

# Relaxations of GF(4)-representable matroids

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Submitted: Apr 11, 2017; Accepted: Jun 7, 2018; Published: Jun 22, 2018

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## Abstract

We consider the GF(4)-representable matroids with a circuit-hyperplane such that the matroid obtained by relaxing the circuit-hyperplane is also GF(4)-representable. We characterize the structure of these matroids as an application of structure theorems for the classes of  $U_{2,4}$ -fragile and  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids. In addition, we characterize the forbidden submatrices in GF(4)-representations of these matroids.

**Mathematics Subject Classifications:** 05B35

## 1 Introduction

Lucas [9] determined the binary matroids that have a circuit-hyperplane whose relaxation yields another binary matroid. Truemper [16], and independently, Oxley and Whittle [13], did the same for ternary matroids. In this paper, we solve the corresponding problem for quaternary matroids. We give both a structural characterization and a characterization in terms of forbidden submatrices.

Truemper [16] used the structure of circuit-hyperplane relaxations of binary and ternary matroids to give new proofs of the excluded-minor characterizations for the classes of binary, ternary, and regular matroids. It is natural to ask if Truemper's techniques can be extended to give excluded-minor characterizations for classes of quaternary matroids. The main results of this paper can be viewed as a first step towards answering this question.

Our structural characterization can be summarized as follows. A matroid has *path width* 3 if there is an ordering  $(e_1, e_2, \dots, e_n)$  of its ground set such that  $\{e_1, e_2, \dots, e_t\}$  is a 3-separating set for all  $t \in \{1, 2, \dots, n\}$ .

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\*Supported by the National Science Foundation, grant 1500343

**Theorem 1.** *Let  $M$  and  $M'$  be  $\text{GF}(4)$ -representable matroids such that  $M'$  is obtained from  $M$  by relaxing a circuit-hyperplane. Then  $M'$  has path width 3.*

In fact, our main result, Theorem 35, describes precisely how the matroids in Theorem 1 of path width 3 can be constructed using the *generalized  $\Delta$ - $Y$  exchange* of [12] and the notion of *gluing a wheel onto a triangle* from [2]. Our description uses the structure of  $U_{2,4}$ -fragile matroids from [10] and the structure of  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids from [3].

In future work, we hope to obtain a description of these matroids that is independent of the notion of fragility. Specifically, we would like to characterize the representations of these matroids. As a step in this direction, we describe minimal  $\text{GF}(4)$ -representations of matroids with a circuit-hyperplane whose relaxation is not  $\text{GF}(4)$ -representable. Note that the proof uses the excluded-minor characterization of the class of  $\text{GF}(4)$ -representable matroids. The setup for this result is as follows.

Let  $M$  be a  $\text{GF}(4)$ -representable matroid on  $E$  with a circuit-hyperplane  $X$ . Choose  $e \in X$  and  $f \in E - X$  such that  $(X - e) \cup f$  is a basis of  $M$ . Then  $M = M[I|C]$  for a quaternary matrix  $C$  of the following block form.

$$C = \begin{array}{c} \begin{array}{c} X-e \\ f \end{array} \begin{array}{cc} \begin{array}{c} (E-X)-f \\ e \end{array} \\ \begin{bmatrix} A & \underline{1} \\ \underline{1}^T & 0 \end{bmatrix} \end{array} \end{array}.$$

In the above matrix,  $A$  is an  $(X - e) \times ((E - X) - f)$  matrix, and we have scaled so that every non-zero entry in the row labelled by  $f$  and the column labelled by  $e$  is 1. Let  $M'$  be the matroid obtained from  $M$  by relaxing the circuit-hyperplane  $X$ . We call the matrix  $C$  a *reduced representation* of  $M$ . If  $M'$  is  $\text{GF}(4)$ -representable, then we can find a reduced representation  $C'$  of  $M'$  in the following block form.

$$C' = \begin{array}{c} \begin{array}{c} X-e \\ f \end{array} \begin{array}{cc} \begin{array}{c} (E-X)-f \\ e \end{array} \\ \begin{bmatrix} A' & \underline{1} \\ \underline{1}^T & \omega \end{bmatrix} \end{array} \end{array}.$$

We have scaled the rows and columns of the matrix such that the entry  $C'_{fe} = \omega \in \text{GF}(4) - \{0, 1\}$ , and the remaining entries in row  $f$  and column  $e$  are all 1. The following theorem is our characterization in terms of forbidden submatrices.

**Theorem 2.** *Let  $M$  and  $C$  be constructed as described above. There is a reduced representation  $C'$  of the above form for  $M'$  if and only if, up to permuting rows and columns,  $A$  and  $A^T$  have no submatrix in the following list, where  $x, y, z$  denote distinct non-zero elements of  $\text{GF}(4)$ .*

$$\begin{aligned} & \begin{bmatrix} x & y & z \end{bmatrix}, \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \begin{bmatrix} x & x \\ y & z \end{bmatrix}, \begin{bmatrix} x & y \\ z & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & y \end{bmatrix}, \\ & \begin{bmatrix} x & x & 0 \\ y & 0 & y \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & y \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}.$$

This paper is organized as follows. In the next section, we collect some results on connectivity and circuit-hyperplane relaxation. In Section 3, we prove a fragility theorem. In Section 4, we describe the structure of the  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids. In Section 5, we prove the structural characterization. In Section 6, we reduce the proof of Theorem 2 to a finite computer check. This check, carried out using SageMath, can be found in the Appendix [4].

## 2 Circuit-hyperplane relaxations and connectivity

We assume the reader is familiar with the fundamentals of matroid theory. Any undefined matroid terminology will follow Oxley [11]. Let  $M$  be a matroid on  $E$ , and let  $\mathcal{B}(M)$  denote the collection of bases of  $M$ . If  $M$  has a circuit-hyperplane  $X$ , then  $\mathcal{B}(M') = \mathcal{B}(M) \cup \{X\}$  is the collection of bases of a matroid  $M'$  on  $E$ . We say that  $M'$  is obtained from  $M$  by *relaxing the circuit-hyperplane*  $X$ . We list here a number of useful results on circuit-hyperplane relaxation.

**Lemma 3.** [11, Proposition 2.1.7] *If  $M'$  is obtained from  $M$  by relaxing the circuit-hyperplane  $X$  of  $M$ , then  $(M')^*$  is obtained from  $M^*$  by relaxing the circuit-hyperplane  $E(M) - X$  of  $M^*$ .*

The following elementary results are originally from [8].

**Lemma 4.** [11, Proposition 3.3.5] *Let  $X$  be a circuit-hyperplane of a matroid  $M$ , and let  $M'$  be the matroid obtained from  $M$  by relaxing  $X$ . When  $e \in E(M) - X$ ,*

- (i)  *$M/e = M'/e$  and, unless  $M$  has  $e$  as a coloop,  $M' \setminus e$  is obtained from  $M \setminus e$  by relaxing the circuit-hyperplane  $X$  of the latter.*

*Dually, when  $f \in X$ ,*

- (ii)  *$M \setminus f = M' \setminus f$  and, unless  $M$  has  $f$  as a loop,  $M'/f$  is obtained from  $M/f$  by relaxing the circuit-hyperplane  $X - f$  of the latter.*

For a set  $\mathcal{N}$  of matroids, we say that a matroid  $M$  has an  $\mathcal{N}$ -minor if  $M$  has an  $N$ -minor for some  $N \in \mathcal{N}$ . We say  $M$  is  $\mathcal{N}$ -fragile if  $M$  has an  $\mathcal{N}$ -minor and, for each element  $e$  of  $M$ , at most one matroid in  $\{M \setminus e, M/e\}$  has an  $\mathcal{N}$ -minor. We say an element  $e$  of an  $\mathcal{N}$ -fragile matroid  $M$  is *nondeletable* if  $M \setminus e$  has no  $\mathcal{N}$ -minor; the element  $e$  is *noncontractible* if  $M/e$  has no  $\mathcal{N}$ -minor.

The following lemma is an immediate consequence of Lemma 4.

**Lemma 5.** *Let  $X$  be a circuit-hyperplane of a matroid  $M$ , and let  $M'$  be the matroid obtained from  $M$  by relaxing  $X$ . If  $\mathcal{N}$  is a set of matroids such that  $M'$  has an  $\mathcal{N}$ -minor but  $M$  has no  $\mathcal{N}$ -minor, then  $M'$  is  $\mathcal{N}$ -fragile. Moreover,  $X$  is a basis of  $M'$  whose elements are nondeletable such that the elements of the cobasis  $E(M') - X$  are noncontractible.*

We use the following connectivity result.

**Lemma 6.** *[11, Proposition 8.4.2] Let  $M'$  be a matroid that is obtained by relaxing a circuit-hyperplane of a matroid  $M$ . If  $M$  is  $n$ -connected, then  $M'$  is  $n$ -connected.*

Kahn [8] proved the following result on the representability of a circuit-hyperplane relaxation.

