cycles. There are already results on Hamilton circles in general infinite locally finite graphs, see [11, 13, 14, 15].

It is worth remarking that the weaker version of the Lovasz's conjecture does not hold true for infinite groups. For example it is straight forward to check that the Cayley graph of any free group with the standard generating set does not contain Hamilton circles, as they are trees.

It is a known fact that every locally finite graph needs to be 1-tough to contain a Hamilton circle, see [11]. Thus a way to obtain infinitely many Cayley graphs with no Hamilton circle is to amalgamate more than k groups over a subgroup of order k. In 2009, Georgakopoulos [11] asked if avoiding this might be enough to force the existence of Hamilton circles in locally finite graphs and proposed the following problem:

Problem 1. [11, Problem 2] Let Γ be a connected Cayley graph of a finitely generated group. Then Γ has a Hamilton circle unless there is a $k \in \mathbb{N}$ such that the Cayley graph of Γ is the amalgamated product of more than k groups over a subgroup of order k.

In Section 4.1 we provide a counterexample to this statement.

For a one-ended graph Γ it suffices to find a spanning two-way infinite path, a *double ray*, to find a Hamilton circle of $|\Gamma|$. In 1959 Nash-Williams [19] showed that any Cayley graph of any infinite finitely generated abelian group admits a spanning double ray. In the case of one-ended graphs, such a double ray is a Hamilton circle in our sense. So Nash-Williams [19] shows that every Cayley graph of an abelian group has a Hamilton circle. We extend this result by showing that any Cayley graph of any finitely generated abelian group, besides $\mathbb{Z}$ generated by $\{\pm 1\}$, contains a Hamilton circle in Section 3.1. We extend this result also to an even larger class of infinite groups. We will show that the Cayley graph of the free product with amalgamation over the subgroup of index two of a Dedekind group and an arbitrary group possesses a Hamilton circle.

2 Preliminaries

For the notations and the terminologies of group theory and topology and graph theory, see [21], [18] and [4], respectively.
In the following we will recall the most important definitions and notations for the readers convenience.

2.1 Topology

In 1931, Freudenthal [10] defined the concept of topological ends for topological spaces and topological groups for the first time. Let X be a locally compact Hausdorff space. In order to define ends of the topological space X, we look at infinite sequence $U_1 \supseteq U_2 \supseteq \cdots$ of non-empty connected open subsets of X such that the boundary of each U_i is compact and $\bigcap \overline{U_i} = \emptyset$. He called two sequences $U_1 \supseteq U_2 \supseteq \cdots$ and $V_1 \supseteq V_2 \supseteq \cdots$ to be *equivalent* if for every $i \in \mathbb{N}$, there are $j, k \in \mathbb{N}$ in such a way that $U_i \supseteq V_j$ and $V_i \supseteq U_k$. The equivalence classes[1] of those sequences are *topological ends* of X and the set of all ends of X is denoted by $\Omega(X)$. The *Freudenthal compactification* of the space X is defined as topology generated by the following open sets:

$$\{O \cup \{[U_i] \in \Omega(X) \mid U_i \subseteq O\} \mid O \textit{ is an open set in } X\}$$

[1] We denote the equivalence class of U_i by $[U_i]$.

We denote the Freudenthal compactification of the topological space X by $|X|$.
In 1964 Halin [12] introduced the vertex ends of infinite graphs. A *ray* is a one-way infinite path in a graph. It's subrays are it's *tails*. He defined two rays R_1 and R_2 of a given graph Γ are equivalent if for every finite set of vertices S of Γ there is a component of $\Gamma \setminus S$ which contains both a tail of R_1 and of R_2. The classes of the equivalent rays is called *vertex ends* and just for abbreviation we say *end*. Diestel and Kühn [7] have investigated the connection between vertex ends by Halin and topological ends by Freudenthal. They have shown that if we consider a locally finite graph Γ as 1-complex with the corresponding topology, then topological ends and vertex ends coincide.

For a graph Γ we denote the Freudenthal compactification of Γ by $|\Gamma|$. A homeomorphic image of $[0,1]$ in the topological space $|\Gamma|$ is called *arc*. A *Hamilton arc* in $|\Gamma|$ is an arc including all vertices of Γ. So a Hamilton arc in a graph always contains all ends of the graph. By a *Hamilton circle* in $|\Gamma|$, we mean a homeomorphic image of the unit circle in $|\Gamma|$ containing all vertices of Γ. A Hamilton arc whose image in a graph is a double ray is a *Hamilton double ray*. It is worth mentioning that an uncountable graph cannot contain a Hamilton circle. To illustrate, let C be a Hamilton circle of graph Γ. Since C is homeomorphic to S^1, we can assign to every edge of C a rational number. Thus we can conclude that $V(C)$ is countable and so Γ is countable. Hence in this paper, we assume that all groups are countable. In addition we will only consider groups with locally finite Cayley graphs in this paper so we assume that all generating sets S will be finite.

2.2 Graphs

Throughout this paper Γ will be reserved for graphs. In addition to the notation of paths and cycles as sequences of vertices such that there are edges between successive vertices we use the notation of [16, 23] for constructing Hamilton paths and Hamilton cycles and circles which uses edges rather than vertices. For that let g and $s_i,\ i \in \mathbb{Z}$, be elements of some group. In this notation $g[s_1]^k$ denotes the concatenation of k copies of s_1 from the right starting from g which translates to the path $g, (gs_1), \ldots, (gs_1^k)$ in the usual notation. Analogously $[s_1]^k g$ denotes the concatenation of k copies of s_1 starting again from g from the left. In addition $g[s_1, s_2, \ldots]$ translates to be the ray $g, (gs_1), (gs_1s_2), \ldots$ and

$$[\ldots, s_{-2}, s_{-1}]g[s_1, s_2, \ldots]$$

translates to be the double ray

$$\ldots, (gs_{-2}s_{-1}), (gs_{-1}), g, (gs), (gs_1s_2), \ldots$$

When discussing rays we extend the notation of $g[s_1, \ldots, s_n]^k$ to k being countably infinite and write $g[s_1, \ldots, s_2]^{\mathbb{N}}$ and the analogue for double rays. Sometimes we will use this notation also for cycles. Stating that $g[c_1, \ldots, c_k]$ is a cycle means that $g[c_1, \ldots, c_{k-1}]$ is a path and that the edge c_k joins the vertices $gc_1 \cdots c_{k-1}$ and g.

