Even subgraph expansions for the flow polynomial of planar graphs with maximum degree at most 4

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Abstract

As projections of links, 4-regular plane graphs are important in combinatorial knot theory. The flow polynomial of 4-regular plane graphs has a close relation with the two-variable Kauffman polynomial of links. F. Jaeger in 1991 provided even subgraph expansions for the flow polynomial of cubic plane graphs. Starting from and based on Jaeger's work, by introducing splitting systems of even subgraphs, we extend Jaeger's results from cubic plane graphs to plane graphs with maximum degree at most 4 including 4-regular plane graphs as special cases. Several consequences are derived and further work is discussed.

Mathematics Subject Classifications: 05C31, 57M27

1 Introduction

Graphs in this paper may have loops and multiple edges. They may also have free loops, i.e. isolated edges which are not incident with any vertex. Let $G = (V, E)$ be a graph and $\nu = |V|$. We denote by $V_i(G)$ the set of vertices of G of degree i, and

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let $\nu_i = |V_i(G)|$. Two graphs will be said to be (topologically) *equivalent* if they are homeomorphic, that is if there exists a third graph which can be obtained from both by sequences of edge subdivisions.

Choose an arbitrary but fixed orientation of G. Let Γ be an additive Abelian group of order λ. A mapping $f : E \longrightarrow \Gamma$ is called a $\Gamma-$*flow* if, for each vertex $v \in V$, the total flow out of v is equal to the total flow into v. A Γ-flow f is called *nowhere-zero* if, for each $e \in E, f(e) \neq 0$. It is not difficult to see that the number of nowhere-zero $\Gamma-$flows does not depend on the chosen orientation of G, or the structure of Γ, but only on the order of Γ. For any positive integer λ, let $F(G, \lambda)$ denote the number of nowhere-zero $\Gamma-$flows of G, where λ is the order of Γ. As we shall see in the following contraction-deletion formula, $F(G, \lambda)$ is always a polynomial in λ, and is called the *flow polynomial* of G.

Let $e \in E$. We denote by G/e and $G - e$ the graphs obtained from G by contracting (that is, deleting e and identifying its two end vertices) and deleting the edge e, respectively. The flow polynomial $F(G, \lambda)$ of a graph G can be evaluated by the following contraction-deletion formula [13, 2].

(1) If G is an empty graph (that is, $E = \emptyset$), then

$$F(G, \lambda) = 1. \tag{1}$$

(2) If G is not empty and $e \in E$, then

   (i) when e is a loop,

$$F(G, \lambda) = (\lambda - 1)F(G - e, \lambda), \tag{2}$$

   (ii) otherwise,

$$F(G, \lambda) = F(G/e, \lambda) - F(G - e, \lambda). \tag{3}$$

In particular, if G has a cut edge (i.e. a bridge), then $F(G, \lambda) = 0$. In addition, the flow polynomial is invariant under equivalence of graphs. Hence, it suffices for us to consider graphs with minimum degree at least 3. In the subsequent figures, we usually do not draw vertices of degree 2. For more properties of the flow polynomial, we refer the reader to [12].

Cubic plane graphs have been studied extensively in graph theory. In [7] Jaeger gave two expansions of the flow polynomial $F(G, \lambda)$ of a cubic plane graph G. The first expansion was in terms of oriented even subgraphs of G, and the second one was an unoriented version of the first. 4-regular plane graphs, as projections of links, are important in combinatorial knot theory. The flow polynomial of 4-regular plane graphs have a close (although not direct) relation with link polynomials and the details will be given in Section 2. Starting from and inspired by Jaeger's work, by introducing splitting systems of even subgraphs and extending the rotation polynomials, in this paper we extend Jaeger's two expansions from cubic plane graphs to plane graphs with maximum degree at most 4 including 4-regular plane graphs. Several consequences are derived and further work is discussed.

The paper is organized as follows. In Section 3 we give the definitions of even subgraphs with splitting systems and extended rotational polynomial of plane graphs with maximum degree at most 4 in terms of oriented even subgraphs (under a given splitting system), and show that extended rotational polynomial is independent of splitting systems. In Section 4 we obtain an oriented even subgraph expansion based on the relationship between flow polynomial and extended rotational polynomial. In Section 5 we obtain the unoriented even subgraph expansion for the flow polynomial by grouping together the contributions of all orientations for each even subgraph. In Section 6, we consider the case of 4-regular plane graphs and several consequences are derived. In the final section further work is discussed.

2 Relation with Kauffman and Vogel's polynomial

The three variable bracket polynomial $[G] = [G](A, B, a)$ for 4-regular plane graphs G can be defined via the following graphical calculus [9, 3].

Graphical calculus for the bracket polynomial

(1) $[\bigcirc] = 1$, where $\bigcirc$ is a free loop.

(2) $[G \sqcup \bigcirc] = \mu[G]$, where $G \sqcup \bigcirc$ is the disjoint union of an unoriented 4-regular plane graph G and $\bigcirc$, and $\mu = \frac{a-a^{-1}}{A-B} + 1$.

(3) Let

$$\begin{aligned} o &= \frac{Aa^{-1} - Ba}{A - B} - (A + B), \\ \gamma &= \frac{B^2a - A^2a^{-1}}{A - B} + AB, \\ \xi &= \frac{B^3a - A^3a^{-1}}{A - B}. \end{aligned}$$

Then identities as shown in Figure 1 hold.

The graphical calculus of the bracket polynomial is a kind of recursive definitions entirely in the category of 4-regular plane graphs appearing in combinatorial knot theory. The following theorem uncovers its relation with the flow polynomial of 4-regular plane graphs.

Theorem 1. *Let G be a 4-regular plane graph. Let $\lambda = -A - A^{-1} + 2$. Then*

$$[G](A, A^{-1}, -A^2) = \frac{1}{\lambda - 1} F(G, \lambda). \tag{4}$$

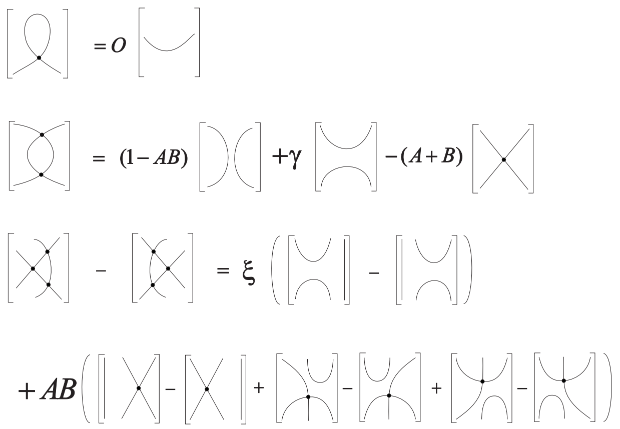

Figure 1: Recursive relations for the bracket polynomial.

Proof. It suffices to check that $\frac{F(G,\lambda)}{\lambda-1}$ satisfies all equations of the graphical calculus of the bracket polynomial. Eqs. (2) and (3), the property that the flow polynomial is invariant under edge subdivisions, and $F(G_1 \cup G_2, \lambda) = F(G_1, \lambda)F(G_2, \lambda)$ if G_1 and G_2 are disjoint or have only one common vertex, will be used. This is routine work and we leave the details to the reader. □

In addition, Carpentier [3] proved the following.

Theorem 2 ([3])**.** *Let G be a 4-regular plane graph. Then*

$$[G](A, A^{-1}, A) = 2^{k(G)-1}(-A - A^{-1})^{\nu(G)}. \tag{5}$$

Definition 3 ([9])**.** A graph diagram is a planar representation of a graph embedded in three dimensional Euclidean space and it is analogous to the link diagram of a link, i.e. disjoint cycles in three dimensional Euclidean space. Let G be a graph diagram of a 4-regular rigid vertex embedded graph. Let $\mathcal{P}$ be the set of 4-regular plane graphs obtained from G by applying to each crossing one of the three types of replacements as shown in Figure 2. Then the Kauffman and Vogel's polynomial is defined as follows.