**Lemma 7.** *Let  $M'$  be a matroid that is obtained by relaxing a circuit-hyperplane of a matroid  $M$ . If  $M$  is connected, then  $M'$  is non-binary.*

We use the following definition of the rank function of the 2-sum from [7]. Let  $M_1$  and  $M_2$  be matroids with at least two elements such that  $E(M_1) \cap E(M_2) = \{p\}$ . Then  $M = M_1 \oplus_2 M_2$  has rank function  $r_M$  defined for all  $A_1 \subseteq E(M_1)$  and  $A_2 \subseteq E(M_2)$  by

$$r_M(A_1 \cup A_2) = r_{M_1}(A_1) + r_{M_2}(A_2) - \theta(A_1, A_2) + \theta(\emptyset, \emptyset)$$

where  $\theta(X, Y) = 1$  if  $r_{M_1}(X \cup p) = r_{M_1}(X)$  and  $r_{M_2}(Y \cup p) = r_{M_2}(Y)$ , and  $\theta(X, Y) = 0$  otherwise.

The next three results on 2-sums and minors of 2-sums are well known.

**Lemma 8.** *[11, Proposition 7.1.20] Let  $M$  and  $N$  be matroids with at least two elements. Let  $E(M) \cap E(N) = \{p\}$  and suppose that neither  $M$  nor  $N$  has  $\{p\}$  as a separator. The set of circuits of  $M \oplus_2 N$  is*

$$\mathcal{C}(M \setminus p) \cup \mathcal{C}(N \setminus p) \cup \{(C \cup D) - p : p \in C \in \mathcal{C}(M) \text{ and } p \in D \in \mathcal{C}(N)\}.$$

**Lemma 9.** *[11, Theorem 8.3.1] A connected matroid  $M$  is not 3-connected if and only if  $M = M_1 \oplus_2 M_2$  for some matroids  $M_1$  and  $M_2$ , each of which has at least three elements and is isomorphic to a proper minor of  $M$ .*

**Lemma 10.** *[11, Proposition 8.3.5] Let  $M, N, M_1, M_2$  be matroids such that  $M = M_1 \oplus_2 M_2$  and  $N$  is 3-connected. If  $M$  has an  $N$ -minor, then  $M_1$  or  $M_2$  has an  $N$ -minor.*

We can now describe the structure of circuit-hyperplanes in matroids of low connectivity. We omit the straightforward proof of the next lemma.

**Lemma 11.** *Let  $M$  be a  $\text{GF}(4)$ -representable matroid with a circuit-hyperplane  $H$ . If  $M$  is not connected, then  $M \cong U_{1,m} \oplus U_{n-1,n}$  for some positive integers  $m$  and  $n$ .*

We now work towards a description of the 2-separations of a connected matroid in which the relaxation of some circuit-hyperplane is  $\text{GF}(4)$ -representable.

**Lemma 12.** *Let  $M$  be a matroid with a circuit-hyperplane  $X$ . If  $A$  is a non-trivial parallel class of  $M$ , then either  $A \subseteq E - X$ , or  $A = X$  and  $|A| = 2$ .*

*Proof.* If  $A \cap X$  and  $A \cap (E - X)$  are both non-empty, then there is a circuit  $\{x, y\}$  contained in  $A$  such that  $x \in X$  and  $y \in E - X$ . But  $E - X$  is a cocircuit of  $M$ , so this is a contradiction to orthogonality. Thus either  $A \cap X$  or  $A \cap (E - X)$  is empty. In the case that  $A \cap (E - X)$  is empty, there is a circuit  $\{x, y\}$  contained in  $A$  that is also contained in the circuit  $X$ , so  $X = A = \{x, y\}$ .  $\square$

For the next result, we say that  $M$  is *3-connected up to series and parallel classes* if  $M$  is connected and, for any 2-separation  $(X, Y)$  of  $M$ , either  $X$  or  $Y$  is a series class or a parallel class.

**Lemma 13.** *Let  $M$  be a  $\text{GF}(4)$ -representable matroid with a circuit-hyperplane  $X$  such that the matroid  $M'$  obtained from  $M$  is also  $\text{GF}(4)$ -representable. If  $M$  is connected but not 3-connected, then  $M$  is 3-connected up to series and parallel classes.*

*Proof.* Assume that  $M$  has a 2-separation  $(S, T)$  where neither side is a series or parallel class. Then  $M$  has a 2-sum decomposition of the form  $M = N \oplus_2 N'$  for some  $N$  and  $N'$  with  $E(N) \cap E(N') = \{p\}$ , where neither  $N$  nor  $N'$  is a circuit or cocircuit.

First suppose that the circuit  $X$  of  $M$  has the form  $(C \cup C') - p$ , where  $C$  is a circuit of  $N$ , and  $C'$  is a circuit of  $N'$  while  $p \in C \cap C'$ . Then

$$r(X) = r(M) - 1, \tag{1}$$

$$r(N) + r(N') - 1 = r(M), \tag{2}$$

and

$$r_M(X) = r_N(C) + r_{N'}(C') - 1. \tag{3}$$

Equation (1) follows from the fact that  $X$  is a hyperplane of  $M$ ; Equations (2) and (3) follow from the definition of the rank function of the 2-sum of  $N$  and  $N'$ . Combining (1) and (2), we see that  $r(X) = r(N) + r(N') - 2$ . Then combining this equation with (3), we see that

$$r(C) + r(C') = r(N) + r(N') - 1.$$

We may therefore assume that  $C$  is a spanning circuit of  $N$ , and hence that  $E(N) = C$  because the hyperplane  $X$  is closed. Therefore  $N$  is a circuit, a contradiction.

By symmetry, it remains to consider the case when  $X$  is a circuit of  $N \setminus p$ . Then  $r(X) \leq r(N')$ . Since  $X$  is a hyperplane of  $M$ , and  $r(M) = r(N) + r(N') - 1$ , it follows that  $r(N) \leq 2$ . Since  $N$  is not a cocircuit, we deduce that  $r(N) = 2$ . Then  $r(M) = r(N') + 1$ , so  $r(X) = r(N') = r(N \setminus p)$ . Since  $N$  is not a circuit we deduce that  $\text{si}(N) \cong U_{2,m}$  for some  $m \geq 4$ . Moreover,  $p$  is not in a non-trivial parallel class in  $N$  otherwise  $X$  is not a hyperplane of  $M$ .

Switching to  $M^*$ , we see that  $r_{M^*}(N') = |X| + r(N) - r(M) = r(N) = 2$ . As above, it follows that  $\text{co}(N') \cong U_{n-1, n+1}$  for some  $n \geq 3$ . Moreover,  $p$  is not in a non-trivial series class in  $N'$ . Let  $X_1$  consist of one representative of each series class of  $N'$ , and let

$Y_1$  consist of one representative of each parallel class of  $N$ . By contracting elements of  $X - X_1$  and deleting elements of  $(E(M) - X) - Y_1$ , we obtain  $U_{n-1, n+1} \oplus_2 U_{2, m}$  as a minor of  $M$  for some  $n \geq 3$  and  $m \geq 4$ . Moreover, by Lemma 4,  $X_1$  is a circuit-hyperplane of this minor whose relaxation is  $\text{GF}(4)$ -representable. Thus  $X_1 \subseteq E(U_{n-1, n+1})$ . Contract  $n - 3$  elements from  $X_1$  and delete  $m - 4$  elements from  $Y_1$  to get  $U_{2, 4} \oplus_2 U_{2, 4}$ . Relaxing a circuit-hyperplane of this minor gives  $P_6$  which is not  $\text{GF}(4)$ -representable (see [11, Proposition 6.5.8]), a contradiction.  $\square$

### 3 A fragility theorem

We will use the following consequence of Geelen, Oxley, Vertigan, and Whittle [6, Theorem 8.4].

**Theorem 14.** *Let  $M$  and  $M'$  be  $\text{GF}(4)$ -representable matroids with the properties that  $M$  is connected,  $M'$  is 3-connected, and  $M'$  is obtained from  $M$  by relaxing a circuit-hyperplane.*

- (i) *If  $M'$  has a  $U_{2, 4}$ -minor but no  $\{U_{2, 5}, U_{3, 5}\}$ -minor, then  $M$  is binary.*
- (ii) *If  $M'$  has a  $\{U_{2, 5}, U_{3, 5}\}$ -minor but no  $U_{3, 6}$ -minor, then  $M$  has no  $\{U_{2, 5}, U_{3, 5}\}$ -minor.*

We can now prove the main result of this section.

**Theorem 15.** *Let  $M$  and  $M'$  be  $\text{GF}(4)$ -representable matroids such that  $M$  is connected,  $M'$  is 3-connected, and  $M'$  is obtained from  $M$  by relaxing a circuit-hyperplane  $X$ . Then  $M'$  is either  $U_{2, 4}$ -fragile or  $\{U_{2, 5}, U_{3, 5}\}$ -fragile. Moreover,  $X$  is a basis of  $M'$  whose elements are nondeletable such that the elements of the cobasis  $E(M') - X$  are noncontractible.*

*Proof.* First assume that  $M'$  has no  $\{U_{2, 5}, U_{3, 5}\}$ -minor. By Lemma 7 and Theorem 14 (i),  $M'$  has a  $U_{2, 4}$ -minor and  $M$  has no  $U_{2, 4}$ -minor. Then it follows from Lemma 5 that  $M'$  is  $U_{2, 4}$ -fragile, and  $M'$  has a basis  $X$  whose elements are nondeletable such that the elements of the cobasis  $E(M') - X$  are noncontractible.

