Random enriched trees with applications to random graphs

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Abstract

We establish limit theorems that describe the asymptotic local and global geometric behaviour of random enriched trees considered up to symmetry. We apply these general results to random unlabelled weighted rooted graphs and uniform random unlabelled k-trees that are rooted at a k-clique of distinguishable vertices. For both models we establish a Gromov–Hausdorff scaling limit, a Benjamini–Schramm limit, and a local weak limit that describes the asymptotic shape near the fixed root.

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Contents

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1 Introduction

The main purpose of this paper is to study the limiting behaviour of randomly selected structures from recursively defined combinatorial classes like (rooted) trees. One of the challenges is to discount symmetries of these objects. The main results provide universal descriptions of the asymptotic local and global shape as the size of the random structures tends to infinity. Specifically, we establish a local weak limit for the vicinity of a fixed root vertex in Theorem 24 and a Benjamini–Schramm limit describing the neighbourhoods of a uniformly at random selected point in Theorem 27. We establish a Gromov–Hausdorff scaling limit for random metrics on these structures in Theorem 28 and provide sharp diameter tail-bounds in Theorem 29. We present applications to random weighted unlabelled connected rooted graphs, random unlabelled front-rooted k-dimensional trees, and simply generated unlabelled rooted trees in Subsections 6.4, 6.5, and 6.6.

The study of large random discrete structures lies at the intersection of probability theory and combinatorics [26]. A combinatorial approach often involves using the framework of combinatorial classes to express the quantities under consideration in terms of coefficients of power series, and applying analytic tools such as singularity analysis or saddle-point methods to obtain very precise limits and concentration results [27, 38, 32, 9, 19, 21]. From a probabilistic viewpoint, the focus is on settings where methods for stochastic processes may be applied, and recent work focuses on graph limits describing the asymptotic shape, either locally in so called local weak limits [7, 66, 22, 20, 13, 58, 15], or globally in Gromov–Hausdorff scaling limits [45, 41, 2, 51, 53, 52, 59], and more recently, also on an intermediate scale in local Gromov–Hausdorff scaling limits [10, 72].

When studying random graphs one usually distinguishes their vertices either explicitly using labels, or implicitly in so called unlabelled graphs. The number of ways to label an unlabelled graph is given by the quotient of its number of vertices and the size of its automorphism group. Hence, if we uniformly select an element from a finite set of

n-vertex graphs that is closed under relabelling, we are more likely to pick a graph with a small automorphism group. That is, the labelling does not influence the geometric shape of the graph, but the measure according to which a graph is drawn. It is natural to wonder how a random unlabelled graph, that is, one that is sampled without this bias, behaves asymptotically. However, this poses a particular challenge. Due to the complexity of the involved symmetries, probabilistic approaches in the past where mostly limited to models of random unlabelled trees [61, 68, 57, 40, 71]. Models of random unlabelled graphs were predominantly studied using analytic methods. In this context we mention the work by Bodirsky, Fusy and Kang [16] and Kraus [47] concerning random unlabelled outerplanar graphs. Drmota, Fusy, Kang, Kraus and Rué [27] took a unified approach to study so called families of subcritical classes of unlabelled graphs, and Drmota and Jin [28] and Gainer-Dewar, Gessel and Ira [36] treated unlabelled k-dimensional trees.

In the present work we aim to combine methods for branching processes with combinatorial tools to establish novel graph limits for models of random graphs considered up to symmetry. The first model considered is that of random unlabelled rooted connected graphs sampled with probability proportional to a product of weights assigned to their 2-connected components. If the weight-sequence satisfies certain conditions, we obtain a local weak limit that describes the asymptotic vicinity near the fixed root, a Benjamini–Schramm limit that describes the asymptotic shape near a random vertex, and a Gromov–Hausdorff scaling limit. Moreover, we obtain sharp tail bounds for the diameter. In the two local limits, we even obtain total variational convergence of arbitrary $o(\sqrt{n})$-sized neighbourhoods of the fixed root and random root, which is best-possible as the convergence fails for neighbourhoods whose radius is comparable to $\sqrt{n}$. The setting we consider explicitly includes uniform random unlabelled rooted graphs from so called subcritical graph classes introduced in [27, Def. 10], such as series-parallel graphs, outerplanar graphs, and cacti graphs. As for extremal graph parameters, our results also establish the correct order of the diameter. The maximum degree and largest 2-connected component are shown to have typically order $O(\log n)$. Our results complement, but neither include or are included by, the results on random labelled graphs from subcritical classes given by Panagiotou, S., and Weller [62], and Drmota and Noy [29]. In [27] additive parameters of these graphs such as the degree distribution were studied using analytic methods. The two local limits add a probabilistic interpretation to the limit degree distributions obtained in [27] for the degree of a random vertex and of the fixed root. Furthermore, general results by Kurauskas [48, Thm. 2.1] and Lyons [56, Thm. 3.2] for Benjamini–Schramm convergent sequences of random graphs may be applied to deduce laws of large numbers for subgraph count asymptotics and spanning tree count asymptotics in terms of the Benjamini–Schramm limit.

Shortly after the limit results of the present work were made available on the arXiv in 2015, Georgakopoulos and Wagner [37] submitted a manuscript to this repository that provides combinatorial proofs of the local weak limit for labelled and unlabelled rooted graphs in the subcritical regime, and a novel Benjamini–Schramm limit for subcritical *unrooted* unlabelled graphs. (See also the footnote on page 2 of [37] for comments on the overlap.) The advantage of the approach via branching processes in the present paper

is that it enables us to describe *both* the local and global asymptotic geometric shape. Previous results by Panagiotou, S., and Weller [62] treat scaling limits of random labelled graphs in a subcriticality regime, but it is important to note that random *labelled* graphs from classes that are subcritical in the sense of [27, Def. 5] and random *unlabelled* rooted graphs that are subcritical in the sense of [27, Def. 10] are different models. This is already evident in the simplest case, the class of trees, which is known to be subcritical in both the labelled and unlabelled sense. The scaling limit of uniform random trees with vertices labelled from 1 to n is due to Aldous [4], but his result does not encompass the case of unlabelled rooted trees. The unlabelled case was conjectured to converge in the global sense in [5, p. 55], and the celebrated proof of this conjecture by Haas and Miermont [40] brought new ideas to the table. Likewise, the generalization from labelled trees to subcritical classes of labelled graphs undertaken in [62] does not encompass the scaling limits for random unlabelled rooted graphs given in the present work, regardless of the fact that both works use the *language* of combinatorial species and approximate the graphs under consideration by conditioned Galton–Watson trees.

The framework of the present paper may also be used to study random unlabelled k-trees that are rooted at a front of distinguishable vertices. A k-tree consists either of a complete graph with k vertices, or is obtained from a smaller k-tree by adding a vertex and connecting it with k distinct vertices of the smaller k-tree. Such objects are interesting from a combinatorial point of view, as their enumeration problem has a long history, see [60, 35, 34, 33, 23, 11, 42]. They are also interesting from an algorithmic point of view, as many NP-hard problems on graphs have polynomial algorithms when restricted to k-trees [8, 39]. Employing recent results for limits of random unlabelled Gibbs partitions [69], we obtain a local weak limit for unlabelled front-rooted k-trees that describes the total variational asymptotic behaviour of arbitrary $o(\sqrt{n})$-neighbourhoods of the root-front. We also obtain a Benjamini–Schramm limit describing the asymptotic shape of $o(\sqrt{n})$-neighbourhoods of a uniformly at random selected vertex. Furthermore, we obtain a Gromov–Hausdorff scaling limit. For all three limits, a concentration result is required that relates the distances of certain points with respect to the k-tree metric and to a tree-metric in the underlying representation by trees endowed with local symmetries. We obtain this intermediate result by locating a hidden Markov chain and applying a large deviation inequality by Lezaud [55] for functions on non-reversible Markov processes.

As a third application, our results also yield local weak limits, Benjamini–Schramm limits and scaling limits for a family of random unordered trees drawn according to weights assigned to the vertex degrees. The scaling limit mildly generalizes previous results for uniform vertex degree restricted trees [57, 40, 61]. From a probabilistic viewpoint it is natural to consider weights and not only the uniform measure, as this constitutes the unordered analogue of Galton–Watson trees conditioned on having a large number of vertices, or more generally so called simply generated plane trees. As demonstrated by Janson [44], random weighted plane trees may be classified into various regimes, each exhibiting characteristic asymptotic behaviour. For this reason, it is also natural to work towards a similar classification for random weighted unordered trees, and the present work makes a small step in this direction.

Due to the length of the present paper one might get the impression that the applications we provide are independent problems that should be treated separately. It is actually one of the main points we are trying to make here, that this is not the case. As for the limits, we are going to use the local weak limits to establish a large deviation result that is crucial for obtaining the scaling limits. Conversely, this result also allows us to establish the local weak limits and Benjamini–Schramm limits in their strong form, featuring total variational convergence of $o(\sqrt{n})$-neighbourhoods.

Moreover, rather than treating each model of random graphs individually, we take a unified approach and establish a set of limit theorems that apply to the abstract family of random unlabelled $\mathcal{R}$-enriched trees, with the class $\mathcal{R}$ ranging over all combinatorial classes. Roughly speaking, given a class $\mathcal{R}$ of combinatorial objects, an $\mathcal{R}$-enriched tree is a rooted tree together with a function that assigns to each vertex an $\mathcal{R}$-structure *on* its offspring. The model we consider is an unlabelled $\mathcal{R}$-enriched tree with n vertices considered up to symmetry, that we sample with probability proportional to a weight formed by taking the product of weights assigned to its local $\mathcal{R}$-structures. The limits are formed as n becomes large, possibly along a shifted sublattice of the integers. We are going to explain in detail how the mentioned models of random unlabelled graphs fit into this framework. Of course it also makes sense to study a labelled analogue, and this endeavour is undertaken in [67].

There are various reasons for taking this general approach. First of all, the method of our proofs requires per se that we consider trees enriched with arbitrary structures, not just the structure that corresponds to the graphs under consideration. The limit objects are defined in terms of plane trees enriched with local symmetries, and then interpreted as random graphs according to combinatorial bijections. The benefit of this is that we may really see how classical probabilistic objects such as Kesten's tree or asymptotic fringe distributions [3] are hidden inside. Furthermore, given the immense variety of combinatorial structures [32], it is interesting that a high-level point of view that covers a large class of seemingly unrelated discrete structures is even possible.

On a final note, we remark that our techniques enable us (in principle) to strengthen the Gromov–Hausdorff scaling limits in the present work to more powerful Gromov–Hausdorff–Prokhorov limits, which entail scaling limits for the distances between multiple independently sampled random points. Due to the already considerable length of the manuscript we resist the temptation to make this extension explicit.

Plan of the paper

Section 1 gives an informal description of the topic and the main applications. Section 2 fixes notation related to graphs, trees and k-trees, and recalls necessary background on local weak convergence, Gromov–Hausdorff convergence and further properties of large critical Galton–Watson trees. Section 3 is a concise introduction to the combinatorial framework of species of structures and Boltzmann distributions, with a focus on the decomposition of symmetries. Section 4 discusses a limit theorem for unlabelled Gibbs partitions, that we are going to use in our applications. Section 5 explicitly states some

probabilistic and combinatorial tools related to random walks and Markov chains. Section 6 presents the contributions of the present paper in detail. Specifically, Subsection 6.1 introduces the model of unlabelled $\mathcal{R}$-enriched trees under consideration, and discusses combinatorial bijections that show how this generalizes various models of random graphs considered up to symmetry, in particular unlabelled block-weighted rooted graphs and unlabelled front-rooted k-trees. Subsection 6.2 builds the framework concerning the local structure of enriched trees and states our main local weak limits for the vicinity of the fixed root vertex in Theorem 24 and for the neighbourhoods of a uniformly at random selected point in Theorem 27. Subsection 6.3 introduces a general model of random metric spaces based on random unlabelled enriched trees, and establishes our main scaling limit in Theorem 28, as well as sharp diameter tail-bounds in Theorem 29. Subsection 6.4 presents applications of our main results to random weighted unlabelled connected rooted graphs. In particular, Theorem 33 states a scaling limit with respect to the first-passage percolation metric and sharp diameter tail-bounds. Theorem 31 states a local weak limit for the vicinity of the fixed root vertex and Theorem 32 gives a Benjamini–Schramm limit. Subsection 6.5 discusses applications to random unlabelled front-rooted k-dimensional trees, for which a scaling limit is established in Theorem 37 and a Benjamini–Schramm limit is given in Theorem 36. Subsection 6.6 presents further applications to a family of simply generated unlabelled rooted trees. In Section 7 we collect all proofs.

2 Preliminaries

2.1 Notation

Throughout, we set

$$\mathbb{N} = \{1, 2, \ldots\}, \qquad \mathbb{N}_0 = \{0\} \cup \mathbb{N}, \qquad [n] = \{1, 2, \ldots, n\}, \qquad n \in \mathbb{N}_0.$$

The set of non-negative real numbers is denoted by $\mathbb{R}_{\geqslant 0}$. We usually assume that all considered random variables are defined on a common probability space whose measure we denote by $\mathbb{P}$, and let $\mathbb{L}_p$ denote the corresponding space of p-integrable real-valued functions. All unspecified limits are taken as n becomes large, possibly taking only values in a subset of the natural numbers. We write $\xrightarrow{d}$ and $\xrightarrow{p}$ for convergence in distribution and probability, and $\overset{(d)}{=}$ for equality in distribution. An event holds with high probability, if its probability tends to 1 as n tends to infinity. We let $O_p(1)$ denote an unspecified random variable X_n of a stochastically bounded sequence $(X_n)_n$, and write $o_p(1)$ for a random variable X_n with $X_n \xrightarrow{p} 0$. We write $\mathcal{L}(X)$ to denote the law of a random variable X. The total variation distance of measures and random variables is denoted by d_{TV}. Given a power series $f(z)$, we let $[z^n]f(z)$ denote the coefficient of z^n in $f(z)$.

2.2 Graphs, trees and k-trees

We are going to consider *simple* graphs, that have no loops or parallel edges. The vertices that are adjacent to a vertex v in a graph G are its *neighbourhood*. The cardinality of

its neighbourhood is called the *degree* of the vertex v, and denoted by $d_G(v)$. We say G is *locally finite*, if all its vertices have finite degree. A graph is called *connected*, if any two vertices may be joined by a *path*. More generally, for $k \geqslant 1$ we say a graph G is k-connected, if it has at least $k+1$ vertices and deleting any $k-1$ of them does not disconnect the graph. A *cutvertex* is a vertex whose removal disconnects the graph. Hence 2-connected graphs are graphs without cutvertices and size at least three.