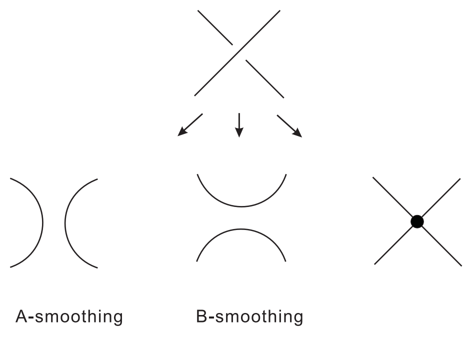

Figure 2: Three types of replacements of an unoriented crossing.

$$[G] = \sum_{P \in \mathcal{P}} A^{i(P)} B^{j(G)} [P](A, B, a), \tag{6}$$

where $i(P)$ and $j(P)$ are the numbers of crossings of G of A-smoothings and B-smoothings used to form P, respectively.

In the case that G has no vertices (i.e. it is a link diagram) $[G]$ reduces to the "Dubrovnik" version of the two-variable Kauffman polynomial of links [8].

3 Even subgraphs with splitting systems and the extended rotational polynomial

We shall always assume that $G = (V, E)$ is a plane graph with $3 \leqslant d(v) \leqslant 4$ for each $v \in V$ unless otherwise specified in the subsequent sections.

3.1 Even subgraphs with splitting systems

Let $C \subset E$. We shall call C an *even subgraph* of G if every vertex of the spanning subgraph (V, C) has even degree. The set of even subgraphs of G will be denoted by $\mathscr{C}(G)$. For convenience, we shall usually identify C with the induced subgraph $G[C]$.

An even subgraph C is said to be *trivial* if $C = \emptyset$, i.e. (V, C) consists of ν isolated vertices. A non-trivial even subgraph C of G without vertices of degree 4 is a disjoint union of cycles. For a non-trivial even subgraph C with vertices of degree 4 and for each $v \in V_4(C)$, there are two ways to split its incident four half-edges into two pairs (with planarity kept) as shown in Figure 3.

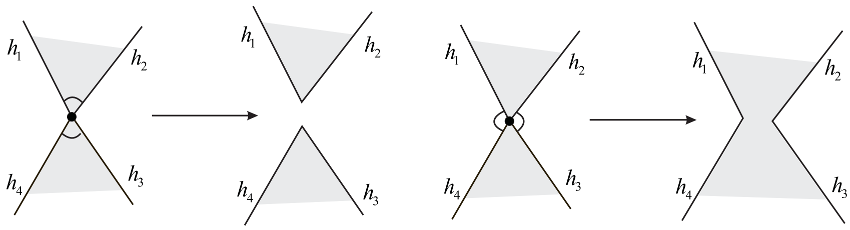

Figure 3: Two ways of splitting a 4-degree vertex into two vertices of degree 2: $\{\{h_1, h_2\}, \{h_3, h_4\}\}$ and $\{\{h_1, h_4\}, \{h_2, h_3\}\}$.

If we assign a splitting $s(v)$ from its two ways of splitting for each vertex $v \in V_4(C)$, that is, placing a pair of opposite angles near $v \in V_4(C)$ as drawn in Figure 3, then we obtain a splitting system of C. We denote by C^s the even subgraph C with a splitting system $s(C) = \{s(v) : v \in V_4(C)\}$. There will be $2^{\nu_4(C)}$ different splitting systems for an even subgraph C. We shall take the convention that $s(C) = \emptyset$ in the case of $\nu_4(C) = 0$. We shall denote by $S(G) = \{s(C) : C \in \mathscr{C}(G)\}$ a splitting system of G, where $s(C)$ is one of the splitting systems of C.

We denote by $c(C^s)$ the number of cycles of C after decomposing C based on the splitting system $s = s(C)$. If $\nu_4(C) = 0$ then $c(C^{\emptyset})$ is exactly the number of cycles of C. In particular, $c(\emptyset^{\emptyset}) = 0$. We can give each cycle a clockwise or an anticlockwise orientation. Thus for each even subgraph $C \in \mathscr{C}(G)$ with a given splitting $s = s(C)$ we obtain $2^{c(C^s)}$ oriented even subgraphs. In particular, the trivial even subgraph produces only one trivial oriented even subgraph. We shall denote by $\mathscr{O}\mathscr{C}(G, C^s)$ the set of oriented even subgraphs of G based on C^s.

Given a $C \in \mathscr{C}(G)$, in the following figures, we shall always use dotted lines to represent edges of G not in C and ordinary lines to represent edges in C. In the case of oriented even subgraphs, arrows are used to indicate orientations.

3.2 The extended rotational polynomial

In [7], Jaeger introduced a *rotational polynomial* $R(G, x)$ for a cubic plane graph G. We extend $R(G, x)$ of a cubic plane graph G to $r(G, S; x)$, where $S = S(G) = \{s(C) : C \in \mathscr{C}(G)\}$ is a splitting system of a plane graph G with $3 \leqslant d(v) \leqslant 4$ for each $v \in V$, and call it the *extended rotational polynomial*.

Let $C' \in \mathscr{O}\mathscr{C}(G, C^s)$. The *rotation number* of C' is defined to be the sum of signs of the cycles of C' after applying the splitting $s(C)$ to C and a cycle of C' has sign $+1$ if it is oriented counterclockwise and sign -1 otherwise. In particular, $r(C') = 0$ if $C = \emptyset$. To define the *weight* $< C' >$ of C', we first define the weight $< v|C' >$ for each $v \in V$ which are given in Figures 4 and 5, respectively. Note that it belongs to the ring $\mathbf{Z}[x^{\pm 1}]$ of Laurent polynomials in the variable x and is determined by the local behavior of C' at v. Then $< C' >= \prod_{v \in V} < v|C' >$.

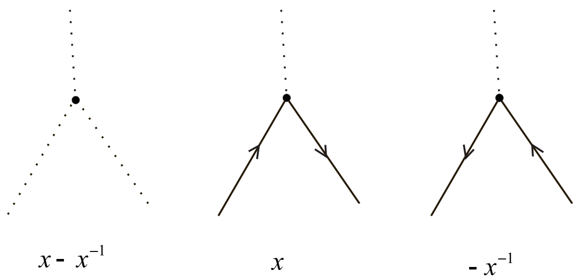

Figure 4: The weight $< v|C' >$ of a vertex v of degree 3.

Now we are in a position to define the extended rotational polynomial which is dependent on a fixed splitting system S of G.

Definition 4.

$$r(G, S; x) = \sum_{C \in \mathscr{C}(G)} \sum_{C' \in \mathscr{O}\mathscr{C}(G, C^s)} < C' > x^{2r(C')}. \tag{7}$$

In the case that G is a cubic plane graph, there is only one unique splitting system of G and $r(G, S; x)$ reduces to $R(G, x)$. In the following, we shall prove that $r(G, S; x)$ is in fact independent of the choice of the splitting system S.

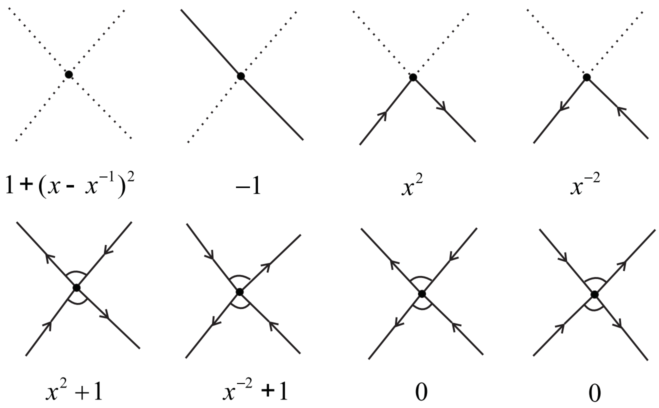

Figure 5: The weight $< v|C' >$ of a vertex v of degree 4 (the orientation of the second diagram is irrelevant).

Theorem 5. *For any two splitting systems* $S(G) = \{s(C) : C \in \mathscr{C}(G)\}$ *and* $T(G) = \{t(C) : C \in \mathscr{C}(G)\}$ *of* G*, we have* $r(G, S; x) = r(G, T; x)$.