We may now assume that  $M'$  has a  $\{U_{2, 5}, U_{3, 5}\}$ -minor. Suppose that  $M$  also has a  $\{U_{2, 5}, U_{3, 5}\}$ -minor, and assume that  $M$  is a minor-minimal matroid with respect to the hypotheses; that is, we assume that  $M$  has no proper minor  $M_0$  such that  $M_0$  is connected,  $M_0$  has a  $\{U_{2, 5}, U_{3, 5}\}$ -minor, and  $M_0$  has a circuit-hyperplane whose relaxation  $M'_0$  is 3-connected,  $\text{GF}(4)$ -representable, and has a  $\{U_{2, 5}, U_{3, 5}\}$ -minor.

**Claim 16.**  *$M$  is  $\{U_{2, 5}, U_{3, 5}\}$ -fragile.*

*Proof of 16.* Suppose that  $M$  has an element  $e \in E(M) - X$  such that  $M \setminus e$  has a  $\{U_{2, 5}, U_{3, 5}\}$ -minor. If  $M \setminus e$  is 3-connected, then we have a contradiction to the minimality of  $M$ . Therefore, by Lemma 13,  $M \setminus e$  is 3-connected up to series and parallel pairs. Suppose that  $A$  is a non-trivial parallel class of  $M \setminus e$ . Suppose  $A \subseteq X$ . Then  $A = X$  and  $|A| = 2$  by Lemma 12, so we deduce that  $M \setminus e$  is a parallel extension of  $U_{2, 5}$  and hence

that  $M \setminus e$  has a  $U_{2,6}$ -minor, a contradiction to the fact that the matroid  $M'$  obtained from  $M$  by relaxing  $X$  is  $\text{GF}(4)$ -representable. Thus  $A \subseteq E(M \setminus e) - X$  by Lemma 12. By duality, any non-trivial series class of  $M \setminus e$  must be contained in  $X$ . Then, by Lemma 10, the matroid  $M_0$  obtained from  $M \setminus e$  by deleting all but one element of every non-trivial parallel class and contracting all but one element of every non-trivial series class has a  $\{U_{2,5}, U_{3,5}\}$ -minor. We deduce from Lemma 13 that  $M_0$  is 3-connected. Then  $M_0$  contradicts the minimality of  $M$ . Therefore  $M \setminus e$  has no  $\{U_{2,5}, U_{3,5}\}$ -minor for all  $e \in E(M) - X$ , and, by duality,  $M/e$  has no  $\{U_{2,5}, U_{3,5}\}$ -minor for all  $e \in X$ , so  $M$  is  $\{U_{2,5}, U_{3,5}\}$ -fragile. This completes the proof of 16.

Since  $M$  has a  $\{U_{2,5}, U_{3,5}\}$ -minor, it follows from Theorem 14 (ii) that  $M'$  has a  $U_{3,6}$ -minor, that is,  $M'/C \setminus D \cong U_{3,6}$  for some subsets  $C$  and  $D$ . If  $C \subseteq X$  and  $D \subseteq E(M') - X$ , then it follows from Lemma 4 that  $U_{3,6}$  can be obtained from  $M/C \setminus D$  by relaxing the circuit-hyperplane  $X - C$ . Hence  $M/C \setminus D \cong P_6$ , a contradiction because  $M/C \setminus D$  is  $\text{GF}(4)$ -representable but  $P_6$  is not. Therefore  $C \cap (E(M') - X)$  or  $D \cap X$  is nonempty, so  $M/C \setminus D = M'/C \setminus D \cong U_{3,6}$  by Lemma 4. This is a contradiction to 16 because any minor of  $M$  must also be  $\{U_{2,5}, U_{3,5}\}$ -fragile, but for any  $e$ , both  $U_{3,6} \setminus e$  and  $U_{3,6}/e$  have a  $\{U_{2,5}, U_{3,5}\}$ -minor. We conclude that  $M$  has no  $\{U_{2,5}, U_{3,5}\}$ -minor. It now follows from Lemma 5 that  $M'$  is  $\{U_{2,5}, U_{3,5}\}$ -fragile, and that  $M'$  has a basis  $X$  whose elements are nondeletable such that the elements of the cobasis  $E(M') - X$  are noncontractible.  $\square$

## 4 The structure of $\{U_{2,5}, U_{3,5}\}$ -fragile matroids

### 4.1 Partial Fields and Constructions

We briefly state the necessary material on partial fields. For a more thorough treatment, we refer the reader to [14].

A *partial field* is a pair  $\mathbb{P} = (R, G)$ , where  $R$  is a commutative ring with unity, and  $G$  is a subgroup of the units of  $R$  with  $-1 \in G$ . A matrix with entries in  $G$  is a  $\mathbb{P}$ -matrix if  $\det(D) \in G \cup \{0\}$  for any square submatrix  $D$  of  $A$ . We use  $\langle X \rangle$  to denote the multiplicative subgroup of  $R$  generated by the subset  $X$ .

A rank- $r$  matroid  $M$  on the ground set  $E$  is  $\mathbb{P}$ -representable if there is an  $r \times |E|$   $\mathbb{P}$ -matrix  $A$  such that, for each  $r \times r$  submatrix  $D$ , the determinant of  $D$  is nonzero if and only if the corresponding subset of  $E$  is a basis of  $M$ . When this occurs, we write  $M = M[A]$ .

The 2-regular partial field is defined as follows.

$$\mathbb{U}_2 = (\mathbb{Q}(\alpha, \beta), \langle -1, \alpha, \beta, 1 - \alpha, 1 - \beta, \alpha - \beta \rangle),$$

where  $\alpha, \beta$  are indeterminates.

It is well-known that any  $\mathbb{U}_2$ -representable matroid is  $\text{GF}(4)$ -representable [12]. On the other hand, there are  $\text{GF}(4)$ -representable matroids that are not  $\mathbb{U}_2$ -representable. We now define three such matroids. The matroid  $P_8$  has a unique pair of disjoint circuit-hyperplanes; we let  $P_8^-$  denote the unique matroid obtained by relaxing one of these

circuit-hyperplanes. We denote by  $F_7^-$  the matroid obtained from the non-Fano matroid  $F_7^-$  by relaxing a circuit-hyperplane. The GF(4)-representable matroids  $P_8^-, F_7^-, (F_7^-)^*$  are not  $\mathbb{U}_2$ -representable. We note that this can be deduced from [1] since  $P_8^-, F_7^-, (F_7^-)^*$  are  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids. Since these matroids are not  $\mathbb{U}_2$ -representable, we have the following lemma.

**Lemma 17.** *The class of  $\mathbb{U}_2$ -representable matroids is contained in the class of GF(4)-representable matroids with no  $\{P_8^-, F_7^-, (F_7^-)^*\}$ -minor.*

To describe the structure of  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids as in [3], we need two constructions: the generalized  $\Delta$ - $Y$  exchange, and gluing on wheels. For a more thorough treatment of these constructions, we refer the reader to [12] and [2].

Loosely speaking, the operations of generalized  $\Delta$ - $Y$  exchange and gluing on wheels both involve gluing matroids together along a common restriction. Let  $M_1$  and  $M_2$  be matroids with a common restriction  $A$ , where  $A$  is a modular flat of  $M_1$ . The *generalized parallel connection* of  $M_1$  and  $M_2$  along  $A$ , denoted  $P_A(M_1, M_2)$ , is the matroid obtained by gluing  $M_1$  and  $M_2$  along  $A$ . It has ground set  $E(M_1) \cup E(M_2)$ , and a set  $F$  is a flat of  $P_A(M_1, M_2)$  if and only if  $F \cap E(M_i)$  is a flat of  $M_i$  for each  $i$  (see [11, Section 11.4]).