A *graph isomorphism* between graphs G and H is a bijection between their vertex sets such any two vertices in G are joined by an edge if and only if their images in H are. In this case we say the two graphs are *isomorphic*. We say a graph is *rooted*, if one of its vertices is distinguished. Graph isomorphisms between rooted graphs are required to map the roots to each other. More generally, we may consider graphs with an ordered set of distinguishable root-vertices, that must be respected by graph isomorphisms. A graph considered up to isomorphism is an *unlabelled graph*. That is, any two unlabelled graphs are distinct if they are not isomorphic. Formally, unlabelled graphs are defined as isomorphism classes of graphs. Unlabelled rooted graphs are defined analogously.

A *tree* is a graph that is connected and does not contain cycles. In a rooted tree, we say the vertices lying on the path between a vertex v and the root are the *ancestors* of v. The vertices that are joined to v by an edge but are not ancestors are its *offspring* or *children*. The collection of the children of a vertex is its *offspring set*. The cardinality of the offspring set of a vertex v in a rooted tree A is its *outdegree* and denoted by $d_A^+(v)$. Unlabelled rooted trees are also called *Pólya*-trees, in honour of George Póya.

The complete graph with n vertices is denoted by K_n. That is, in K_n any two distinct vertices are connected. A subgraph of a graph is called an *n-clique*, if its isomorphic to K_n. A *k-tree* is a graph that may be constructed by starting with a k-clique, and adding in each step a new vertex that gets connected with k arbitrarily chosen distinct vertices of the previously constructed graph. The k-cliques of a k-tree are also called its *fronts*, and the $k+1$-cliques its *hedra*. In the present work, we are only considering k-trees that are rooted at a front of distinguishable vertices.

A *block* B of a graph G is a subgraph that is inclusion maximal with the property of being either an isolated vertex, a 2-clique, or 2-connected. Connected graphs have a tree-like block-structure, whose details are explicitly given for example in Diestel's book [25, Ch. 3.1]. We mention a few properties, that we are going to use. Any two blocks of G overlap in at most one vertex. The cutvertices of G are precisely the vertices that belong to more than one block.

Any connected graph C is naturally equipped with the *graph-metric* on its vertex set, that assigns to any two vertices the minimum number of edges required to join them. We usually denote the graph metric by $d_C(\cdot,\cdot)$. Given a vertex $v \in C$ and an integer $k \geqslant 0$, the *k-neighbourhood* $V_k(C,v)$ is the subgraph induced by all vertices with distance at most k from v. We regard $V_k(C,v)$ as rooted at the vertex v. The *diameter* $\mathrm{D}(C)$ is the supremum of all distances between pairs of vertices. For a rooted graph $C^\bullet$, we may also consider its *height* $\mathrm{H}(C^\bullet)$, which is the supremum of all distances of vertices from the root of $C^\bullet$. Given a vertex v, we let $\mathrm{h}_{C^\bullet}(v)$ denote its distance from the root.

Another metric on C is the *block-metric* d_{BLOCK}. The block-distance between any two

vertices of C is given by the minimum number of blocks required to cover any joining path. By standard properties of the block-structure of connected graphs, the choice of the joining path does not matter. For any vertex $v \in C$ and any integer $k \geqslant 0$ we let $U_k(C, v)$ denote the *k-block-neighbourhood*, that is, the subgraph induced by all vertices with block-distance at most k. We regard $U_k(C, v)$ as rooted at the vertex v.

2.3 Local weak convergence

Let $G^\bullet = (G, v_G)$ and $H^\bullet = (H, v_H)$ be two connected, rooted, and locally finite graphs. We may consider the distance

$$d(G^\bullet, H^\bullet) = 2^{-\sup\{k \in \mathbb{N}_0 \,|\, V_k(G^\bullet) \simeq V_k(H^\bullet)\}}$$

with $V_k(G^\bullet) \simeq V_k(H^\bullet)$ denoting isomorphism of rooted graphs. This defines a premetric on the collection of all rooted locally finite connected graphs. Two such graphs have distance zero, if and only if they are isomorphic. Hence this defines a metric d_{BS} on the collection $\mathbb{B}$ of all unlabelled, connected, rooted, locally finite graphs. The space $(\mathbb{B}, d_{\mathrm{BS}})$ is known to be complete and separable, that is, a Polish space.

A random rooted graph $\mathsf{G}^\bullet \in \mathbb{B}$ is the the *local weak limit* of a sequence $\mathsf{G}_n^\bullet = (\mathsf{G}_n, v_n)$, $n \in \mathbb{N}$ of random elements of $\mathbb{B}$, if it is the weak limit with respect to this metric. That is, if

$$\lim_{n\to\infty} \mathbb{E}\left[f(G_n^\bullet)\right] = \mathbb{E}\left[f(G^\bullet)\right]$$

for any bounded continuous function $f : \mathbb{B} \to \mathbb{R}$. This is equivalent to stating

$$\lim_{n\to\infty} \mathbb{P}\left(V_k(\mathsf{G}_n^\bullet) \simeq G^\bullet\right) = \mathbb{P}\left(V_k(\mathsf{G}^\bullet) \simeq G^\bullet\right).$$

for any rooted graph $G^\bullet$. If the conditional distribution of v_n given the graph G_n is uniform on the vertex set $V(\mathsf{G}_n)$, then the limit $G^\bullet$ is often also called the *Benjamini–Schramm limit* of the sequence $(\mathsf{G}_n)_n$. This kind of convergence often yields laws of large numbers for additive graph parameters [56, 48].

2.4 Gromov–Hausdorff convergence

Let $X^\bullet = (X, d_X, x_0)$ and $Y^\bullet = (Y, d_Y, y_0)$ denote pointed compact metric spaces. A *correspondence* between $X^\bullet$ and $Y^\bullet$ is a subset $R \subset X \times Y$ containing (x_0, y_0) such that for any $x \in X$ there is a point $y \in Y$ with $(x, y) \in R$, and conversely for any $y \in Y$ there is a point $x \in X$ with $(x, y) \in R$. The *distortion* of the correspondence is defined as the supremum

$$\mathrm{dis}(R) = \sup\{|d_X(x_1, x_2) - d_Y(y_1, y_2) \mid (x_1, y_1), (x_2, y_2) \in R\}.$$

The *(pointed) Gromov–Hausdorff distance* between the pointed spaces $X^\bullet$ and $Y^\bullet$ may be defined by

$$d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_R \mathrm{dis}(R)$$

with the index R ranging over all correspondences between $X^\bullet$ and $Y^\bullet$. The factor $1/2$ is only required in order to stay consistent with an alternative definition of the Gromov–Hausdorff distance via the Hausdorff distance of embeddings of $X^\bullet$ and $Y^\bullet$ into common metric spaces, see [54, Prop. 3.6] and [18, Thm. 7.3.25]. This distance satisfies the axioms of a premetric on the collection of all compact rooted metric spaces. Two such spaces have distance zero from each other, if and only if they are isometric. That is, if there is a distance preserving bijection between the two that also preserves the root vertices. Hence this yields a metric on the collection $\mathbb{K}^\bullet$ of isometry classes of pointed compact metric spaces. The space $(\mathbb{K}^\bullet, d_{\mathrm{GH}})$ is known to be Polish (complete and separable), see [54, Thm. 3.5] and [18, Thm. 7.3.30 and 7.4.15].

2.5 Large critical Galton–Watson trees

In this section we let $\mathcal{T}_n$ denote a Galton–Watson tree conditioned on having n vertices, such that the offspring distribution ξ has average value $\mathbb{E}[\xi] = 1$ and finite non-zero variance σ^2. We refer the reader to Drmota's book [26, Sec. 1.2.7] for a detailed introduction to this model of random trees.

2.5.1 Convergence toward the CRT

The (Brownian) continuum random tree (CRT) is a random metric space constructed by Aldous in his pioneering papers [4, 5, 6]. Its construction is as follows. To any continuous function $f : [0,1] \to [0,\infty[$ satisfying $f(0) = f(1) = 0$ we may associate a premetric d on the unit interval $[0,1]$ given by

$$d(u,v) = f(u) + f(v) - 2 \inf_{u \leqslant s \leqslant v} f(s)$$

for $u \leqslant v$. The corresponding quotient space $(\mathcal{T}_f, d_{\mathcal{T}_f}) = ([0,1]/\!\sim, \bar{d})$, in which points with distance zero from each other are identified, is considered as rooted at the coset $\bar{0}$ of the point zero. This pointed metric space is an $\mathbb{R}$-tree, see [31, 54] for the definition of $\mathbb{R}$-trees and further details. The CRT may be defined as the random pointed metric space $(\mathcal{T}_{\mathsf{e}}, d_{\mathcal{T}_{\mathsf{e}}}, \bar{0})$ corresponding to Brownian excursion $\mathsf{e} = (\mathsf{e}_t)_{0 \leqslant t \leqslant 1}$ of duration one.

The famous invariance principle,

$$(\mathcal{T}_n, \frac{\sigma}{2} n^{-1/2} d_{\mathcal{T}_n}, \emptyset) \xrightarrow{d} (\mathcal{T}_{\mathsf{e}}, d_{\mathcal{T}_{\mathsf{e}}}, \bar{0}) \tag{1}$$

in the Gromov–Hausdorff sense, is due to Aldous [6] and there exist various extensions, see for example Duquesne [30], Duquesne and Le Gall [31], Haas and Miermont [40]. We are going to use a popular shortened notation for rescaled version of metric spaces, writing for example simply

$$\frac{\sigma}{2} n^{-1/2} \mathcal{T}_n \xrightarrow{d} \mathcal{T}_{\mathsf{e}}$$

for Equation (1).

2.5.2 Bounds for the height and level width

Addario-Berry, Devroye and Janson [1, Thm. 1.2] showed that there are constants $C, c > 0$ such that for all n and $h \geqslant 0$

$$\mathbb{P}\left(\mathrm{H}(\mathcal{T}_n) \geqslant h)\right) \leqslant C \exp(-ch^2/n). \tag{2}$$

The first moment of the number $L_k(\mathcal{T}_n)$ of all vertices v with height $h_{\mathcal{T}_n}(v) = k$ admits a bound of the form

$$\mathbb{E}\left[L_k(\mathcal{T}_n)\right] \leqslant Ck \exp(-ck^2/n). \tag{3}$$

for all n and $k \geqslant 1$. See [1, Thm. 1.5].

3 Combinatorial species and weighted Boltzmann distributions

In order to study combinatorial objects up to symmetry, it is convenient to use the language of combinatorial species developed by Joyal [46]. It provides a clean and powerful framework in which complex combinatorial bijections may be stated using simple algebraic terms. In order to make the present work accessible to a large audience, we recall the notions and results required to state and prove our main results. The theory admits an elegant and concise description using the language of category theory, but we will avoid this terminology and assume no knowledge by the reader in this regard. The exposition of the combinatorial and algebraic aspects in the subsequent subsections follows mainly Joyal [46] and Bergeron, Labelle and Leroux [14]. The probabilistic aspects in the Boltzmann sampling framework is based on a recent result by Bodirsky, Fusy, Kang and Vigerske [17, Prop. 38].

3.1 Weighted combinatorial species

A *combinatorial species* $\mathcal{F}$ is a functor or "rule" that produces for each finite set U a finite set $\mathcal{F}[U]$ of $\mathcal{F}$*-objects* (or $\mathcal{F}$*-structures*) and for each bijection $\sigma : U \to V$ a bijective map $\mathcal{F}[\sigma] : \mathcal{F}[U] \to \mathcal{F}[V]$ such that the following properties hold.

1) $\mathcal{F}$ preserves identity maps, that is for any finite set U it holds that

$$\mathcal{F}[\mathrm{id}_U] = \mathrm{id}_{\mathcal{F}[U]}.$$

2) $\mathcal{F}$ preserves composition of maps, that is, for any bijections of finite sets $\sigma : U \to V$ and $\sigma' : V \to W$ we require that

$$\mathcal{F}[\sigma'\sigma] = \mathcal{F}[\sigma']\mathcal{F}[\sigma].$$

The idea behind this is that finite combinatorial objects are composed out of atoms, and relabelling these atoms yields structurally equivalent objects. For example, we may

consider the species $\mathcal{J}$ of graphs where $\mathcal{J}[U]$ is the collection of all graphs with vertex set U. The bijection $\mathcal{J}[\sigma]$ is given by the relabelling of vertices according to σ.

We say a combinatorial species $\mathcal{F}$ maps any finite set U of *labels* to the finite set $\mathcal{F}[U]$ of $\mathcal{F}$*-objects* and any bijection $\sigma : U \to V$ to the *transport function* $\mathcal{F}[\sigma]$. For any two $\mathcal{F}$-objects $F_U \in \mathcal{F}[U]$ and $F_V \in \mathcal{F}[V]$ that satisfy $\mathcal{F}[\sigma](F_U) = F_V$, we say F_U and F_V are *isomorphic* and σ is an *isomorphism* between them. The object F_U has *size* $|F_U| = |U|$ and U is its *underlying set*. An *unlabelled* $\mathcal{F}$*-object* or *isomorphism type* is an isomorphism class of $\mathcal{F}$-objects. That is, a maximal collection of pairwise isomorphic objects. By abuse of notation, we treat unlabelled objects as if they were regular objects. In our example $\mathcal{J}$, the species of graphs, the size of a graph equals its number of vertices. An isomorphism between $\mathcal{J}$-objects is a graph isomorphism, and unlabelled $\mathcal{J}$-objects are unlabelled graphs in the usual sense.

An $\mathbb{R}_{\geqslant 0}$*-weighted species* $\mathcal{F}^\omega$ consists of a species $\mathcal{F}$ and a *weighting* ω that produces for any finite set U a map

$$\omega_U : \mathcal{F}[U] \to \mathbb{R}_{\geqslant 0}$$

such that $\omega_U = \omega_V \circ \mathcal{F}[\sigma]$ for any bijection $\sigma : U \to V$. Any object $F \in \mathcal{F}[U]$ has *weight* $\omega_U(F)$. By abuse of notation we will often drop the index and write $\omega(F)$ instead of $\omega_U(F)$. Isomorphic structures have the same weight, hence we may define the weight of an unlabelled $\mathcal{F}$-object to be the weight of any representative. The inventory $|\tilde{\mathcal{F}}[n]|_\omega$ is defined as the sum of weights of all unlabelled $\mathcal{F}$-objects of size n. Any species may be considered as a weighted species by assigning weight 1 to each structure, and in this case the inventory counts the number of $\mathcal{F}$-objects. If we do not specify any weighting for a species, then we assume that it is equipped with this canonical weighting.