Proof. We shall prove, for each $C \in \mathscr{C}(G)$,

$$\sum_{C' \in \mathscr{O}\mathscr{C}(G,C^s)} < C' > x^{2r(C')} = \sum_{C'' \in \mathscr{O}\mathscr{C}(G,C^t)} < C'' > x^{2r(C'')}. \tag{8}$$

It suffices to show that Eq. (8) holds when $s(C)$ and $t(C)$ are only distinct at a single vertex $u \in V_4(C)$. Note that $|c(C^s) - c(C^t)| = 1$, without loss of generality, we assume that $c(C^s) - c(C^t) = 1$, which means that there are two (resp. one) cycles containing u in C^s (resp. C^t) after splitting based on s (resp. t). There are two different cases for $s(C)$: non-nested as shown in Figure 6 (1) and nested as shown in Figure 6 (6). The corresponding two cases of $t(C)$ are shown in Figures. 6 ($1'$) and ($6'$), respectively. Let $C' \in \mathscr{O}\mathscr{C}(G, C^s)$. Let C_1 and C_2 be two cycles of C' (after splitting) containing u. If the two cycles are oriented as shown in Figure 6 (4),(5),(9) and (10), then $< u|C' >= 0$ and such C''s have no contributions to $\sum_{C' \in \mathscr{O}\mathscr{C}(G,C^s)} < C' > x^{2r(C')}$. We denote by $\mathscr{O}\mathscr{C}^*(G, C^s)$ the set of oriented even subgraph C''s such that $< u|C' > \neq 0$. Then there is a one-to-one correspondence between oriented even subgraphs of $\mathscr{O}\mathscr{C}^*(G, C^s)$ and oriented even subgraphs of $\mathscr{O}\mathscr{C}(G, C^t)$. Let $C' \in \mathscr{O}\mathscr{C}^*(G, C^s)$ correspond to $C'' \in \mathscr{O}\mathscr{C}(G, C^t)$ such that orientations of C'' are the same to C' as shown in Figure 6 (i) and (i'), $i = 2, 3, 7, 8$. Let C_u be the cycle of C'' containing u. Now we only need to prove that $< C' > x^{2r(C')} =< C'' > x^{2r(C'')}$ for each $C' \in \mathscr{O}\mathscr{C}^*(G, C^s)$. Note that for each $v \neq u$, we have $< v|C' >=< v|C'' >$ and cycles not containing u contribute the same value to $x^{2r(C')}$ and $x^{2r(C'')}$. Hence, it suffices to prove that $< u|C' > x^{2(sgn(C_1)+sgn(C_2))} =< u|C'' > x^{2sgn(C_u)}$.

Case 1. Non-nested case as in Figure 6 (i), $i = 2, 3$.

If the two cycles C_1 and C_2 are both oriented clockwise as in Figure 6 (2), then C_u is oriented clockwise as in Figure 6 ($2'$). We have

$$< u|C' > x^{2(sgn(C_1)+sgn(C_2))} = (x^2 + 1)x^{-4} = x^{-2} + x^{-4},$$
$$< u|C'' > x^{2sgn(C_u)} = (x^{-2} + 1)x^{-2} = x^{-2} + x^{-4}.$$

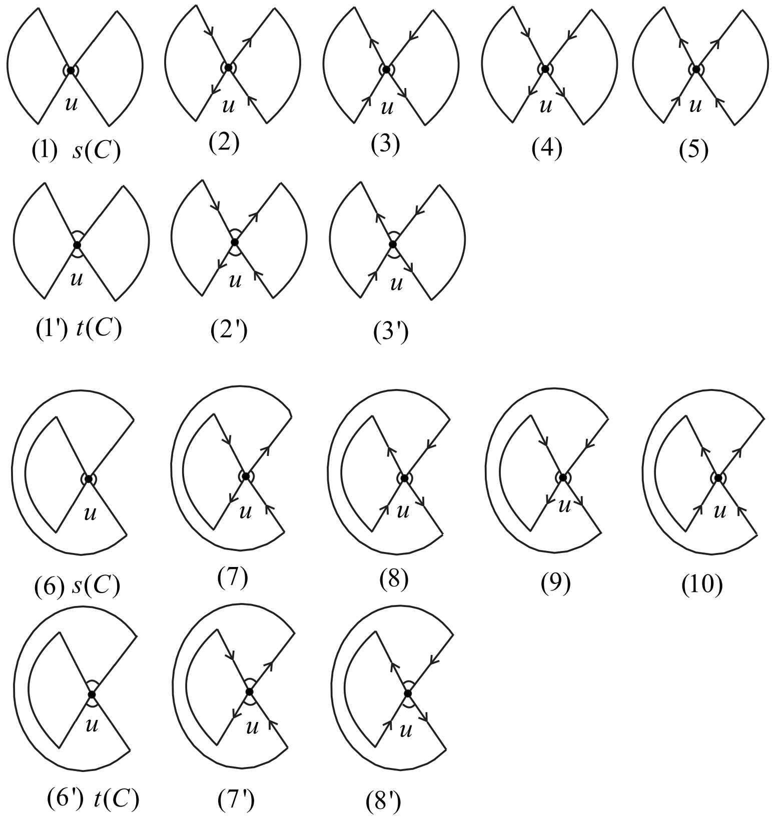

Figure 6: Illustrations of the Proof of Theorem 5.

If the two cycles C_1 and C_2 are both oriented counterclockwise as Figure 6 (3), then C_u is oriented counterclockwise as Figure 6 (3′). We have

$$< u|C' > x^{2(sgn(C_1)+sgn(C_2))} = (x^{-2}+1)x^4 = x^2+x^4,$$
$$< u|C'' > x^{2sgn(C_u)} = (x^2+1)x^2 = x^2+x^4.$$

Case 2. Nested case as in Figure 6 (i), $i = 7, 8$.

If the inner cycle C_1 is oriented clockwise and the outer cycle C_2 is oriented counterclockwise as Figure 6 (7), then C_u is oriented counterclockwise as Figure 6 (7′). We have

$$< u|C' > x^{2(sgn(C_1)+sgn(C_2))} = (x^2+1)x^0 = x^2+1,$$
$$< u|C'' > x^{2sgn(C_u)} = (x^{-2}+1)x^2 = x^2+1.$$

If the inner cycle C_1 is oriented counterclockwise and the outer cycle C_2 is oriented clockwise as Figure 6 (8), then C_u is oriented clockwise as Figure 6 (8′). We have

$$< u|C' > x^{2(sgn(C_1)+sgn(C_2))} = (x^{-2}+1)x^0 = x^{-2}+1,$$
$$< u|C'' > x^{2sgn(C_u)} = (x^2+1)x^{-2} = x^{-2}+1.$$

This completes the proof of Theorem 5. □

4 Oriented even subgraph expansion

A *degree-4 vertex-splitting* of G is a way of splitting each vertex $v_i \in V_4(G)$ into two new vertices of degree 2 in one of the two ways given in Figure 3. In this section, let u be a fixed degree-4 vertex-splitting of G. Let $H = H^u$ be the plane graph obtained from G by splitting each $v_i \in V_4(G)$ into two vertices v_{i_1}, v_{i_2} in the way given by u and adding a new edge e_i between v_{i_1} and v_{i_2} for all $i = 1, 2, \cdots, \nu_4$ as shown in Figure 7. For any integer i with $0 \leqslant i \leqslant 2^{\nu_4} - 1$, there is a unique expression $i = \sum_{j=1}^{\nu_4} 2^{j-1} x_j$, where $x_j = 0, 1$ for each j. Let H_i $(i = 0, 1, \cdots, 2^{\nu_4} - 1)$ be the plane graph obtained from H^u by deleting all edges e_j with $x_j = 0$. Then $H_0 = H - e_1 \cdots - e_{\nu_4}$ and $H_{2^{\nu_4}-1} = H$.

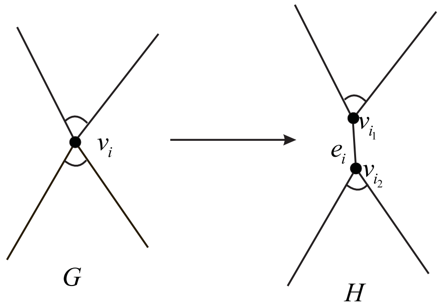

Figure 7: Splitting a vertex v_i to two vertices v_{i_1} and $v_{i_2}$of degree 3 by adding an edge e_i.

