

A New Approach for Examining q -Steiner Systems

Tuvi Etzion*

Department of Computer Science
Technion
Haifa 3200003, Israel
etzion@cs.technion.ac.il

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Abstract

One of the most intriguing problems in q -analogs of designs and codes is the existence question of an infinite family of q -analog of Steiner systems (spreads not included) in general, and the existence question for the q -analog of the Fano plane in particular.

We exhibit a completely new method to attack this problem. In the process we define a new family of designs whose existence is implied by the existence of q -Steiner systems, but could exist even if the related q -Steiner systems do not exist.

The method is based on a possible system obtained by puncturing all the subspaces of the q -Steiner system several times. We define the punctured system as a new type of design and enumerate the number of subspaces of various types that it might have. It will be evident that its existence does not imply the existence of the related q -Steiner system. On the other hand, this type of design demonstrates how close can we get to the related q -Steiner system.

Necessary conditions for the existence of such designs are presented. These necessary conditions will be also necessary conditions for the existence of the related q -Steiner system. Trivial and nontrivial direct constructions and a nontrivial recursive construction for such designs are given. Some of the designs have a symmetric structure, which is uniform in the dimensions of the existing subspaces in the system. Most constructions are based on this uniform structure of the design or its punctured designs.

Mathematics Subject Classifications: 94B25, 05B40, 51E10

1 Introduction

Let \mathbb{F}_q be the finite field with q elements and let \mathbb{F}_q^n be the set of all vectors of length n over \mathbb{F}_q . \mathbb{F}_q^n is a vector space with dimension n over \mathbb{F}_q . For a given integer k , $0 \leq k \leq n$,

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let $\mathcal{G}_q(n, k)$ denote the set of all k -dimensional subspaces (k -subspaces in short) of \mathbb{F}_q^n . $\mathcal{G}_q(n, k)$ is often referred to as a Grassmannian. It is well known that

$$|\mathcal{G}_q(n, k)| = \begin{bmatrix} n \\ k \end{bmatrix}_q \stackrel{\text{def}}{=} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q -binomial coefficient (known also as the *Gaussian coefficient* [33, pp. 325-332]).

A *Grassmannian code* (known better as a *constant dimension code*) \mathbb{C} is a subset of $\mathcal{G}_q(n, k)$. In recent years there has been an increasing interest in Grassmannian codes as a result of their application to error-correction in random network coding which was demonstrated in the seminal work by Koetter and Kschischang [26]. This work has motivated lot of research on coding for Grassmannian codes (see for example [19, 20] and references therein). But the interest in these codes has been also before this application, since Grassmannian codes are the q -analogs of the well studied constant weight codes [6]. The Grassmann scheme is the q -analog of the Johnson scheme, where q -analogs replace concepts of subsets by concepts of subspaces when problems on sets are transferred to problems on subspaces over the finite field \mathbb{F}_q . For example, the size of a set is replaced by the dimension of a subspace, the binomial coefficients are replaced by the Gaussian coefficients, etc. One example of such q -analog problem in coding theory is the nonexistence of nontrivial perfect codes in the Grassmann scheme which was proved in [9, 27]. This problem is the q -analog for the nonexistence problem of perfect codes in the Johnson scheme, which is a well-known open problem [11, 13, 17]. Also, the q -analogs of other various combinatorial objects are well known [33, pp. 325-332]. The work of Koetter and Kschischang [26] has motivated also increasing interest and lot of research work on these related q -analog of designs (see for example [4, 15, 21] and references therein). The most intriguing question is the existence of q -analogs for Steiner systems which is the topic of the research in this paper.

A *Steiner system* $S(t, k, n)$ is a set S of k -subsets (called *blocks*) from an n -set \mathcal{N} such that each t -subset of \mathcal{N} is contained in exactly one block of S . Steiner systems were subject to an extensive research in combinatorial designs [10]. A Steiner system is also equivalent to an optimal constant weight code in the Hamming scheme. It is well-known that if a Steiner system $S(t, k, n)$ exists, then for all $0 \leq i \leq t - 1$, $\frac{\binom{n-i}{t-i}}{\binom{k-i}{t-i}}$ must be integers. It was proved only recently that these necessary conditions for the existence of a Steiner system $S(t, k, n)$ are also sufficient for each t and k such that $0 < t < k$, except for a finite number of values of n [22, 23].

Cameron [7, 8] and Delsarte [12] have extended the notions of block design and Steiner systems to vector spaces. A q -*Steiner system* $\mathbb{S}_q(t, k, n)$ is a set \mathbb{S} of k -subspaces of \mathbb{F}_q^n (called *blocks*) such that each t -subspace of \mathbb{F}_q^n is contained in exactly one block of \mathbb{S} . A q -Steiner system $\mathbb{S}_q(t, k, n)$ is an optimal constant dimension code [20, 21]. Similarly, to Steiner systems, simple necessary divisibility conditions for the existence of a given q -Steiner system were developed [30, 31].

Theorem 1. *If a q -Steiner system $\mathbb{S}_q(t, k, n)$ exists, then for each i , $1 \leq i \leq t - 1$, a q -Steiner system $\mathbb{S}_q(t - i, k - i, n - i)$ exists.*

Corollary 2. *If a q -Steiner system $\mathbb{S}_q(t, k, n)$ exists, then for all $0 \leq i \leq t - 1$,*

$$\frac{\begin{bmatrix} n-i \\ t-i \end{bmatrix}_q}{\begin{bmatrix} k-i \\ t-i \end{bmatrix}_q}$$

must be integers.

While a lot of information is known about the existence of Steiner systems [10, 22, 23], our knowledge about the existence of q -Steiner systems is quite limited. Until recently, the only known q -Steiner systems $\mathbb{S}_q(t, k, n)$ were either trivial or for $t = 1$, where such systems exist if and only if k divides n . These systems are known as *spreads* in finite geometries and they will be used in Section 5. Thomas [32] showed that certain kind of q -Steiner systems $\mathbb{S}_2(2, 3, 7)$ cannot exist. Metsch [29] conjectured that nontrivial q -Steiner systems with $t \geq 2$ do not exist. The concept of q -Steiner systems appeared also in connection of diameter perfect codes in the Grassmann scheme. It was proved in [1] that the only diameter perfect codes in the Grassmann scheme are the q -Steiner systems. Recently, the first q -Steiner system $\mathbb{S}_q(t, k, n)$ with $t \geq 2$ was found. This is a q -Steiner system $\mathbb{S}_2(2, 3, 13)$ which have a large automorphism group [4]. Using q -analog of derived and residual designs it was proved that sometimes the necessary conditions for the existence of a q -Steiner system $\mathbb{S}_q(t, k, n)$ are not sufficient [24]. The first set of parameters (t , k , and n) for which the existence question of q -Steiner systems is not settled is the parameters for the q -analog of the Fano plane, i.e. the q -Steiner systems $\mathbb{S}_q(2, 3, 7)$, which will be called also in this paper the *q -Fano plane*. There was a lot of effort to find whether the q -Fano plane, especially for $q = 2$, exists or does not exist, e.g. [5, 21, 32] All these attempts didn't provide any answer for the existence question. It was proved recently in [5] that if such system exists for $q = 2$, then its automorphism group has a small order.

In this paper we present a completely new approach to examine the existence of q -Steiner systems. This approach is based on the structure obtained by puncturing some coordinates from all the subspaces of the possible q -Steiner system. This is equivalent to say that the projection of the other coordinates is considered for all the subspaces of the system. This idea was suggested first in [14] and this paper completes and proves all the ideas mentioned in [14]. The sizes of punctured subspaces and the numbers involved are sometimes the same as those in [25], where intersection numbers of combinatorial designs were considered. But, the objects considered in these two papers are different and the results are different. We consider (and define) this structure, obtained by puncturing all the subspaces of the system, as a new type of design, which exists if the related q -Steiner system exists, but could exist even if the related q -Steiner system does not exist. If this design does not exist, then the related q -Steiner system does not exist. To highlight our main contributions in this paper are:

1. A definition of a new method to examine the existence of a q -Steiner system.

2. A definition for a new type of designs which are close to q -Steiner systems and their construction might be a first step to find the related q -Steiner system. Some constructions for these designs are given.
3. A method which hopefully will improve our understanding of the structure of the q -Fano plane for $q \geq 2$, and maybe will help to find such a structure or to prove its nonexistence. Indeed, the method in this paper led to a breakthrough, in this area, presented in [16], i.e. a construction for a punctured q -Fano plane (or so called residual q -Fano plane) for all prime powers $q \geq 2$.

The rest of this paper is organized as follows. In Section 2, the definition of punctured q -Steiner systems and other related definitions, are presented. We prove some properties of punctured systems, define the new type of design, and examine some of its properties. An inverse operation for puncturing and related operations are also presented and the number of subspaces which are generated by the inverse operation are computed. In Section 3, a system of equations, which form the necessary conditions for the existence of the related punctured q -Steiner system, is presented. These systems of equations are obtained by precise enumeration of covering t -subspaces by k -subspaces in the the q -Steiner system as reflected by the punctured system. In Section 4, a sequence of examples for punctured q -Steiner systems $\mathbb{S}_q(k-1, k, n)$, i.e. a sequence of examples for the new defined design, is presented. In Section 5, a recursive construction for punctured q -Steiner systems $\mathbb{S}_q(2, 3, n)$ is presented. One of the important ingredients, for this construction, is a large set of spreads. Conclusions are given in Section 6.