To any weighted species $\mathcal{F}^\omega$ we may associate its *ordinary generating series*

$$\begin{aligned}\tilde{\mathcal{F}}^\omega(z) &= \sum_{n\geqslant 0} |\tilde{\mathcal{F}}[n]|_\omega z^n \\ &= \sum_{F \text{ unlabelled } \mathcal{F}\text{-object}} \omega(F) z^{|F|}.\end{aligned}$$

We may form the species $\mathrm{Sym}(\mathcal{F})$ of $\mathcal{F}$*-symmetries* by letting $\mathrm{Sym}(\mathcal{F})[U]$ be the set of all pairs (F, σ) with $F \in \mathcal{F}[U]$ an $\mathcal{F}$-structure and σ an *automorphism* of F, that is, a bijection $\sigma : U \to U$ with $\mathcal{F}[\sigma](F) = F$. For any bijection $\gamma : U \to V$ the corresponding transport function $\mathrm{Sym}(\mathcal{F})[\gamma]$ maps a symmetry $(F, \sigma) \in \mathrm{Sym}(\mathcal{F})[U]$ to the symmetry $(\mathcal{F}[\gamma](F), \gamma\sigma\gamma^{-1})$ in $\mathrm{Sym}(\mathcal{F})[V]$. For example, in the species $\mathcal{J}$ of graphs a $\mathcal{J}$-symmetry consists of a pair (G, σ) of a labelled graph G together with one of its graph automorphisms σ. If we relabel the symmetry according to the bijection γ we change the labels of G accordingly and form the graph automorphism $\gamma\sigma\gamma^{-1}$ of the relabelled graph.

There is a canonical weighting on $\mathrm{Sym}(\mathcal{F})$ with weights in the power series ring $\mathbb{R}[[s_1, s_2, \ldots]]$. By abuse of notation, we also denote this weighting by ω. It is given by

$$\omega(F, \sigma) = \omega(F) s_1^{\sigma_1} s_2^{\sigma_2} \cdots$$

with σ_i denoting the number of cycles of length i of the permutation σ. Here we count fixed points as 1-cycles. The *cycle index sum* $Z_{\mathcal{F}^\omega}$ of $\mathcal{F}^\omega$ may be defined by

$$Z_{\mathcal{F}^\omega} = \sum_{n\geqslant 0} \sum_{(F,\sigma)\in \mathrm{Sym}(\mathcal{F})[n]} \omega(F,\sigma)/n! \in \mathbb{R}[[s_1, s_2, \ldots]].$$

The generating series are related by

$$\tilde{\mathcal{F}}^\omega(z) = Z_{\mathcal{F}^\omega}(z, z^2, z^3, \ldots). \tag{4}$$

See Chapter 2.3 in the book by Bergeron, Labelle and Leroux [14] for details. In the example $\mathcal{J}$ of graphs the initial summands of the cycle index sum are given by

$$Z_{\mathcal{J}}(s_1, s_2, \ldots) = 1 + s_1 + (s_1^2 + s_2) + \frac{1}{6}(5s_1^3 + 9s_1s_2 + 4s_3) + \cdots$$

and the ordinary generating series starts with

$$\tilde{\mathcal{J}}(z) = 1 + z + 2z^2 + 3z^3 + \cdots .$$

The main reason for considering symmetries is the following enumerative fact.

Lemma 1 (Number of symmetries)**.** *For each unlabelled $\mathcal{F}$-object t with size n there are precisely $n!$ symmetries $(F,\sigma) \in Sym(\mathcal{F})[n]$ such that the isomorphism type $t(F)$ of the $\mathcal{F}$-object F is equal to t.*

This follows from basic properties of group operations, see for example Joyal [46, Sec. 3]. In particular, if we draw a symmetry (F,σ) from the set $\mathrm{Sym}(\mathcal{F})[n]$ at random with probability proportional to the weight of its $\mathcal{F}$-object, then

$$\mathbb{P}\left(t(\mathsf{F}) = t\right) = \omega(t) / \sum_{s\in\tilde{\mathcal{F}}[n]} \omega(s) \tag{5}$$

for any unlabelled $\mathcal{F}$-object t of size n.

We say that two species $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic*, denoted by $\mathcal{F} \simeq \mathcal{G}$, if there is a family $(\alpha_U)_U$ of bijections $\alpha_U : \mathcal{F}[U] \to \mathcal{G}[U]$, with the index U ranging over all finite sets, such that the following diagram commutes for any bijection $\sigma : U \to V$ of finite sets.

$$\begin{array}{ccc} \mathcal{F}[U] & \xrightarrow{\mathcal{F}[\sigma]} & \mathcal{F}[V] \\ \downarrow{\scriptstyle \alpha_U} & & \downarrow{\scriptstyle \alpha_V} \\ \mathcal{G}[U] & \xrightarrow{\mathcal{G}[\sigma]} & \mathcal{G}[V] \end{array}$$

The family $(\alpha_U)_U$ is then called a *species isomorphism* from $\mathcal{F}$ to $\mathcal{G}$.

Two weighted species $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ are called isomorphic, if there exists a species isomorphism $(\alpha_U)_U$ from $\mathcal{F}$ to $\mathcal{G}$ that preserves the weights, that is, with $\nu(\alpha_U(F)) = \omega(F)$

for each finite set U and $\mathcal{F}$-object $F \in \mathcal{F}[U]$. In this case the cycle index sums and hence also the ordinary generating series of $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ coincide.

There are some natural examples of species that we are going to encounter frequently. The species SET with $\mathrm{SET}[U] = \{U\}$ has only one structure of each size and its cycle index sum is given by

$$Z_{\mathrm{SET}}(s_1, s_2, \ldots) = \exp\left(\sum_{i \geqslant 1} \frac{s_i}{i}\right). \tag{6}$$

The species SEQ of linear orders assigns to each finite set U the set $\mathrm{SEQ}[U]$ of tuples $(u_1, \ldots, u_t)$ of distinct elements with $U = \{u_1, \ldots, u_t\}$. Its cycle index sum is given by

$$Z_{\mathrm{SEQ}}(s_1, s_2, \ldots) = \frac{1}{1 - s_1}.$$

The species $\mathcal{X}$ is given by $\mathcal{X}[U] = \emptyset$ if $|U| \neq 1$ and $\mathcal{X}[U] = \{U\}$ if U is a singleton. The *empty species* 0 is given by $0[U] = \emptyset$ for all U, and the species 1 by $1[\emptyset] = \{\emptyset\}$ and $1[U] = \emptyset$ for all finite non-empty sets U.

3.2 Operations on species

Species may be combined in several ways to form new species.

3.2.1 Products

The *product* $\mathcal{F} \cdot \mathcal{G}$ of two species $\mathcal{F}$ and $\mathcal{G}$ is the species given by

$$(\mathcal{F} \cdot \mathcal{G})[U] = \bigsqcup_{(U_1, U_2)} \mathcal{F}[U_1] \times \mathcal{G}[U_2]$$

with the index ranging over all ordered 2-partitions of U, that is, ordered pairs of (possibly empty) disjoint sets whose union equals U. The transport of the product along a bijection is defined componentwise. Given weightings ω on $\mathcal{F}$ and ν on $\mathcal{G}$, there is a canonical weighting on the product given by

$$\mu(F, G) = \omega(F)\nu(G).$$

This defines the product of weighted species

$$(\mathcal{F} \cdot \mathcal{G})^\mu = \mathcal{F}^\omega \cdot \mathcal{G}^\nu.$$

The corresponding cycle index sum satisfies

$$Z_{(\mathcal{F} \cdot \mathcal{G})^\mu} = Z_{\mathcal{F}^\omega} Z_{\mathcal{G}^\nu}.$$

We also define the powers of a species by

$$(\mathcal{F}^\omega)^i = \mathcal{F}^\omega \cdot \ldots \cdot \mathcal{F}^\omega$$

with i factors in total, and define $(\mathcal{F}^\omega)^0 = 0$ to be the empty species having no objects at all.

Example 2. The species $\mathcal{X}^i$ represents ordered sequences of length i.

3.2.2 Sums

Let $(\mathcal{F}_i)_{i\in I}$ be a family of species such that for any finite set U only finitely many indices i with $\mathcal{F}_i[U] \neq \emptyset$ exist. Then the *sum* $\sum_{i\in I} \mathcal{F}_i$ is a species defined by

$$\left(\sum_{i\in I} \mathcal{F}_i\right)[U] = \bigsqcup_{i\in I} \mathcal{F}_i[U].$$

Given weightings ω_i on $\mathcal{F}_i$, there is a canonical weighting μ on the sum given by

$$\mu(F) = \omega_i(F)$$

for any i and $F \in \mathcal{F}_i[U]$. This defines the sum of the weighted species

$$\left(\sum_{i\in I} \mathcal{F}_i\right)^{\mu} = \sum_{i\in I} \mathcal{F}_i^{\omega_i}.$$

The corresponding cycle index sum is given by

$$Z_{\sum_i \mathcal{F}_i^{\omega_i}} = \sum_i Z_{\mathcal{F}_i^{\omega_i}}.$$

Example 3. The species SEQ of linear orders admits an isomorphism $\mathrm{SEQ} \simeq \sum_{i\geqslant 0} \mathcal{X}^i$.

3.2.3 Derived species

Given a species $\mathcal{F}$, the corresponding *derived* species $\mathcal{F}'$ is defined by

$$\mathcal{F}'[U] = \mathcal{F}[U \cup \{*_U\}]$$

with $*_U$ referring to an arbitrary fixed element not contained in the set U. For example, we could set $*_U = \{U\}$. By abuse of notation, we are often going to drop the index and just refer to the $*$-atom. Any weighting ω on $\mathcal{F}$ may also be viewed as a weighting on $\mathcal{F}'$, by letting the weight of a derived object $F \in \mathcal{F}'[U]$ be given by $\omega_{U\cup\{*_U\}}(F)$. The transport along a bijection $\sigma : U \to V$ is done by applying the transport $\mathcal{F}[\sigma']$ of the bijection $\sigma' : U \cup \{*_U\} \to V \cup \{*_V\}$ with $\sigma'|U = \sigma$. This defines the weighted derived species $(\mathcal{F}')^{\omega}$. Its cycle index sum is given by

$$Z_{(\mathcal{F}')^{\omega}} = \frac{\partial}{\partial s_1} Z_{\mathcal{F}^{\omega}}.$$

Example 4. The derivation of the species SEQ of linear orders admits an isomorphism $\mathrm{SEQ}' \simeq \mathrm{SEQ}^2$, since a derived sequence of atoms may be decomposed into the two sequences of atoms that come before and after the $*$-placeholder.

3.2.4 Pointing

For any species $\mathcal{F}$ we may form the *pointed* species $\mathcal{F}^\bullet$. It is given by the product of species

$$\mathcal{F}^\bullet = \mathcal{X} \cdot \mathcal{F}'.$$

(Recall that $\mathcal{X}$ denotes the species consisting of single object of size 1.) In other words, an $\mathcal{F}^\bullet$-object is a pair (F, v) of an $\mathcal{F}$-object F and a distinguished label v which we call the root of the object. Any weighting ω on $\mathcal{F}$ may also be considered as a weighting on $\mathcal{F}^\bullet$, by letting the weight of (F, v) be given by $\omega(F)$. This choice of weighting is consistent with the natural weighting given by the product and derivation operation $\mathcal{X} \cdot \mathcal{F}'$, if we assign weight 1 to the unique object of $\mathcal{X}$. The corresponding cycle index sum is consequently given by

$$Z_{(\mathcal{F}^\bullet)^\omega} = s_1 \frac{\partial}{\partial s_1} Z_{\mathcal{F}^\omega}.$$

3.2.5 Substitution

Given species $\mathcal{F}$ and $\mathcal{G}$ with $\mathcal{G}[\emptyset] = \emptyset$, we may form the *substitution* or *composition* $\mathcal{F} \circ \mathcal{G}$ as the species with object sets

$$(\mathcal{F} \circ \mathcal{G})[U] = \bigcup_\pi \left(\{\pi\} \times \mathcal{F}[\pi] \times \prod_{Q \in \pi} \mathcal{G}[Q] \right),$$

with the index π ranging over all unordered partitions of the set U. Here the transport $(\mathcal{F} \circ \mathcal{G})[\sigma]$ along a bijection $\sigma : U \to V$ is done as follows. For any object $(\pi, F, (G_Q)_{Q \in \pi})$ in $(\mathcal{F} \circ \mathcal{G})[U]$ define the partition

$$\hat{\pi} = \{\sigma(Q) \mid Q \in \pi\},$$

and let

$$\hat{\sigma} : \pi \to \hat{\pi}$$

denote the induced bijection between the partitions. Then set

$$(\mathcal{F} \circ \mathcal{G})[\sigma](\pi, F, (G_Q)_{Q \in \pi}) = (\hat{\pi}, \mathcal{F}[\hat{\sigma}](F), (\mathcal{G}[\sigma|_Q](g_Q))_{\sigma(Q) \in \hat{\pi}}).$$

That is, the transport along the induced bijection of partitions gets applied to the $\mathcal{F}$-object and the transports along the restrictions $\sigma|_Q$, $Q \in \pi$ get applied to the $\mathcal{G}$-objects. Often, we are going to write $\mathcal{F}(\mathcal{G})$ instead of $\mathcal{F} \circ \mathcal{G}$. Given a weighting ω on $\mathcal{F}$ and a weighting ν on $\mathcal{G}$, there is a canonical weighting μ on the composition given by

$$\mu(\pi, F, (G_Q)_{Q \in \pi}) = \omega(F) \prod_{Q \in \pi} \nu(Q).$$

This defines the composition of weighted species

$$(\mathcal{F} \circ \mathcal{G})^\mu = \mathcal{F}^\omega \circ \mathcal{G}^\nu.$$

The corresponding cycle index sum is given by

$$Z_{(\mathcal{F}\circ\mathcal{G})^\mu}(s_1, s_2, \ldots) = Z_{\mathcal{F}^\omega}(Z_{\mathcal{G}^\nu}(s_1, s_2, \ldots), Z_{\mathcal{G}^{\nu^2}}(s_2, s_4, \ldots), Z_{\mathcal{G}^{\nu^3}}(s_3, s_6, \ldots), \ldots). \tag{7}$$

Here ν^i denotes the weighting with $(\nu^i)(G) = \nu(G)^i$ for all $\mathcal{G}$-structures G.