Lemma 6. *Let H_i be the plane graph described above for $i = 0, 1, 2, \cdots, 2^{\nu_4} - 1$. Then*

$$F(G, \lambda) = \sum_{i=0}^{2^{\nu_4}-1} F(H_i, \lambda).$$

Proof. Note that $G = H/e_1/\cdots/e_{\nu_4}$. By using the contraction-deletion formula (3) repeatedly, we obtain

$$\begin{aligned} F(G, \lambda) &= F(H/e_1 \cdots /e_{\nu_4-1}, \lambda) + F(H/e_1 \cdots /e_{\nu_4-1} - e_{\nu_4}, \lambda) \\ &= [F(H/e_1 \cdots /e_{\nu_4-2}, \lambda) + F(H/e_1 \cdots /e_{\nu_4-2} - e_{\nu_4-1}, \lambda)] + \\ &\qquad [F(H/e_1 \cdots /e_{\nu_4-2} - e_{\nu_4}, \lambda) + F(H/e_1 \cdots /e_{\nu_4-2} - e_{\nu_4-1} - e_{\nu_4}, \lambda)] \\ &= \cdots \\ &= \sum_{i=0}^{2^{\nu_4}-1} F(H_i, \lambda). \end{aligned}$$

□

Theorem 7 ([7])**.** *For every cubic plane graph G,*

$$R(G, x) = F(G, (x + x^{-1})^2). \tag{9}$$

Remark 8. If G is homeomorphic to a cubic plane graph, but contains vertices of degree 2, then Theorem 7 will still be valid under the assumption that $< v|C' >= 1$ for each vertex $v \in V_2(G)$. See Figure 8.

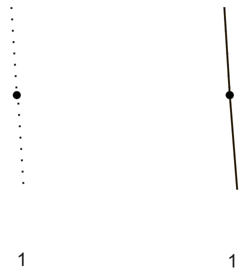

Figure 8: The weight $< v|C' >$ of a vertex v of degree 2 (the orientation of the second diagram is irrelevant).

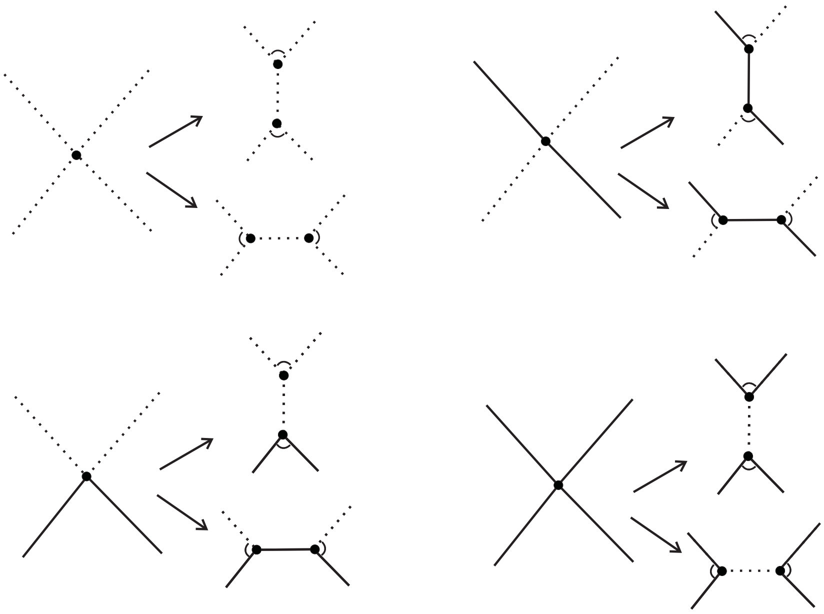

Figure 9: The local correspondence between C and $B_{2^{\nu_4}-1}$.

For any $C \in \mathscr{C}(G)$, let C^u denote the graph obtained from C by splitting each vertex $v \in V_4(C)$ in the way given by u. This will induce uniformly a splitting system for each $C \in \mathscr{C}(G)$. Thus u actually gives a splitting system of G, denoted by U. Let $C' \in \mathscr{O}\mathscr{C}(G, C^u)$.

We first define the even subgraph $B_{2^{\nu_4}-1}$, corresponding to C, of $H_{2^{\nu_4}-1}$ in a natural way depending on u as shown in Figure 9. Note that cycles of C^u are one-to-one correspondent to cycles of $B_{2^{\nu_4}-1}$. Then let $B_i = B_{2^{\nu_4}-1} \cap E(H_i)$, corresponding to C, for $i = 0, 1, \cdots, 2^{\nu_4} - 2$. But B_i of H_i may not be even, we shall call such a *fake even subgraph* of H_i. For C', B'_i is similarly defined by inheriting orientations of C' (orientations of e_i's are irrelevant). Then cycles of C' will correspond to (fake) cycles of B'_i. We define $r(B'_i) = r(C')$. Conversely, for any even subgraph $C'_i \in \mathscr{O}\mathscr{C}(H_i, C_i)$, there exists one unique $C' \in \mathscr{O}\mathscr{C}(G, C^u)$ such that $B'_i = C'_i$. In fact C' can be obtained from C'_i by contracting all edges $e_j \in C'_i$. We take the 4-regular plane graph shown in Figure 10 as an example to illustrate the above correspondences. See Figure 11.

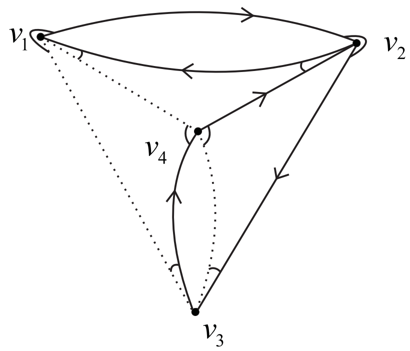

Figure 10: A 4-regular plane graph G with a degree-4 vertex-splitting u and one of its oriented even subgraphs C' with only one cycle.

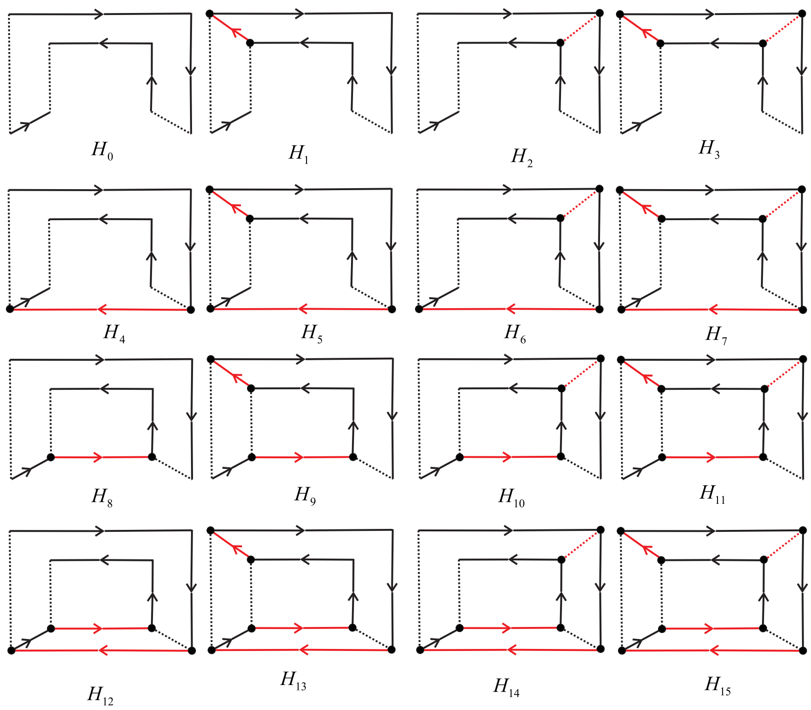

Figure 11: H_i (red lines, dashed or ordinary) correspond to vertices of G, and B'_i (ordinary lines) correspond to C' for $i = 0, 1, \ldots, 15$. Note that only B'_{13} and B'_{16} are real cycles.