2 Punctured q -Steiner Systems

Given an $n \times m$ matrix A , the *punctured* matrix A' is an $n \times (m-1)$ matrix obtained from A by deleting one of the columns from A . Codes and punctured codes in the Hamming space are well established in coding theory [28, pp. 27-32]. A q -analog of punctured codes for subspace codes (codes whose codewords are subspaces such as the Grassmannian codes), was defined in [18], but this is not the puncturing considered in this paper.

A subspace $X \in \mathcal{G}_q(n, k)$, i.e. a k -subspace of \mathbb{F}_q^n , consists of q^k vectors of length n with elements taken from \mathbb{F}_q . The punctured subspace X' by the i th coordinate is defined as the subspace obtained from X by deleting coordinate i in all the vectors of X . The result of this puncturing is a new subspace of \mathbb{F}_q^{n-1} . If X does not contain the unity vector with an *one* in the i th coordinate, e_i , then X' is a subspace in $\mathcal{G}_q(n-1, k)$. If X contains the unity vector with a *one* in the i th coordinate, then X' is a subspace in $\mathcal{G}_q(n-1, k-1)$. Assume that we are given a set \mathbb{S} of subspaces from $\mathcal{G}_q(n, k)$. The *punctured set* \mathbb{S}' is defined as $\mathbb{S}' = \{X' : X \in \mathbb{S}\}$, where all subspaces are punctured in the same coordinate, and it can contain subspaces only from $\mathcal{G}_q(n-1, k)$ or from $\mathcal{G}_q(n-1, k-1)$. The set \mathbb{S}' is regarded as multi-set and hence $|\mathbb{S}'| = |\mathbb{S}|$ (since two distinct k -subspaces of \mathbb{F}_q^n can be punctured into the same k -subspaces of \mathbb{F}_q^{n-1}). When the punctured coordinate is not mentioned it will be assumed that the last coordinate was punctured. A subspace can be punctured several times. A k -subspace X , of \mathbb{F}_q^n , is punctured p times (*p-punctured*)

to a p -punctured subspace Y of \mathbb{F}_q^{n-p} . The subspace Y can be an s -subspace for any s such that $\max\{0, k-p\} \leq s \leq \min\{k, n-p\}$. Similarly we define a p -punctured set. We summarize this brief introduction on punctured subspaces with the main observation.

Lemma 3. *A k -subspace of \mathbb{F}_q^n , $k > 0$, is punctured either into a k -subspace or into a $(k-1)$ -subspace of \mathbb{F}_q^{n-1} . A k -subspace of \mathbb{F}_q^n is p -punctured into an s -subspace such that $\max\{0, k-p\} \leq s \leq \min\{k, n-p\}$.*

If \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$, we would like to know if the p -punctured system \mathbb{S}' has some interesting properties, i.e. \mathbb{S}' has some uniqueness properties related to the punctured k -subspaces and the punctured (contained) t -subspaces. The motivation is to define a new set of designs which must exist if the related q -Steiner systems $\mathbb{S}_q(t, k, n)$ exist. But, these new designs can exist even if the related q -Steiner systems do not exist. If the nonexistence of such designs can be proved, then the related q -Steiner systems won't exist too. On the other hand, the existence of such a design might lead to a construction for the related q -Steiner system.

For puncturing there is an inverse operation called *extension*. A related operation for our discussion, which is completely of a different nature from extension, is the *expansion*. These two operations are defined next.

A t -subspace X of \mathbb{F}_q^m is *extended* to a t' -subspace Y of $\mathbb{F}_q^{m'}$, where $t' \geq t$, $m' > m$, and $m' - m \geq t' - t$, if X is the subspace obtained from Y by puncturing Y $m' - m$ times. Note, that the *extension* of a subspace is not always unique, as we will prove in the sequel (see lemma 4), but there is always a unique outcome for puncturing. In other words, a p -punctured t -subspace X of \mathbb{F}_q^m can be obtained from a few different t' -subspaces of \mathbb{F}_q^{m+p} . The new columns are added as the last columns of extended subspace. But, similarly to puncturing, they can added theoretically between any set of columns. To make the paper consistent, we will make the extensions only to the end of the columns of the related subspaces, unless otherwise is specifically stated.

To prove our claims we will need some general way to represent subspaces, in such a way that the representation of the punctured subspace X' will be derived directly from the representation of the subspace X . For this purpose two different representations will be used for a k -subspace X of \mathbb{F}_q^m . The first representation is by an $(q^k - 1) \times m$ matrix which contains the $q^k - 1$ nonzero vectors of X . Each nonzero vector of X is a row in this matrix. The second representation is by a $k \times m$ matrix which is the generator matrix for X in reduced row echelon form. A $k \times n$ matrix with rank k is in *reduced row echelon form* if the following conditions are satisfied.

- The leading coefficient of a row is always to the right of the leading coefficient of the previous row.
- All leading coefficients are *ones*.
- Every leading coefficient is the only nonzero entry in its column.

The next definition of virtual subspace is essential in understanding our exposition. An r -subspace Y of \mathbb{F}_q^m is called a *virtual k -subspace* of \mathbb{F}_q^m , $r \leq k$, if Y was punctured

from a k -subspace X of \mathbb{F}_q^n , $n > m$ and it is represented by a $(q^k - 1) \times m$ matrix which represents the actual outcome when Y was punctured. In other words the representation of Y is obtained by the deleting the last $n - m$ columns from the $(q^k - 1) \times n$ matrix which represents X . It is important to understand that Y has exactly one representation as a virtual k -subspace, no matter from which subspace it was punctured. Using this representation, Y has $\binom{k}{t}_q$ t -subspaces for each $0 \leq t \leq k$, some of them are identical and some of them are virtual (see Example 7). This will be used later in our enumerations. Note, that with the virtual k -subspace of the punctured subspace, it is easier to see how the t -subspaces were punctured. This is easily seen in Example 7. Finally, note that a k -subspace of \mathbb{F}_q^m is always also a virtual k -subspace of \mathbb{F}_q^m , where both have the same matrix representation.

An r -subspace X of \mathbb{F}_q^m is *expanded* to a (virtual) k -subspace Y of \mathbb{F}_q^m , $r \leq k$, if X can be obtained by puncturing a k -subspace Z , and Y is the representation of X by a $(q^k - 1) \times m$ matrix after the puncturing. The virtual k -subspace Y is also called the *k-expansion* of X .

Lemma 4 (from a t -subspace to a t -subspace, extension of one column). *If X is a t -subspace of \mathbb{F}_q^m , then it can be extended in exactly q^t distinct ways to a t -subspaces of \mathbb{F}_q^{m+1} .*

Proof. Any one of the q^t distinct linear combinations of the columns of X can be appended as the new $(m + 1)$ th column and each such linear combination, appended as the $(m + 1)$ th column, yields a different t -subspace of \mathbb{F}_q^{m+1} . \square

Example 5. Let X be the following 2-subspace of \mathbb{F}_2^4 represented by the 3×4 matrix,

$$X = \begin{pmatrix} 0100 \\ 0010 \\ 0110 \end{pmatrix},$$

It can be extended in exactly 4 distinct ways to 2-subspaces of \mathbb{F}_2^5 , represented by the 3×5 matrices,

$$\begin{pmatrix} 01000 & 01001 & 01000 & 01001 \\ 00100 & 00100 & 00101 & 00101 \\ 01100 & 01101 & 01101 & 01100 \end{pmatrix}.$$

Lemma 6 (from t -subspace to $(t + 1)$ -subspace, extension of one column). *If X is a t -subspace of \mathbb{F}_q^m , then it can be extended in exactly one way to a $(t + 1)$ -subspace of \mathbb{F}_q^{m+1} .*

Proof. Any extension which increase the dimension by one is equivalent to first adding a column of zeroes to the existing t -subspace X to form the t -subspace \hat{X} of \mathbb{F}_q^{m+1} . Then the only extension to a $(t + 1)$ -subspace is $\langle \hat{X}, e_{m+1} \rangle$, where $\langle Z \rangle$ denotes the linear span of Z . \square

Example 7. Let X be the following 2-subspace of \mathbb{F}_2^4 represented by the 3×4 matrix,

$$X = \begin{pmatrix} 0100 \\ 0010 \\ 0110 \end{pmatrix},$$

and let

$$Y = \begin{pmatrix} 01000 \\ 00100 \\ 01100 \end{pmatrix},$$

be one of its extensions to a 2-subspace of \mathbb{F}_2^5 represented by the 3×5 matrix. X can be extended in a unique way to a 3-subspace of \mathbb{F}_2^5 , and Y has a unique representation as a virtual 3-subspace (up to permutations of rows). They are represented by the two 7×5 matrices, respectively,

$$\begin{pmatrix} 01000 & 01000 \\ 00100 & 00100 \\ 01100 & 01100 \\ 01001 & 01000 \\ 00101 & 00100 \\ 01101 & 01100 \\ 00001 & 00000 \end{pmatrix}.$$

The extended 3-subspace contains the following seven 2-subspaces

$$\begin{pmatrix} 01000 & 01000 & 00100 & 01100 & 01000 & 00100 & 01100 \\ 00100 & 00101 & 01001 & 01001 & 01001 & 00101 & 01101 \\ 01100 & 01101 & 01101 & 00101 & 00001 & 00001 & 00001 \end{pmatrix},$$

while the virtual 3-subspace contains the following related seven 2-subspaces, from which three are virtual (note that the distinction between the two sets of seven subspaces is the last column),

$$\begin{pmatrix} 01000 & 01000 & 00100 & 01100 & 01000 & 00100 & 01100 \\ 00100 & 00100 & 01000 & 01000 & 01000 & 00100 & 01100 \\ 01100 & 01100 & 01100 & 00100 & 00000 & 00000 & 00000 \end{pmatrix}.$$

Finally, in this section we consider the outcome of puncturing a q -Steiner systems and some properties of the related puncturing. For this purpose we need to use the following well-known equation [28, p. 444].