A subset  $S$  of  $E(M)$  is a *segment* of  $M$  if every three-element subset of  $S$  is a triangle of  $M$ . Let  $M$  be a matroid with a  $k$ -element segment  $A$ . Intuitively, a generalized  $\Delta$ - $Y$  exchange on  $A$  turns the segment  $A$  into a  $k$ -element cosegment. To define the generalized  $\Delta$ - $Y$  exchange formally, we first recall the following definition of a family of matroids  $\Theta_k$  from [12]. For  $k \geq 3$ , fix a basis  $B = \{b_1, b_2, \dots, b_k\}$  of the rank- $k$  projective geometry  $PG(k-1, \mathbb{R})$ , and choose a line  $L$  of  $PG(k-1, \mathbb{R})$  that is freely placed relative to  $B$ . It follows from modularity that, for each  $i$ , the hyperplane spanned by  $B - \{b_i\}$  meets  $L$ ; we let  $a_i$  be the point of intersection. Let  $A = \{a_1, a_2, \dots, a_k\}$ , and let  $\Theta_k$  be the matroid obtained by restricting  $PG(k-1, \mathbb{R})$  to the set  $A \cup B$ . Note that the matroid  $\Theta_k$  has  $A$  as a modular  $k$ -point segment  $A$ , so the generalized parallel connection of  $\Theta_k$  and  $M$  along  $A$  is well-defined. If the  $k$ -element segment  $A$  is coindependent in  $M$ , then we define the matroid  $\Delta_A(M)$  to be the matroid obtained from  $P_A(\Theta_k, M) \setminus A$  by relabeling the elements of  $E(\Theta_k) - A$  by  $A$  in the natural way, and we say that  $\Delta_A(M)$  is obtained from  $M$  by performing a *generalized  $\Delta$ - $Y$  exchange* on  $A$ . For a matroid  $M$  with an independent cosegment  $A$ , a *generalized  $Y$ - $\Delta$  exchange on  $A$* , denoted by  $\nabla_A(M)$ , is defined to be the matroid  $(\Delta_A(M^*))^*$ .

We use the following results on representability and the minor operations.

**Lemma 18.** [12, Lemma 3.7] *Let  $\mathbb{P}$  be a partial field. Then  $M$  is  $\mathbb{P}$ -representable if and only if  $\Delta_A(M)$  is  $\mathbb{P}$ -representable.*

**Lemma 19.** [12, Lemma 2.13] *Suppose that  $\Delta_A(M)$  is defined. If  $x \in A$  and  $|A| \geq 3$ , then  $\Delta_{A-x}(M \setminus x)$  is also defined, and  $\Delta_A(M)/x = \Delta_{A-x}(M \setminus x)$ .*

**Lemma 20.** [12, Lemma 2.16] *Suppose that  $\Delta_A(M)$  is defined.*

- (i) *If  $x \in E(M) - A$  and  $A$  is coindependent in  $M \setminus x$ , then  $\Delta_A(M \setminus x)$  is defined and  $\Delta_A(M) \setminus x = \Delta_A(M \setminus x)$ .*



(ii) If  $x \in E(M) - \text{cl}(A)$ , then  $\Delta_A(M/x)$  is defined and  $\Delta_A(M)/x = \Delta_A(M/x)$ .

**Lemma 21.** [12, Lemma 2.15] Suppose that  $x \in \text{cl}(A) - A$  and let  $a$  be an arbitrary element of the  $k$ -element segment  $A$ . Then  $\Delta_A(M)/x$  equals the 2-sum, with basepoint  $p$ , of a copy of  $U_{k-1, k+1}$  with groundset  $A \cup p$  and the matroid obtained from  $M/x \setminus (A - a)$  by relabeling  $a$  as  $p$ .

The next result implies that every  $\{U_{2,5}, U_{3,5}\}$ -fragile matroid is 3-connected up to series and parallel classes.

**Lemma 22.** [10, Proposition 4.3] Let  $M$  be a matroid with a 2-separation  $(A, B)$ , and let  $N$  be a 3-connected minor of  $M$ . Assume  $|E(N) \cap A| \geq |E(N) \cap B|$ . Then  $|E(N) \cap B| \leq 1$ . Moreover, unless  $B$  is a parallel or series class, there is an element  $x \in B$  such that both  $M \setminus x$  and  $M/x$  have a minor isomorphic to  $N$ .

The following is an easy consequence of the property that  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids are 3-connected up to parallel and series classes.

**Lemma 23.** Let  $M$  be a  $\{U_{2,5}, U_{3,5}\}$ -fragile matroid with at least 8 elements. If  $S$  is a triangle or 4-element segment of  $M$  such that  $E(M) - S$  is not a series or parallel class of  $M$ , then  $S$  is coindependent in  $M$ . If  $C$  is a triad or 4-element cosegment of  $M$  such that  $E(M) - C$  is not a series or parallel class of  $M$ , then  $C$  is independent.

Let  $M$  be a  $\{U_{2,5}, U_{3,5}\}$ -fragile matroid. A segment  $S$  of  $M$  is *allowable* if  $S$  is coindependent and some element of  $S$  is nondeletable. A cosegment  $C$  of  $M$  is *allowable* if the segment  $C$  of  $M^*$  is allowable. In [3], it was shown that we can obtain a new  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\mathbb{U}_2$ -representable matroid from an old  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\mathbb{U}_2$ -representable matroid by performing a generalized  $\Delta$ - $Y$  exchange on an allowable segment. We will prove an analogous result for  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroids with no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor.

Let  $\mathcal{U}$  be the class of  $\text{GF}(4)$ -representable matroids with no  $\{U_{2,5}, U_{3,5}\}$ -minor. The class of *sixth-root-of-unity* matroids is the class of matroids that are representable over both  $\text{GF}(3)$  and  $\text{GF}(4)$ . Semple and Whittle [15, Theorem 5.2] showed that  $\mathcal{U}$  is the class of matroids that can be obtained by taking direct sums and 2-sums of binary and sixth-root-of-unity matroids.

**Lemma 24.** Let  $M$  be a matroid in the class  $\mathcal{U}$ . If  $M'$  is obtained from  $M$  by performing a generalized  $\Delta$ - $Y$  exchange or a generalized  $Y$ - $\Delta$  exchange, then  $M' \in \mathcal{U}$ .

*Proof.* Suppose that there exists a matroid  $M \in \mathcal{U}$  with a coindependent segment  $A$  such that  $\Delta_A(M) \notin \mathcal{U}$ . Among all counterexamples, suppose that  $M$  has been chosen so that  $|E(M)|$  is as small as possible. Suppose  $M$  is 3-connected. Since any 3-connected member of  $\mathcal{U}$  is either a binary or sixth-root-of-unity matroid, this also holds for  $\Delta_A(M)$  by Lemma 18. Hence  $\Delta_A(M) \in \mathcal{U}$ , contradicting the assumption that  $M$  is a counterexample. Therefore  $M$  is not 3-connected.

Now either  $M = M_1 \oplus M_2$  or  $M = M_1 \oplus_2 M_2$  for some  $M_1, M_2 \in \mathcal{U}$  with  $|E(M_i)| < |E(M)|$  for each  $i \in \{1, 2\}$ . Moreover, we may assume that  $M_1$  and  $M_2$  have been chosen

so that the segment  $A$  of  $M$  is contained in  $E(M_1)$ . Now either  $\Delta_A(M) = \Delta_A(M_1) \oplus M_2$  or  $\Delta_A(M) = \Delta_A(M_1) \oplus_2 M_2$ . Since  $|E(M_1)| < |E(M)|$ , it follows that  $\Delta_A(M_1) \in \mathcal{U}$ . Hence  $\Delta_A(M) \in \mathcal{U}$ . Since  $\mathcal{U}$  is closed under duality, the result follows.  $\square$

**Lemma 25.** *Let  $M$  be a  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroid with no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. If  $A$  is an allowable segment of  $M$  with  $|A| \in \{3, 4\}$ , then  $\Delta_A(M)$  is a  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroid with no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. Moreover,  $A$  is an allowable cosegment of  $\Delta_A(M)$ .*

*Proof.* The proof that  $\Delta_A(M)$  is a  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroid where  $A$  is an allowable cosegment of  $\Delta_A(M)$  closely follows the proof of [3, Lemma 2.21]. The only difference is where the proof of [3, Lemma 2.21] uses the fact that a  $\mathbb{U}_2$ -representable matroid with no  $\{U_{2,5}, U_{3,5}\}$ -minor is near-regular and the class of near-regular matroids is closed under the generalized  $\Delta$ - $Y$  exchange, we instead use Lemma 24.

We must also show that  $\Delta_A(M)$  has no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. This follows for  $|E(M)| \leq 9$  from the generation of the 3-connected  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroids with no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor on at most 9 elements (see the Appendix [4]), since all such matroids are  $\mathbb{U}_2$ -representable. Suppose that  $M$  is a minimum-sized counterexample, so  $\Delta_A(M)$  has a  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor and  $\Delta_A(M)$  has at least ten elements. Then  $\Delta_A(M)$  has a minor  $N$ , obtained by deleting or contracting an element  $x$  say, that also has a  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. Since  $\Delta_A(M)$  is  $\{U_{2,5}, U_{3,5}\}$ -fragile it follows that the minor  $N$  is also  $\{U_{2,5}, U_{3,5}\}$ -fragile. Suppose that  $N = \Delta_A(M)/x$ . Suppose that  $x \in A$ . Then  $\Delta_A(M)/x = \Delta_{A-x}(M \setminus x)$  by Lemma 19, a contradiction since  $M$  is a minimum-sized counterexample. Next suppose that  $x \in \text{cl}(A) - A$ . Since  $N$  is  $\{U_{2,5}, U_{3,5}\}$ -fragile it follows from Lemma 21 and Proposition 22 that  $|A| = 4$  and  $M/x \setminus (A - a) \cong U_{1,n}$  for some  $n \geq 2$ . Hence  $\Delta_A(M)$  has no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor, a contradiction. We may now assume  $x \in E(M) - \text{cl}(A)$ . Then  $\Delta_A(M)/x = \Delta_A(M/x)$  by Lemma 20, a contradiction since  $M$  is a minimum-sized counterexample. We deduce that  $N = \Delta_A(M) \setminus x$ , and we may assume that any minor obtained from  $\Delta_A(M)$  by contracting an element has no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. Now if  $x \in A$ , then  $A - x$  is a series class of  $\Delta_A(M) \setminus x$ , so there is some  $y \in A$  such that  $\Delta_A(M)/y$  has a  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor, a contradiction. Therefore  $x \notin A$ . If  $A$  is not coindependent in  $M \setminus x$ , then it follows from Lemma 23 that  $\Delta_A(M)$  has no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor, a contradiction. Therefore  $A$  is coindependent in  $M \setminus x$ , so  $\Delta_A(M) \setminus x = \Delta_A(M \setminus x)$  by Lemma 20, a contradiction since  $M$  is a minimum-sized counterexample.  $\square$