For a graph Γ let the induced subgraph on the vertex set X be called $\Gamma[X]$.

2.3 Groups

Throughout this paper G will be reserved for groups. For a group G with respect to generating set S, i.e. $G = \langle S \rangle$, we denote the Cayley graph of G with respect to S by $\Gamma(G, S)$ unless explicitly stated otherwise. The Cayley graph associated with (G, S) is a graph having one vertex associated with each element of G and edges (g_1, g_2) whenever $g_1 g_2^{-1}$ lies in S. For a set $T \subseteq G$ we set $T^{\pm} := T \cup T^{-1}$. Through out this paper we assume that all generating sets are symmetric, i.e. whenever $s \in S$ then $s^{-1} \in S$. Thus if we add an element s to a generating set S, we always also add the inverse of s to S as well.

Suppose that G is an abelian group. A finite set of elements $\{g_i\}_{i=1}^n$ of G is called *linearly dependent* if there exist integers λ_i for $i = 1, \ldots, n$, not all zero, such that $\sum_{i=1}^n \lambda_i g_i = 0$. A system of elements that does not have this property is called *linearly independent*. It is an easy observation that a set containing elements of finite order is linearly dependent. The *rank* of an abelian group is the size of a maximal independent set. This is exactly the rank the torsion free part, i.e if $G = \mathbb{Z}^n \oplus G_0$ where G_0 is the torsion part of G, then rank of G is n.

Let G_1 and G_2 be two groups with subgroups H_1 and H_2 respectively such that there is an isomorphism $\phi\colon H_1 \to H_2$. The *free product with amalgamation* is defined as

$$G_1 *_{H_1} G_2 := \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup H_1 \phi^{-1}(H_1) \rangle.$$

A way to present elements of a free product with amalgamation is the Britton's Lemma:

Lemma 1. [2, Theorem 11.3] *Let G_1 and G_2 be two groups with subgroups $H_1 \cong H_2$ respectively. Let T_i be a left transversal*[2] *of H_i for $i = 1, 2$. Any element $x \in G_1 *_H G_2$ can be uniquely written in the form $x = x_0 x_1 \cdots x_n$ with the following:*

(i) $x_0 \in H_1$.

(ii) $x_j \in T_1 \setminus \{1\}$ *or* $x_i \in T_2 \setminus \{1\}$ *for* $j \geqslant 1$ *and the consecutive terms* x_j *and* x_{j+1} *lie in distinct transversals.*

Let $G = \langle S \mid R \rangle$ be a group with subgroups H_1 and H_2 in such a way that there is an isomorphism $\phi\colon H_1 \to H_2$. We now insert a new symbol t not in G and we define the *HNN-extension* of $G *_{H_1}$ as follows:

$$G *_{H_1} := \langle S, t \mid R \cup t^{-1} H_1 t \phi(H_1)^{-1} \rangle.$$

Throughout this paper we assume that any generating set $S = \{s_1, \ldots, s_n\}$ is *minimal* in the following sense: Each $s_i \in S$ cannot be generated by $S \setminus \{s_i\}$, i.e. we have that $s_i \notin \langle s_j \rangle_{j \in \{1,\ldots,n\} \setminus \{i\}}$. We may do so because say $S' \subseteq S$ is a minimal generating set of G. If we can find a Hamilton circle C in $\Gamma(G, S')$, then this circle C will still be a Hamilton circle in $\Gamma(G, S)$. For this it is important to note that the number of ends

[2]A *transversal* is a system of representatives of left cosets of H_i in G_i and we always assume that 1 belongs to it.

of G and thus of $\Gamma(G, S')$ does not change with changing the generating set to S by [17, Theorem 11.23], as long as S is finite, which will always be true in this paper.

We now cite a structure theorem for finitely generated groups with two ends.

Theorem 2. [20, Theorem 5.12] *Let G be a finitely generated group. Then the following statements are equivalent.*

(i) *The number of ends of G is 2.*

(ii) *G has an infinite cyclic subgroup of finite index.*

(iii) *$G = A *_C B$ and C is finite and $[A : C] = [B : C] = 2$ or $G = C *_C$ with C is finite.*

Throughout this paper we use Theorem 2 to characterize the structure of two ended groups, see Section 3 for more details. It is still important to pay close attention to the generating sets for those groups though, as the following example shows. Take two copies of $\mathbb{Z}_2$, with generating sets $\{a\}$ and $\{b\}$, respectively. Now consider the free product of them. It is obvious that this Cayley graph with generating set $\{a, b\}$ does not contain a Hamilton circle. Again consider $\mathbb{Z}_2 * \mathbb{Z}_2$ with generating set $\{a, ab\}$ which is isomorphic to $D_\infty = \langle x, y \mid x^2 = (xy)^2 = 1\rangle$. It is easy to see that the Cayley graph of D_∞ with this generating set contains a Hamilton circle.