Lemma 8. *If $1 \leq k \leq n - 1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$, where $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$.*

Theorem 9. *If \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$, then the punctured system \mathbb{S}' contain exactly $\frac{\begin{bmatrix} n-1 \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q}$ distinct $(k-1)$ -subspaces which form a q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$, denoted by $\tilde{\mathbb{S}}$. Each t -subspace of \mathbb{F}_q^{n-1} which is contained in a $(k-1)$ -subspace of $\tilde{\mathbb{S}}$ is not contained in any of the k -subspaces of \mathbb{S}' . Each t -subspace of \mathbb{F}_q^{n-1} which is not contained in a $(k-1)$ -subspace of $\tilde{\mathbb{S}}$, appears exactly q^t times in the k -subspaces of \mathbb{S}' .*

Proof. By Lemma 3, \mathbb{S}' can contain only $(k-1)$ -subspaces and k -subspaces. A k -subspace which contains the unity vector with a *one* in the last coordinate is punctured into a $(k-1)$ -subspace, while other k -subspaces are punctured into k -subspaces.

Let X be a $(t-1)$ -subspace of \mathbb{F}_q^{n-1} . By Lemma 6, there is a unique way to extend it to a t -subspace Y of \mathbb{F}_q^n . Therefore, since \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$, it follows that Y is contained in exactly one k -subspace Z of \mathbb{S} . Clearly, Z is a k -subspace in \mathbb{F}_q^n which contains the unity vector with a *one* in the last coordinate. Hence, Z is punctured into a $(k-1)$ -subspace Z' which is the only $(k-1)$ -subspace of \mathbb{S}' containing X . Therefore, all the k -subspaces of \mathbb{S} which contain the unity vector with a *one* in the last coordinate are punctured into a q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$.

Let Z be a k -subspace of \mathbb{F}_q^n which contains $\langle e_n \rangle$. The virtual k -subspace, obtained from the punctured $(k-1)$ -subspace Z' , contains $\begin{bmatrix} k \\ t \end{bmatrix}_q$ t -subspaces, some of them are identical and some of them are virtual t -subspaces (see Example 7). Since $\langle e_n \rangle \subset Z$ it follows that Z has $\begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q$ t -subspaces which contains $\langle e_n \rangle$ and hence $\begin{bmatrix} k \\ t \end{bmatrix}_q - \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q$ t -subspaces which do not contain $\langle e_n \rangle$. These are the only t -subspaces contained in Z . Since $\langle e_n \rangle \subset Z$ it follows that Z' is a $(k-1)$ -subspace and hence it has $\begin{bmatrix} k-1 \\ t \end{bmatrix}_q$ distinct t -subspaces. The $\begin{bmatrix} k \\ t \end{bmatrix}_q - \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q$ t -subspaces of Z which do not contain $\langle e_n \rangle$ are punctured, and each $(t-1)$ -subspace obtained in this way is obtained the same amount of times in the set of punctured $(t-1)$ -subspaces. Therefore, each such $(t-1)$ -subspace is contained $\frac{\begin{bmatrix} k \\ t \end{bmatrix}_q - \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ t \end{bmatrix}_q}$ times in the punctured t -subspaces of \mathbb{F}_q^{n-1} obtained from \mathbb{S} . By Lemma 8 we have $\frac{\begin{bmatrix} k \\ t \end{bmatrix}_q - \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ t \end{bmatrix}_q} = q^t$ (by Lemma 4 these are exactly all their appearances and hence no one could have appeared more times).

By Lemma 4, each t -subspace of \mathbb{F}_q^{n-1} can be extended in q^t distinct ways to a t -subspace of \mathbb{F}_q^n . This implies, that each t -subspace which is not contained in $\tilde{\mathbb{S}}$, appears exactly q^t times in the other k -subspaces of \mathbb{S}' . Simple counting shows that we have covered all the subspaces of the punctured system, which completes the proof of the theorem. \square

Corollary 10. *If \mathbb{S} is a q -Steiner system $\mathbb{S}_q(k-1, k, n)$, then the punctured system \mathbb{S}' has a set $\tilde{\mathbb{S}}$ with $\frac{\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ k-2 \end{bmatrix}_q}$ different $(k-1)$ -subspaces which form a q -Steiner system $\mathbb{S}_q(k-2, k-1, n-1)$. Each other $(k-1)$ -subspace which is not contained in $\tilde{\mathbb{S}}$, appears exactly q^{k-1} times in the k -subspaces of \mathbb{S}' .*

We note, that the q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$ of Theorem 9 is the same as the one constructed in [30]. In [30], two subspaces U_1 and U_{n-1} are chosen, where U_i has dimension i and $U_{n-1} + U_1 = \mathbb{F}_q^n$. If \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$, then the subset $\{X \cap U_{n-1} : X \in \mathbb{S}, U_1 \subset X\}$ is q -Steiner system $\mathbb{S}_q(t-1, k-1, n-1)$. Our punctured system is obtained when $e_n \in U_1$ and U_{n-1} is the $(n-1)$ -subspace of \mathbb{F}_q^n which contains all the vectors whose last coordinate is a *zero*. The extra factor of Theorem 9 are the k -subspaces which do not belong to the q -Steiner system. We note that puncturing

is equivalent to the projection $X \rightarrow (X + U)/U$, where U is a fixed 1-subspace. If H is a hyperplane in \mathbb{F}_q^n and U is a 1-subspace not contained in H , then puncturing of a subspace X is equivalent to $H \cap \langle X, U \rangle$.

The first goal of this paper is to define a new type of design $\mathbb{S}_q(t, k, n; m)$ which contains the possible subspaces punctured from a q -Steiner system $\mathbb{S}_q(t, k, n)$. This is done in the following definition.

Definition 11. A p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $m = n - p$, is a multi-set \mathbb{S} of subspaces of \mathbb{F}_q^m , satisfying the following two requirements.

1. The number of subspaces in \mathbb{S} is the same as the number of subspaces in a q -Steiner system $\mathbb{S}_q(t, k, n)$.
2. Let $\tilde{\mathbb{S}}$ be a system which contains the virtual k -subspaces of the subspaces in \mathbb{S} (\mathbb{S} and $\tilde{\mathbb{S}}$ have the same size). Let \mathbb{T} be the set of all t -subspaces of \mathbb{F}_q^n and \mathbb{T}' be the set of all p -punctured t -subspaces of \mathbb{F}_q^m . For each subspace $X \in \mathbb{T}'$ let

$$\lambda(X) = |\{Y : Y \in \mathbb{T}, X \text{ is a } p\text{-punctured } t\text{-subspace of } Y\}| .$$

It is required that for each $X \in \mathbb{T}'$, X will be appear $\lambda(X)$ times as a virtual t -subspace in the virtual k -subspaces of $\tilde{\mathbb{S}}$.

Example 12. Let \mathbb{S} be a system which consists of 336 1-subspaces and 45 0-subspaces of \mathbb{F}_2^1 . There are $\begin{bmatrix} 6 \\ 2 \end{bmatrix}_2 = 651$ 2-subspaces of \mathbb{F}_2^7 whose first column is the all-zero column. Each extension of an 1-subspace of \mathbb{F}_2^1 will contribute one 2-subspaces of \mathbb{F}_2^7 whose first column is all-zero, while each extension of a 0-subspace of \mathbb{F}_2^1 will contribute seven 2-subspaces of \mathbb{F}_2^7 whose first column is all-zero. Hence, the extension produces $336 + 45 \cdot 7 = 651$ such subspaces as required. The same goes for the other 2-subspaces and hence \mathbb{S} is a 6-punctured q -Steiner system $\mathbb{S}_2(2, 3, 7; 1)$.

The following lemma can be easily verified.

Lemma 13. *If there exists a p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $1 < m \leq n$, then there exists a $(p + 1)$ -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m - 1)$.*

The following theorem is given for a punctured q -Steiner system $\mathbb{S}_q(t, k, n; n - 1)$. It can be generalized to other p -punctured q -Steiner systems, $p > 1$. For simplicity and since the case $p = 1$ is the most informative we prove only this case.