Let  $M$  be a matroid, and  $(a, b, c)$  an ordered subset of  $E(M)$  such that  $T = \{a, b, c\}$  is a triangle. Let  $r \geq 3$  be a positive integer, and, when  $r = 3$ , we fix a vertex of  $\mathcal{W}_3$  to be the center, so we can refer to rim and spoke elements of  $M(\mathcal{W}_3)$ . Let  $N$  be obtained from  $M(\mathcal{W}_r)$  by relabeling some triangle as  $\{a, b, c\}$ , where  $a, c$  are spoke elements, and let  $X \subseteq \{a, b, c\}$  such that  $b \in X$ . We say the matroid  $M' := P_T(M, N) \setminus X$  is obtained from  $M$  by *gluing an  $r$ -wheel onto  $(a, b, c)$* . We also say that  $M^*$  is obtained from  $N^*$  by gluing a wheel onto the triad  $T$ . Suppose that  $T_1, T_2, \dots, T_n$  are ordered triples whose underlying sets are triangles of  $M$ . We say  $M'$  can be obtained from  $M$  by *gluing wheels*

onto  $T_1, T_2, \dots, T_n$  if, for some subset  $J$  of  $\{1, 2, \dots, n\}$ ,  $M'$  can be obtained from  $M$  by a sequence of moves, where each move consists of gluing an  $r_j$ -wheel onto  $T_j$  for  $j \in J$ . Note that the spoke elements of a triangle in this sequence may only be deleted as part of the gluing operation when they do not appear in any subsequent triangle in the sequence.

**Lemma 26.** *Let  $M$  be a  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroid with no  $\{P_8^-, F_7^-, (F_7^-)^*\}$ -minor. Let  $A = \{a, b, c\}$  be an allowable triangle of  $M$ , where  $b$  is nondeletable. If  $M'$  is obtained from  $M$  by gluing an  $r$ -wheel onto  $(a, b, c)$ , where  $X \subseteq \{a, b, c\}$  is such that  $b \in X$ , then  $M'$  is a  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroid with no  $\{P_8^-, F_7^-, (F_7^-)^*\}$ -minor. Moreover,  $F = E(W_r) - X$  is the set of elements of a fan, the spoke elements of  $F$  are noncontractible in  $M'$ , and the rim elements of  $F$  are nondeletable in  $M'$ .*

*Proof.* The proof is the same as [3, Lemma 2.22] except that we use Lemma 25 instead of [3, Lemma 2.21].  $\square$

## 4.2 Path sequences

We can now describe a family of  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroids with no  $\{P_8^-, F_7^-, (F_7^-)^*\}$ -minor obtained by performing generalized  $\Delta$ - $Y$  exchanges and gluing on wheels. In fact, the matroids in this family are  $\mathbb{U}_2$ -representable and were first described in [3]. Each matroid in this family has a  $\{X_8, Y_8, Y_8^*\}$ -minor, and an associated path of 3-separations that we need to describe in order to define the family.

We call the set  $X \subseteq E(M)$  *fully closed* if  $X$  is closed in both  $M^*$  and  $M$ . The *full closure* of  $X$ , denoted  $\text{fcl}_M(X)$ , is the intersection of all fully closed sets containing  $X$ . The full closure of  $X$  can be obtained from  $X$  by repeatedly taking closure and coclosure until no new elements are added. We call  $X$  a *path-generating* set if  $X$  is a 3-separating set of  $M$  such that  $\text{fcl}_M(X) = E(M)$ . A path-generating set  $X$  thus gives rise to a natural path of 3-separating sets  $(P_1, \dots, P_m)$ , where  $P_1 = X$  and each step  $P_i$  is either the closure or coclosure of the 3-separating set  $P_1 \cup \dots \cup P_{i-1}$ .

Let  $X$  be an allowable cosegment of the  $\{U_{2,5}, U_{3,5}\}$ -fragile matroid  $M$ . A matroid  $Q$  is an *allowable series extension of  $M$  along  $X$*  if  $M = Q/Z$  and, for every element  $z$  of  $Z$ , there is some element  $x$  of  $X$  such that  $x$  is  $\{U_{2,5}, U_{3,5}\}$ -contractible in  $M$  and  $z$  is in series with  $x$  in  $Q$ . We also say that  $Q^*$  is an *allowable parallel extension of  $M^*$  along  $X$* .

Let  $N$  be a matroid with a path-generating allowable segment or cosegment  $A$ . We say that  $M$  is obtained from  $N$  by a  $\Delta$ - $\nabla$ -*step* along  $A$  if, up to duality,  $M$  is obtained from  $N$  by performing a non-empty allowable parallel extension along  $A$ , followed by a generalized  $\Delta$ - $Y$  exchange on  $A$ .

Let  $X_8$  be the matroid obtained from  $U_{2,5}$  by choosing a 4-element segment  $C$ , adding a point in parallel with each of three distinct points of  $C$ , then performing a generalized  $\Delta$ - $Y$ -exchange on  $C$  (see Figure 1). In what follows,  $S$  will be the elements of the 4-element segment of  $X_8$ , and  $C$  the elements of the 4-element cosegment of  $X_8$ , so  $E(X_8) = S \cup C$ . We will build matroids from  $X_8$  by performing a sequence of  $\Delta$ - $\nabla$ -steps along  $A \in \{S, C\}$ . Note that, in such matroids, each of  $S$  and  $C$  can be either a segment or a cosegment.

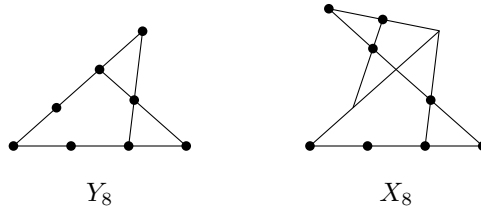


Figure 1: The matroids  $Y_8$  and  $X_8$ .

A sequence of matroids  $M_1, \dots, M_n$  is called a *path sequence* if the following conditions hold:

- (P1)  $M_1 = X_8$ ; and
- (P2) For each  $i \in \{1, \dots, n-1\}$ , there is some 4-element path-generating segment or cosegment  $A \in \{S, C\}$  of  $M_i$  such that either:
  - (a)  $M_{i+1}$  is obtained from  $M_i$  by a  $\Delta$ - $\nabla$ -step along  $A$ ; or
  - (b)  $M_{i+1}$  is obtained from  $M_i$  by gluing a wheel onto an allowable subset  $A'$  of  $A$ .

Note in (P2) that each  $\Delta$ - $\nabla$ -step described in (a) increases the number of elements by at least one, and that the wheels in (b) are only glued onto allowable subsets of 4-element segments or cosegments.

We say that a path sequence  $M_1, \dots, M_n$  *describes* a matroid  $M$  if  $M_n \cong M$ . We also say that  $M$  is a matroid *described by* a path sequence if there is some path sequence that describes  $M$ . Let  $\mathcal{P}$  denote the class of matroids such that  $M \in \mathcal{P}$  if and only if there is some path sequence  $M_1, \dots, M_n$  that describes a matroid  $M'$  such that  $M$  can be obtained from  $M'$  by some, possibly empty, sequence of allowable parallel and series extensions. Since  $X_8$  is self-dual, it is easy to see that the sequence of dual matroids  $M_1^*, \dots, M_n^*$  of a path sequence  $M_1, \dots, M_n$  is also a path sequence. Thus the class  $\mathcal{P}$  is closed under duality.

We denote by  $Y_8$  the unique matroid obtained from  $X_8$  by performing a  $Y$ - $\Delta$ -exchange on an allowable triad (see Figure 1). We will prove the following result.