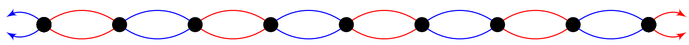

Figure 1: The Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2$ with the generating set $\{a, b\}$

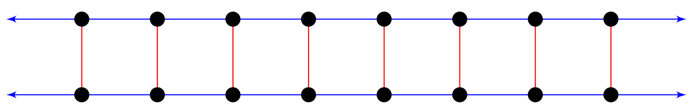

Figure 2: The Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2$ with the generating set $\{a, ab\}$

3 Hamilton circles

In this section we prove sufficient conditions for the existence of Hamilton circles in Cayley graphs. In Section 3.1 we take a look at abelian groups. Section 3.2 contains basic lemmas and structure theorems used to prove our main results which we prove in the Section 3.3.

3.1 Abelian Groups

In the following we will examine abelian groups as a simple starting point for studying Hamilton circles in infinite Cayley graphs. Our main goal in this section is to extend a well-known theorem of Nash-Williams from one-ended abelian groups to two ended abelian groups by a simple combinatorial argument. First, we cite a known result for finite abelian groups.

Lemma 3. [22, Corollary 3.2] *Let G be a finite abelian group with at least three elements. Then any Cayley graph of G has a Hamilton cycle.*

Next we state the theorem of Nash-Williams.

Theorem 4. [19, Theorem 1] *Let G be a finitely generated abelian group.*

(i) *If G has exactly one end, then any Cayley graph of G has a Hamilton circle.*

(ii) *If G has exactly two ends, then any Cayley graph of G has a spanning double ray.*

It is obvious that the maximal class of groups to extend Theorem 4 to cannot contain $\Gamma(\mathbb{Z}, \{\pm 1\})$, as this it cannot contain a Hamilton circle. In Theorem 5 we prove that this is the only exception.

Theorem 5. *Let $G = \langle S \rangle$ be an infinite finitely generated abelian group. Then $\Gamma(G, S)$ has a Hamilton circle unless $G = \mathbb{Z}$ and $S = \{\pm 1\}$.*

Proof. Let $G = \langle S \rangle$ be an infinite finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups [21, 5.4.2], one can see that $G \cong \mathbb{Z}^n \oplus G_0$ where G_0 is the torsion part of G and $n \in \mathbb{N}$. It follows from [20, lemma 5.6] that the number of ends of $\mathbb{Z}^n$ and G are equal. We know that the number of ends of $\mathbb{Z}^n$ is one if $n \geqslant 2$ and two if $n = 1$. By Theorem 4 we are done if $n \geqslant 2$. So we can assume that G has exactly two ends.

Since $\Gamma(\mathbb{Z}, \{\pm 1\})$ is not allowed, we may assume that S contains at least two elements. Let $S_{\text{inf}} \subseteq S$ be the set of generators of infinite order of G. Note that $S_{\text{inf}} \neq \emptyset$ as G is infinite and abelian. There is an $s \in S_{\text{inf}}$ such that $\langle s \rangle \neq G$. Otherwise the only generator of infinite order already generates G, which implies that $G \cong \mathbb{Z}$. This yields that $S \setminus S_{\text{inf}} = \emptyset$, which contradicts our starting assumption. Let us say $S = \{s_1, \ldots s_\ell\}$ with $s = s_1$. In the following we define a sequence of double rays. We start with the double ray $R_1 = [s_1^{-1}]^{\mathbb{N}} 1 [s_1]^{\mathbb{N}}$.

Now inductively assume that we have defined elements $s_1, \ldots, s_i$ in S such that $\langle s_1, \ldots, s_i \rangle \neq G$ and the double ray R_i spanning $\langle s_1, \ldots, s_i \rangle$. Now we choose $s_{i+1} \in S \setminus \langle s_1, \ldots, s_i \rangle$. If $\langle s_1, \ldots, s_{i+1} \rangle \neq G$, then we define R_{i+1} as in the previous step. More precisely assume that $R_i = [\ldots, y_{-2}, y_{-1}] 1 [y_1, y_2, \ldots]$. Let $j \in \mathbb{N}$ be minimal such that $s_{i+1}^{j+1} \in \langle s_1, \ldots, s_i \rangle$. We now define the double ray

$$R_{i+1} = \cdots [s_{i+1}^{-1}]^j [y_{-2}] [s_{i+1}]^j [y_{-1}] 1 [s_{i+1}]^j [y_1] [s_{i+1}^{-1}]^j [y_2] \cdots .$$

If $\langle s_1, \ldots, s_{i+1} \rangle = G$, then we define the following two disjoint double rays: Suppose that j is the smallest natural number such that $s_\ell^{j+1} \in \langle s_1, \ldots, s_i \rangle$. Now, put

$$\mathcal{P}_1 = \cdots [s_{i+1}^{-1}]^{j-1} [y_{-2}] [s_{i+1}]^{j-1} [y_{-1}] 1 [s_{i+1}]^{j-1} [y_1] [s_{i+1}^{-1}]^{j-1} [y_2] \cdots$$

and

$$\mathcal{P}_2 = [\ldots, y_{-2}, y_{-1}]s_{i+1}^{j}[y_1, y_2, \ldots].$$

It is not hard to see that $\mathcal{P}_1 \cup \mathcal{P}_2$ is a Hamilton circle of $\Gamma(G, S)$. $\square$

Remark 6. One can prove Theorem 4 by same the arguments used in the above proof of Theorem 5.

3.2 Structure Tools

In this section we assemble all the most basic tools to prove our main results. The most important tools are Lemma 8 and Lemma 9. In both Lemmas we prove that a given graph Γ contains a Hamilton circle if Γ admits a partition of its vertex set fulfilling the following nice properties. All partition classes are finite and of the same size. And each partition class contain some special cycle and between two consecutive partition classes there are edges in Γ connecting those cycles in a useful way, see Lemma 8 and 9 for details.