Theorem 14. *If \mathbb{S}' , a punctured q -Steiner system $\mathbb{S}_q(t, k, n; n - 1)$, $1 < t < k < n$, was obtained by puncturing a q -Steiner system $\mathbb{S}_q(t, k, n)$, then all subspaces of \mathbb{S}' are distinct.*

Proof. Let \mathbb{S} be a q -Steiner system $\mathbb{S}_q(t, k, n)$. By Theorem 9, all the $(k - 1)$ -subspaces of \mathbb{S}' , the punctured q -Steiner systems $\mathbb{S}_q(t, k, n; n - 1)$, are distinct since they form a q -Steiner system $\mathbb{S}_q(t - 1, k - 1, n - 1)$. Hence, we only have to prove that there are no two equal k -subspaces in \mathbb{S}' .

Let $X, Y \in \mathbb{S}$, H an hyperplane in \mathbb{F}_q^n and U a 1-subspace not contained in X , Y , and H . Clearly,

$$\dim((H \cap \langle X, U \rangle) \cap (H \cap \langle Y, U \rangle)) = \dim(\langle X, U \rangle \cap \langle Y, U \rangle) - 1 \leq \dim(X \cap Y) + 1 \leq t < k,$$

which completes the proof. □

3 System of Equations

Corollary 2 yields a set of necessary conditions for the existence of a q -Steiner system $\mathbb{S}_q(t, k, n)$. Similar necessary conditions can be derived for any p -punctured q -Steiner system. Some of these conditions yield new interesting equations which must be satisfied. In this section we will derive these new necessary conditions.

Let $\tilde{\mathbb{S}}$ be an $(n - m)$ -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$ and let $p = n - m$, i.e. $\mathbb{S}_q(t, k, n; m)$ is a p -punctured q -Steiner system. We start with two simple lemmas which are implied by our previous discussion on punctured subspaces. The first lemma is an immediate consequence of Lemma 3.

Lemma 15 (dimension of subspaces to be covered). *Suppose \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$. Let \mathbb{T} be the set of t -subspaces which are covered by the blocks of \mathbb{S} . Then in the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $p = n - m$, derived from \mathbb{S} , each element of \mathbb{T} corresponds to a p -punctured s -subspace, where $\max\{0, t - p\} \leq s \leq \min\{t, m\}$.*

Lemma 16 (dimension of subspaces which cover the p -punctured s -subspaces). *Suppose \mathbb{S} is a q -Steiner system $\mathbb{S}_q(t, k, n)$. Let \mathbb{T} be the t -subspaces which are covered by the blocks of \mathbb{S} . Then in the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $p = n - m$, derived from \mathbb{S} , each element of \mathbb{T} corresponds to a p -punctured s -subspace, covered by a p -punctured r -subspace, where $\max\{k - p, s\} \leq r \leq \min\{k - t + s, m\}$.*

Proof. The lower bound is a consequence of Lemma 3 and the fact that an s -subspace cannot be covered by a subspace of a smaller dimension. Since the subspaces of $\mathbb{S}_q(t, k, n; m)$ are subspaces of \mathbb{F}_q^m , it follows that $r \leq m$. Finally, if a t -subspace X was punctured to an s -subspace then the k -subspace Y which covers X must also be reduced by at least $t - s$ times in its dimension and hence $r \leq k - t + s$. \square

We are now in a position to describe a set of equations, related to the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, which must be satisfied if the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$ exists. Each s -subspace X of \mathbb{F}_q^m , $\max\{0, t - p\} \leq s \leq \min\{t, m\}$, yields one equation related to the way it is covered by $\mathbb{S}_q(t, k, n; m)$. Each r -subspace Y of \mathbb{F}_q^m , $\max\{k - p, 0\} \leq r \leq \min\{k, m\}$, yields one nonnegative integer variable, a_Y , which is the number of appearances of Y in the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$. In the equation for the s -subspace X we have a linear combination of the variables for the r -subspaces of \mathbb{F}_q^m which contain X .

Example 17. Assume that we want to examine the 5-punctured q -Steiner system $\mathbb{S}_2(2, 3, 7; 2)$. Clearly, by Lemma 15, the 2-subspaces of \mathbb{F}_2^7 were punctured to s -subspaces of \mathbb{F}_2^2 , where $0 \leq s \leq 2$. There is exactly one 0-subspace, three 1-subspaces, and one 2-subspaces, of \mathbb{F}_2^2 , represented as virtual 2-subspaces by the following five 3×2 matrices.

$$\begin{array}{ccccc} 00 & 10 & 01 & 11 & 01 \\ 00, & 10, & 01, & 11, & 10 \\ 00 & 00 & 00 & 00 & 11 \end{array} .$$

Clearly, by Lemma 16 these s -subspaces of \mathbb{F}_2^7 are covered by r -subspaces of \mathbb{F}_2^2 , where $0 \leq r \leq 2$. There is exactly one 0-subspace, three 1-subspaces, and one 2-subspaces, of \mathbb{F}_2^2 , represented as virtual 3-subspaces by the following five 7×2 matrices.

$$\begin{array}{ccccc}
 & 00 & 10 & 01 & 11 & 01 \\
 & 00 & 10 & 01 & 11 & 01 \\
 & 00 & 10 & 01 & 11 & 10 \\
 X = & 00 & 10 & 01 & 11 & 10 \\
 & 00 & 00 & 00 & 00 & 11 \\
 & 00 & 00 & 00 & 00 & 11 \\
 & 00 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 & 00
 \end{array}
 , \quad
 \begin{array}{ccccc}
 & 10 & 01 & 11 & 10 \\
 & 10 & 01 & 11 & 10 \\
 & 10 & 01 & 11 & 10 \\
 Y = & 10 & 01 & 11 & 10 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00
 \end{array}
 , \quad
 \begin{array}{ccccc}
 & 01 & 11 & 10 & 10 \\
 & 01 & 11 & 10 & 10 \\
 & 01 & 11 & 10 & 10 \\
 Z = & 01 & 11 & 10 & 10 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00
 \end{array}
 , \quad
 \begin{array}{ccccc}
 & 11 & 10 & 10 & 10 \\
 & 11 & 10 & 10 & 10 \\
 & 11 & 10 & 10 & 10 \\
 U = & 11 & 10 & 10 & 10 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00
 \end{array}
 , \quad
 \begin{array}{ccccc}
 & 01 & 10 & 10 & 10 \\
 & 01 & 10 & 10 & 10 \\
 & 01 & 10 & 10 & 10 \\
 V = & 01 & 10 & 10 & 10 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00 \\
 & 00 & 00 & 00 & 00
 \end{array}
 .$$

Therefore, we have 5 variables $a_X, a_Y, a_Z, a_U,$ and a_V . The system of equations consists of the following five equations:

$$\begin{aligned}
 155 &= 7 \cdot a_X + a_Y + a_Z + a_U \\
 496 &= 6 \cdot a_Y + a_V \\
 496 &= 6 \cdot a_Z + a_V \\
 496 &= 6 \cdot a_U + a_V \\
 1024 &= 4 \cdot a_V
 \end{aligned}
 .$$

The first equation is constructed as follows: there are $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_2 = 155$ 2-subspaces whose first two columns are *zeroes*. For all the seven 2-subspaces resulting from the virtual 3-subspace X the first two columns are *zeroes*. Only for one such 2-subspaces resulting from $Y, Z,$ or U the first two columns are *zeroes*. This explains the first equation. The other four equations are constructed in a similar way. There is a unique solution for this set of equations, $a_X = 5, a_Y = a_Z = a_U = 40,$ and $a_V = 256,$ and hence the 5-punctured q -Steiner system $\mathbb{S}_2(2, 3, 7; 2)$ exists.

The variables which appear in each equation and their coefficients in the equation should be computed in advance as will be done next. First we have to compute the number of t -subspaces in \mathbb{F}_q^n which are formed by extending an s -subspace X of \mathbb{F}_q^m . Let $N_{(s,m),(t,n)}$ be the number of distinct t -subspaces in \mathbb{F}_q^n which are formed by extending a given s -subspace X of \mathbb{F}_q^m .

Example 18. Let X be the following 2-subspace of \mathbb{F}_2^5 represented by the 3×5 matrix,

$$\begin{array}{ccccc}
 & 01001 & & & \\
 X = & 00101 & & & \\
 & 01100 & & &
 \end{array}
 .$$

It can be extended to the following $N_{(2,5),(3,7)} = 12$ 3-subspaces of \mathbb{F}_2^7 represented by the

7×7 matrices,

0100100	0100110	0100110	0100100	0100100	0100101
0010100	0010110	0010100	0010110	0010100	0010100
0110000	0110000	0110010	0110010	0110000	0110001
0100101 ,	0100111 ,	0100111 ,	0100101 ,	0100110 ,	0100111 ,
0010101	0010111	0010101	0010111	0010110	0010110
0110001	0110001	0110011	0110011	0110010	0110011
0000001	0000001	0000001	0000001	0000010	0000010
0100100	0100101	0100100	0100101	0100100	0100101
0010101	0010101	0010100	0010100	0010101	0010101
0110001	0110000	0110000	0110001	0110001	0110000
0100110 ,	0100111 ,	0100111 ,	0100110 ,	0100111 ,	0100110 .
0010111	0010111	0010111	0010111	0010110	0010110
0110011	0110010	0110011	0110010	0110010	0110011
0000010	0000010	0000011	0000011	0000011	0000011

Some of the claims in Lemmas 19, 21, and 22, which follow, can be proved by using coordinate-free notation. But, we will continue with our coordinate notation since it is simpler at this point of our exposition.