Theorem 9. *For any splitting system $S(G)$ of G, $r(G,S;x) = F(G,(x+x^{-1})^2)$.*

Proof. By Theorem 5, we can take S to be the splitting system U of G given by the degree-4 vertex-splitting u. By Lemma 6, $F(G,(x+x^{-1})^2) = \sum_{i=0}^{2^{\nu_4}-1} F(H_i,(x+x^{-1})^2)$. By Theorem 7 and Remark 8, $R(H_i,x) = F(H_i,(x+x^{-1})^2)$. Thus $F(G,(x+x^{-1})^2) = \sum_{i=0}^{2^{\nu_4}-1} R(H_i,x)$. It remains to prove that $r(G,U;x) = \sum_{i=0}^{2^{\nu_4}-1} R(H_i,x)$. For each $v \in V_2(H_i)$, take $< v|B_i' >= 0$ if exactly one of two edges incident with v is contained in B_i' and $< v|B_i' >= 1$ otherwise. Then

$$\begin{aligned} R(H_i,x) &= \sum_{C_i\in\mathscr{C}(H_i)} \sum_{C_i'\in\mathscr{OC}(H_i,C_i)} < C_i' > x^{2r(C_i')} \\ &= \sum_{C\in\mathscr{C}(G)} \sum_{C'\in\mathscr{OC}(G,C^u)} < B_i' > x^{2r(B_i')}. \end{aligned}$$

According to Definition 4, $r(G,U;x) = \sum_{C\in\mathscr{C}(G)} \sum_{C'\in\mathscr{OC}(G,C^u)} < C' > x^{2r(C')}$. Recalling that $r(C') = r(B_i')$, it will thus be enough to show that, for every oriented even subgraph $C' \in \mathscr{OC}(G,C^u)$,

$$< C' >= \sum_{i=0}^{2^{\nu_4}-1} < B_i' > .$$

First, if $v \in V_3(G)$, then $< v|C' >=< v|B_i' >$. It suffices to show that

$$\prod_{v\in V_4(G)} < v|C' >= \sum_{i=0}^{2^{\nu_4}-1} \prod_{v\in V(H_i)\setminus V_3(G)} < v|B_i' > .$$

Note that

$$\begin{aligned} & \sum_{i=0}^{2^{\nu_4}-1} \prod_{v\in V(H_i)\setminus V_3(G)} < v|B_i' > \\ = & \sum_{i=0}^{2^{\nu_4}-1} \prod_{v_j\in V_4(G)} < v_{j_1}|B_i' >< v_{j_2}|B_i' > \\ = & \prod_{v_j\in V_4(G)} \left[< v_{j_1}|B_0' >< v_{j_2}|B_0' > + < v_{j_1}|B_{2^{\nu_4}-1}' >< v_{j_2}|B_{2^{\nu_4}-1}' >\right]. \end{aligned}$$

Now we only need to show that $< v_j|C' >=< v_{j_1}|B_0' >< v_{j_2}|B_0' > + < v_{j_1}|B_{2^{\nu_4}-1}' >< v_{j_2}|B_{2^{\nu_4}-1}' >$ for each $v_j \in V_4(G)$. All cases are checked in Figures. 12 and 13. This completes the proof of Theorem 9. □

5 Unoriented even subgraph expansion

In [7], Jaeger expressed $R(G,x)$ as a polynomial in the variable $\lambda = (x+x^{-1})^2$. Motivated by Jaeger's work we shall express $r(G,S;x)$ as a polynomial in the variable $\lambda = (x+x^{-1})^2$.

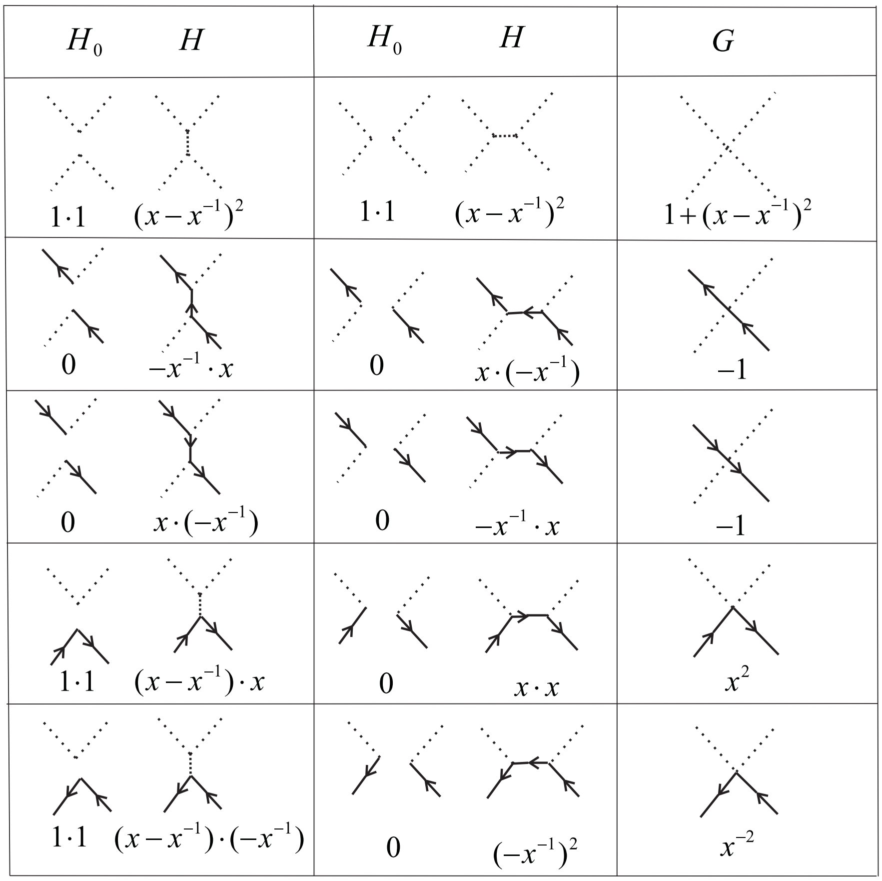

Figure 12: The case $d_C(v) = 0, 2$:$< v_{j_1}|B'_0 >< v_{j_2}|B'_0 >$, $< v_{j_1}|B'_{2^{\nu_4}-1} >< v_{j_2}|B'_{2^{\nu_4}-1} >$ and $< v_j|C' >$.

Let $\mathscr{S}(G)^\circ$ be the set of splitting systems $S(G)$ of G such that $c(C^s)$ is exactly the number of connected components of C for each $C \in \mathscr{C}(G)$. We consider a fixed such splitting system $S(G) \in \mathscr{S}(G)^\circ$. Note that for such a splitting system the zero weight in Figure 5 will never happen.

Let $C \in \mathscr{C}(G)$. We first suppose that C is connected and in this case C^s (after splitting) is a single cycle.

(i) Let $v \in V_2(C) \cap V_3(G)$. We call v an *inlet vertex* (resp. *outlet vertex*) if the edge incident to v of G which does not belong to C lies inside (resp. outside) C^s (after splitting). We denote by $i(C)$ (resp. $o(C)$) the number of inlet vertices (resp. outlet vertices) of C.

(ii) Let $v \in V_2(C) \cap V_4(G)$. We call v an *inlet half vertex* (resp. *outlet half vertex*) if the two edges incident to v which do not belong to C lie inside (resp. outside) C^s. We denote by $ih(C)$ (resp. $oh(C)$) the number of inlet half vertices (resp. outlet half vertices) of C. We call v a *straight vertex* if one of the two edges incident to v which does not belong to C lies inside C^s and the other outside C^s. We denote by $sv(C)$ the number of straight vertices of C.

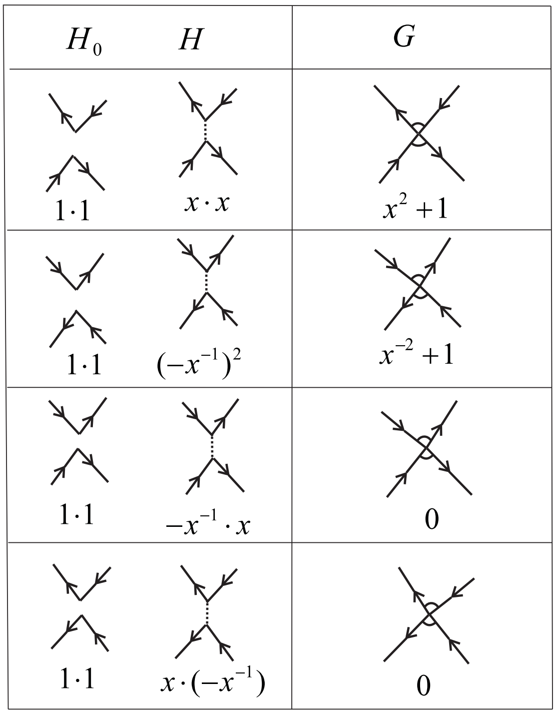

Figure 13: The case $d_C(v) = 4$:$< v_{j_1}|B'_0 >< v_{j_2}|B'_0 >$, $< v_{j_1}|B'_{2^{\nu_4}-1} >< v_{j_2}|B'_{2^{\nu_4}-1} >$ and $< v_j|C' >$.