But first we cite the work of Diestel in the following lemma as a tool to finding Hamilton circles in two-ended graphs.

Lemma 7. [5, Theorem 2.5] *Let $\Gamma = (V, E)$ be a two-ended graph. And let R_1 and R_2 be two doubles rays such that the following holds:*

(i) $R_1 \cap R_2 = \emptyset$

(ii) $V = R_1 \cup R_2$

(iii) *For each $\omega \in \Omega(\Gamma)$ both R_i have a tail that belongs to ω.*

Then $R_1 \sqcup R_2$ is a Hamilton circle of Γ.

Lemma 8. *Let Γ be a graph that admits a partition of its vertex set into finite sets X_i, $i \in \mathbb{Z}$, fulfilling the following conditions:*

(i) *$\Gamma[X_i]$ contains a Hamilton cycle C_i or $\Gamma[X_i]$ is isomorphic to K_2.*

(ii) *For each $i \in \mathbb{Z}$ there is a perfect matching between X_i and X_{i+1}.*

(iii) *There is a $k \in \mathbb{N}$ such that for all $i, j \in \mathbb{Z}$ with $|i - j| \geqslant k$ there is no edge in Γ between X_i and X_j.*

Then Γ has a Hamilton circle.

Proof. By (i) we know that each X_i is connected and so we conclude from the structure given by (ii) and (iii) that Γ has exactly two ends. In addition note that $|X_i| = |X_j|$ for all $i, j \in \mathbb{Z}$. First we assume that $\Gamma[X_i]$ is just a K_2. It follows directly that Γ is spanned by the double ladder, which is well-known to contain a Hamilton circle. As this double ladder shares its ends with Γ, this Hamilton circle is also a Hamilton circle of Γ.

Now we assume that $|X_i| \geqslant 3$. Fix an orientation of each C_i. The goal is to find two disjoint spanning doubles rays in Γ. We first define two disjoint rays belonging to same end, say for all the X_i with $i \geqslant 1$. Pick two vertices u_1 and w_1 in X_1. For R_1 we start with u_1 and move along C_1 in the fixed orientation of C_1 till the next vertex on C_1 would be w_1. Then, instead of moving along C_1, we move to X_2 by the given matching edge. We take this to be a the initial part of R_1. We do the analogue for R_2 by starting with w_1 and moving also along C_1 in the fixed orientation till the next vertex would be u_1, then move to X_2. We repeat the process of starting with in two vertices u_i and w_i contained in some X_i, where u_i is the first vertex of R_1 on X_i and w_i the analogue for R_2. We follow along the fixed orientation on C_i till the next vertex would be u_i or w_i, respectively. Then we move to X_{i+1} by the giving matching edges. One can easily see that each vertex of X_i for $i \geqslant 1$ is contained exactly either in R_1 or R_2. By moving from u_1 and w_1 to X_0 by the matching edges and then using the same process but moving from X_i to X_{i-1} extents the rays R_1 and R_2 into two double rays. Obviously those double rays are spanning and disjoint. As Γ has exactly two ends it remains to show that R_1 and R_2 have a tail in each end, see Lemma 7. By (iii) there is a k such that there is no edge between any X_i and X_j with $|i-j| \geqslant k$. The union $\bigcup_{i=\ell}^{\ell+k} X_i$, $\ell \in \mathbb{Z}$, separates Γ into two components such that R_i has a tail in each component, which is sufficient. □

Next we prove a slightly different version of Lemma 8. In this version we split each X_i into an "upper" and "lower" part, X_i^+ and X_i^-, and assume that we only find a perfect matching between upper and lower parts of adjacent partition classes, see Lemma 9 for details.

Lemma 9. *Let Γ be a graph that admits a partition of its vertex set into finite sets $X_i, i \in \mathbb{Z}$ with $|X_i| \geqslant 4$ fulfilling the following conditions:*

(i) $X_i = X_i^+ \cup X_i^-$, *such that* $X_i^+ \cap X_i^- = \emptyset$ *and* $|X_i^+| = |X_i^-|$

(ii) $\Gamma[X_i]$ *contains an Hamilton cycle C_i which is alternating between X_i^- and X_i^+.*[3]

(iii) *For each $i \in \mathbb{Z}$ there is a perfect matching between X_i^+ and X_{i+1}^-.*

(iv) *There is a $k \in \mathbb{N}$ such that for all $i, j \in \mathbb{Z}$ with $|i-j| \geqslant k$ there is no edge in Γ between X_i and X_j.*

Then Γ has a Hamilton circle.

The proof of Lemma 9 is very closely related to the proof of Lemma 8. We still give the complete proof for completeness.

Proof. By (ii) we know that each X_i is connected and so we conclude from the structure given by (iii) and (iv) that Γ has exactly two ends. In addition note that $|X_i| = |X_j|$ for all $i, j \in \mathbb{Z}$.

[3]Exactly every other element of C_i is contained in X_i^-.