Lemma 19. *If $0 < m < n$ and $0 \leq s \leq t$, then $N_{(s,m),(t,n)} = q^{s(n-m-t+s)} \begin{bmatrix} n-m \\ t-s \end{bmatrix}_q$.*

Proof. The s -subspace X of \mathbb{F}_q^m is represented by an $s \times m$ matrix G_1 in reduced row echelon form. A t -subspace Y formed by extending X to a t -subspace of \mathbb{F}_q^n is represented by a $t \times n$ generator matrix G in reduced row echelon form. The upper left $s \times m$ matrix of G is the generator matrix G_1 of X and hence G has the following structure.

$$\begin{bmatrix} G_1 & B \\ \mathbf{0} & G_2 \end{bmatrix}.$$

The new $n - m$ columns (in G relatively to G_1), restricted to the last $t - s$ rows, forms a generator matrix for a $(t - s)$ -subspace of \mathbb{F}_q^{n-m} . This $(t - s) \times (n - m)$ generator matrix G_2 is in reduced row echelon form, where the columns with leading *ones* are the ones in which the dimension is increased during the extension (see Lemma 6). This generator matrix can be chosen in $\begin{bmatrix} n-m \\ t-s \end{bmatrix}_q$ distinct ways since it forms a $(t - s)$ -subspace of \mathbb{F}_q^{n-m} . In the other new $n - m - t + s$ columns of G (columns with no leading *ones*) the dimension is not increased compared to the original s -subspace X and hence they can be chosen (after the columns with the leading *ones* of G_2 are fixed; see Lemma 4) in $q^{s(n-m-t+s)}$ distinct ways. The reason is that B has s rows and $n - m$ columns from which the entries of $n - m - t + s$ columns can be chosen arbitrarily from \mathbb{F}_q . It leads to a total of $q^{s(n-m-t+s)} \begin{bmatrix} n-m \\ t-s \end{bmatrix}_q$ distinct ways to form this extension. Note, that the columns of G_2 do not (and need not) contribute to these extensions. Each $(t - s)$ -subspace of \mathbb{F}_q^{n-m} is combined with the related s -subspace in \mathbb{F}_q^m to form a t -subspace of \mathbb{F}_q^n . \square

For a given s -subspace X of \mathbb{F}_q^m , $N_{(s,m),(t,n)}$ should be equal to the number of r -subspaces in the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $\max\{k-p, s\} \leq r \leq \min\{k-t+s, m\}$, which contains X . Note, that if $r < k$, then there are r -subspaces in $\mathbb{S}_q(t, k, n; m)$ which contain X more than once, since we should look on the k -expansion of the r -subspace. Clearly, for this purpose we also have to consider the t -expansion of the related s -subspace X . Let $C_{(s,t),(r,k)}$ be the number of copies of the t -expansion \tilde{X} obtained from the s -subspace X in \mathbb{F}_q^m (note that X is t -expanded in a unique way), which are contained in the k -expansion \tilde{Y} of an r -subspace Y in \mathbb{F}_q^m , such that X is a subspace of Y .

Example 20. Let X be the following 2-subspace of \mathbb{F}_2^4 represented by the 3×4 matrix,

$$X = \begin{pmatrix} 0100 \\ 0010 \\ 0110 \end{pmatrix} .$$

If $Y = X$, then the 3-expansion \tilde{Y} of Y is represented by 7×4 matrix, and one of its extensions \hat{Y} for a 3-subspace is represented by a 7×7 matrix as follows:

$$\tilde{Y} = \begin{pmatrix} 0100 \\ 0010 \\ 0110 \\ 0100 \\ 0010 \\ 0110 \\ 0000 \end{pmatrix} , \quad \hat{Y} = \begin{pmatrix} 0100000 \\ 0010000 \\ 0110000 \\ 0100101 \\ 0010101 \\ 0110101 \\ 0000101 \end{pmatrix} .$$

\hat{Y} contains $C_{(2,2),(2,3)} = 4$ 2-subspaces extended from X represented by the 3×7 matrices,

$$\begin{pmatrix} 0100000 \\ 0010000 \\ 0110000 \end{pmatrix} , \quad \begin{pmatrix} 0100000 \\ 0010101 \\ 0110101 \end{pmatrix} , \quad \begin{pmatrix} 0100101 \\ 0010000 \\ 0110101 \end{pmatrix} , \quad \begin{pmatrix} 0100101 \\ 0010101 \\ 0110000 \end{pmatrix} .$$

Lemma 21. If $0 \leq s \leq t < k$ and $s \leq r \leq k - t + s$, then $C_{(s,t),(r,k)} = \begin{bmatrix} k-r \\ t-s \end{bmatrix}_q q^{s(k-r-t+s)}$.

Proof. An r -subspace Y of \mathbb{F}_q^m can be represented by a $(q^r - 1) \times m$ matrix whose rows are the nonzero vectors of Y . This r -subspace Y is k -expanded in \mathbb{F}_q^m by writing in a $(q^k - 1) \times m$ matrix vertically q^{k-r} copies of Y and after them $(q^{k-r} - 1) \times m$ all-zero matrix Z . This forms a $(q^k - 1) \times m$ matrix which represents the k -expansion of Y . If Y was p -punctured from a k -subspace W of \mathbb{F}_q^n , then W is formed from the k -expansion of Y by concatenating to Z a $(q^{k-r} - 1) \times (n - m)$ matrix \tilde{Z} which represent a $(k - r)$ -subspace of \mathbb{F}_q^{n-m} . The k -subspace of \mathbb{F}_q^n is a direct sum of this $(k - r)$ -subspace in \mathbb{F}_q^n with an extension of Y to an r -subspace in \mathbb{F}_q^n ,

Similarly, the s -subspace X , which is a subspace of Y , is extended and expanded to a $(k - r + s)$ -subspace by writing in a $(q^k - 1) \times m$ matrix vertically q^{k-r} copies of X and after them $(q^{k-r} - 1) \times m$ all-zero matrix Z . Each t -subspace in \mathbb{F}_q^n which is

extended from X , in the q -Steiner system $\mathbb{S}_q(t, k, n)$, is constructed by first choosing a $(t - s)$ -subspace from \tilde{Z} , which can be done in $\begin{bmatrix} k-r \\ t-s \end{bmatrix}_q$ different ways. The $(t - s)$ -subspace is completed to a t -subspace by performing direct sum with the extension of X . The s -subspace X can be chosen in a few distinct ways from the k -expansion of Y . Each vector from a given basis of X can be chosen in q^{k-r} distinct ways (since q^{k-r} copies of Y were written). But, since each vector of X appears q^{t-s} times in the t -expansion of X , it follows that each choice of X is chosen in $(\frac{q^{k-r}}{q^{t-s}})^s$ distinct ways. This implies that $C_{(s,t),(r,k)} = \begin{bmatrix} k-r \\ t-s \end{bmatrix}_q (\frac{q^{k-r}}{q^{t-s}})^s = \begin{bmatrix} k-r \\ t-s \end{bmatrix}_q q^{s(k-r-t+s)}$ \square

Now, for a given s -subspace X of \mathbb{F}_q^m , $N_{(s,m),(t,n)}$ should be equal to the sum over all r -subspaces which contain X , where $C_{(s,t),(r,k)}$, for a given r -subspace Y , is multiplied by a_Y (see the definitions after Lemma 16), the number of appearance of Y in the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$. For a given r , $\max\{k - p, s\} \leq r \leq \min\{k - t + s, m\}$, let $D_{s,r,m}$ be the number of r -subspaces which contain a given s -subspace in \mathbb{F}_q^m . In other words, $D_{s,r,m}$ is the number of variables for r -subspaces which appear in the equation for any given s -subspace.

Lemma 22. *If $0 \leq s \leq r \leq m$, then $D_{s,r,m} = \begin{bmatrix} m-s \\ r-s \end{bmatrix}_q$.*

Proof. Let X be an s -subspace of \mathbb{F}_q^m . We enumerate the number of distinct r -subspaces which contain X in \mathbb{F}_q^m , by adding linearly independent vectors one by one to X . The first vector can be chosen in $q^m - q^s$ distinct ways, the second in $q^m - q^{s+1}$ distinct ways and the last in $q^m - q^{r-1}$ different ways, for a total of $\prod_{i=r-1}^s (q^m - q^i)$ different ways. Similarly, a given r -subspace Y which is formed in this way can be constructed in $\prod_{i=r-1}^s (q^r - q^i)$ distinct ways (the first vector can be chosen in $q^r - q^s$ distinct ways and so on). Hence, the total number of distinct r -subspaces formed in this way is

$$\frac{\prod_{i=r-1}^s (q^m - q^i)}{\prod_{i=r-1}^s (q^r - q^i)} = \prod_{i=r-1}^s \frac{q^m - q^i}{q^r - q^i} = \prod_{i=r-1}^s \frac{q^{m-i} - 1}{q^{r-i} - 1} = \begin{bmatrix} m-s \\ m-r \end{bmatrix}_q = \begin{bmatrix} m-s \\ r-s \end{bmatrix}_q \quad \square$$

So far, we have described the computation of the components in the equations that should be satisfied if a p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$ exists. The solution for the variables must be nonnegative integers. Before we describe the specific equations, and before we reduce the number of equations in some cases, we compute the total number of equations and the total number of variables in the equations for the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$.