(iii) Let $v \in V_4(C)$. We call v an *inlet total vertex* (resp. *outlet total vertex*) if both the two angles near v under $S(G)$ lie inside (resp. outside) C^s. We denote by $it(C)$ (resp. $ot(C)$) the number of inlet total vertices (resp. outlet total vertices) of C.

See Figure 14 for an example. If C is not connected, each of the above parameters of C is defined to be the sum of the corresponding parameters of each component of C.

We denote by $p_3(C)$ (resp. $p_4(C)$) the number vertices of G of degree 3 (resp. 4) which do not belong to $V(C)$. We call a component of (V, C) an *odd component* if it has odd number of vertices of G of degree 3. We denote by $q(C)$ the number of odd components of the graph (V, C).

Lemma 10. *(1)* $\nu_3(G) = i(C) + o(C) + p_3(C)$.

(2) $\nu_4(G) = it(C) + ot(C) + ih(C) + oh(C) + sv(C) + p_4(C)$.

(3) $q(C)$ *is even.*

Proof. (1) is trivial. (2) follows from the fact that there are no vertices of degree 4 in C with one angle inside a cycle of C^s and the other outside the same cycle of C^s. (3) follows from the fact that $\nu_3(G)$ is even. □

Lemma 11. *All parameters* $i(C)$, $o(C)$, $it(C)$, $ot(C)$, $ih(C)$, $oh(C)$, *and* $sv(C)$ *do not depend on the splitting system* $S(G) \in \mathscr{S}(G)^\circ$.

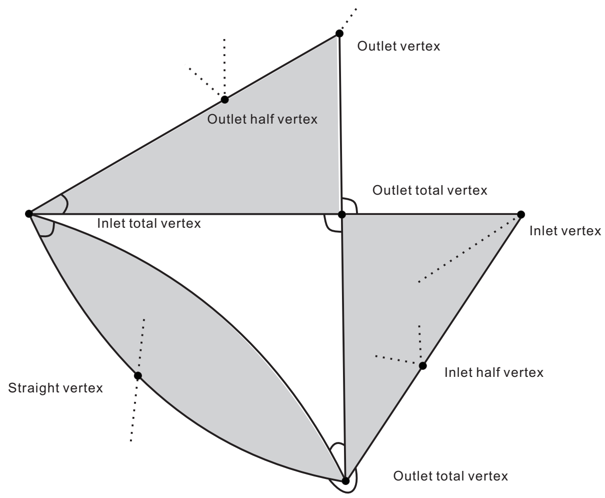

Figure 14: Illustration of vertex types: the interior of C^s is shaded.

Proof. Without loss of generality, we assume that C is connected. Shade C in a checkerboard fashion such that the unbounded face is unshaded. Then for any $S(G) \in \mathscr{S}(G)^\circ$, the interior of C^s (after splitting) consists exactly of shaded faces of C. In fact, the unicycle splitting s corresponds to exactly one spanning tree of the shaded face graph of C, where the spanning tree connects all shaded faces of C together to form the interior of C^s. A simple example is given in Figure 15. Hence $i(C)$, $o(C)$, $ih(C)$, $oh(C)$, and $sv(C)$ do not depend on $S(G)$. In other words, $i(C)$ is the number of vertices of degree 3 of G with an incident dashed edge inside the shaded face of C, and so on. Let $b(C)$ and $w(C)$ be the numbers of shaded and unshaded faces (including the unbounded face) of C, respectively. Then it is not difficult to see that $it(C) = w(C) - 1$ and $ot(C) = b(C) - 1$. Thus $it(C)$ and $ot(C)$ do not depend on $S(G)$ either. $\square$

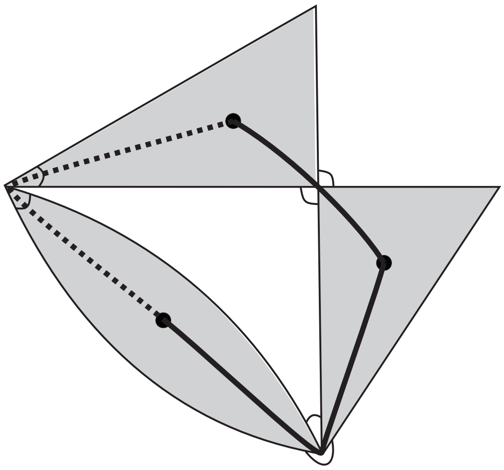

Figure 15: A connected even subgraph C and its shaded face graph (bold and dotted lines).

Let C be an even subgraph. We define the *sign* of C as follows:

$$\sigma(C) = (-1)^{sv(C)+o(C)}. \tag{10}$$

In particular, $\sigma(\emptyset) = 1$. To define the *surrounding polynomial* $S(C,\lambda)$ of C, we shall need the sequence of polynomials A_n in $\mathbf{Z}[u]$ $(n \geqslant 0)$, which are defined by $A_n(u) = 2T_n(u/2)$, where T_n is the Chebyshev polynomial of the first kind. In addition, A_n can be recursively defined as follows:

$$A_{n+2}(u) = uA_{n+1}(u) - A_n(u) \quad (n \geqslant 0), \tag{11}$$

$$A_0(u) = 2, A_1(u) = u. \tag{12}$$

For all $n \geqslant 0$, we have

$$A_n(t + t^{-1}) = t^n + t^{-n}. \tag{13}$$

By Eq. (13), we can get $A_n(2) = 2$ for any $n \geqslant 0$ and $A_{2k}(-2) = 2, A_{2k+1}(-2) = -2$ for any $k \geqslant 0$.

We first define the surrounding polynomial of a connected C. Let

$$m(i,j) = i(C) - o(C) + 2(1 + ih(C) - oh(C) - i + j),$$

where $0 \leqslant i \leqslant it(C), 0 \leqslant j \leqslant ot(C)$.

(i) If $i(C) + o(C)$ is even, then

$$S(C,\lambda) = \sum_{i,j} \binom{it(C)}{i} \binom{ot(C)}{j} A_{n(i,j)}(\lambda - 2), \tag{14}$$

where $n(i,j) = |m(i,j)|/2$.

(ii) If $i(C) + o(C)$ is odd, then

$$S(C,\lambda) = \sum_{i,j} \binom{it(C)}{i} \binom{ot(C)}{j} sgn(m(i,j)) \left[1 + \sum_{h=1}^{n(i,j)} A_h(\lambda - 2) \right], \tag{15}$$

where $sgn(x)$ is the sign function of the real number x defined by $sgn(x) = 1$ if $x > 0$, $sgn(0) = 0$ and $sgn(x) = -1$ if $x < 0$, and $n(i,j) = (|m(i,j)| - 1)/2$.

If C is not connected with $k(C)$ connected components $C_1, C_2, \cdots, C_{k(C)}$, we define its surrounding polynomial as the product of the surrounding polynomials of its connected components, i.e.

$$S(C,\lambda) = \prod_{l=1}^{k(C)} S(C_l, \lambda). \tag{16}$$

We take the convention that $S(\emptyset, \lambda) = 1$.

Theorem 12.

$$F(G,\lambda) = \sum_{C\in\mathscr{C}(G)} (\lambda-4)^{\frac{q(C)}{2}}(\lambda-3)^{p_4(C)}\sigma(C)S(C,\lambda). \tag{17}$$

Proof. Recall that we write $\lambda = (x+x^{-1})^2$. It follows from Theorem 9 that

$$F(G,\lambda) = \sum_{C\in\mathscr{C}(G)} \sum_{C'\in\mathscr{O}\mathscr{C}(G,C^s)} < C' > x^{2r(C')}.$$

Consider a fixed splitting system $S(G) \in \mathscr{S}(G)^\circ$. Let $C_1, C_2, \ldots, C_k$ be components of C. Recall that C_l^s will become a single cycle after splitting for $l = 1, 2, \cdots, k$. If C_l is oriented counterclockwise in C', then:

(i) each inlet vertex (resp. outlet vertex) of C_l contributes a term x^1 (resp. $-x^{-1}$) to $< C' >$,

(ii) each inlet total vertex (resp. outlet total vertex) of C_l contributes a term $x^{-2}+1$ (resp. x^2+1) to $< C' >$,

(iii) each inlet half vertex (resp. outlet half vertex) of C_l contributes a term x^2 (resp. x^{-2}) to $< C' >$ and

(iv) each straight vertex of C_l contributes a term -1 to $< C' >$.