Fix an orientation of each C_i. The goal is to find two disjoint double rays whose union is spanning in Γ. We first define two disjoint rays belonging to the same end, say for all the X_i with $i \geqslant 0$. Pick two vertices u_1 and w_1 in X_1^-. For R_1 we start with u_1 and move along C_1 in the fixed orientation of C_1 till the next vertex on C_1 would be w_1, then instead of moving along C_1 we move to X_2^- by the given matching edges. Note that as w_1 is in X_1^- and because each C_i is alternating between X_i^- and X_i^+ this is possible. We take this to be a the initial part of R_1. We do the analog for R_2 by starting with w_1 and moving also along C_1 in the fixed orientation till the next vertex would be u_1, then move to X_2^-. We repeat the process of starting with some X_i in two vertices u_i and w_i, where u_i is the first vertex of R_1 on X_i and w_i the analog for R_2. We follow along the fixed orientation on C_i till the next vertex would be u_i or w_i, respectively. Then we move to X_{i+1} by the giving matching edges. One can easily see that each vertex of X_i for $i \geqslant 1$ is contained exactly either in R_1 or R_2. By moving from u_1 and w_1 to X_0^+ by the matching edges and then using the same process but moving from X_i^- to X_{i-1}^+ extends the rays R_1 and R_2 into two double rays. Obviously those double rays are spanning and disjoint. As Γ has exactly two ends it remains to show that R_1 and R_2 have a tail in each end, see Lemma 7. By (iv) there is a k such that there is no edge between any X_i and X_j with $|i-j| \geqslant k$ the union $\bigcup_{i=l}^{\ell+k} X_i,\ l \in \mathbb{Z}$ separates a Γ into two components such that R_i has a tail in each component, which is sufficient. $\square$

Remark 10. It is easy to see that one can find a Hamilton double ray instead of a Hamilton circle in Lemma 8 and Lemma 9. Instead of starting with two vertices and following in the given orientation to define the two double rays, one just starts in a single vertex and follows the same orientation.

The following lemma is one of our main tools in proving the existence of Hamilton circles in Cayley graphs. It is important to note that the restriction, that $S \cap H = \emptyset$, which looks very harsh at first glance, will not be as restrictive in the later parts of this paper. In most cases we can turn the case $S \cap H \neq \emptyset$ into the case $S \cap H = \emptyset$ by taking an appropriate quotient.

Lemma 11. *Let $G = \langle S \rangle$ and $\widetilde{G} = \langle \widetilde{S} \rangle$ be finite groups with non-trivial subgroups $H \cong \widetilde{H}$ of indices two such that $S \cap H = \emptyset$ and such that $\Gamma(G,S)$ contains a Hamilton cycle. Then the following statements are true.*

(i) $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ *has a Hamilton circle.*

(ii) $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ *has a Hamilton double ray.*

To prove Lemma 11 we start by finding some general structure given by our assumptions. This structure will make it possible to use Lemma 9 and Remark 10 to prove the statements (i) and (ii).

Proof. First we define $\Gamma := \Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$. Let $s \in S \setminus H$ and let $\widetilde{s}$ be in $\widetilde{S} \setminus \widetilde{H}$. By our assumptions $\Gamma(G,S)$ contains a Hamilton cycle, say $C_0 = 1[c_1, \ldots, c_k]$. It follows from $S \cap H = \emptyset$ that C_0 is alternating between H and the right coset Hs. For each $i \in \mathbb{Z}$

we now define the graph $\Gamma_i := \Gamma[(s\tilde{s})^i G]$. If i is a positive number, then one can see that $H(s\tilde{s})^i \cup H(s\tilde{s})^i s$ is equal to $(s\tilde{s})^i H \cup (s\tilde{s})^i Hs$, as H is a normal subgroup of G. So $H(s\tilde{s})^i \cup H(s\tilde{s})^i s = (s\tilde{s})^i G$ if i is positive. Now suppose that i is negative. We note that $Hs = Hs^{-1}$ and also $H\tilde{s} = H\tilde{s}^{-1}$. Analogously we are able to show that

$$\begin{aligned}
&H\tilde{s}(s\tilde{s})^{-i-1} \cup H(s\tilde{s})^{-i} \\
=&H\tilde{s}(\tilde{s}^{-1}s^{-1})^{i+1} \cup H(\tilde{s}^{-1}s^{-1})^{i} \\
=&(s\tilde{s})^i Hs \cup (s\tilde{s})^i H \\
=&(s\tilde{s})^i G
\end{aligned}$$

Now let us define the cycle $C_i := (s\tilde{s})^i[c_1, \ldots, c_k]$ which means the translation of C_0 into Γ_i. Since C_0 is a Hamilton cycle in $\Gamma[G]$, we are able to conclude that the cycle C_i is a Hamilton cycle of the graph Γ_i

In the following we give some easy observations on the structure of the C_i's. First note that $C_i \cap C_j = \emptyset$ for $i \neq j$ and also that the union of all C_i's contains all the vertices of Γ. In addition note that each C_i is alternating between two copies of H as C_0 was alternating between cosets of Γ_0. Finally note that by the structure of Γ there is no edge between any Γ_i and Γ_j with $|i-j| \geqslant 2$ in Γ.

By the structure of Γ for $i \geqslant 0$ we get a perfect matching between $C_i \cap H(s\tilde{s})^i s$ and $C_{i+1} \cap H(s\widetilde{s})^{i+1}$ by $\widetilde{s}$.

By a similar argument one can show that for $i < 0$ we get a similar structure and the desired perfect matchings.

The statement (i) now follows by Lemma 9. Analog statement (ii) follows by Remark 10. □

We now recall two known statements about Hamilton cycles on finite groups, which we then will first combine and finally generalize to infinite groups. For that let us first recall some definitions. A group G is called *Dedekind*, if every subgroup of G is normal in G. If a Dedekind groups G is also non-abelian, it is called a *Hamilton group*.

Lemma 12. [3] *Any Cayley graph of a Hamilton group G has a Hamilton cycle.*

In addition we know that all finite abelian groups also contain Hamilton cycles by Lemma 3. In the following remark we combine these two facts.

Remark 13. Any Cayley graph of a finite Dedekind group of order at least three contains a Hamilton cycle.