Lemma 23. *The number of equations for the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $m = n - p$, is*

$$\sum_{s=\max\{0, t-p\}}^{\min\{t, m\}} \begin{bmatrix} m \\ s \end{bmatrix}_q,$$

i.e. $\begin{bmatrix} m \\ s \end{bmatrix}_q$ equations for all the s -subspaces of \mathbb{F}_q^m , where $\max\{0, t - p\} \leq s \leq \min\{t, m\}$.

Proof. The range of s is a direct consequence from Lemma 15. For each s -subspaces of \mathbb{F}_q^m we have one equation and hence there are $\begin{bmatrix} m \\ s \end{bmatrix}_q$ equations for each s . \square

Lemma 24. *The number of variables for the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $m = n - p$, is*

$$\sum_{r=\max\{0, k-p\}}^{\min\{k, m\}} \begin{bmatrix} m \\ r \end{bmatrix}_q,$$

i.e. $\begin{bmatrix} m \\ r \end{bmatrix}_q$ variables for all the r -subspaces of \mathbb{F}_q^m , where $\max\{0, k - p\} \leq r \leq \min\{k, m\}$.

Proof. The range of r is a direct consequence from Lemma 16 by noting that either s gets the value of t for a given t and if $m < t$ then also $m < k$ and the value of r is at most m . For each r -subspaces of \mathbb{F}_q^m we have one variable and hence there are $\begin{bmatrix} m \\ r \end{bmatrix}_q$ variables for each r . \square

Corollary 25. *If $m \leq t$, then the number of variables is equal to the number of equations, for the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$. This number is equal to*

$$\sum_{e=0}^m \begin{bmatrix} m \\ e \end{bmatrix}_q.$$

If the equations are linearly independent, then there is a unique solution to the set of equations in this case (when the variables are not constrained). If the solution consists of nonnegative integers then the p -punctured q -Steiner system $\mathbb{S}_q(t, t + 1, n; m)$ exists.

Corollary 26. *If $m = t + 2$, then the number of variables for the p -punctured q -Steiner system $\mathbb{S}_q(t, t + 1, n; m)$, $m = n - p$, is equal to*

$$\sum_{r=0}^{t+1} \begin{bmatrix} t+2 \\ r \end{bmatrix}_q.$$

The number of equations in this case is equal to

$$\sum_{s=0}^t \begin{bmatrix} t+2 \\ s \end{bmatrix}_q.$$

If we set the value of the variable which corresponds to the 0 -subspace of \mathbb{F}_q^m to be $\frac{\begin{bmatrix} n-m \\ t \end{bmatrix}_q}{\begin{bmatrix} t \end{bmatrix}_q}$ and the equations are linearly independent, then there is a unique solution to the set of equations in this case (when the variables are not constrained). If the solution consists of nonnegative integers then the p -punctured q -Steiner system $\mathbb{S}_q(t, t + 1, n; m)$ exists.

Proof. If the variable related to the 0-subspace of \mathbb{F}_q^m is set to $\frac{\begin{bmatrix} n-m \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}$, then all the $\begin{bmatrix} t+2 \\ 1 \end{bmatrix}_q$ variables related to the 1-subspaces of \mathbb{F}_q^m are equal to 0. Therefore, the number of equations in the new set of equations is

$$\sum_{s=1}^t \begin{bmatrix} t+2 \\ s \end{bmatrix}_q.$$

The number of variables which are not assigned with values is this new set of equations is

$$\sum_{r=2}^{t+1} \begin{bmatrix} t+2 \\ r \end{bmatrix}_q.$$

Clearly, these two summations are equal and the claim follows. \square

Corollary 27. *If $m = t + 1$, then the number of variables for the p -punctured q -Steiner system $\mathbb{S}_q(t, t + 1, n; m)$, $m = n - p$, is equal to*

$$\sum_{r=0}^{t+1} \begin{bmatrix} t+1 \\ r \end{bmatrix}_q.$$

The number of equations in this case is equal to

$$\sum_{s=0}^t \begin{bmatrix} t+1 \\ s \end{bmatrix}_q.$$

If we set the value of the variable which corresponds to the 0-subspace of \mathbb{F}_q^m to be $\frac{\begin{bmatrix} n-m \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}$ and the equations are linearly independent, then there is a unique solution to the set of equations in this case (when the variables are not constrained). If the solution consists of nonnegative integers then the p -punctured q -Steiner system $\mathbb{S}_q(t, t + 1, n; m)$ exists.

Note, that if there is a unique solution to the set of equations, then the existence of the related design, i.e. p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$ is not guaranteed yet. Only if the unique solution is a nonnegative integer solution, then the design exists. We also did not consider the linear independence of the equations, although it can be proved in some cases. It is also important to understand that the number of equations and the number of variables can be large, and in most cases the number of variables is much larger than the number of equations. In this case there are many free variables, which usually make it even harder to find if the set of equations have a solution with nonnegative integer values for the variables.

In the sequel we will examine cases, where the set of equations have a solution with nonnegative integers. In these cases it will be proved that the p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$ exists. In most cases we will consider *uniform solutions*, i.e. solutions

in which for each r , the number of r -subspaces in the systems is equal for any two r -subspaces of \mathbb{F}_q^m , i.e. the related variables have the same value. The related design will be called a *uniform design*. For such systems we can reduce the number of variables and the number of equations. The choice of uniform solution is usually a good choice when the equations are linearly independent. In such a case the solution is uniform in many cases.

Let \mathbb{S} be a uniform p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $m = n - p$. Let Z be an r -subspace of \mathbb{F}_q^m and let $X_{r,m}$ be the number of appearances of Z in \mathbb{S} . The conclusion of our discussion is the following set of equations for uniform designs.

Theorem 28 (For a uniform p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$).

Let \mathbb{S} be a uniform p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$, $m = n - p$. For each s , $\max\{0, t - p\} \leq s \leq \min\{t, m\}$, the following equation must be satisfied.

$$N_{(s,m),(t,n)} = \sum_{r=\max\{k-p,s\}}^{\min\{k-t+s,m\}} D_{s,r,m} \cdot C_{(s,t),(r,k)} \cdot X_{r,m} .$$

Proof. The left side of the equation is the number of distinct t -subspaces in \mathbb{F}_q^m which are formed by extending a given s -subspace Y of \mathbb{F}_q^m . The right hand side is summing over all the r -subspaces of \mathbb{F}_q^m which contain Y (the range is obtained from Lemma 16), where $D_{s,r,m}$ is the number of r -subspaces which contain Y , $C_{(s,t),(r,k)}$ is the number of appearances of Y in such a given r -subspace, and $X_{r,m}$ is the number of appearances of each r -subspace in \mathbb{S} . \square

4 Examples for Existing Systems

In this section we will give examples of p -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$ for various parameters. We start with the 3-punctured q -Fano plane $\mathbb{S}_q(2, 3, 7; 4)$ and continue with $\mathbb{S}_q(3, 4, 8; 4)$, $\mathbb{S}_2(3, 4, 8; 5)$, $\mathbb{S}_q(4, 5, 11; 6)$, and $\mathbb{S}_q(5, 6, 12; 6)$. We conclude with a more general example for the k -punctured q -Steiner system $\mathbb{S}_q(3, 4, 2k; k)$, $k \equiv 2$ or $4 \pmod{6}$, $k \geq 4$.

The 3-punctured q -Steiner system $\mathbb{S}_q(2, 3, 7; 4)$:

There are $\begin{bmatrix} 4 \\ s \end{bmatrix}_q$ equations for each $0 \leq s \leq 2$, for a total of $1 + (q^3 + q^2 + q + 1) + (q^2 + 1)(q^2 + q + 1)$ equations. There are $\begin{bmatrix} 4 \\ r \end{bmatrix}_q$ variables for each $0 \leq r \leq 3$, for a total of $1 + (q^3 + q^2 + q + 1) + (q^2 + 1)(q^2 + q + 1) + (q^3 + q^2 + q + 1)$ variables.

For $s = 0$, there is a unique equation for the 0-subspace (the null space) given by

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = (q^2 + q + 1)a + b_{i_1} + b_{i_2} + \cdots + b_{i_{q^3+q^2+q+1}} ,$$

where a is the unique variable related to the 0-subspace, while b_{i_j} is a variable for an 1-subspace. To have a unique solution we must have linearly independent equations in

which the number of variables equals the number of equations. Hence, we set $a = 1$ which implies that $b_{i_j} = 0$ for each j .

For $s = 1$, there are $q^3 + q^2 + q + 1$ equations related to the 1-subspaces, where each equation is of the form

$$q^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = (q^2 + q)b + c_{i_1} + c_{i_2} + \cdots + c_{i_{q^2+q+1}} ,$$

where b is a variable related to an 1-subspace and hence $b = 0$, while c_{i_j} is a variable for a 2-subspace.