**Theorem 27.** *If  $M$  is a 3-connected  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroid that has an  $\{X_8, Y_8, Y_8^*\}$ -minor but no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor, then there is some path sequence that describes  $M$ .*

The proof of Theorem 27 closely follows the proof of [3, Corollary 4.3]. The strategy is to show that a minor-minimal counterexample has at most 12 elements. Let  $M$  be a  $\text{GF}(4)$ -representable  $\{U_{2,5}, U_{3,5}\}$ -fragile matroid  $M$  with an  $\{X_8, Y_8, Y_8^*\}$ -minor but no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. Suppose that  $M$  is a minimum-sized matroid that is not in the class  $\mathcal{P}$ . Then  $M$  is 3-connected because  $\mathcal{P}$  is closed under series and parallel extensions. Moreover, the dual  $M^*$  is also not in  $\mathcal{P}$  because  $\mathcal{P}$  is closed under duality. Thus, by the Splitter Theorem and duality, we may assume there is some element  $x$  of  $M$  such that  $M \setminus x$

is also a 3-connected GF(4)-representable  $\{U_{2,5}, U_{3,5}\}$ -fragile matroid with an  $\{X_8, Y_8, Y_8^*\}$ -minor but no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. By the assumption that  $M$  is minimum-sized with respect to being outside the class  $\mathcal{P}$ , it follows that  $M \setminus x \in \mathcal{P}$ . Thus  $M \setminus x$  is described by a path sequence  $M_1, \dots, M_n$ . The next lemma [3, Lemma 6.3] identifies the three possibilities for the position of  $x$  in  $M$  relative to the path of 3-separations associated with  $M_1, \dots, M_n$ .

**Lemma 28.** *Let  $M$  and  $M \setminus x$  be 3-connected  $\{U_{2,5}, U_{3,5}\}$ -fragile matroids. If  $M \setminus x$  is described by a path sequence with associated path of 3-separations  $\mathbf{P}$ , then either:*

- (i) *there is some 3-separation  $(X, Y)$  displayed by  $\mathbf{P}$  such that  $x \in \text{cl}(X)$  and  $x \in \text{cl}(Y)$ ;*  
or
- (ii) *there is some 3-separation  $(X, Y)$  displayed by  $\mathbf{P}$  such that  $x \notin \text{cl}(X)$  and  $x \notin \text{cl}(Y)$ ;*  
or
- (iii) *for each 3-separation  $(R, G)$  of  $M$  displayed by  $\mathbf{P}$ , there is some  $X \in \{R, G\}$  such that  $x \in \text{cl}_M(X)$  and  $x \in \text{cl}_M^*(X)$ .*

The proofs of the next three lemmas follow the proofs of [3, Lemma 7.4], [3, Lemma 8.7], and [3, Lemma 9.7] but use Lemma 25 above instead of [3, Lemma 2.21].

**Lemma 29.** *Lemma 28 (i) does not hold.*

**Lemma 30.** *If Lemma 28 (ii) holds, then  $|E(M \setminus x)| \leq 10$ .*

**Lemma 31.** *If Lemma 28 (iii) holds, then  $|E(M \setminus x)| \leq 11$ .*

*Proof of Theorem 27.* In view of the last three lemmas, it suffices to verify that  $\mathcal{P}$  contains each 3-connected  $\{U_{2,5}, U_{3,5}\}$ -fragile GF(4)-representable matroid with an  $\{X_8, Y_8, Y_8^*\}$ -minor and no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor having at most 12 elements. This is done in the Appendix [4]. □

### 4.3 Fan extensions

The following theorem describes the structure of the matroids with no  $\{X_8, Y_8, Y_8^*\}$ -minor. Note that  $M_{9,9}$  is the rank-4 matroid on 9 elements in Figure 2. The matroid  $M_{7,1}$  is the 7-element matroid that is obtained from  $Y_8$  by deleting the unique point that is contained in the two 4-element segments of  $Y_8$ . We label the points of a triangle of  $M_{7,1}$  by  $\{1, 2, 3\}$  as in Figure 2.

**Theorem 32.** *Let  $M'$  be a 3-connected  $\{U_{2,5}, U_{3,5}\}$ -fragile GF(4)-representable matroid with no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor. Then  $M'$  is isomorphic to a matroid  $M$  for which at least one of the following holds:*

- (i)  *$M$  has an  $\{X_8, Y_8, Y_8^*\}$ -minor;*
- (ii)  *$M \in \{M_{9,9}, M_{9,9}^*\}$ ;*

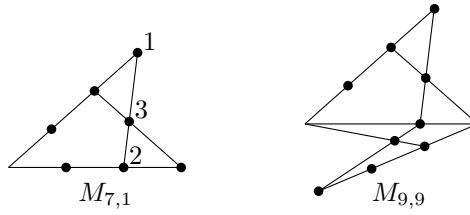


Figure 2: The matroids  $M_{7,1}$  and  $M_{9,9}$ .

- (iii)  $M$  or  $M^*$  can be obtained from  $U_{2,5}$  (with ground set  $\{a, b, c, d, e\}$ ) by gluing wheels to  $(a, c, b), (a, d, b), (a, e, b)$ ;
- (iv)  $M$  or  $M^*$  can be obtained from  $U_{2,5}$  (with ground set  $\{a, b, c, d, e\}$ ) by gluing wheels to  $(a, b, c), (c, d, e)$ ;
- (v)  $M$  or  $M^*$  can be obtained from  $M_{7,1}$  by gluing a wheel to  $(1, 3, 2)$ .

*Proof.* Assume  $M$  has no  $\{X_8, Y_8, Y_8^*\}$ -minor. For (ii), we show in Lemma 1 of the Appendix [4] that the matroids  $M_{9,9}$  and  $M_{9,9}^*$  are splitters for the class of 3-connected  $\{U_{2,5}, U_{3,5}\}$ -fragile  $\text{GF}(4)$ -representable matroids with no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor.

We may therefore assume  $M$  has no  $\{M_{9,9}, M_{9,9}^*, X_8, Y_8, Y_8^*\}$ -minor. To show that (iii), (iv), or (v) holds, we use the main result of [2] called the ‘‘Fan Lemma’’, which reduces the proof to showing that extensions and coextensions of the 9-element matroids with this structure also have this structure. These verifications are completed in Lemmas 2 through 7 of the Appendix [4].  $\square$

## 5 From fragility to relaxations

We use the following result of Mayhew, Whittle, and Van Zwam [10, Lemma 8.2].

**Lemma 33.** *Let  $M$  be a 3-connected  $U_{2,4}$ -fragile matroid that has no  $\{U_{2,6}, U_{4,6}\}$ -minor. Then exactly one of the following holds.*

- (i)  $M$  has rank or corank two;
- (ii)  $M$  has an  $\{F_7^-, (F_7^-)^*\}$ -minor;
- (iii)  $M$  has rank and corank at least 3 and is a whirl.

We show next that  $P_8^-, F_7^=, (F_7^=)^*$  do not arise from circuit-hyperplane relaxation of a  $\text{GF}(4)$ -representable matroid.

**Lemma 34.** *Let  $M$  and  $M'$  be  $\text{GF}(4)$ -representable matroids such that  $M$  is connected,  $M'$  is 3-connected, and  $M'$  is obtained from  $M$  by relaxing a circuit-hyperplane  $X$ . Then  $M'$  has no  $\{P_8^-, F_7^=, (F_7^=)^*\}$ -minor.*

*Proof.* Assume that  $M'$  has a  $\{P_8^-, F_7^-, (F_7^-)^*\}$ -minor. Since  $M'$  is obtained from  $M$  by relaxing  $X$ , it follows from Theorem 15 and Lemma 33 that  $M'$  is  $\{U_{2,5}, U_{3,5}\}$ -fragile. Each of the matroids in  $\{P_8^-, F_7^-, (F_7^-)^*\}$  has a  $\{U_{2,5}, U_{3,5}\}$ -minor, so if  $C$  and  $D$  are such that  $M'/C \setminus D \cong N'$  for some  $N' \in \{P_8^-, F_7^-, (F_7^-)^*\}$ , then  $C \subseteq X$  and  $D \subseteq E(M') - X$  since the elements of  $X$  are nondeletable and the elements of  $E(M') - X$  are noncontractible by Theorem 15. But then it follows from Lemma 4 that  $N'$  can be obtained from  $M'/C \setminus D$  by relaxing the circuit-hyperplane  $X - C$ . It follows that  $M'/C \setminus D \cong N$  for some  $N \in \{P_8, F_7^-, (F_7^-)^*\}$ , a contradiction because  $M$  is GF(4)-representable.  $\square$

We can now describe the structure of the GF(4)-representable matroids that are circuit-hyperplane relaxations of GF(4)-representable matroids.

**Theorem 35.** *Let  $M$  and  $M'$  be GF(4)-representable matroids such that  $M$  is connected,  $M'$  is 3-connected, and  $M'$  is obtained from  $M$  by relaxing a circuit-hyperplane. Then at least one of the following holds.*

- (a)  $M'$  is a whirl;
- (b)  $M' \in \{M_{9,9}, M_{9,9}^*\}$ ;
- (c)  $M'$  or  $(M')^*$  can be obtained from  $U_{2,5}$  (with groundset  $\{a, b, c, d, e\}$ ) by gluing wheels to  $(a, c, b), (a, d, b)$ ;
- (d)  $M'$  or  $(M')^*$  can be obtained from  $U_{2,5}$  (with groundset  $\{a, b, c, d, e\}$ ) by gluing wheels to  $(a, b, c), (c, d, e)$ ;
- (e)  $M'$  or  $(M')^*$  can be obtained from  $M_{7,1}$  by gluing a wheel to  $(1, 3, 2)$ ;
- (f) there is some path sequence that describes  $M'$ .