3.3 Main Results

In this section we prove our main results. For that let us recall that by Theorem 2 we know that there every two ended group either a free product with amalgamation over a finite subgroup of index two or an HNN-extension over a finite subgroup. Now we prove our first main result, Thereom 14, which deals with the first type of groups. To be more precise we use Remark 13 to prove that there is a Hamilton circle in the free product with amalgamation over the subgroup of index two of a Dedekind group and an arbitrary group.

Theorem 14. *Let $G = \langle S \rangle$ and $\widetilde{G} = \langle \widetilde{S} \rangle$ be two finite groups with non-trivial subgroups $H \cong \widetilde{H}$ of indices two and such that G is a Dedekind group. Then $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ has a Hamilton circle.*

Proof. For easier reading let us define $\Gamma := \Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$. First, it follows from Remark 13 that $\Gamma(G, S)$ has a Hamilton cycle. If all generators of $S = \{s_1, \ldots, s_n\}$ lie outside H, then Lemma 11 completes the proof. So let $s_n \in S \setminus H$ and let $\widetilde{s} \in \widetilde{S} \setminus \widetilde{H}$. Suppose that $S' := \{s_1, \ldots, s_i\}$ is a maximal set of generators of S contained in H and set $L := \langle S' \rangle$. First note that L is a normal subgroup of G as G is a Dedekind group. We now have two cases, either $H = L$ or $L \neq H$. We first assume that $H = L$. If $|H| = 2$ we are done by Lemma 8 so we assume that $|H| \geqslant 3$. Thus we can apply Remark 13 to H and find a Hamilton cycle of $\Gamma(H, S')$. We conclude that $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ contains a Hamilton circle by Lemma 8 which finishes this case.

In the following we now only need to consider the case that $H \neq L$. Since G is a Dedekind group, the quotient group G/L is a Dedekind group as well. It follows from Remark 12 that G/L has a Hamilton cycle $\overline{C} := L[x_1, \ldots, x_t]$. For now we only consider the case that $|L| \geqslant 3$. The case $|L| = 2$ is an easier version of the proof below.[4] Since L is a Dedekind group, we assume that L has a Hamilton cycle C_1 by Remark 13. We select two vertices in L, say v_1 and v_2 and we fix an orientation of C_1. We start at v_1 and follow the orientation of C_1 until we reach the last vertex before v_2, say v_1'. Starting at v_2 we also follow C_1 until we reach the last vertex before v_1, say v_2'. Then we move to the next coset along $\overline{C}$. More precisely, we use x_1 to move to Lx_1. We again fix an orientation of the Hamilton cycle of Lx_1 and start in the vertices $v_1'x_1$ and $v_2'x_2$ and collect all vertices of Lx_1. We iterate this process until we reach $Lx_1 \cdots x_t$. Suppose that the last vertices of the disjoint paths in the last step are u_1 and u_2. We notice that we cover all vertices of $G *_H \tilde{G}$ belonging to G.

We now use $\tilde{s}$ to go the next layer. Note that H is normal in both the groups G and $\tilde{G}$ and so $Hs\tilde{s} = s\tilde{s}H$ and $Hs\tilde{s}s = s\tilde{s}sH$. On the other hand $G = \cup_{j=0}^{t} Lx_0 \cdots x_j$, where $x_0 = 1$. Thus we deduce that

$$Hs\tilde{s} \cup Hs\tilde{s}s = (\cup_{i \in I} s\tilde{s}Lx_i') \bigcup (\cup_{j \in J} s\tilde{s}sLx_j'),$$

where x_i' and x_j' are some product of the $x_1, \ldots x_t$. Note that the above equation gives us a partition of $Hs\tilde{s} \cup Hs\tilde{s}s$. Now we have two cases: either $u_1\tilde{s}$ and $u_2\tilde{s}$ lies in the same set $s\tilde{s}Lx_i'$ or they belong to different sets $s\tilde{s}Lx_i'$ and $s\tilde{s}Lx_j'$. If they belong to the same set, then we repeat the process we used above for the cosets of L in G.

Thus we assume $u_1\tilde{s}$ and $u_2\tilde{s}$ belong to $s\tilde{s}Lx_i'$ and $s\tilde{s}Lx_j'$, where $i \neq j$. Since G/L has a Hamilton cycle $\overline{C}$, there is a Hamilton cycle of $(Hs\tilde{s} \cup Hs\tilde{s}s)/L$ in Γ. We denote this

[4] The main difference is, that one can omit the Hamilton cycle of $|L|$. Instead one can, dependent on the part of the proof, either directly leave each copy of L if we entered with two disjoint paths or collect both vertices of L by moving along the one edge in L.

cycle by $C_2 := s\tilde{s}L[x_1, \ldots, x_t]$. Fix an orientation of C_2. We start at $u_1\tilde{s}$ and use the cycle in L and cover all vertices in $s\tilde{s}Lx'_i$. We move along the orientation of C_2 and enter to the next partition class of the above mentioned partition. Again since L has a Hamilton cycle we can cover all vertices of this class. Continue this process until the next partition class is $s\tilde{s}Lx'_j$. We do the same starting at $u_2\tilde{s}$ stopping before the partition class $s\tilde{s}Lx'_i$. So far we have covered all vertices in $G \cup Hs\tilde{s} \cup Hs\tilde{s}s$. Iterate this construction to cover all vertices of Γ and we end up with two disjoint double rays, as desired. $\square$

The following Theorem 16 proves that the second type of two ended groups also contains a Hamilton circle, given some conditions.