For $s = 2$, there are $(q^2 + 1)(q^2 + q + 1)$ equations for 2-subspaces, where each equation is of the form

$$q^6 \begin{bmatrix} 3 \\ 0 \end{bmatrix}_q = q^2 c + d_{i_1} + d_{i_2} + \cdots + d_{i_{q+1}} ,$$

where c is a variable related to a 2-subspace, while d_{i_j} is a variable related to a 3-subspace.

This system of equations has a unique solution, which is also a solution for a uniform design (uniform punctured system), $X_{0,4} = 1$, $X_{1,4} = 0$, $X_{2,4} = q^2$, and $X_{3,4} = q^4(q - 1)$.

The 4-punctured q -Steiner system $\mathbb{S}_q(3, 4, 8; 4)$:

There are $\begin{bmatrix} 4 \\ s \end{bmatrix}_q$ equations for each $0 \leq s \leq 3$, and there are $\begin{bmatrix} 4 \\ r \end{bmatrix}_q$ variables for each $0 \leq r \leq 4$. To have a unique solution, which also forms a uniform design, we set $X_{0,4} = 1$ which implies that $X_{1,4} = 0$, and the system of equations has the unique solution, $X_{2,4} = q^2(q^2 + 1)$, $X_{3,4} = q^4(q^4 - 1)$, and $X_{4,4} = q^{12} - q^{11} + q^7$.

The 3-punctured q -Steiner system $\mathbb{S}_q(3, 4, 8; 5)$:

It is left for the reader to verify that the following set \mathbb{T} is a 3-punctured q -Steiner system $\mathbb{S}_q(3, 4, 8; 5)$. contains:

1. One 1-subspace which is punctured into the unique 0-subspace of \mathbb{F}_q^4 .
2. The $q^2(q^2 + 1)(q^2 + q + 1)$ distinct 2-subspaces of \mathbb{F}_q^5 , which are punctured into a 2-subspace of \mathbb{F}_q^4 , each one is contained exactly once in \mathbb{T} .
3. The $(q^2 + q + 1)(q^2 + 1)$ distinct 3-subspaces of \mathbb{F}_q^5 , which are punctured into a 2-subspace of \mathbb{F}_q^4 , each one is contained q^4 times in \mathbb{T} .
4. The $q^3(q^3 + q^2 + q + 1)$ distinct 3-subspaces of \mathbb{F}_q^5 , which are punctured into a 3-subspace of \mathbb{F}_q^4 , each one is contained $q(q^3 - 1)$ times in \mathbb{T} .
5. The $q^3 + q^2 + q + 1$ distinct 4-subspaces of \mathbb{F}_q^5 , which are punctured into a 3-subspace of \mathbb{F}_q^4 , each one is contained $q^7(q - 1)$ times in \mathbb{T} .
6. The q^4 distinct 4-subspaces of \mathbb{F}_q^5 , which are punctured into the unique 4-subspace of \mathbb{F}_q^4 , each one is contained $q^8 - q^7 + q^3$ times in \mathbb{T} .

The 5-punctured q -Steiner system $\mathbb{S}_q(4, 5, 11; 6)$:

There are $\begin{bmatrix} 6 \\ s \end{bmatrix}_q$ equations for each $0 \leq s \leq 4$ and there are $\begin{bmatrix} 6 \\ r \end{bmatrix}_q$ variables for each $0 \leq r \leq 5$. To have a uniform design we set $X_{0,6} = 1$ which implies that $X_{1,6} = 0$ and the system of equations will have a unique solution $X_{2,6} = q^2(q^2 + 1)$, $X_{3,6} = q^9 + q^7 - q^4$, $X_{4,6} = q^{14} - q^9 + q^7$, and $X_{5,6} = (q^{18} + q^{11})(q - 1)$.

The 6-punctured q -Steiner system $\mathbb{S}_q(5, 6, 12; 6)$:

There are $\begin{bmatrix} 6 \\ s \end{bmatrix}_q$ equations for each $0 \leq s \leq 5$ and there are $\begin{bmatrix} 6 \\ r \end{bmatrix}_q$ variables for each $0 \leq r \leq 6$. A solution for a uniform design for the system of equations is $X_{0,6} = 1$, $X_{1,6} = 0$, $X_{2,6} = q^2(q^4 + q^2 + q)$, $X_{3,6} = q^4(q^8 + q^6 + q^5 - 1)$, $X_{4,6} = q^7(q^{11} + q^9 + q^7 - q^6 + 1)$, $X_{5,6} = q^{11}(q^{13} - q^7 + q^6 - 1)$, and $X_{6,6} = q^{16}(q^{14} - q^{13} + q^7 - q^6 + 1)$.

The k -punctured q -Steiner system $\mathbb{S}_q(3, 4, 2k; k)$, $k \equiv 2$ or $4 \pmod{6}$, $k \geq 4$:

In this case, we will consider only a possible uniform design. For this design we have that $X_{0,k} = \frac{\begin{bmatrix} k \\ 3 \end{bmatrix}_q}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}_q}$, $X_{1,k} = 0$, $X_{2,k} = q^{k-2} \frac{q^k - 1}{q^2 - 1}$, $X_{3,k} = q^k(q^k - 1)$, and $X_{4,k} = \frac{(q^{3k} - q^{2k+3} + q^{k+3})(q-1)}{q^{k-3} - 1}$. We note that the remainder in the division of the polynomials in $X_{4,k}$ is $q^7 - q^6$ and hence $X_{4,k}$ is an integer only for $k = 4$ and all q 's. This solution was given in a previous example for $\mathbb{S}_q(3, 4, 8; 4)$.

The k -punctured q -Steiner system $\mathbb{S}_q(2, 3, 2k + 1; k + 1)$, $k \equiv 1$ or $3 \pmod{6}$, $k \geq 3$:

The number of equations in the system is $\sum_{s=0}^2 \begin{bmatrix} k+1 \\ s \end{bmatrix}_q$. The number of variables is $\sum_{r=0}^3 \begin{bmatrix} k+1 \\ r \end{bmatrix}_q$. We will consider only uniform designs and hence we only have 3 equations and 4 variables.

The first equation for the 0-subspace of \mathbb{F}_q^{k+1} is $N_{(0,k+1),(2,2k+1)} = D_{0,0,k+1} \cdot C_{(0,2)(0,3)} \cdot X_{0,k+1} + D_{0,1,k+1} \cdot C_{(0,2)(1,3)} \cdot X_{1,k+1}$ which is equal to $\begin{bmatrix} k \\ 2 \end{bmatrix}_q = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \cdot X_{0,k+1} + \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q \cdot X_{1,k+1}$. If we set $X_{0,k+1} = \frac{\begin{bmatrix} k \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q}$, then we have $X_{1,k+1} = 0$.

The second equation for 1-subspaces is $N_{(1,k+1),(2,2k+1)} = D_{1,1,k+1} \cdot C_{(1,2)(1,3)} \cdot X_{1,k+1} + D_{1,2,k+1} \cdot C_{(1,2)(2,3)} \cdot X_{2,k+1}$. Since $N_{(1,k+1),(2,2k+1)} = q^{k-1} \begin{bmatrix} k \\ 1 \end{bmatrix}_q = q^{k-1} \frac{q^k - 1}{q - 1}$ and $D_{1,2,k+1} = \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \frac{q^k - 1}{q - 1} = q^{k-1} + q^{k-2} + \dots + q + 1$, it follows that

$$q^{k-1} \begin{bmatrix} k \\ 1 \end{bmatrix}_q = q^{k-1} \frac{q^k - 1}{q - 1} = (q^2 + q)X_{1,k+1} + (q^{k-1} + q^{k-2} + \dots + q + 1)X_{2,k+1} .$$

The third equation for 2-subspaces is $N_{(2,k+1),(2,2k+1)} = D_{2,2,k+1} \cdot C_{(2,2)(2,3)} \cdot X_{2,k+1} + D_{2,3,k+1} \cdot C_{(2,2)(3,3)} \cdot X_{3,k+1}$. Since $N_{(2,k+1),(2,2k+1)} = q^{2k}$ and $D_{2,3,k+1} = \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q = \frac{q^{k-1} - 1}{q - 1} = q^{k-2} + q^{k-3} + \dots + q + 1$, it follows that

$$q^{2k} = q^2 X_{2,k+1} + (q^{k-2} + q^{k-3} + \dots + q + 1)X_{3,k+1} .$$

The solution for this set of equations is $X_{0,k+1} = \frac{\begin{bmatrix} k \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q}$, $X_{1,k+1} = 0$, $X_{2,k+1} = q^{k-1}$, and $X_{3,k+1} = q^{k+1}(q-1)$.

5 A Recursive Construction

In this section we present a recursive construction for a p -punctured q -Steiner system $\mathbb{S}_q(2, 3, 2k+1; k+1 + \lfloor \frac{k+1}{3} \rfloor)$, $p = k - \lfloor \frac{k+1}{3} \rfloor$, where $k \equiv 1$ or $3 \pmod{6}$. The basis for the construction is the trivial q -Steiner system $\mathbb{S}_q(2, 3, 3)$.