*Proof.* It follows from Theorem 15 that  $M'$  is either  $U_{2,4}$ -fragile or  $\{U_{2,5}, U_{3,5}\}$ -fragile. If  $M'$  is  $U_{2,4}$ -fragile, then it follows from Lemma 33 that  $M'$  is a whirl. We may therefore assume that  $M'$  is  $\{U_{2,5}, U_{3,5}\}$ -fragile. It follows from Lemma 34 that  $M'$  has no  $\{P_8^-, F_7^-, (F_7^-)^*\}$ -minor. Then, by Theorem 32 and Theorem 15, one of (b) through (e) holds or else  $M'$  has an  $\{X_8, Y_8, Y_8^*\}$ -minor. Note that outcome (iii) of Theorem 32 corresponds to outcome (c) here, since a matroid or its dual that is obtained from  $U_{2,5}$  (with groundset  $\{a, b, c, d, e\}$ ) by gluing wheels onto all three of the triangles  $(a, c, b), (a, d, b), (a, e, b)$  does not have a basis of nondeletable elements and a cobasis of noncontractible elements, and therefore cannot be obtained by relaxing a circuit-hyperplane. We can see this by the following counting argument. Observe that the rank of a matroid obtained from  $U_{2,5}$  (with groundset  $\{a, b, c, d, e\}$ ) by gluing wheels  $A, B$  and  $C$  onto the triangles  $(a, c, b), (a, d, b), (a, e, b)$  is  $r(A) + r(B) + r(C) - 4$ . But the nondeletable elements of this matroid are precisely the rim elements of the wheels of which there are  $r(A) + r(B) + r(C) - 3$ . Hence any cobasis must contain a nondeletable element  $e$ . Since this matroid has  $M_{9,18}$  as a minor (see Appendix [4, Lemma 2]),  $M$  has no essential elements, which implies that  $e$  must be contractible.

Finally, if  $M'$  has an  $\{X_8, Y_8, Y_8^*\}$ -minor, then it follows from Theorem 27 that (f) holds.  $\square$

We can now show that if  $M$  and  $M'$  are  $\text{GF}(4)$ -representable matroids such that  $M'$  is obtained from  $M$  by relaxing a circuit-hyperplane, then  $M'$  has path width 3.

*Proof of Theorem 1.* If  $M$  is not connected, then it follows from Lemma 11 that  $M'$  has path width 3. We may therefore assume that  $M$  is connected. Then, by Lemma 13,  $M'$  can be obtained from a matroid in Theorem 35 (a) - (f) by performing some, possibly empty, sequence of series or parallel extensions. The result now follows from the fact that all the matroids in Theorem 35 (a) - (f) have path width 3.  $\square$

## 6 Forbidden submatrices

In this section, we will prove our second characterization, Theorem 2. Let  $M$  be a  $\text{GF}(4)$ -representable matroid with a circuit-hyperplane  $X$ . Choose  $e \in X$  and  $f \in E - X$  such that  $B = (X - e) \cup f$  is a basis of  $M$ . Then we can find a reduced  $\text{GF}(4)$ -representation of  $M$  in block form,

$$C = \begin{matrix} & & \begin{matrix} (E-X)-f & e \end{matrix} \\ \begin{matrix} X-e \\ f \end{matrix} & \left[ \begin{array}{cc} A & \underline{1} \\ \underline{1}^T & 0 \end{array} \right]. \end{matrix}$$

Here  $A$  is an  $(X - e) \times ((E - X) - f)$  matrix over  $\text{GF}(4)$ , and we have scaled so that every non-zero entry in the row labelled by  $f$  and the column labelled by  $e$  is 1. We denote by  $A_{ij}$  the entry in row  $i$  and column  $j$  of  $A$ .

Let  $M'$  be the matroid obtained from  $M$  by relaxing the circuit-hyperplane  $X$ . If  $M'$  is  $\text{GF}(4)$ -representable, then we can find a reduced representation of  $M'$  in block form,

$$C' = \begin{matrix} & & \begin{matrix} (E-X)-f & e \end{matrix} \\ \begin{matrix} X-e \\ f \end{matrix} & \left[ \begin{array}{cc} A' & \underline{1} \\ \underline{1}^T & \omega \end{array} \right]. \end{matrix}$$

We have scaled the rows and columns of the matrix such that the entry in the row labelled by  $f$  and column labelled by  $e$  is  $\omega \in \text{GF}(4) - \{0, 1\}$ , and every remaining entry in row  $e$  and column  $f$  is a 1.

We omit the straightforward proof of the following lemma.

**Lemma 36.**  $A_{ij} = 0$  if and only if  $A'_{ij} = 0$ .

Next we show that the only non-zero entries of  $A'$  are 1 and  $\omega$ .

**Lemma 37.**  $A'_{ij} \neq \omega + 1$ .

*Proof.* Suppose  $A'_{ij} = \omega + 1$ . Then  $C'$  has a submatrix

$$C''[\{i, f\}, \{e, j\}] = \begin{matrix} & \begin{matrix} j & e \end{matrix} \\ \begin{matrix} i \\ f \end{matrix} & \left[ \begin{array}{cc} \omega + 1 & 1 \\ 1 & \omega \end{array} \right], \end{matrix}$$



which has determinant zero. Therefore  $B\Delta\{e, f, i, j\}$  is not a basis of the matroid  $M[I|C']$ . But the corresponding submatrix of  $C$  is

$$C[\{i, f\}, \{e, j\}] = \begin{matrix} & j & e \\ \begin{matrix} i \\ f \end{matrix} & \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \end{matrix},$$

for some non-zero  $x$ . Since  $C[\{i, f\}, \{e, j\}]$  has non-zero determinant,  $B\Delta\{e, f, i, j\}$  is a basis of  $M$ , and hence of  $M'$ . Therefore  $M' \neq M[I|C']$ .  $\square$

**Lemma 38.**  $A_{ij} = A_{ik}$  if and only if  $A'_{ij} = A'_{ik}$ . Similarly,  $A_{ij} = A_{kj}$  if and only if  $A'_{ij} = A'_{kj}$

*Proof.* We show that  $A_{ij} = A_{ik}$  implies that  $A'_{ij} = A'_{ik}$ . The proof of the converse, and the proof of the second statement proceed by similar easy arguments. Suppose that  $A_{ij} = A_{ik}$ . Then  $C$  has a submatrix

$$C[\{i, f\}, \{j, k\}] = \begin{matrix} & j & k \\ \begin{matrix} i \\ f \end{matrix} & \begin{bmatrix} x & x \\ 1 & 1 \end{bmatrix} \end{matrix},$$

for some non-zero  $x$ . Since  $C[\{i, f\}, \{j, k\}]$  has zero determinant,  $B\Delta\{f, i, j, k\}$  is not a basis of  $M$ , and hence not a basis of  $M' = M[I|C']$ . Therefore  $\det(C'[\{i, f\}, \{j, k\}]) = 0$ , so it follows that  $A'_{ij} = A'_{ik}$ .  $\square$

The following lemma on diagonal submatrices will be used frequently.

**Lemma 39.** *Let*

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

*be corresponding submatrices of  $A$  and  $A'$  respectively, where  $x, y, a, b$  are non-zero entries. Then  $x = y$  if and only if  $a \neq b$ .*

*Proof.* Adjoining  $e$  and  $f$  to the specified  $2 \times 2$  submatrices, we get the  $3 \times 3$  submatrices

$$\begin{bmatrix} x & 0 & 1 \\ 0 & y & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & \omega \end{bmatrix}.$$

These matrices have determinants  $x + y$  and  $ab\omega + a + b$ . Thus if  $x = y$ , then  $a \neq b$ . Conversely, if  $a \neq b$ , then  $\{a, b\} = \{1, \omega\}$  by Lemma 37 so  $ab\omega + a + b = \omega^2 + \omega + 1 = 0$ . Hence  $x = y$ .  $\square$

We can now identify all of the forbidden submatrices. We use Lemma 38 to identify the first such matrix in the following lemma.

**Lemma 40.** *Neither  $A$  nor  $A^T$  has a submatrix of the form*

$$\begin{bmatrix} x & y & z \end{bmatrix},$$

where  $x, y, z$  are distinct non-zero entries.

*Proof.* By Lemma 38, the corresponding submatrix of  $A'$  must have the form

$$\begin{bmatrix} a & b & c \end{bmatrix},$$

where  $a, b, c$  are distinct non-zero entries, which is a contradiction to Lemma 37.  $\square$

We now use Lemma 38 and Lemma 39 to find several more forbidden submatrices.