Moreover, C_l contributes a term x^2 to the rotation factor $x^{2r(C')}$. Thus if C_l is oriented counterclockwise, its contribution to $< C' > x^{2r(C')}$ equals

$$\begin{aligned} & (-1)^{sv(C_l)}x^{i(C_l)}(-x^{-1})^{o(C_l)}(x^{-2}+1)^{it(C_l)}(x^2+1)^{ot(C_l)}x^{2(ih(C_l)-oh(C_l))}x^2 \\ = \; & (-1)^{sv(C_l)+o(C_l)}\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}x^{m_l(i,j)}, \end{aligned}$$

where $m_l(i,j) = i(C_l) - o(C_l) + 2(1 + ih(C_l) - oh(C_l) - i + j)$. Similarly, if C_l is oriented clockwise in C', its contribution to $< C' > x^{2r(C')}$ equals

$$\begin{aligned} & (-1)^{sv(C_l)}x^{o(C_l)}(-x^{-1})^{i(C_l)}(x^{-2}+1)^{ot(C_l)}(x^2+1)^{it(C_l)}x^{2(oh(C_l)-ih(C_l))}x^{-2} \\ = \; & (-1)^{sv(C_l)+i(C_l)}\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}x^{-m_l(i,j)}. \end{aligned}$$

The even subgraph C has 2^k orientations, thus its contribution equals

$$\sum_{C'\in\mathscr{O}\mathscr{C}(G,C^s)} < C' > x^{2r(C')} = (x-x^{-1})^{p_3(C)}(\lambda-3)^{p_4(C)}\prod_{l=1}^{k} H(C_l),$$

where $H(C_l) = \sigma(C_l)\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}\left[x^{m_l(i,j)} + (-1)^{i(C_l)-o(C_l)}x^{-m_l(i,j)}\right]$.

Case 1. If $i(C_l)+o(C_l)$ is even, writing $|m_l(i,j)|=2n_l(i,j)$, we obtain

$$\begin{aligned}
H(C_l) &= \sigma(C_l)\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}(x^{2n_l(i,j)}+x^{-2n_l(i,j)}) \\
&= \sigma(C_l)\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}A_{n_l(i,j)}(x^2+x^{-2}) \\
&= \sigma(C_l)\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}A_{n_l(i,j)}(\lambda-2) \\
&= \sigma(C_l)S(C_l,\lambda).
\end{aligned}$$

Case 2. If $i(C_l)+o(C_l)$ is odd, writing $|m_l(i,j)|=2n_l(i,j)+1$, we have

$$\begin{aligned}
H(C_l) &= \sigma(C_l)\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}sgn(m_l(i,j))\left[x^{2n_l(i,j)+1}-x^{-2n_l(i,j)-1}\right] \\
&= \sigma(C_l)\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}sgn(m_l(i,j))\times \\
&\quad \left[(x-x^{-1})\sum_{h\in\{-n_l(i,j),\dots,n_l(i,j)\}}x^{2h}\right] \\
&= \sigma(C_l)(x-x^{-1})\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}sgn(m_l(i,j))\times \\
&\quad (1+\sum_{h=1}^{n_l(i,j)}A_h(\lambda-2)) \\
&= \sigma(C_l)(x-x^{-1})S(C_l,\lambda).
\end{aligned}$$

Note that $i(C_l)+o(C_l)$ is odd $\iff C_l$ is an odd component of C. In addition, isolated vertices of (V,C) are also odd components of C. It follows that

$$\begin{aligned}
\sum_{C'\in\mathscr{O}\mathscr{C}(G,C^s)} < C' > x^{2r(C')} &= (x-x^{-1})^{q(C)}(\lambda-3)^{p_4(C)}\prod_{l=1}^{k}\sigma(C_l)S(C_l,\lambda) \\
&= (\lambda-4)^{\frac{q(C)}{2}}(\lambda-3)^{p_4(C)}\sigma(C)S(C,\lambda).
\end{aligned}$$

This completes the proof. □

Example 13. To illustrate Theorem 12, let us compute the flow polynomial of the graph K_4^+ obtained from K_4 by adding a single parallel edge. It has 16 even subgraphs and each of their corresponding contributions are listed in Figure 16. Thus,

$$F(K_4^+,\lambda)=\lambda^4-7\lambda^3+19\lambda^2-23\lambda+10.$$

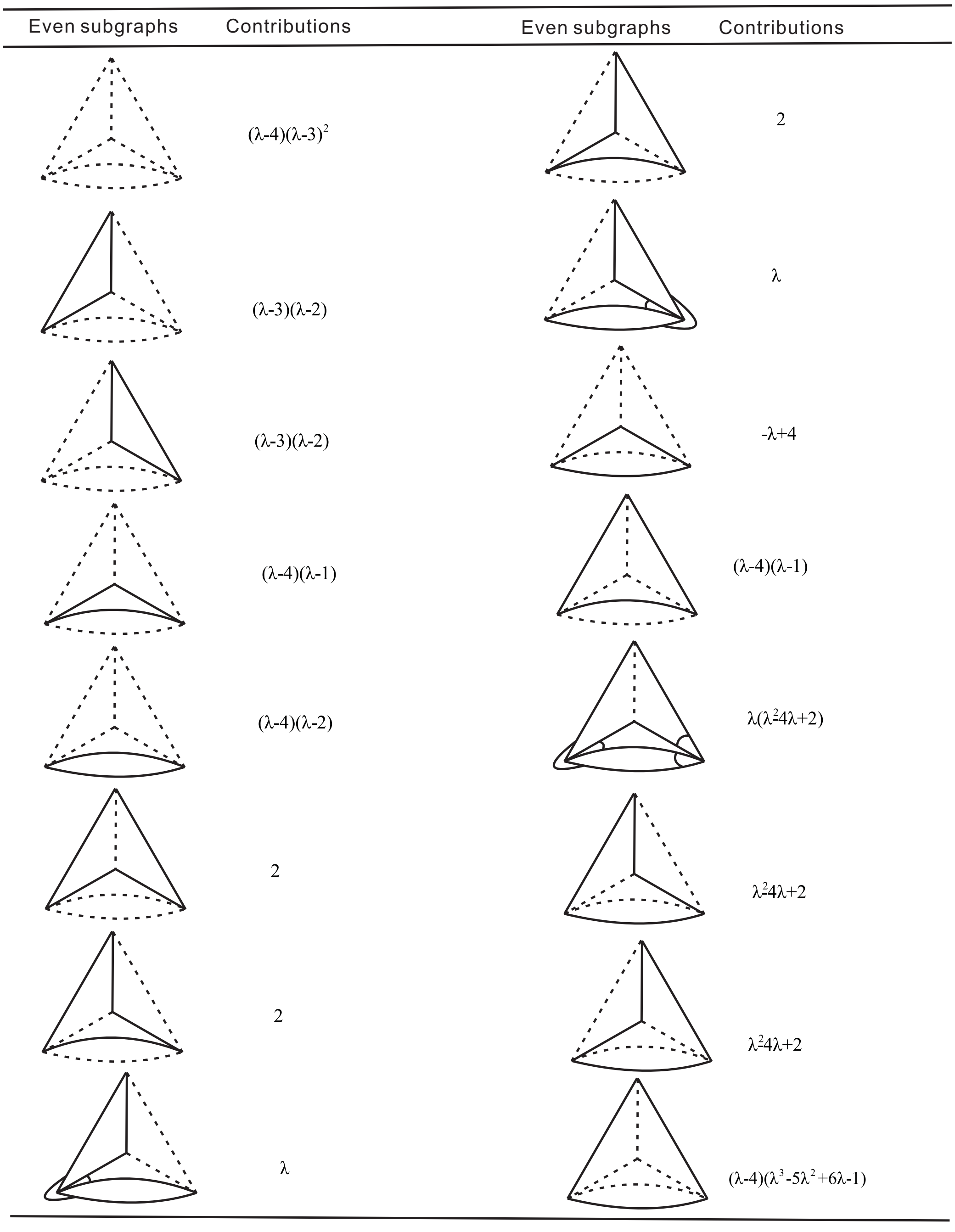

Figure 16: An example: summands of the flow polynomial of K_4^+.