Example 5. Any graph may viewed as the collection of its connected components. Hence the species $\mathcal{J}$ of graphs admits an isomorphism $\mathcal{J} \simeq \mathrm{SET} \circ \mathcal{C}$ with $\mathcal{C}$ denoting the species of connected graphs. By Equation (6) for the cycle index sum of the species SET and Equation (4) for the relation of cycle index sums with ordinary generating sums this yields

$$\tilde{\mathcal{J}}(z) = \exp\left(\sum_{i\geqslant 1} \frac{\tilde{\mathcal{C}}(z^i)}{i}\right).$$

3.2.6 Restriction

For any subset $\Omega \subset \mathbb{N}_0$ we may restrict a weighted species $\mathcal{F}^\omega$ to objects whose size lies in Ω and denote the result by $\mathcal{F}^\omega_\Omega$.

Example 6. The restriction SEQ_Ω admits an isomorphism $\mathrm{SEQ}_\Omega \simeq \sum_{i\in\Omega} \mathcal{X}^i$.

3.2.7 Relations between the different operations

The interplay of the operations discussed in this section is described by a variety of natural isomorphisms. The two most important are the *product rule* and the *chain rule*.

Proposition 7 ([46])**.** *Let $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ be weighted species.*

1. *There is a canonical choice for an isomorphism*

$$(\mathcal{F}^\omega \cdot \mathcal{G}^\nu)' \simeq (\mathcal{F}^\omega)' \cdot \mathcal{G}^\nu + \mathcal{F}^\omega \cdot (\mathcal{G}^\nu)'.$$

2. *Suppose that $\mathcal{G}[\emptyset] = \emptyset$. Then there is also a canonical isomorphism*

$$(\mathcal{F}^\omega \circ \mathcal{G}^\nu)' \simeq ((\mathcal{F}^\omega)' \circ \mathcal{G}^\nu) \cdot (\mathcal{G}^\nu)'.$$

We may easily verify the product rule, as the $*$-label in $(\mathcal{F}^\omega \cdot \mathcal{G}^\nu)'$ may either belong the $\mathcal{F}$-structure, accounting for the summand $(\mathcal{F}^\omega)' \cdot \mathcal{G}^\nu$, or to the $\mathcal{G}$-structure, accounting for the second summand. The chain rule also has an intuitive explanation. The idea is that the partition class or $\mathcal{G}$-structure containing the $*$-label in an $(\mathcal{F}^\omega \circ \mathcal{G}^\nu)'$-structure distinguishes an atom of the $\mathcal{F}$-structure. Hence the $(\mathcal{F}^\omega \circ \mathcal{G}^\nu)'$-structure consists of a $(\mathcal{F}^\omega)' \circ \mathcal{G}^\nu)$ composite structure, where all atoms of the $\mathcal{F}$-structure receive a regular $\mathcal{G}$-structure, except for a marked $*$-atom, to which we assign a derived $\mathcal{G}$-structure, which accounts for the extra factor $(\mathcal{G}^\nu)'$.

3.3 Symmetries of the substitution operation

We are going to need detailed information on the structure of the symmetries of the composition $\mathcal{F} \circ \mathcal{G}$. The exposition of this section follows Joyal [46, Sec. 3.2] and Bergeron, Labelle and Leroux [14, Sec. 4.3]. We are going to discuss the following result.

Lemma 8 (Parametrization of the symmetries of the substitution)**.** *Up to isomorphism of symmetries, any $\mathcal{F} \circ \mathcal{G}$-symmetry may be constructed as described below from an $\mathcal{F}$-symmetry (F, σ) together with a family of $\mathcal{G}$-symmetries $(G_\tau, \sigma_\tau)_\tau$ with the index τ ranging over all cycles of the permutation σ.*

There is much more to this result, as it lies at the heart of the proof of Equation (7). We refer the inclined reader to the mentioned literature for details. For our purposes, it is sufficient to understand how the symmetry and in particular its cycles get assembled. We are going to use this later in order to define random unlabelled structures based on a tree-like decomposition of symmetries.

The method of construction referred to in Lemma 8 is a bit involved, hence let us first recall what an $\mathcal{F} \circ \mathcal{G}$-symmetry is by definition. Let U be a finite set. Any element of $\mathrm{Sym}(\mathcal{F} \circ \mathcal{G})[U]$ consists of the following objects: a partition π of the set U, an $\mathcal{F}$-structure $F \in \mathcal{F}[\pi]$, a family of $\mathcal{G}$-structures $(G_Q)_{Q \in \pi}$ with $G_Q \in \mathcal{G}[Q]$ and a permutation $\sigma : U \to U$. The permutation σ is required to permute the partition classes and induce an automorphism

$$\bar{\sigma} : \pi \to \pi, \ Q \mapsto \sigma(Q)$$

of the $\mathcal{F}$-object F. Moreover, for any partition class $Q \in \pi$ the restriction $\sigma|_Q : Q \to \sigma(Q)$ is required to be an isomorphism from G_Q to $G_{\sigma(Q)}$.

Note that for any cycle

$$\tau = (Q_1, \dots, Q_\ell)$$

of $\bar{\sigma}$, it follows that

$$\sigma^\ell|_{Q_1} : Q_1 \to Q_1$$

is an automorphism of G_{Q_1}. Hence

$$(G_\tau, \sigma_\tau) := (G_{Q_1}, \sigma^\ell|_{Q_1})$$

is a $\mathcal{G}$-symmetry. The symmetry S together with the bijections

$$\gamma_i := \sigma|_{Q_i}$$

for $1 \leqslant i \leqslant \ell - 1$ already contain all information about the $\mathcal{G}$-objects $G_{Q_1}, \dots, G_{Q_\ell}$ and the restriction $\sigma|_{Q_1 \cup \dots \cup Q_\ell}$. Indeed, it holds that

$$G_{Q_{i+1}} = \mathcal{G}[\gamma_i \cdots \gamma_1](G_\tau)$$

for all $1 \leqslant i \leqslant \ell - 1$, hence we may reconstruct the $\mathcal{G}$-objects. The bijections γ_i contain all information about $\sigma_{Q_1 \cup \dots \cup Q_{\ell-1}}$, and it holds that

$$\sigma|_{Q_\ell} = \gamma_{\ell-1}^{-1} \cdots \gamma_1^{-1} \sigma_\tau.$$

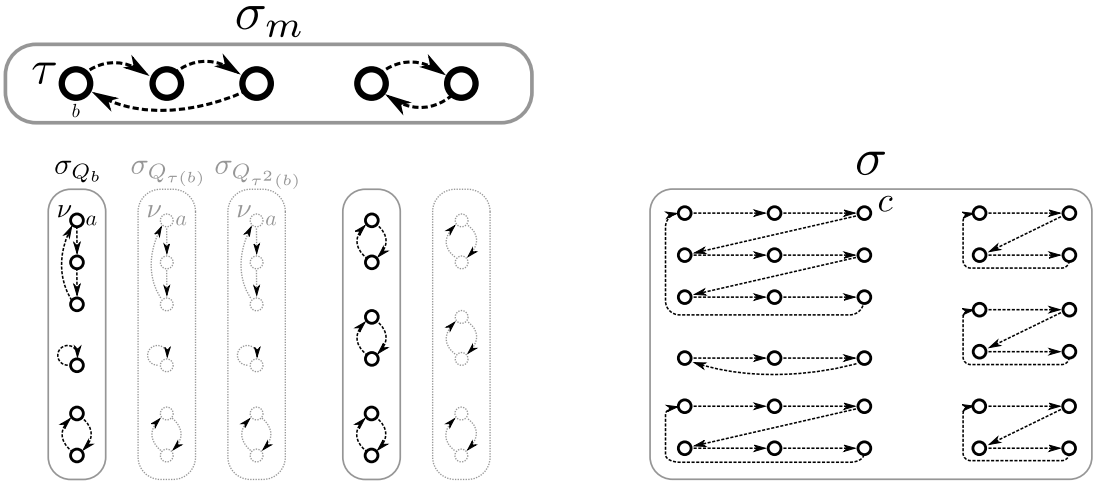

Figure 1: Composition of cycles.

In particular, any k-cycle $(a_1, \ldots, a_k)$ of the permutation σ_τ corresponds to the $k\ell$-cycle

$$(a_1, \sigma(a_1), \ldots, \sigma^{\ell-1}(a_1), a_2, \sigma(a_2), \ldots, \sigma^{\ell-1}(a_2), \ldots, a_k, \sigma(a_k), \ldots, \sigma^{\ell-1}(a_k))$$

of the permutation $\sigma|_{Q_1 \cup \ldots \cup Q_\ell}$.

This implies that any $\mathcal{F} \circ \mathcal{G}$-symmetry is isomorphic to a symmetry

$$((\pi, F, (G_Q)_{Q \in \pi}), \sigma)$$

constructed in the following way, which is illustrated in Figure 1. Start with choosing an $\mathcal{F}$-symmetry (m, σ_m). For any cycle τ of the permutation σ_m choose a $\mathcal{G}$-symmetry (G_τ, σ_τ) and let Q_τ denote its set of labels. For every atom e of the cycle τ set

$$Q_e := Q_\tau \times \{e\}$$

and

$$(G_{Q_e}, \sigma_{Q_e}) := \mathrm{Sym}(\mathcal{G})[f_e](G_\tau, \sigma_\tau)$$

with $f_e : Q_\tau \to Q_e$ the canonical bijection. For any label e of the $\mathcal{F}$-structure m set

$$f(e) := Q_e$$

and let π denote the set of all sets Q_e. Thus

$$F := \mathcal{F}[f](m)$$

is an $\mathcal{F}$-structure with label set π and

$$C := (\pi, F, (G_Q)_{Q \in \pi})$$

is an $\mathcal{F} \circ \mathcal{G}$-structure. Let τ be a cycle of σ_m and ν a cycle of σ_τ. Fix an atom $b = b(\tau)$ of τ and an atom $a = a(\nu)$ of ν. Let ℓ denote the length of τ and k the length of ν. Form the composed cycle c by

$$((a,b),\ \dots\ ,(a,\tau^{\ell-1}(b)),\ (\nu(a),b),\ \dots\ ,(\nu(a),\tau^{\ell-1}(b)),\ \dots\ ,\\ (\nu^{k-1}(a),b),\ \dots\ ,(\nu^{k-1}(a),\tau^{\ell-1}(b))).$$

Then the product σ of all composed cycles (one for each choice of (c,ν)) is an automorphism of the $\mathcal{F} \circ \mathcal{G}$-structure C. The composed cycles are pairwise disjoint, hence it does not matter in which order we take the product. Note that σ does not depend on the choice of the a's but different choices of the b's result in a different automorphism σ. More precisely, if for a given cycle τ of σ_m we choose $\tau(b)$ instead of b, then the resulting automorphism is given by the conjugation

$$(\mathrm{id},\tau)\sigma(\mathrm{id},\tau)^{-1}$$

instead of σ. But (id,τ) is an automorphism of the $\mathcal{F} \circ \mathcal{G}$-structure C, hence the resulting symmetry

$$(C,(\mathrm{id},\tau)\sigma(\mathrm{id},\tau)^{-1})$$

is isomorphic to (C,σ). This implies that the isomorphism type of (C,σ) does not depend on the choices of the a's and b's. Fixing any canonical way of making these choices yields the construction of Lemma 8.

3.4 Weighted Boltzmann distributions and samplers

Boltzmann distributions crop up in the study of local limit of random discrete structures and in the limit of convergent unlabelled Gibbs partitions. A *Boltzmann sampler* is a possibly recursive procedure involving random choices that generates a structure according to a Boltzmann distribution. For example, a subcritical or critical Galton–Watson tree may be considered as a Boltzmann distributed plane tree. A recursive sampler in this setting is a procedure that draws the offspring of the root and then calls itself for each offspring vertex.

3.4.1 Boltzmann distributions

Let $\mathcal{F}^\omega$ be a weighted species. For any $y \geqslant 0$ satisfying $0 < \tilde{\mathcal{F}}^\omega(y) < \infty$, the corresponding *Boltzmann distribution for unlabelled $\mathcal{F}$-objects* is given by

$$\mathbb{P}_{\tilde{\mathcal{F}}^\omega,y}(\tilde{F}) = \tilde{\mathcal{F}}^\omega(y)^{-1}\omega(\tilde{F})y^{|\tilde{F}|}, \quad \tilde{F} \text{ an unlabelled } \mathcal{F}\text{-object.} \tag{8}$$

Given a sequence $\mathbf{y} = (y_j)_{j\in\mathbb{N}}$ of non-negative parameters y_j satisfying $0 < Z_{\mathcal{F}^\omega}(\mathbf{y}) < \infty$, the corresponding *Pólya-Boltzmann distribution* is given by

$$\mathbb{P}_{\mathrm{Sym}(\mathcal{F})^\omega,\mathbf{y}}(F,\sigma) = Z_{\mathcal{F}^\omega}(\mathbf{y})^{-1}\omega(F)\frac{y_1^{\sigma_1}y_2^{\sigma_2}\cdots}{|(F,\sigma)|!}, \quad (F,\sigma) \in \sum_{m\geqslant 0}\mathrm{Sym}(\mathcal{F})[m], \tag{9}$$

with σ_i denoting the number of i-cycles of a permutation σ. By Lemma 1 and Equation (4), the Boltzmann distribution for unlabelled objects may be considered as the marginal distribution of the $\mathcal{F}$-object in special cases of the Pólya-Boltzmann distribution. That is, the $\mathcal{F}$-object of a $\mathbb{P}_{\mathrm{Sym}(\mathcal{F})^\omega,(y,y^2,y^3,\ldots)}$-distributed $\mathcal{F}$-symmetry follows a $\mathbb{P}_{\tilde{\mathcal{F}}^\omega,y}$-distribution.

3.4.2 Boltzmann samplers

The following lemma allows us to construct Boltzmann distributed random variables in the unlabelled setting for the sum, product and composition of species. The results of this section have been established by Bodirsky, Fusy, Kang and Vigerske [17, Prop. 38] for species without weights, and the generalization to the weighted setting is straight-forward.