Remark 15. Let us have a closer look at an HNN-extension of a finite group C. For that let $C = \langle S \mid R\rangle$ be a finite group. It is important to notice that every automorphism $\phi\colon C \to C$ gives us an HNN-extension $G = C *_C$. In particular every such HNN-extension comes from an automorphism $\phi\colon C \to C$. Therefore C is a normal subgroup of G with the quotient $\mathbb{Z}$, as the presentation of HNN-extension $G = C *_C$ is

$$\langle S, t \mid R,\, t^{-1}ct = \phi(c)\, \forall c \in C\rangle.$$

Hence G can be expressed by a semidirect product $C \rtimes \mathbb{Z}$ which is induced by ϕ. To summarize; every two ended group with a structure of HNN-extension is a semidirect product of a finite group with the infinite cyclic group.

Theorem 16. *Let $G = (H \rtimes F, X \cup Y)$ with $F = \mathbb{Z} = \langle Y\rangle$ and $H = \langle X\rangle$ and such that non-trivial group H is finite and H contains a Hamilton cycle. Then G has a Hamilton circle.*

Proof. Let $C = [c_1, \ldots, c_t]$ be a Hamilton cycle in $\Gamma(H, X)$. First we notice that H is a normal subgroup of G and moreover $G/H \cong Y$. Let $\pi\colon G \to Y$ be a such isomorphism. It follows from Part (ii) Theorem 4 that Y has a spanning double ray. We denote this spanning double ray $[\ldots, y_{-2}, y_{-1}]1[y_1, y_2, \ldots]$. On the other hand, the generators y_i for $i \in \mathbb{Z}$ gives a perfect matching between each consecutive cosets of H in G. More precisely $\pi^{-1}(y_i)$ for $i \in \mathbb{Z}$ is a perfect matching between Hx and $Hx\pi^{-1}(y_i)$. In addition the translation C by g is a cycle in gH. Thus we are ready to invoke Theorem 8 nd it finishes the proof. $\square$

4 Multiended groups

In this section we give a few insights into the problem of finding Hamilton circles in groups with more than two ends, as well as showing a counter example for Problem 1. We call a group to be a *multiended group* if is has more than two ends. In 1993 Diestel, Jung and Möller [6] proved that any transitive graph with more than two ends has infinitely many ends[5] and as all Cayley graphs are transitive it follows that the number of ends of any group is either zero, one, two or infinite. This yields completely new challenges for

[5]In this case the number of ends is uncountably infinite.

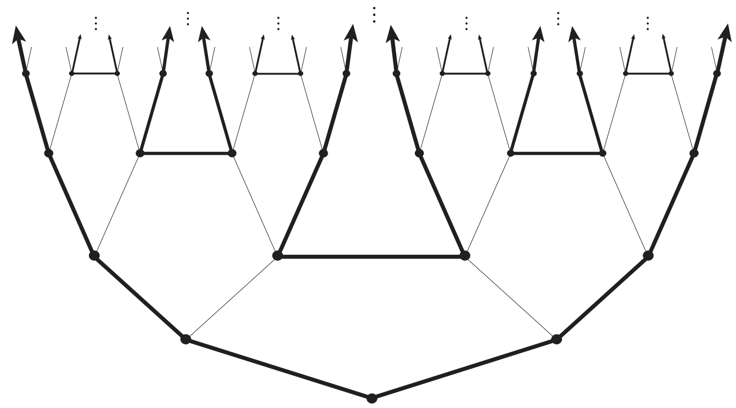

Figure 3: Hamilton circle in the Wild Circle.

finding a Hamilton circle in groups with more than two ends. In the following we provide the reader with an example to illustrate the problems of finding a Hamilton circles in an infinite graph with unaccountably many ends. In Figure 3 we illustrate the graph which is known as the Wild Circle, for more details see [4, Figure 8.5.1]. The thick edges of this locally finite connected graph form a Hamilton circle which uses only countably many edges and vertices while visiting all unaccountably many ends. Thus studying graph with more than two ends to find Hamilton circles is more complicated than just restricting one-self to two-ended groups.

4.1 A counterexample of Problem 1

We now give a counterexample to Problem 1. Define

$$G_1 := G_2 := \mathbb{Z}_3 \times \mathbb{Z}_2.$$

Let $\Gamma := \Gamma(G_1 *_{\mathbb{Z}_2} G_2)$. Let $G_1 = \langle a, b \rangle$ and $G_2 = \langle a, c \rangle$ where the order of a is two and the orders of b and c, respectively, are three. In the following we show that the assertion of Problem 1 holds for Γ and we show that $|\Gamma|$ does not contain a Hamilton circle.

For that we use the following well-known lemma and theorem.

Lemma 17. [4, Lemma 8.5.5] *If Γ is a locally finite connected graph, then a standard subspace* [6] *of $|\Gamma|$ is topologically connected (equivalently: arc-connected) if and only if it contains an edge from every finite cut of Γ of which it meets both sides.*

Theorem 18. [5, Theorem 2.5] *The following statements are equivalent for sets $D \subseteq E(\Gamma)$:*

(i) *Every vertex and every end has even degree in D.*

(ii) *D meets every finite cut in an even number of edges.*

[6] A standard subspace of $|\Gamma|$ is a subspace of $|\Gamma|$ that is a closure of a subgraph of Γ.

Assume for a contradiction that there is a Hamilton circle in Γ and let D be its edge set. Clearly D contains precisely two edges incident to every vertex. Theorem 18 tells us that D meets every finite cut in an even number and every vertex twice. Since circles are connected and arc connected we can, by Lemma 17, conclude that D meets every finite cut in at least one edge. We will now show that there is no set $D \subseteq E$ with these properties. For this purpose we study two cases: In each case we will consider a few finite cuts in Γ that show that such a D cannot exist. Figures 4 and 5 display induced subgraphs of Γ. The relevant cuts in those figures are the edges that cross the thick lines. The cases we study are that D contains the dashed edges of the appropriate figure corresponding to the case, see Figures 4 and 5. For easier reference we call the two larger vertices the *central vertices*.