6 Consequences

It is well known that $F(G,1)=0$ and $F(G,2)=1$ for an Eulerian graph G. Now we consider expressions of $F(G,4)$ and $F(G,0)$ for 4-regular plane graph G.

Theorem 14. *Let G be a 4-regular plane graph G. Then*

$$F(G,\lambda)=\sum_{C\in\mathscr{C}(G)}(\lambda-3)^{p_4(C)}S(C,\lambda). \tag{18}$$

Proof. For every 4-regular plane graph G and for any even subgraph $C\in\mathscr{C}(G)$, it is clear that $q(C)=0$ and $i(C)+o(C)=0$. Now we prove that $sv(C)$ is even. Without loss of generality we assume that C is connected. By contracting C^s and its interior to a new vertex, we obtain a graph in which the degree of the new vertex is $sv(C)$ and the degree of all other vertices is 4. This implies that $sv(C)$ is even. Hence, $\sigma(C)=1$. □

Corollary 15. *Let G be a 4-regular plane graph. Then*

(1) $F(G,4)=\sum_{C\in\mathscr{C}(G)}2^{\nu_4(C)+k(C)}$, *where $k(C)$ is the number of connected components of C;*

(2) $F(G,0)=\sum_{C:\nu_4(C)=0}(-1)^{e(C)}(-3)^{p_4(C)}2^{k(C)}$, *where $e(C)$ is the number of even cycles of C.*

Proof. Consider a splitting system $S(G)\in\mathscr{S}(G)^\circ$ and $S(G)=\{s(C):C\in\mathscr{C}(G)\}$. Recall that $S(\emptyset,\lambda)=1$. Let C be a nontrivial even subgraph of G with $k(C)$ components $C_1,C_2,\cdots,C_{k(C)}$. Then

$$\begin{aligned}
S(C,4) &= \prod_{l=1}^{k(C)}S(C_l,4)\\
&= \prod_{l=1}^{k(C)}\left[\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}A_{n_l(i,j)}(2)\right]\\
&= \prod_{l=1}^{k(C)}2^{it(C_l)+ot(C_l)+1}\\
&= 2^{\nu_4(C)+k(C)}.
\end{aligned}$$

If C is nontrivial then

$$\begin{aligned}
S(C,0) &= \prod_{l=1}^{k(C)}S(C_l,0)\\
&= \prod_{l=1}^{k(C)}\left[\sum_{i,j}\binom{it(C_l)}{i}\binom{ot(C_l)}{j}A_{n_l(i,j)}(-2)\right].
\end{aligned}$$

If $it(C) > 0$ or $ot(C) > 0$ then $S(C, 0) = 0$. Note that $\nu_4(C) = it(C) + ot(C)$. We only need to consider the case that $\nu_4(C) = 0$ and for such a C, we have

$$S(C, 0) = \prod_{l=1}^{k(C)} A_{|1+ih(C_l)-oh(C_l)|}(-2).$$

Recall that $sv(C_l)$ is even. Hence $|1 + ih(C_l) - oh(C_l)|$ is odd if and only if $\nu(C_l) = ih(C_l) + oh(C_l) + sv(C_l)$ is even. Thus,

$$\begin{aligned} F(G, 0) &= \sum_{C \in \mathscr{C}(G)} (-3)^{p_4(C)} S(C, 0) \\ &= (-3)^{\nu} + \sum_{C:\text{2-regular}} (-3)^{p_4(C)} S(C, 0) \\ &= (-3)^{\nu} + \sum_{C:\text{2-regular}} (-1)^{e(C)} (-3)^{p_4(C)} 2^{k(C)} \\ &= \sum_{C:\nu_4(C)=0} (-1)^{e(C)} (-3)^{p_4(C)} 2^{k(C)}. \end{aligned}$$

□

It is immediate from Corollary 15 that $F(G, 0)$ (i.e. the constant term of the flow polynomial) and $F(G, 4)$ are both odd. In fact, Hong [5] proved that $a_1(G)$ (the coefficient of λ in the chromatic polynomial of G) is odd if and only if G is connected and bipartite. By duality we know that the constant term of the flow polynomial of 4-regular plane graphs is odd. Since $F(G, 4) \equiv F(G, 0) \pmod 4$, $F(G, 4)$ for 4-regular plane graphs is also odd.

Corollary 16. *Let G be a graph diagram of a 4-regular graph. Let*

$$f_G(A) = [G](A, A^{-1}, -A^2)$$

be the Kauffman-Vogel polynomial of G with $B = A^{-1}$ and $d = -A^2$. Then $f_G(1)$ is odd.

Proof. We first consider the case when G is a 4-regular plane graph. Then, by Theorem 1, we have:

$$[G](A, A^{-1}, -A^2) = \frac{1}{1 - A - A^{-1}} F(G, 2 - A - A^{-1}). \tag{19}$$

Thus $f_G(1) = -F(G, 0)$ and hence $f_G(1)$ is odd. Otherwise, let $c(G)$ be the number of crossings of G. By Definition 3, $f_G(1)$ is the summation of $3^{c(G)}$ odd numbers and hence it is also odd. □

7 Concluding remarks

Jaeger's work cannot be generalized to non-planar cubic graphs since in his expansion the interior and the exterior of a planar cycle were used. Our generalization depends heavily on Jaeger's work on cubic plane graphs, and hence it is impossible to drop the 'planar' condition. In the case of 4-regular plane graphs, it is possible to prove Theorems 9 and 12 independent of Jaeger's work by using the recursive relations in Figure 1, but still in the category of plane graphs. How about 4-regular cellularly embedded graphs with 2-face colorings? We note however that their even subgraphs may not be 2-face colorable, and so Theorem 12 cannot be generalized to 4-regular cellularly embedded graphs with 2-face colorings.

By using two different special splitting systems Theorems 9 and 12 are obtained. Can our work be generalized to plane graphs with maximum degree greater than 4? For a vertex of degree greater than 4, there will be many more degree selections and splittings for even subgraphs than in the case that $d(v) = 3$ or 4. We think it will be difficult to find suitable vertex weights to make Theorem 5 hold, i.e. the extended rotational polynomial is independent of splitting systems.

Let G be a plane graph with $d(v) = 3, 4$ for each $v \in V(G)$. Theorem 12 implies that

$$F(G,(x+x^{-1})^2) = \sum_{(V,C):C\in\mathscr{C}(G)} \prod_{H_i:components\ of\ (V,C)} w_{H_i}(x), \tag{20}$$

where $w_{H_i}(x)$ is a Laurent polynomial in x, depending on H_i, G and their planar embedding. It is very different from transition polynomials defined for abstract 4-regular graphs or embedded graphs via their medial graphs, where the summation is over 3^{ν} transition systems and where the counting is of the number of edge-disjoint cycles that the 4-regular graph is decomposed into under the transition system. For details of the transition polynomial, see [11, 10, 14, 6] and [1, 4].

Let G be a 4-regular plane graph. By Theorem 14, we obtain

$$F(G,3) = \sum_{Spanning\ even\ subgraphs\ C} \prod_{C_l} \left[\sum_{i,j} \binom{it(C_l)}{i} \binom{ot(C_l)}{j} A_{n(i,j)}(1) \right], \tag{21}$$

where the product is over all components C_l of C and $n(i,j) = |ih(C_l) - oh(C_l) + 1 - i + j| \pmod 6$ and $A_0(1) = 2, A_1(1) = 1, A_2(1) = -1, A_3(1) = -2, A_4(1) = -1, A_5(1) = 1$. For an Eulerian graph G, it is known that $F(G,3)$ (i.e. $|T_G(0,-2)|$) enumerates the number of Eulerian orientations of G [16]. Eq. (21) may be used to obtain new upper or lower bounds [15] of the number of Eulerian orientations of 4-regular plane graphs.

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