Let $k \equiv 1$ or $3 \pmod{6}$, which implies that $2k+1 \equiv 3$ or $7 \pmod{12}$, and assume that there exists a p -punctured q -Steiner system $\mathbb{S}_q(2, 3, k; \lfloor \frac{k+1}{3} \rfloor)$, $p = k - \lfloor \frac{k+1}{3} \rfloor$. Let \mathbb{S} be a k -punctured q -Steiner system $\mathbb{S}_q(2, 3, 2k+1; k+1)$ presented in Section 4. For \mathbb{S} we have that $X_{0,k+1} = \frac{\begin{bmatrix} k \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q}$, $X_{1,k+1} = 0$, $X_{2,k+1} = q^{k-1}$, and $X_{3,k+1} = q^{k+1}(q-1)$. In the recursive construction, we will generate a system \mathbb{T} , a p -punctured q -Steiner system $\mathbb{S}_q(2, 3, 2k+1; k+1 + \lfloor \frac{k+1}{3} \rfloor)$, $p = k - \lfloor \frac{k+1}{3} \rfloor$.

Let $r = \lfloor \frac{k+1}{3} \rfloor$ be the number of columns that should be appended to the subspaces (of dimension 0, 2, and 3) of \mathbb{S} to form \mathbb{T} . To each one of the $\begin{bmatrix} k+1 \\ 3 \end{bmatrix}_q$ distinct 3-subspaces of \mathbb{S} we append the q^{3r} possible combinations of r columns. Each column has q^3 possible combinations by Lemma 4. Since $X_{3,k+1} = q^{k+1}(q-1)$, it follows that each such combination (a 3-subspace of \mathbb{F}_q^{k+1+r}), whose r -punctured subspace is also a 3-subspace, will appear $q^{k+1-3r}(q-1)$ times in \mathbb{T} . To the $\frac{\begin{bmatrix} k \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q}$ 0-subspaces of \mathbb{S} we append the subspaces of a $(k-r)$ -punctured q -Steiner system $\mathbb{S}_q(2, 3, k; r)$ system which exists by our assumption. Hence, we have completed the extension of the 0-subspaces and 3-subspaces of \mathbb{S} . To complete our construction we have to extend the 2-subspaces of \mathbb{S} .

For the extension of the 2-subspaces we need two more concepts, namely spreads and large sets in $\mathcal{G}_q(k+1, 2)$ (known as 1-spreads and 1-parallelisms in $\text{PG}(k, q)$). A *spread* in $\mathcal{G}_q(k+1, 2)$ is a set of 2-subspaces whose nonzero elements form a partition of all the elements of $\mathbb{F}_q^{k+1} \setminus \{0\}$, i.e. each nonzero vector of \mathbb{F}_q^{k+1} appears in exactly one 2-subspace of the spread. In other words, a spread in $\mathcal{G}_q(k+1, 2)$ is a q -Steiner system $\mathbb{S}_q(1, 2, k+1)$. A *large set* (1-parallelism) of q -Steiner systems $\mathbb{S}_q(1, 2, k+1)$ is a partition of all 2-subspaces of $\mathcal{G}_q(k+1, 2)$ into q -Steiner systems $\mathbb{S}_q(1, 2, k+1)$ (spreads). If $q = 2$, then such large sets are known to exist whenever $k+1$ is even [2].

We continue by considering the case of $q = 2$. Note, that $k+1$ is even and hence there exists a spread in $\mathcal{G}_2(k+1, 2)$. The size of such spread is $\frac{2^{k+1}-1}{3}$, i.e. it contains $\frac{2^{k+1}-1}{3}$ subspaces. The total number of subspaces in $\mathcal{G}_2(k+1, 2)$ is $\begin{bmatrix} k+1 \\ 2 \end{bmatrix}_2 = \frac{(2^{k+1}-1)(2^k-1)}{3}$. There exists a partition (large set) of these 2-subspaces into disjoint spreads and hence there are $2^k - 1$ disjoint spreads in such a large set. We continue and arbitrarily partition these $2^k - 1$ disjoint spreads into 2^r sets of spreads, one set with $2^{k-r} - 1$ spreads and $2^r - 1$ sets each one with 2^{k-r} spreads. To each one of these $2^r - 1$ sets we assign arbitrarily a different nonzero row vector of length r , and the all-zero vector of length r is assigned to

the set of size $2^{k-r} - 1$.

For demonstration of the construction, each 2-subspace of \mathbb{S} is represented by a $3 \times (k + 1)$ matrix, each 2-subspace of \mathbb{T} is represented by a $3 \times (k + 1 + r)$ matrix, and each 3-subspace of \mathbb{T} is represented by a $7 \times (k + 1 + r)$ matrix.

Consider now these two sets of spreads:

1. For the set which contains $2^{k-r} - 1$ spreads, each 2-subspace X from each spread is contained 2^{k-1} times in the \mathbb{S} . The 2-subspaces X is extended to several 2-subspaces in \mathbb{F}_2^{k+1+r} as follows. The first $k + 1$ columns which represent these 2-subspaces are equal to the $3 \times (k + 1)$ matrix which represents X . In the last r columns there are 4 possible options in each column and thus 2^{2r} distinct combinations of r columns. Each such combination will appear 2^{k-1-2r} times in \mathbb{T} .
2. For a set which contains 2^{k-r} spreads (there are $2^r - 1$ such sets in the partition), each 2-subspace X from each spread is contained 2^{k-1} times in \mathbb{S} . There is a nonzero vector v of length r which is assigned to this set. The 2-subspace X is extended to several 3-subspaces in \mathbb{F}_2^{k+1+r} as follows. The first three rows in the first $k + 1$ columns which represent these subspaces are equal to the $3 \times (k + 1)$ matrix which represents X . The next three rows in these $k + 1$ columns are also equal to the $3 \times (k + 1 + r)$ matrix which represents X . The seventh and the last row in these $k + 1$ columns is a row of zeroes. We turn now to complete the last r columns in the $7 \times (k + 1 + r)$ matrices which represents the 3-subspaces extended from X . The entries of the last (seventh) row in these columns are assigned with the values of v . The first column in which v has a *one* has values which corresponds to the unique extension from a 2-subspace to a 3-subspace as proved in Lemma 6. Finally, in each other column there are 4 possible distinct combinations: if the related entry in v is a *zero*, it relates to the extension from 2-subspace to 2-subspace; and if the related entry in v is a *one* it relates to the extension from 3-subspace to 3-subspace in which there are 4 combinations, out of the 8 combinations, with a *one* in a given coordinate. In total there are $2^{2(r-1)}$ distinct combinations for these r columns. Each such combination will appear $2^{k-1-2(r-1)}$ times in \mathbb{T} .

The proof that \mathbb{T} is p -punctured q -Steiner system $\mathbb{S}_q(2, 3, 2k + 1; k + 1 + r)$, $p = k - r$, follows immediately from the described construction.

The generalization for $q > 2$ is similar, as explained before the requirement is the existence of large set of q -Steiner system $\mathbb{S}_q(1, 2, k + 1)$, where $k \equiv 1$ or $3 \pmod{6}$. Such a large set is known to exist for $q > 2$ only if $k + 1$ is a power of 2 [3], making the possible generalizations for $q > 2$ with limited number of parameters. The recursive construction, with the basis of $\mathbb{S}_q(2, 3, 3)$, leads to the following theorem.

Theorem 29. *There exists a p -punctured q -Steiner system $\mathbb{S}_q(2, 3, 2^\ell - 1; 2^\ell - 1 - \lfloor \frac{2^\ell - 1}{3} \rfloor)$, $p = \lfloor \frac{2^\ell - 1}{3} \rfloor$, $\ell \geq 3$.*

For $q = 2$ the construction can be applied also by starting with the q -Steiner system $\mathbb{S}_2(2, 3, 13)$ [4].

6 Conclusions

We have presented a new framework to examine the existence of q -Steiner systems. Based on this framework we have defined a new set of q -designs which are punctured q -Steiner systems. Necessary conditions for the existence of such designs were presented. Several parameters where these new designs exist, were given. A recursive construction for one set of parameter for such designs was given. For future research the main problem in this direction is to find lower bounds on m for any given $(n - m)$ -punctured q -Steiner system $\mathbb{S}_q(t, k, n; m)$.

The new framework was as a basis to find a punctured q -Fano plane (also known as a residual q -Fano plane), i.e. a punctured q -Steiner system $\mathbb{S}_q(2, 3, 7; 6)$, for each prime power $q \geq 2$ [16]. The construction is based on extending the 3-punctured q -Steiner system $\mathbb{S}_q(2, 3, 7; 4)$. This system is very interesting. It consists of 2-subspaces of \mathbb{F}_q^6 , which form a spread. Each 2-subspace of \mathbb{F}_q^6 which is not part of the spread is contained in exactly q^2 3-subspaces of the system. These new findings give us a renewed hope to construct a q -Fano plane for $q > 2$, as the system cannot be extended for $q = 2$.

The new framework can be used to prove that in the q -Fano plane (if exists) we can force three 3-subspaces with four columns of zeroes - one in the first four columns, one in the last four columns, and one in the first two and the last two columns. Such a result as well as other derived properties are left as an exercise to the interested reader.

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