Lemma 9 (Weighted Pólya-Boltzmann samplers)**.**

1. *Let $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ be weighted species, and let X and Y be independent random variables with distributions $\mathcal{L}(X) = \mathbb{P}_{Sym(\mathcal{F})^\omega,\mathbf{y}}$ and $\mathcal{L}(Y) = \mathbb{P}_{Sym(\mathcal{G})^\omega,\mathbf{y}}$. Then (X,Y) may be interpreted as an $(\mathcal{F}\cdot\mathcal{G})$-symmetry over the set $[|X|]\sqcup[|Y|]$. If α denotes a uniformly at random drawn bijection from this set to $[|X|+|Y|]$, then*

$$\mathcal{L}\left((\mathcal{F}\cdot\mathcal{G})[\alpha](X,Y)\right) = \mathbb{P}_{Sym(\mathcal{F}\cdot\mathcal{G})^\omega,\mathbf{y}}.$$

2. *Let $(\mathcal{F}_i^{\omega_i})_{i\in I}$ be a family of weighted species, and $(X_i)_{i\in I}$ a family of independent random variables with distributions $\mathbb{P}_{Sym(\mathcal{F}_i)^{\omega_i},\mathbf{y}}$ such that $\sum_i Z_{\mathcal{F}_i^{\omega_i}}(\mathbf{y}) < \infty$. If $K\in I$ gets drawn at random with probability proportional to $Z_{\mathcal{F}_K^{\omega_K}}(\mathbf{y})$, that is*

$$\mathbb{P}\left(K=k\right) = Z_{\mathcal{F}_k^{\omega_k}}(\mathbf{y}) / \sum_i Z_{\mathcal{F}_i^{\omega_i}}(\mathbf{y}),$$

then

$$\mathcal{L}(X_K) = \mathbb{P}_{Sym(\sum_i \mathcal{F}_i^{\omega_i}),\mathbf{y}}.$$

3. *Let $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ be species such that $\mathcal{G}^\nu[\emptyset]=\emptyset$ and let $\mathbf{y}=(y_j)_{j\in\mathbb{N}}$ a family of non-negative parameters with $0 < Z_{\mathcal{G}^\nu}(\mathbf{y}) < \infty$ and $0 < Z_{\mathcal{F}^\omega\circ\mathcal{G}^\nu}(\mathbf{y}) < \infty$. For each k set*

$$\mathbf{y}^k = (y_k, y_{2k}, y_{3k}, \ldots).$$

Let $X=(F,\sigma)$ be a $\mathbb{P}_{Sym(\mathcal{F}^\omega),(Z_{\mathcal{G}^{\nu^1}}(\mathbf{y}),Z_{\mathcal{G}^{\nu^2}}(\mathbf{y^2}),\ldots)}$-distributed random $\mathcal{F}$-symmetry and let $(Y_{i,k})_{i,k\in\mathbb{N}}$ an independent family (that is also independent of X) of random $\mathcal{G}$-symmetries such that $Y_{i,k}$ follows a $\mathbb{P}_{Sym(\mathcal{G}^\nu),\mathbf{y}^k}$-distribution for all k,i. We may canonically assign to each cycle τ of the random permutation σ a unique element Y_τ of the list $(Y_{i,|\tau|})_{i\geqslant 1}$. For example, we could do this by ordering for each ℓ the cycles of σ having length ℓ according to their smallest atom, and assign $Y_{i,\ell}$ to the ith cycle in the ordering. Then $(X,(Y_\tau)_\tau)$ corresponds according to Lemma 8 to an $\mathcal{F}\circ\mathcal{G}$-symmetry over some set M. Draw a bijection $\alpha: M\to[|M|]$ uniformly at random. Then

$$\mathcal{L}((\mathcal{F}\circ\mathcal{G})[\alpha](X,(Y_\tau)_\tau)) = \mathbb{P}_{Sym(\mathcal{F}^\omega\circ\mathcal{G}^\nu),\mathbf{y}}.$$

3.5 Combinatorial specifications and recursive Boltzmann samplers

3.5.1 Motivation

A recursive procedure is a list of instructions that are to be followed step by step and may contain references to the procedure itself. For example, a Galton–Watson tree may be described by the procedure that starts with a root vertex and attaches a random number of independent calls of itself.

Often one encounter species admitting a recursive isomorphism such as $\mathcal{F}^\omega \simeq \mathcal{X} + (\mathcal{F}^\omega)^2$. If this decomposition satisfies a certain property (that is, the assumptions of Theorem 12 below - but let us call it property *(R)* for now), then for any admissible parameter we may apply the rules from Section 3.4.2 for the sum, product and composition of species in order to construct a recursive (Pólya-)Boltzmann sampler for $\mathcal{F}$. That is, a recursive procedure that terminates almost surely and samples objects according to the (Pólya)-Boltzmann distribution.

For the given example $\mathcal{F}$, such a recursive Boltzmann sampler would first, by the sum rule, make a coin flip in order to determine whether it terminates with a single vertex (a Boltzmann sampler for $\mathcal{X}^\kappa$), or creates, by the product rule, an ordered pair of independent calls of itself. In other words, its a Galton–Watson tree. Property *(R)* will guarantee that this process terminates almost surely, so we also know that this Galton–Watson tree must be critical or subcritical. It is clear that not any recursive decomposition can have this desired property. For example, the species 1 which consists of a single object with size zero admits an isomorphism $1 \simeq 1 \cdot 1$, but applying the product rule yields a recursive procedure that never terminates.

Precisely stating the property *(R)* requires us to introduce the complex concepts of weighted multi-sort species and samplers, as well as related operations such as combinatorial composition and partial derivatives in this context.

3.5.2 Combinatorial specifications

One should think of multi-sort species as species whose atoms have different colours, such as vertices of bicoloured graphs. This concept makes sense for arbitrarily many sorts, but for our purposes it will be sufficient to work with only 2 types of atoms.

A 2-*sort species* $\mathcal{H}$ is a functor that maps any pair $U = (U_1, U_2)$ of finite sets to a finite set $\mathcal{H}[U] = \mathcal{H}[U_1, U_2]$ and any pair $\sigma = (\sigma_1, \sigma_2)$ of bijections $\sigma_i : U_i \to V_i$ to a bijection $\mathcal{H}[\sigma] : \mathcal{H}[U] \to \mathcal{H}[V]$ (with $V = (V_1, V_2)$) in such a way, that identity maps and composition of maps are preserved. We may interpret the pair U as a set where any element may either have type 1 or type 2. In this context we often use the term 1-*sort species* for regular combinatorial species in order to avoid confusion.

Example 10. We may interpret the collection of properly bicoloured graphs as a 2-sort species $\mathcal{P}$. Here objects from the set $\mathcal{P}[U_1, U_2]$ are graphs with vertex set given by the outer disjoint union $U_1 \sqcup U_2$. Vertices from $U_1 \subset U_1 \sqcup U_2$ are coloured blue and vertices from $U_2 \subset U_1 \sqcup U_2$ are coloured red in such a way, that there are no monochromatic edges.

That is, for any edge $e = \{x, y\}$ of the graph we require that x and y have distinct colours. Graph isomorphisms of such object are of course required to preserve colours.

The operations of sum and product and composition extend naturally to the multi-sort-context: Let $\mathcal{H}$ and $\mathcal{K}$ be 2-sort species and $U = (U_1, U_2)$ a pair of finite sets. The *sum* is defined by

$$(\mathcal{H} + \mathcal{K})[U] = \mathcal{H}[U] \sqcup \mathcal{K}[U].$$

The transport function $(\mathcal{H} + \mathcal{K})[\sigma]$ is the unique bijection that restricts to $\mathcal{H}[\sigma_1]$ on the subset $\mathcal{H}[U]$, and to $\mathcal{K}[\sigma_2]$ on the subset $\mathcal{K}[U]$ (with $\mathcal{H}[U]$ and $\mathcal{K}[U]$ being interpreted as disjoint subsets of $\mathcal{H}[U] \sqcup \mathcal{K}[U]$). Given another pair $W = (W_1, W_2)$ of finite sets we write $U = V + W$ if $U_i = V_i \cup W_i$ and $V_i \cap W_i = \emptyset$ for all $i \in \{1, 2\}$. The *product* is defined by

$$(\mathcal{H} \cdot \mathcal{K})[U] = \bigsqcup_{V+W=U} \mathcal{H}[V] \times \mathcal{K}[W].$$

The transport functions of the product are defined componentwise. The *partial derivatives* are given by

$$\partial_1\mathcal{H}[U] = \mathcal{H}[U_1 \cup \{*_{U_1}\}, U_2] \quad \text{and} \quad \partial_2\mathcal{H}[U] = \mathcal{H}[U_1, U_2 \cup \{*_{U_2}\}].$$

We define $\partial_1\mathcal{H}[\sigma] = \mathcal{H}[\bar{\sigma}]$ with $\bar{\sigma}$ denoting the unique bijective extension of σ to the set $(U_1 \cup \{*_{U_1}\}) \times U_2$. The transport $\partial_2\mathcal{H}[\sigma]$ is defined analogously.

Given 1-sort species $\mathcal{F}_1$ and $\mathcal{F}_2$ with no objects of size 0 we may form the *composition* $\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$ which is also a 1-sort species. An $\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$-structure over a set M is a quadruple $(\pi, \chi, \alpha, \beta)$ such that:

1. π is a partition of the set M.
2. $\chi : \pi \to \{1, 2\}$ is a function assigning to each class a sort.
3. α a function that assigns to each class $Q \in \pi$ an $\mathcal{F}_{\chi(Q)}$ object $\alpha(Q) \in \mathcal{F}_{\chi(Q)}[Q]$.
4. β an $\mathcal{H}$-structure over the pair $(\chi^{-1}(1), \chi^{-1}(2))$.

That is, we partition the set M and assign one of two types to each partition class in an arbitrary way. Each class Q of type 1 receives an $\mathcal{F}_1$-object from $\mathcal{F}_1[Q]$ and likewise for the type 2 classes. Finally we select an $\mathcal{H}$-structure over the typed collection of partition classes. The transport function is defined in a canonical way, the inclined reader may consult [46, Sec. 5] for details. This construction is *functorial*: any pair of species isomorphisms α_1, α_2 with $\alpha_i : \mathcal{F}_i \simeq \mathcal{G}_i$ *induces* an isomorphism

$$\mathcal{H}[\alpha_1, \alpha_2] : \mathcal{H}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathcal{H}(\mathcal{G}_1, \mathcal{G}_2).$$

If $\mathcal{F}_1^{\omega_1}$ and $\mathcal{F}_2^{\omega_2}$ are weighted species with no objects of size zero, then the composition $\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$ becomes a weighted species that we denote by $\mathcal{H}(\mathcal{F}_1^{\omega_1}, \mathcal{F}_2^{\omega_2})$. The weight of an object $(\pi, \chi, \alpha, \beta)$ is given by

$$\left(\prod_{Q \in \chi^{-1}(1)} \omega_1(\alpha(Q))\right)\left(\prod_{Q \in \chi^{-1}(2)} \omega_2(\alpha(Q))\right).$$

Recall that $\mathcal{X}$ denotes the species with a unique object of size one. We may consider the 2-sort species $\mathcal{X}_1$ that has a unique element over any set consisting of a single atom with type 1. The 2-sort species $\mathcal{X}_2$ is defined analogously.

Example 11. The sum and product of 1-sort species may be seen as special cases of the 2-sort composition: For any 1-sort species $\mathcal{F}_1$ and $\mathcal{F}_2$ it holds that

$$\begin{aligned} \mathcal{F}_1 + \mathcal{F}_2 &\simeq \mathcal{H}_1(\mathcal{F}_1, \mathcal{F}_2) \qquad \text{with} \qquad \mathcal{H}_1 := \mathcal{X}_1 + \mathcal{X}_2 \\ \mathcal{F}_1 \cdot \mathcal{F}_2 &\simeq \mathcal{H}_2(\mathcal{F}_1, \mathcal{F}_2) \qquad \text{with} \qquad \mathcal{H}_2 := \mathcal{X}_1 \cdot \mathcal{X}_2. \end{aligned}$$

Recall that 0 denotes the empty 1-sort species with $0[U] = \emptyset$ for all finite sets U. A solution of the system $\mathcal{Y} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ is pair $(\mathcal{A}, \alpha)$ of a species $\mathcal{A}$ with $\mathcal{A}(0) = 0$ and an isomorphism $\alpha : \mathcal{A} \simeq \mathcal{H}(\mathcal{X}, \mathcal{A})$. An isomorphism of two solutions $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ is an isomorphism of species $u : \mathcal{A} \simeq \mathcal{B}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha} & \mathcal{H}(\mathcal{X}, \mathcal{A}) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathcal{H}(\mathrm{id}, u)} \\ \mathcal{B} & \xrightarrow{\beta} & \mathcal{H}(\mathcal{X}, \mathcal{B}) \end{array}$$

We may now state Joyal's implicit species theorem.

Theorem 12 ([46], Théorème 6)**.** *Let $\mathcal{H}$ be a 2-sort species satisfying $\mathcal{H}(0,0) = 0$. If $(\partial_2 \mathcal{H})(0,0) = 0$, then the system*

$$\mathcal{Y} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$$

has up to isomorphism only one solution. Moreover, between any two given solutions there is exactly one isomorphism.

We say that an isomorphism $\mathcal{F} \simeq \mathcal{H}(\mathcal{X}, \mathcal{F})$ is a *combinatorial specification* for the species $\mathcal{F}$ if the 2-sort species $\mathcal{H}$ satisfies the requirements of Theorem 12. If $\mathcal{F}$ is endowed with a weighting ω we additionally require that $\mathcal{F}^\omega \simeq \mathcal{H}(\mathcal{X}, \mathcal{F}^\omega)$ is weight-preserving.

Example 13. Let $\mathcal{F}$ be the species of binary unordered rooted trees, where each tree receives weight 1. Any such tree is either a single root vertex, or root vertex with two binary trees dangling from it. This yields an isomorphism

$$\mathcal{F} \simeq \mathcal{X} + \mathcal{X} \cdot \mathcal{F}^2,$$

where the two summands correspond to the two described cases. This may be reformulated by

$$\mathcal{F} \simeq \mathcal{H}(\mathcal{X}, \mathcal{F}) \qquad \text{with} \qquad \mathcal{H} := \mathcal{X}_1 + \mathcal{X}_1 \cdot \mathcal{X}_2^2.$$

It holds that $(\partial_2 \mathcal{H})(0,0) = 0$. Hence $\mathcal{F} \simeq \mathcal{H}(\mathcal{X}, \mathcal{F})$ is a combinatorial specification.

3.5.3 Recursive Boltzmann samplers

Given a combinatorial specification

$$\mathcal{F}^\omega \simeq \mathcal{H}(\mathcal{X}, \mathcal{F}^\omega)$$

and parameters $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}_{\geqslant 0}$ with $0 < Z_{\mathcal{F}^\omega}(\mathbf{x}) < \infty$ we may apply the rules of Lemma 9 recursively to construct a recursive procedure that is guaranteed to terminated almost surely and sample according to a Boltzmann distribution $\mathbb{P}_{Z_{\mathcal{F}^\omega},\mathbf{x}}$. A justification of this fact is given by Bodirsky, Fusy, Kang and Vigerske [17, Thm. 40] in a more specific settings for species without weights (and using a slightly different terminology), but the generalization to the weighted setting is straight-forward.

Let us illustrate this construction of recursive Boltzmann samplers with an example.