Case 1: We now consider Figure 4, so we assume that the edges from the central vertices into the 'upper' side are one going to the left and the other to the right. First we note that the cut F ensures that the curvy edge between the central vertices is not contained in D. Also note that F ensures that the remaining two edges leaving the central vertices must go to the 'lower' side of Figure 4. As the cuts B and C have to meet an even number of edges of D we may, due to symmetry, assume that the dotted edge is also contained in D. This yields the contraction that the cut A now cannot meet any edge of D.

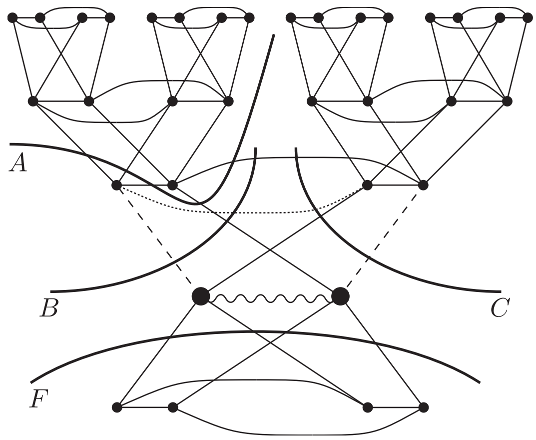

Figure 4: Case 1

Case 2: This case is very similar to Case 1. Again we may assume that the there are two edges leaving the central into the 'upper' and the 'lower' side, each. The cut C ensures that D must contain both dotted edges. But this again yields the contraction that A cannot meet any edge in D.

It remains to show that $G_1 *_{\mathbb{Z}_2} G_2$ cannot be obtained as a free product with amalgamation over subgroups of size k of more than k groups. If $G_1 *_{\mathbb{Z}_2} G_2$ were fulfilling the premise of Problem 1 then there would be a finite $W \subset V(\Gamma)$, say $|W| = k$, such that $\Gamma \setminus W$ has more than k components.

We will now use induction on the size of W. For a contraction we assume that such a set W exists. For that we now introduce some notation to make the following arguments

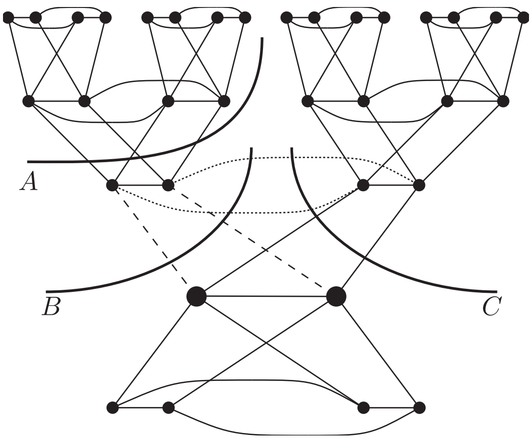

Figure 5: Case 2

easier. In the following we will consider each group element as its corresponding vertex in Γ. As Γ is transitive we may assume that 1 is contained in W. Furthermore we may even assume that no vertex which has a representation starting with c is contained in W. Let X_i be the set of vertices in Γ that have distance exactly i from $\{1, a\}$. We set $W_i := X_i \cap W$. For $x_i \in W_i$ let x_i^- be its neighbour in X_{i-1}, note that this is unique. For a vertex $x \in X_i$ let $\bar{x}$ be the neighbour of x in X_i which is not xa, note this will always be either xb or xc. For a set Y of vertices of Γ let C_Y be the number of components of $\Gamma \setminus Y$.

As Γ is obviously 2-connected the induction basis for $|W| = 0$ or $|W| = 1$ holds trivially.

We now assume that $|W| = k$ and that for each W' with $|W'| \leqslant |W| - 1$ we know that $C_{W'} \leqslant |W'|$. In our argument we will remove sets of vertices of size ℓ from W while decreasing C_W by at most ℓ.

Let $x \in W$ be a vertex with the maximum distance to $\{1, a\}$ in Γ. Say $x \in X_j$ and define $W_j := W \cap X_j$.

Suppose that $xa \notin W$. The set $\{xb, xb^2\}$ intersects at most one component of $\Gamma \setminus W$, as the two vertices are connected by an edge. We can use the same argument for $\{xc, xc^2\}$. If $xa \notin W$, then it lies in one these components as well. If xb further away from from $\{1, a\}$, then it is connected to xb by the path $xb, xba = xab, xa$, otherwise we can argue analogously with c instead of b. Hence x has neighbor in at most two component of $\Gamma \setminus W$, so removing x reduces C_W by at most one.

4.2 Closing Words

We still believe that it should be possible to find a condition on the size of the subgroup H to amalgamate over relative to the index of H in G_1 and G_2 such that the free product with amalgamation of G_1 and G_2 over H contains a Hamilton circle for the standard generating set. In addition it might be necessary to require some condition on the group G_1/H. We conjecture the following:

Conjecture 1. *There exists a function* $f : \mathbb{N} \to \mathbb{N}$ *such that if* $G_1 = \langle S_1 \rangle$ *and* $G_2 = \langle S_2 \rangle$ *are finite groups with following properties:*

(i) $[G_1 : H] = k$ *and* $[G_2 : H] = 2$.

(ii) $|H| \geqslant f(k)$.

(iii) *Each subgroup of* H *is normal in* G_1 *and* G_2.

(iv) $\Gamma(G_1/H, S/H)$ *contains a Hamilton cycle.*

Then $\Gamma(G_1 *_H G_2, S_1 \cup S_2)$ *contains a Hamilton circle.*

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