**Lemma 41.**  *$A$  has no submatrices of the following forms, where  $x, y,$  and  $z$  are distinct non-zero entries.*

$$\begin{aligned} & (i) \begin{bmatrix} x & x & 0 \\ x & 0 & x \end{bmatrix}; \quad (ii) \begin{bmatrix} x & x & 0 \\ x & 0 & y \end{bmatrix}; \quad (iii) \begin{bmatrix} x & x & 0 \\ y & 0 & y \end{bmatrix}; \quad (iv) \begin{bmatrix} x & y & 0 \\ x & 0 & y \end{bmatrix}; \\ & (v) \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \end{bmatrix}; \quad (vi) \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}; \quad (vii) \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}; \quad (viii) \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}. \end{aligned}$$

*Proof.* Suppose  $A$  has the submatrix (i). By applying Lemma 38 to the rows and the first column, we deduce that the corresponding submatrix of  $A'$  has the form

$$\begin{bmatrix} a & a & 0 \\ a & 0 & a \end{bmatrix},$$

where  $a$  is a non-zero entry, a contradiction of Lemma 39.

Suppose  $A$  has the submatrix (ii). By applying Lemma 38 to the rows and the first column, and since  $A'$  has at most two distinct non-zero entries by Lemma 37, we deduce that the corresponding submatrix of  $A'$  has the form

$$\begin{bmatrix} a & a & 0 \\ a & 0 & b \end{bmatrix},$$

where  $a$  and  $b$  are the two non-zero entries of  $A'$ , a contradiction to Lemma 39.

The proofs for (iii) and (iv) are similar to that for (ii). We omit the details.

Suppose  $A$  has the submatrix (v). Then, by two applications of Lemma 39, the corresponding submatrix of  $A'$  must have the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & a \end{bmatrix},$$

for some non-zero entry  $a$ . This is a contradiction to Lemma 38.

Suppose  $A$  has the submatrix (vi). By Lemma 39, the corresponding submatrix of  $A'$  must be a diagonal matrix with distinct non-zero entries, a contradiction to Lemma 37.

Suppose  $A$  has the submatrix (vii). Applying Lemma 39 to the two submatrices the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix},$$

it follows that the corresponding submatrix of  $A'$  is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

for some  $a$ , which is a contradiction to Lemma 39.

Suppose  $A$  has the submatrix (viii). Then the corresponding submatrix of  $A'$  is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

for some  $a$ . Adjoining  $e$  and  $f$ , we have a submatrix of  $C$ ,

$$\begin{bmatrix} x & 0 & 0 & 1 \\ 0 & y & 0 & 1 \\ 0 & 0 & z & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

which has zero determinant, while the corresponding submatrix of  $C'$ ,

$$\begin{bmatrix} a & 0 & 0 & 1 \\ 0 & a & 0 & 1 \\ 0 & 0 & a & 1 \\ 1 & 1 & 1 & \omega \end{bmatrix},$$

has non-zero determinant, a contradiction. □

**Lemma 42.** *A has no submatrices of the following forms, where  $x$ ,  $y$ , and  $z$  are distinct non-zero entries:*

$$(i) \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}; (ii) \begin{bmatrix} x & y \\ y & x \end{bmatrix}; (iii) \begin{bmatrix} x & x \\ y & z \end{bmatrix}; (iv) \begin{bmatrix} x & y \\ z & x \end{bmatrix}; (v) \begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}.$$

*Proof.* Suppose  $A$  has the submatrix (i). Then, adjoining  $e$  and  $f$ , we see that  $C$  has the following submatrix with non-zero determinant.

$$\begin{bmatrix} x & y & 1 \\ 0 & x & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

But then, by Lemma 38, the corresponding submatrix of  $C'$  must have the following form.

$$\begin{bmatrix} a & b & 1 \\ 0 & a & 1 \\ 1 & 1 & \omega \end{bmatrix},$$

where  $\{a, b\} = \{1, \omega\}$  by Lemma 37. This gives a contradiction because this submatrix of  $C'$  has zero determinant. A similar proof handles (ii).

Suppose  $A$  has the submatrix (iii). Then, by Lemma 38, in the corresponding submatrix of  $A'$ , the entries in the first row are the same and the entries in the second row are different. But, by Lemma 37, there are only two distinct non-zero entries in  $A'$ , so the entries are the same in one of the columns of  $A'$ , which is a contradiction to Lemma 38.

Suppose  $A$  has the submatrix (iv). Note that this submatrix has zero determinant. By Lemma 38, the corresponding submatrix of  $A'$  must have the following form.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

where  $\{a, b\} = \{1, \omega\}$  by Lemma 37. But this submatrix of  $A'$  has non-zero determinant, a contradiction.

Suppose  $A$  has the submatrix (v). Then  $C$  contains the following submatrix, which does not use its last column:

$$\begin{bmatrix} x & y & 0 \\ x & 0 & z \\ 1 & 1 & 1 \end{bmatrix}.$$

This matrix has determinant 0. By Lemmas 37, 38, and 39, the corresponding submatrix of  $C'$  is

$$\begin{bmatrix} a & b & 0 \\ a & 0 & b \\ 1 & 1 & 1 \end{bmatrix},$$

where  $\{a, b\} = \{1, \omega\}$ . This matrix has non-zero determinant, a contradiction.  $\square$

Finally, we find two more  $3 \times 3$  forbidden submatrices of  $A$ .

**Lemma 43.** *A has no submatrices of the following forms, where  $x, y,$  and  $z$  are distinct non-zero entries:*

$$(i) \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}; (ii) \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}.$$

*Proof.* Suppose that  $A$  has the submatrix (i). Then, adjoining  $e$  and  $f$ , we see that  $C$  has the submatrix

$$\begin{bmatrix} x & y & x & 1 \\ y & y & 0 & 1 \\ x & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

which has zero determinant. The corresponding submatrix of  $C'$  is

$$\begin{bmatrix} a & b & a & 1 \\ b & b & 0 & 1 \\ a & 0 & 0 & 1 \\ 1 & 1 & 1 & \omega \end{bmatrix},$$

for distinct  $a, b \in \{1, \omega\}$ . This submatrix of  $C$  has non-zero determinant, a contradiction.

Suppose that  $A$  has the submatrix (ii). Note that the determinant of this submatrix is not zero. By Lemma 37 and Lemma 38, the corresponding submatrix of  $A'$  is

$$\begin{bmatrix} a & b & a \\ b & b & 0 \\ a & 0 & b \end{bmatrix},$$

for distinct  $a, b \in \{1, \omega\}$ . This submatrix of  $A'$  has zero determinant, which is a contradiction.  $\square$

To prove the main theorem of this section, we need the following theorem [5, Theorem 5.1].

**Theorem 44.** *Minor-minimal non-GF(4)-representable matroids have rank and corank at most 4.*

We can now prove the main theorem, which we repeat for convenience.

**Theorem 45.** *There is some matrix  $C'$  representing  $M'$  if and only if, up to permuting rows and columns,  $A$  and  $A^T$  have no submatrix in the following set, where  $x, y, z$  are distinct non-zero elements of GF(4):*

$$\begin{aligned} & \begin{bmatrix} x & y & z \end{bmatrix}, \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \begin{bmatrix} x & x \\ y & z \end{bmatrix}, \begin{bmatrix} x & y \\ z & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ x & 0 & y \end{bmatrix}, \\ & \begin{bmatrix} x & x & 0 \\ y & 0 & y \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & y \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \end{bmatrix}, \begin{bmatrix} x & y & 0 \\ x & 0 & z \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}, \\ & \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & 0 \end{bmatrix}, \begin{bmatrix} x & y & x \\ y & y & 0 \\ x & 0 & z \end{bmatrix}. \end{aligned}$$

*Proof.* It follows from Lemmas 40, 41, 42, and 43 that both  $A$  and  $A^T$  have no submatrix on the above list.

Conversely, suppose that the GF(4)-representable matroid  $M$  is chosen to be minimal subject to the property that the relaxation  $M'$  is not GF(4)-representable. Then  $M'$  has a minor  $N$  isomorphic to one of the excluded minors for the class of GF(4)-representable matroids. Assume that  $N = M'/C \setminus D$  for some subsets  $C$  and  $D$ . If there is an element

$g$  in both  $D$  and the circuit-hyperplane  $X$  of  $M$ , then  $M \setminus g = M' \setminus g$  by Lemma 4, so  $M$  also has an  $N$ -minor, contradicting the fact that  $M$  is  $\text{GF}(4)$ -representable. We deduce that  $D \subseteq E(M) - X$ , and dually,  $C \subseteq X$ . Now if  $|D| \geq 2$ , then there is some element  $g$  in both  $D$  and  $E(M') - (X \cup f)$ , so relaxing the circuit-hyperplane  $X$  of  $M \setminus g$  gives  $M' \setminus g$  that is not  $\text{GF}(4)$ -representable, which contradicts the minimality of  $M$ . Therefore  $|D| \leq 1$ , and by a dual argument, there is no element  $g$  in both  $C$  and  $X - e$ , so  $|C| \leq 1$ . Since we know, by Theorem 44, that  $|E(N)| \leq 8$ , it now follows that  $|E(M')| \leq 10$ . The computations in the Appendix [4] show that  $M'$  must have a submatrix from the above list.  $\square$

## Acknowledgements

The authors thank the anonymous referee for some helpful suggestions that improved the exposition.

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