Example 14. As discussed in Example 13, the species $\mathcal{F}$ of binary unordered rooted trees admits a combinatorial specification

$$\mathcal{F} \simeq \mathcal{X} + \mathcal{X} \cdot \mathcal{F}^2$$

that we may rewrite as

$$\mathcal{F} \simeq \mathcal{X} \cdot \mathcal{G}(\mathcal{F}) \qquad \text{with} \qquad \mathcal{G} := 1 + \mathcal{X}^2$$

Hence for each parameter $\mathbf{x} = (x_i)_i \in \mathbb{R}^{\mathbb{N}}_{\geqslant 0}$ with $0 < Z_{\mathcal{F}}(\mathbf{x}) < \infty$ we may apply the rules of Lemma 9 to obtain a recursive procedure that terminates almost surely and whose output follows a $\mathbb{P}_{Z_{\mathcal{F}},\mathbf{x}}$-distribution. Briefly described, the procedure starts with a root-vertex and then draws a $\mathcal{G}$-symmetry according to a Boltzmann distribution $\mathbb{P}_{Z_{\mathcal{G}},(Z_{\mathcal{F}}(\mathbf{x}^i))_{i\geqslant 1}}$, with $\mathbf{x}^i = (x_i, x_{2i}, \ldots)$. There are two different outcomes. Either the $\mathcal{G}$-symmetry has size 0, in which the sampler stops. Or it has size 2, in which case the sampler calls itself recursively twice to form the subtrees of the root.

4 Unlabelled Gibbs partitions and subexponential sequences

The term Gibbs partition was introduced by Pitman [63] in his comprehensive survey on combinatorial stochastic processes. It describes a model of random partitions of sets, where the collection of classes as well as each individual partition class are endowed with a weighted structure.

Many structures such as classes of graphs may also be viewed up to symmetry. The symmetric group acts in a canonical way on the collection of composite structures over a fixed set, and its orbits may be identified with the unlabelled objects. Sampling such an isomorphism class with probability proportional to its weight is the natural unlabelled version of the Gibbs partitions model.

Let $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ be weighted species such that $\mathcal{G}[\emptyset] = \emptyset$, and such that the ordinary generating series $\widetilde{\mathcal{F}^\omega \circ \mathcal{G}^\nu}(z)$ is not a polynomial. An *unlabelled Gibbs partition* is a random composite structure

$$\mathsf{S}_n = (\pi_n, \mathsf{F}_n, (\mathsf{G}_Q)_{Q\in\pi_n})$$

sampled from the set of all n-sized unlabelled $\mathcal{F} \circ \mathcal{G}$-objects with probability proportional to its weight. We are going to describe the asymptotic behaviour of the remainder R_n obtained by deleting "the" largest component from S_n. More specifically, we make a uniform choice of a component $Q_0 \in \pi_n$ having maximal size, and let F'_n denote the $\mathcal{F}'$-object obtained from the $\mathcal{F}$-object F_n by relabelling the Q_0 atom of F_n to a $*$-placeholder.

This yields an unlabelled $\mathcal{F}' \circ \mathcal{G}$-object

$$\mathsf{R}_n := (\pi_n \setminus \{Q_0\}, \mathsf{F}'_n, (\mathsf{G}_Q)_{Q \in \pi_n \setminus \{Q_0\}}).$$

In the so called convergent case, the remainder R_n is stochastically bounded and even converges in total variation toward a limit object.

Theorem 15 ([69, Thm. 3.1])**.** *Suppose that the ordinary generating series $\tilde{\mathcal{G}}^\nu(z)$ has positive radius of convergence ρ, and that the coefficients $g_i = [z^i]\tilde{\mathcal{G}}^\nu(z)$ satisfy*

$$\frac{g_n}{g_{n+1}} \sim \rho, \qquad \frac{1}{g_n} \sum_{i+j=n} g_i g_j \sim 2\tilde{\mathcal{G}}^\nu(\rho) < \infty.$$

Suppose further that

$$Z_{\mathcal{F}^\omega}(\tilde{\mathcal{G}}^\nu(\rho) + \epsilon, \tilde{\mathcal{G}}^{\nu^2}((\rho+\epsilon)^2), \tilde{\mathcal{G}}^{\nu^3}((\rho+\epsilon)^3), \ldots) < \infty$$

for some $\epsilon > 0$. Then

$$[z^n]\widetilde{\mathcal{F}^\omega \circ \mathcal{G}^\nu}(z) \sim (\widetilde{\mathcal{F}')^\omega \circ \mathcal{G}^\nu}(\rho)[z^n]\mathcal{G}^\nu(z).$$

and

$$d_{\mathrm{TV}}(\mathsf{R}_n, \mathsf{R}) \to 0, \qquad n \to \infty$$

with R denoting a random unlabelled $\mathcal{F}' \circ \mathcal{G}$-element that follows a $\mathbb{P}_{\widetilde{(\mathcal{F}')^\omega \circ \mathcal{G}^\nu}, \rho}$-Boltzmann distribution.

Note that the requirements of the theorem are satisfied if $g_n = f(n)n^{-\beta}\rho^{-n}$ for some constant $\beta > 1$ and a function f that varies slowly at infinity.

5 Probabilistic and combinatorial tools

5.1 The lattice case of the multivariate central local limit theorem

We will make frequent use of the classical local limit theorem for random walks.

Lemma 16 (Local central limit theorem for lattice distributions)**.** *Let $\mathbf{Y}$ be a random vector in $\mathbb{R}^d$ whose support is contained in the lattice $\mathbf{a} + \mathbf{D}\mathbb{Z}^d$ with $\mathbf{D} \in \mathrm{GL}_d(\mathbb{R})$, $\mathbf{a} \in \mathbb{R}^d$,*

and in no proper sublattice. Suppose that $\mathbf{Y}$ *has a finite non-zero covariance matrix* $\mathbf{\Sigma}$*, and let* $(\mathbf{Y}_i)_{i\geqslant 1}$ *be a family of independent copies of* $\mathbf{Y}$*. For all* n *and* $\mathbf{y}$ *we set*

$$p_n(\mathbf{y}) := \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{Y_i} = \mathbf{y}\right),$$

and as our assumptions imply that Σ *is positive-definite, we may also set*

$$\bar{p}_n(\mathbf{y}) := \frac{|\det \mathbf{D}|}{(2\pi n)^{d/2}\sqrt{\det \mathbf{\Sigma}}} \exp\left(-\frac{1}{2n}(\mathbf{y} - n\mathbb{E}\left[\mathbf{Y}\right])^{\intercal}\mathbf{\Sigma}^{-1}(\mathbf{y} - n\mathbb{E}\left[\mathbf{Y}\right])\right).$$

Then

$$\sup_{\mathbf{y}\in\mathbf{a}+\mathbf{D}\mathbb{Z}^d} |p_n(\mathbf{y}) - \bar{p}_n(\mathbf{y})| = o(n^{-d/2}).$$

In particular,

$$p_n(\mathbf{y}) \sim \bar{p}_n(\mathbf{y})$$

uniformly for $(\mathbf{y} - n\mathbb{E}\left[\mathbf{Y}\right])/\sqrt{n}$ *bounded.*

5.2 A large deviation inequality for functions on finite Markov chains

The following Chernoff-type bound for finite Markov chains was established by Lezaud [55] and does not require the chain to be reversible.

Lemma 17 ([55, Thm. 3.3])**.** *Let* $(X_n)_{n\geqslant 1}$ *denote an irreducible Markov chain on a finite state space* S *with transition matrix* $\mathbf{P}$ *and stationary distribution* π*. Suppose that the multiplicative symmetrization* $\mathbf{K} = \mathbf{P}^{\intercal}\mathbf{P}$ *is irreducible, and let* $\epsilon(\mathbf{K})$ *denote its spectral gap. Let*

$$f : S \to [-1, 1]$$

denote a function whose expected value with respect to the distribution π *equals* $\mathbb{E}_\pi[f] = 0$*. Let* $b > 0$ *be a constant such that* $0 < \|f\|_2 \leqslant b$*. Then for each initial distribution* $q = \mathcal{L}(X_1)$ *and each* $0 < \delta \leqslant 1$ *and* $n \geqslant 1$ *it holds that*

$$\mathbb{P}(|f(X_1) + \ldots + f(X_n)| \geqslant \delta n) \leqslant 2N_q \exp\left(-\frac{n\delta^2\epsilon(\mathbf{K})}{8b^2(1 + h(5\delta/b^2))}\right)$$

where $N_q = \|q/\pi\|_2$ *(the* ℓ_2*-norm of the density of* q *related to the stationary distribution* π*) and*

$$h(x) = \frac{1}{2}\left(\sqrt{1+x} - 1 + \frac{x}{2}\right).$$

5.3 A deviation inequality for random walk

The following deviation inequality is found in most textbooks on the subject.

Lemma 18 (Medium deviation inequality for one-dimensional random walks)**.** *Let $(X_i)_{i\in\mathbb{N}}$ be an i.i.d. family of real-valued random variables with*

$$\mathbb{E}[X_1] = 0 \qquad \text{and} \qquad \mathbb{E}\left[e^{tX_1}\right] < \infty$$

for all t in some interval around zero. Then there are constants $\delta, c > 0$ such that for all $n \in \mathbb{N}$, $x \geqslant 0$ and $0 \leqslant \lambda \leqslant \delta$ it holds that

$$\mathbb{P}(|X_1 + \ldots + X_n| \geqslant x) \leqslant 2\exp(cn\lambda^2 - \lambda x).$$

The proof is by observing that

$$\mathbb{E}\left[e^{\lambda|X_1|}\right] \leqslant 1 + c\lambda^2$$

for some constant c and sufficiently small λ, and applying Markov's inequality to the random variable

$$\exp(\lambda(|X_1| + \ldots + |X_n|)).$$

5.4 The cycle lemma

The following combinatorial result is given for example in Takács [70].

Lemma 19 (The cycle lemma)**.** *For each sequence $k_1, \ldots, k_n \geqslant -1$ of integers with*

$$\sum_i k_i = -r \leqslant 0$$

there exist precisely r values of $0 \leqslant j \leqslant n-1$ such that the cyclic shift

$$(k_{1,j}, \ldots, k_{n,j}) := (k_{1+j}, \ldots, k_n, k_1, \ldots, k_j)$$

satisfies

$$\sum_{i=1}^{u} k_{i,j} > -r$$

for all $1 \leqslant u \leqslant n-1$.

6 Random unlabelled weighted enriched trees with applications

In this section we develop a framework for random enriched trees considered up to symmetry and present our main applications to different models of random unlabelled graphs. We start in Subsection 6.1 by introducing a general model of random weighted unlabelled $\mathcal{R}$-enriched trees and showing how this encompasses various models of random unlabelled graphs. We describe the asymptotic local structure of random unlabelled $\mathcal{R}$-enriched in Subsection 6.2 by the local weak limits stated in Theorems 24 and 27. The global structure is characterized in Subsection 6.3 by the scaling limit in Theorem 28 and the sharp diameter tail-bounds in Theorem 29. In Subsections 6.4, 6.5, and 6.6 we apply our main results to random unlabelled block-weighted graphs, random unlabelled front-rooted k-trees, and simply generated unlabelled rooted trees. The proofs of all results may be found in in Section 7.

Index of notation The following list summarizes frequently used terminology.

$\mathcal{R}^{\kappa}$	κ-weighted species of $\mathcal{R}$-structures, page 30
$\mathcal{A}^{\omega}_{\mathcal{R}}$	ω-weighted species of $\mathcal{R}$-enriched trees, page 30
ρ	radius of convergence of the ordinary generating series $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(z)$, page 33
$\tilde{\mathsf{A}}^{\mathcal{R}}_n$	random n-sized unlabelled $\mathcal{R}$-enriched tree, page 30
C^{ω}_n	random n-sized unlabelled rooted block-weighted connected graph, page 31
$(\mathcal{C}^{\bullet})^{\omega}$	block-weighted species of rooted connected graphs, page 31
$\mathcal{B}^{\gamma}$	weighted species of 2-connected graphs, page 31
$\mathcal{K}$	species of k-dimensional front-rooted trees, page 32
K_n	random unlabelled front-rooted k-tree, page 32
$\mathcal{K}^{\circ}$	subspecies of $\mathcal{K}$ of k-trees where the root front lies in precisely one hedron, page 32
K°_n	random unlabelled k-tree from the class $\mathcal{K}^{\circ}$, page 32
$d_G(\cdot,\cdot)$	graph-metric on a connected graph G, page 8
$d_{\text{BLOCK}}(\cdot,\cdot)$	block-metric, page 8
$V_k(\cdot)$	graph-distance k-neighbourhood, page 8
$U_k(\cdot)$	block-distance k-neighbourhood, page 9
$L_k(\cdot)$	number of vertices at height k in a rooted tree, page 11
$f(A,v)$	enriched fringe subtree of an enriched tree A at a vertex v, page 33
$(\mathcal{T},\beta)$	random Sym($\mathcal{R}$)-enriched plane tree, page 34
$\mathcal{T}^f$	fixed point subtree corresponding to $(\mathcal{T},\beta)$, page 35
$(\mathcal{T}_n,\beta_n)$	the random Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}_n,\beta_n)$ conditioned on having n vertices, page 35
$\mathcal{T}^f_n$	fixed point subtree corresponding to $(\mathcal{T}_n,\beta_n)$, page 35
$(\mathcal{T}^{(\ell)},\beta^{(\ell)})$	size-biased Sym($\mathcal{R}$)-enriched tree, page 38
$(\mathcal{T}^{(\infty)},\beta^{(\infty)})$	local weak limit of the Sym($\mathcal{R}$)-enriched tree $(\mathcal{T}_n,\beta_n)$, page 39
$\tau^{[k]}$	plane tree trimmed at height k, page 39
$(\tau,\gamma)^{<k>}$	$\mathcal{G}$-enriched tree trimmed at height k, page 39
G	random $\mathcal{G}$-object, page 35
$\hat{\mathsf{G}}$	random $\mathcal{G}$-object with a bias on the number of fixed points, page 40
$\bar{\mathsf{G}}$	random $\mathcal{G}$-object with a bias on the number of non-fixed points, page 41
$\hat{\mathsf{H}}^{\bullet}=(\hat{\mathsf{H}},u^*)$	the local limit of $(\mathcal{T}_n,\beta_n)$ near a random vertex, page 42

6.1 Random weighted $\mathcal{R}$-enriched trees

The concept of $\mathcal{R}$-enriched trees was introduced by Labelle [49], and facilitates the unified treatment of a large variety of tree-like combinatorial structures.

Given a species of structures $\mathcal{R}$, the corresponding species of $\mathcal{R}$-*enriched trees* $\mathcal{A}_{\mathcal{R}}$ is constructed as follows. For each finite set U let $\mathcal{A}_{\mathcal{R}}[U]$ be the set of all pairs (A, α) with $A \in \mathcal{A}[U]$ a rooted unordered tree with labels in U, and α a function that assigns to each vertex v of A with offspring set M_v an $\mathcal{R}$-structure $\alpha(v) \in \mathcal{R}[M_v]$. The transport along a bijection $\sigma : U \to V$ relabels the vertices of the tree and the $\mathcal{R}$-structures on the offspring sets accordingly. That is, $\mathcal{A}_{\mathcal{R}}[\sigma]$ maps the enriched tree (A, α) to the tree (B, β) with $B = \mathcal{A}[\sigma](A)$ and $\beta(\sigma(v)) = \mathcal{R}[\sigma|_{M_v}](\alpha(v))$ for each $v \in A$. The species of $\mathcal{R}$-enriched trees admits the combinatorial specification

$$\mathcal{A}_{\mathcal{R}} \simeq \mathcal{X} \cdot \mathcal{R}(\mathcal{A}_{\mathcal{R}}), \tag{10}$$

as any $\mathcal{R}$-enriched tree consists of a root vertex (corresponding to the factor $\mathcal{X}$) together with an $\mathcal{R}$-structure, in which each atom is identified with the root of a further $\mathcal{R}$-enriched tree. By Theorem 12 it holds that given any species $\mathcal{F}$ with an isomorphism $\mathcal{F} \simeq \mathcal{X} \cdot \mathcal{R}(\mathcal{F})$, there is a natural choice of an isomorphism $\mathcal{F} \simeq \mathcal{A}_{\mathcal{R}}$. Hence a large variety of combinatorial structures have a natural interpretation as enriched trees.

Given a weighting κ on the species $\mathcal{R}$, we obtain a weighting ω on the species $\mathcal{A}_{\mathcal{R}}$ given by

$$\omega(A, \alpha) = \prod_{v \in A} \kappa(\alpha(v)). \tag{11}$$

This weighting is consistent with the isomorphism in (10), that is,

$$\mathcal{A}^{\omega}_{\mathcal{R}} \simeq \mathcal{X} \cdot \mathcal{R}^{\kappa}(\mathcal{A}^{\omega}_{\mathcal{R}}). \tag{12}$$

We are going to study the random unlabelled enriched tree $\tilde{\mathsf{A}}^{\mathcal{R}}_n$, drawn with probability proportional to its weight among all unlabelled objects with size n. In the following, we illustrate how this model of random enriched trees generalizes various models of random graphs. The list is of course non-exhaustive and a large variety of other special cases of $\tilde{\mathsf{A}}^{\mathcal{R}}_n$ may be found in the literature.

6.1.1 Simply generated Pólya trees

An unordered rooted tree consists of a root vertex with a set of unordered rooted trees dangling from it. There is no additional structure imposed on the offspring of the root, which is why the class of unordered rooted trees admits a decomposition of the form (11) with $\mathcal{R}$ given by the species SET that we defined at the end of Section 3.1. The corresponding unlabelled objects are called Pólya trees, see Section 2.2 for details.

Let $\mathbf{w} = (\omega_k)_{k \geqslant 0}$ be a sequence of non-negative weights with $\omega_i > 0$ for $i = 0$ and at least one $i \geqslant 2$. For each $k \geqslant 0$ we may assign weight ω_k to the unique k-sized unlabelled $\mathcal{R} = \mathrm{SET}$ structure. Then $\tilde{\mathsf{A}}^{\mathcal{R}}_n$ is the random unordered unlabelled tree such that any

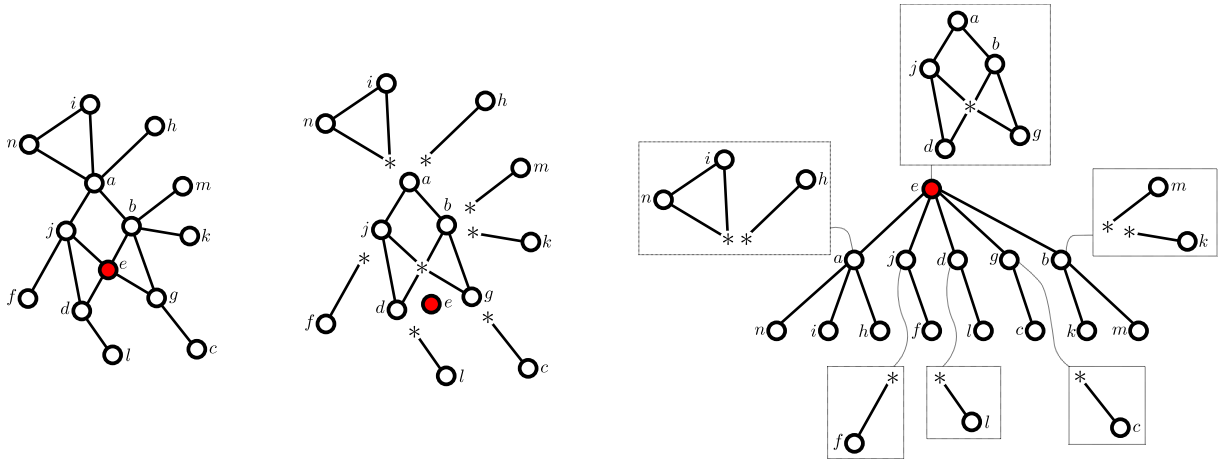

Figure 2: Correspondence of rooted connected graphs and enriched trees.

Pólya tree A with n vertices gets drawn with probability proportional to $\prod_{v\in A}\omega_{d_A^+(v)}$, with $d_A^+(v)$ denoting the outdegree of a vertex v. Note that setting weights to zero allows us to impose arbitrary degree restrictions.

Janson [44] classified weight-sequences for simply generated plane trees into categories, each characterized by a common asymptotic behaviour. It is natural to pursue a similar classification for simply generated Pólya trees.

6.1.2 Random unlabelled connected rooted graphs with weights on the blocks

The species $\mathcal{C}$ of connected graphs admits a decomposition in terms of the species $\mathcal{B}$ of graphs that are 2-connected or consist of two distinct vertices joined by an edge. The well-known combinatorial specification

$$\mathcal{C}^\bullet \simeq \mathcal{X}\cdot \mathrm{SET}(\mathcal{B}'(\mathcal{C}^\bullet)) \tag{13}$$

is illustrated in Figure 2. The isomorphism can be found for example in Harary and Palmer [42, 1.3.3, 8.7.1], Robinson [64, Thm. 4], and Labelle [50, 2.10]. The idea is that in any rooted connected graph, the root-vertex is incident to an unordered set of blocks, where each non-root vertex gets identified with the root of a further rooted connected graph.

Equation (13) allows us to identify the species $\mathcal{C}^\bullet$ of rooted connected graphs with $\mathrm{SET}\circ\mathcal{B}'$-enriched trees. That is, rooted trees, in which each offspring set gets partitioned, and each partition class Q carries a $\mathcal{B}'$-structure, that has $|Q|+1$ vertices, as the $*$-vertex receives no label.

Let γ be a weighting on the species $\mathcal{B}$. We may consider the weighting ω on $\mathcal{C}$ that assigns weight

$$\omega(C)=\prod_B \gamma(B)$$

to any graph C, with the index B ranging over the blocks of C. The random graph C_n^ω drawn from the unlabelled n-sized $\mathcal{C}^\bullet$-objects with probability proportional to its ω-

weight is distributed like the random unlabelled enriched tree $\tilde{\mathsf{A}}_n^{\mathcal{R}}$ for the weighted species $\mathcal{R}^\kappa = (\mathrm{SET} \circ \mathcal{B}')^\kappa$, with κ assigning the product of the γ-weights of the individual classes to any assembly of $\mathcal{B}'$-structures.

If all γ-weights are equal either to 0 or 1, we obtain random connected graphs from so called *block-classes* (or *block-stable* classes), that is, classes of graphs defined by placing constraints on the allowed blocks. For example, any class of graphs $\mathbf{Ex}(\mathcal{M})$ that may be defined by excluding a set $\mathcal{M}$ of 2-connected minors is also block-stable. Here a *minor* of a graph G refers to any graph that may be obtained from G by repeated deletion and contraction of edges. Prominent examples are outerplanar graphs (the class $\mathbf{Ex}(K_4, K_{2,3})$), that may be drawn in the plane such that each vertex lies on the frontier of the infinite face, and series-parallel graphs (the class $\mathbf{Ex}(K_4)$), that may be constructed similar to electric networks in terms of repeated serial and parallel composition. These two classes fall under the more general setting of random graphs from *subcritical* block-classes in the sense of Drmota, Fusy, Kang, Kraus and Rué [27], which also are special cases of the random graph C_n^ω.

6.1.3 Random unlabelled front-rooted k-trees

Let $\mathcal{K}$ denote the species of k-dimensional trees that are rooted at a front of distinguishable $*$-placeholder vertices. We let K_n denote the random unlabelled front-rooted k-tree that gets drawn uniformly at random among all unlabelled $\mathcal{K}$-objects with n hedra.

We consider the subspecies $\mathcal{K}^\circ$ front-rooted k-trees where the root-front is contained in precisely one hedron, and let K_n° be sampled uniformly at random from the unlabelled $\mathcal{K}$-objects with n hedra.

Any element from $\mathcal{K}$ may be obtained in a unique way by glueing an arbitrary unordered collection of $\mathcal{K}^\circ$-objects together at their root-fronts. Hence

$$\mathcal{K} \simeq \mathrm{SET}(\mathcal{K}^\circ).$$

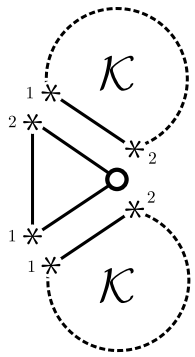

Figure 3: Decomposition of the class $\mathcal{K}^\circ$ for $k = 2$.

As illustrated in Figure 3, any $\mathcal{K}_1$-object may be constructed in a unique way by starting with a hedron H consisting of the root-front and a vertex v, and then choosing, for each front M of H that contains v, a k-tree from $\mathcal{K}$ whose root-front gets identified in a canonical way with M. This yields an isomorphism

$$\mathcal{K}^\circ \simeq \mathcal{X} \cdot \mathrm{SEQ}_{\{k\}}(\mathcal{K}).$$

Combining the isomorphisms yields

$$\mathcal{K}^\circ \simeq \mathcal{X} \cdot (\mathrm{SEQ}_{\{k\}} \circ \mathrm{SET})(\mathcal{K}^\circ).$$

This identifies the species $\mathcal{K}^\circ$ as $\mathrm{SEQ}_{\{k\}} \circ \mathrm{SET}$-enriched trees, and the species $\mathcal{K}$ as unordered forest of enriched trees. In particular, K_n° corresponds to the random unlabelled enriched tree $\tilde{\mathsf{A}}_n^\omega$.

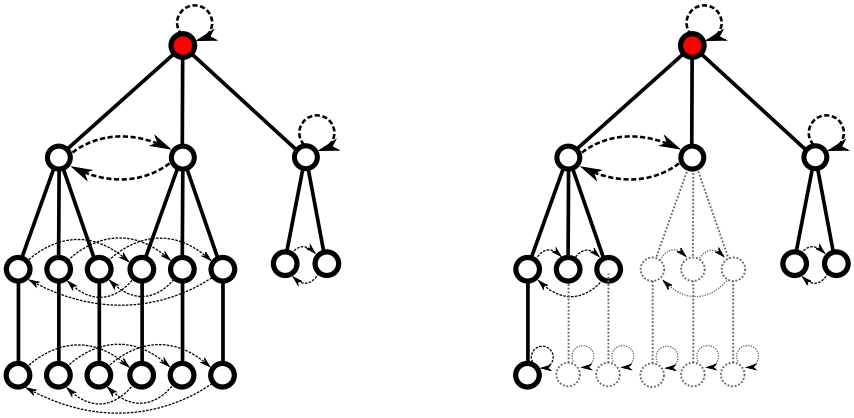

Figure 4: The encoding of $\mathcal{A}_{\mathcal{R}}$-symmetries as $\mathrm{Sym}(\mathcal{R})$-enriched trees.

6.2 Local convergence of random unlabelled enriched trees

In the following, we let $\mathcal{R}^{\kappa}$ denote an arbitrary weighted species such that the inventory $|\mathcal{R}[k]|_{\kappa}$ is positive for $k = 0$ and for at least one $k \geqslant 2$. We let $\mathcal{A}^{\omega}_{\mathcal{R}}$ denote the corresponding species of weighted $\mathcal{R}$-enriched trees. The radius of convergence of the ordinary generating series $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(z)$ will be denoted by ρ.

6.2.1 A coupling with a random $\mathcal{G}$-enriched (plane) tree

Our first observation is that any symmetry $S = ((T, \alpha), \sigma)$ of an $\mathcal{R}$-enriched tree $A = (T, \alpha)$ admits a tree-like decomposition in form of a $\mathrm{Sym}(\mathcal{R})$-enriched tree (T, β). Indeed, the automorphism σ fixes the root o of T and permutes the roots of the $\mathcal{R}$-enriched trees dangling from o in such a way, that the induced permutation $\sigma(o)$ on the offspring of o is an automorphism of the $\mathcal{R}$-structure $\alpha(o)$. This yields an $\mathcal{R}$-symmetry $\beta(o) := (\alpha(o), \sigma(o))$. For each fixed point v of the permutation $\sigma(o)$ it holds that the restriction of σ to the $\mathcal{R}$-enriched fringe subtree $f(A, v)$ (the maximum enriched subtree rooted at the vertex v) yields an $\mathcal{A}_{\mathcal{R}}$-enriched symmetry $(f(A, v), \sigma|_{f(A,v)})$ and we may proceed with the construction of β in the same way. For each cycle $\tau = (v_1, \ldots, v_t)$ of $\sigma(o)$ having length $t \geqslant 2$ the situation is more complicated. We know that σ permutes the $\mathcal{R}$-enriched fringe subtrees $f(A, v_i)$ cyclically. Hence they are all structurally equivalent, and in fact, by the discussion in Section 3.3, up to isomorphism composed out of isomorphic symmetries $(f(A, v_i), \sigma_i)$ with $\sigma_i = \sigma^t|_{f(A,v_i)}$. Hence we may proceed with the construction of β as before, by considering the individual symmetries. This process is illustrated in Figure 4. Note that the $\mathrm{Sym}(\mathcal{R})$-enriched tree (T, β) does not contain all information about the symmetry S, but we may reconstruct S up to relabelling from (T, β).

The fixed points of the automorphism σ form a subtree T^f of T. Each fixed point v has a possibly empty set $f(v)$ of other fixed points as offspring, and the remaining offspring correspond to a forest $F(v)$ of $\mathrm{Sym}(\mathcal{R})$-enriched fringe subtrees $f((T, \beta), v_i)$, which consist of non-fixed points of σ. We are going to say the triple

$$G(v) := (\beta(v), f(v), F(v))$$

is a $\mathcal{G}$-object *on* the fixed points $f(v)$ and define $|f(v)|$ to be its size. Formally, $\mathcal{G}$-objects do not correspond to any species, but the analogy is clear, and we may call $(T^f, (G(v))_{v\in T^f})$ a $\mathcal{G}$-enriched tree.

Similarly, we may define the concept of a *$\mathcal{G}$-enriched plane tree*, in which the label set of each occurring $\mathcal{R}$-symmetry is required to belong to the collection $\{[k] \mid k \geqslant 0\}$. We are going to use the following recursive procedure illustrated in Figure 5 in order to sample random $\mathcal{A}_{\mathcal{R}}$-symmetries according to a weighted Boltzmann-distribution.

Lemma 20 (A coupling of random unlabelled $\boldsymbol{\mathcal{R}}$-enriched trees with random $\boldsymbol{\mathcal{G}}$-enriched trees)**.** *For any parameter $x > 0$ with $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(x) < \infty$ consider the following recursive procedure $\Gamma\mathcal{S}(x)$ which draws a random Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}, \beta)$.*

1. *Start with a root vertex o and draw a random symmetry*

$$\beta(o) = (R(o), \sigma(o))$$

   from the set $\bigcup_{k\geqslant 0} Sym(\mathcal{R})[k]$ such that $\beta(o)$ gets drawn with probability proportional to

$$\frac{\kappa(R(o))}{|R(o)|!}\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(x)^{\sigma_1(o)}\tilde{\mathcal{A}}^{\omega^2}_{\mathcal{R}}(x^2)^{\sigma_2(o)}\cdots .$$

   Here $\sigma_i(o)$ denotes the number of i-cycles of the permutation $\sigma(o)$, with fixed points counting as 1-cycles.

2. *For each cycle τ of the permutation $\sigma(o)$ draw an independent copy $(\mathcal{T}^{\tau}, \beta^{\tau})$ of the recursively called sampler $\Gamma\mathcal{S}(x^{|\tau|})$. Here $|\tau| \geqslant 1$ denotes the length of the cycle. For each atom a of τ make an identical copy $(\mathcal{T}^a, \beta^a)$ of $(\mathcal{T}^{\tau}, \beta^{\tau})$.*

3. *Let k denote the size of the Sym($\mathcal{R}$)-structure $\beta(o)$. For each label $a \in [k]$ add an edge between the root vertex o and the root of the plane tree $\mathcal{T}^a$. The ordering of the offspring set is given by the order on the label set $[k]$. This defines a plane tree $\mathcal{T}$ with root-vertex o. Moreover, for each $a \in [k]$ and each vertex $v \in \mathcal{T}^a$ set $\beta(v) := \beta^a(v)$. This defines a Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}, \beta)$.*

This procedure terminates almost surely and the resulting Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}, \beta)$ corresponds to a symmetry on the vertex set of the plane tree $\mathcal{T}$. Let $\Gamma Z_{\mathcal{A}^{\omega}_{\mathcal{R}}}(x)$ denote the result of relabelling this symmetry uniformly at random with labels from the set $[n]$, with n denoting the number of vertices of the tree $\mathcal{T}$. Then for any symmetry (A, σ) from the set $\bigcup_{k\geqslant 0} Sym(\mathcal{A}_{\mathcal{R}})[k]$ it holds that

$$\mathbb{P}\left(\Gamma Z_{\mathcal{A}^{\omega}_{\mathcal{R}}}(x) = (A, \sigma)\right) = \omega(A)\frac{x^{|A|}}{|A|!}\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(x)^{-1}.$$

If we condition the sampler $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}(x)$ on producing a symmetry with size n, then any symmetry from $\mathrm{Sym}(\mathcal{A}_{\mathcal{R}})[n]$ gets drawn with probability proportional to the ω-weight of its $\mathcal{A}_{\mathcal{R}}$-object. By the discussion in Section 3.1 it follows that the isomorphism class of this $\mathcal{A}_{\mathcal{R}}$-object is distributed like the random unlabelled $\mathcal{R}$-enriched tree $\tilde{\mathsf{A}}_n^{\mathcal{R}}$.

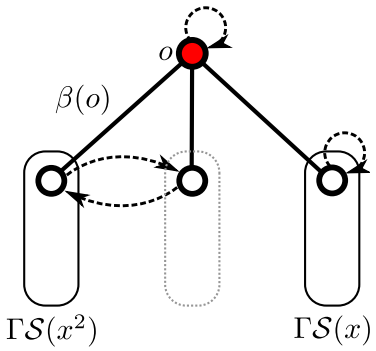

Figure 5: The sampler $\Gamma\mathcal{S}(x)$.

Suppose that the radius of convergence ρ of the ordinary generating series $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(z)$ is positive. As we state below, it holds that $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho)$ is finite and hence we may consider the random $\mathrm{Sym}(\mathcal{R})$-enriched $(\mathcal{T},\beta)$ drawn according to the sampler $\Gamma\mathcal{S}(\rho)$. The vertices of $\mathcal{T}$ that correspond to fixed points of the symmetry $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}(\rho)$ form a subtree $\mathcal{T}^f \subset \mathcal{T}$ containing the root. Note that by the discussion in Section 3.3 the fixed points correspond precisely to the vertices in which the sampler $\Gamma\mathcal{S}$ calls itself with parameter ρ (as opposed to parameter ρ^i for some $i \geqslant 2$).

For each vertex v of $\mathcal{T}^f$ let $G_{\mathcal{T}^f}(v) = (\beta(v), f(v), F(v))$ denote the corresponding $\mathcal{G}$-object. Moreover, let $(\mathcal{T}_n,\beta_n)$, $\mathcal{T}_n^f$ and $G_{\mathcal{T}_n^f}(\cdot) = (\beta_n(\cdot), f_n(\cdot), F_n(\cdot))$ denote the corresponding random variables conditioned on the event $|\mathcal{T}| = n$. Let G be a random variable that is identically distributed to the $\mathcal{G}$-object $G_{\mathcal{T}^f}(o)$ corresponding to the root o of $\mathcal{T}^f$. Moreover, let ξ denote the number of the fixed points of G and ζ the size of the enriched forest corresponding to the non-fixed points.

Lemma 21 (Properties of the coupling with $\boldsymbol{\mathcal{G}}$-enriched trees)**.** *We make the following observations.*

1. *The radius of convergence ρ of $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(z)$ is finite, and the sum $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho)$ is finite as well.*

2. *The size of the tree $\mathcal{T}$ satisfies*

$$|\mathcal{T}| = \sum_{v\in\mathcal{T}^f} (1 + |F(v)|).$$

3. *For any $\mathrm{Sym}(\mathcal{R})$-enriched plane tree $(\mathcal{T}',\beta')$ corresponding to $\mathcal{G}$-objects $G_1,\dots,G_\ell$ it holds that*

$$\mathbb{P}\left((\mathcal{T},\beta) = (\mathcal{T}',\beta')\right) = \prod_{i=1}^{\ell} \mathbb{P}\left(\mathsf{G} = G_i\right).$$

4. *An arbitrary sequence of $\mathcal{G}$-objects $G_i = (S_i, f_i, F_i)$, $i = 1,\dots,\ell$ corresponds to a $\mathrm{Sym}(\mathcal{R})$-enriched tree if and only if*

$$\sum_{i=1}^{\ell} |f_i| = \ell - 1 \quad \text{and} \quad \sum_{i=1}^{m} |f_i| \geqslant m \quad \text{for all} \quad 1 \leqslant m \leqslant \ell - 1.$$

Let $\mathsf{G}_i = (\mathsf{S}_i, \mathsf{f}_i, \mathsf{F}_i)$ *denote independent identical copies of* G*. Let* L *denote depth-first-search ordered list* L *of the* $\mathcal{G}$*-objects of* $\mathcal{T}_n^f$ *and* $|L|$ *its length. Then* $(L \mid |L| = \ell)$ *is distributed like*

$$\left((\mathsf{G}_1, \dots, \mathsf{G}_\ell) \middle| \sum_{i=1}^{\ell} (1 + |\mathsf{F}_i|) = n, \sum_{i=1}^{\ell} |\mathsf{f}_i| = \ell - 1, \sum_{i=1}^{m} |\mathsf{f}_i| \geqslant m \text{ for all } 1 \leqslant m \leqslant \ell - 1\right).$$

5. *The plane tree* $\mathcal{T}^f$ *is distributed like a Galton–Watson tree with offspring distribution* ξ *having probability generating function*

$$\mathbb{E}\left[z^{\xi}\right] = Z_{\mathcal{R}^\kappa}(z\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho), \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega^2}(\rho^2), \dots)\rho/\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho).$$

6. *Given* $\mathcal{T}^f$*, the forests* $(F(v))_{v\in\mathcal{T}^f}$ *are conditionally independent. The conditional distribution of each forest depends only on the outdegree* $d^+_{\mathcal{T}^f}(v)$*. The distribution of the forest size* ζ *is given by its probability generating function*

$$\mathbb{E}\left[z^{\zeta}\right] = Z_{\mathcal{R}^\kappa}(\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho), \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega^2}((\rho z)^2), \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega^3}((\rho z)^3), \dots)\rho/\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho).$$

In a more specific setting, where the random vector (ξ, ζ) has finite exponential moments, even more can be said.

Lemma 22 (Further properties of the coupling with $\boldsymbol{\mathcal{R}}$-enriched trees in a specific setting)**.** *Suppose that* $\rho > 0$ *and that the function*

$$E(z, u) = zZ_{\mathcal{R}^\kappa}(u, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega^2}(z^2), \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega^3}(z^3), \dots)$$

satisfies $E(\rho + \epsilon, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho) + \epsilon) < \infty$ *for some* $\epsilon > 0$*.*

1. *Then the* n*th coefficient of* $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(z)$ *is asymptotically given by*

$$[z^n]\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(z) \sim \mathsf{span}(\mathbf{w})\sqrt{\frac{\rho E_z(\rho, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho))}{2\pi E_{uu}(\rho, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho))}}\rho^{-n}n^{-3/2}$$

   as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ *tends to infinity.*

2. *The series* $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(z)$ *has square root singularities at the points*

$$s_k = \rho \exp(2\pi i k/\mathsf{span}(\mathbf{w})), \quad k = 0, \dots, \mathsf{span}(\mathbf{w}) - 1,$$

   with local expansions as analytic functions of $\sqrt{1 - z/s_k}$*.*

3. *The offspring distribution* ξ *of the Galton–Watson tree* $\mathcal{T}^f$ *and the random variable* ζ *have finite exponential moments. Moreover,*

$$\mathbb{E}[\xi] = E_u(\rho, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho)) = \rho\widetilde{(\mathcal{R}')^\kappa \circ \mathcal{A}}_{\mathcal{R}}^{\omega}(\rho) = 1,$$
$$\mathbb{V}[\xi] = E_{uu}(\rho, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho))\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho), \quad \text{and} \quad \mathbb{E}[\zeta] = E_z(\rho, \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho))\rho/\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho) - 1.$$

   Consequently,

$$\mathbb{P}(|\mathcal{T}| = n) \sim \mathsf{span}(\mathbf{w})n^{-3/2}\sqrt{\frac{1 + \mathbb{E}[\zeta]}{2\pi\mathbb{V}[\xi]}}$$

4. *Suppose that at least one $\mathcal{R}$-structure with positive κ-weight has a non-trivial automorphism. Then the lattice spanned by the support of (ξ, ζ) has a 2-dimensional $\mathbb{Z}$-basis $\mathbf{B} \in \mathbb{Z}^{2\times 2}$, and the covariance matrix $\mathbf{\Sigma}$ of (ξ, ζ) is positive-definite. Set*

$$\mu = \frac{1}{1+\mathbb{E}[\zeta]}, \quad \sigma^2 = \frac{\det \mathbf{\Sigma}}{\mathbb{V}[\zeta](1+\mathbb{E}[\zeta])^3}, \quad \text{and} \quad d = \frac{|\det \mathbf{B}|}{\mathsf{span}(\mathbf{w})}.$$

Then, as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ tends to infinity,

$$\sqrt{n}\mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \sim \frac{d}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})$$

uniformly for all bounded x satisfying

$$\ell := \mu n + x\sqrt{n} \in n + d\mathbb{Z}.$$

In particular,

$$\frac{|\mathcal{T}_n^f| - n\mu}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Properties *(1)* and *(2)* are an application of results by Bell, Burris and Yeats [12]. The requirement in *(4)* that $\mathcal{R}$ has at least one structure (with positive κ-weight) with non-trivial symmetries is not really a restriction. If it fails, then almost surely $|\mathcal{T}_n^f| = n$ for all n, which makes the analysis of $\tilde{\mathsf{A}}_n^\omega$ even easier.

6.2.2 Local convergence around the fixed root

Labelle established in [49, Thm. A] the following decomposition of pointed $\mathcal{R}$-enriched trees, which will aid us studying the behaviour of the $\mathcal{R}$-structures along paths starting from the root in random enriched trees. The weighted species $\mathcal{A}_{\mathcal{R}}^\omega$ satisfies an isomorphism $\mathcal{A}_{\mathcal{R}}^\omega \simeq \mathcal{X} \cdot \mathcal{R}^\kappa(\mathcal{A}_{\mathcal{R}}^\omega)$. The derivative operator satisfies a product rule similar to the product rule for the derivative of smooth functions, see Proposition 7. Hence applying the pointing operator yields a weight-compatible isomorphism

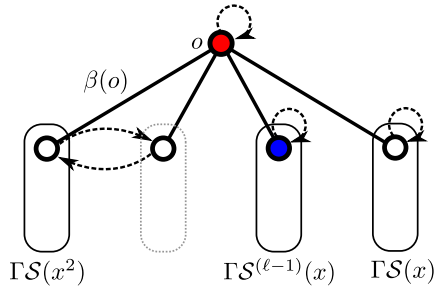

Figure 7: The sampler $\Gamma\mathcal{S}^{(\ell)}(x)$.

$$\mathcal{A}_{\mathcal{R}}^\bullet \simeq \mathcal{A}_{\mathcal{R}} + \mathcal{X} \cdot \mathcal{R}'(\mathcal{A}_{\mathcal{R}}) \cdot \mathcal{A}_{\mathcal{R}}^\bullet.$$

We may apply Joyal's implicit species theorem [46, Th. 6] in order to unwind this recursion and obtain an isomorphism

$$\mathcal{A}_{\mathcal{R}}^\bullet \simeq \sum_{\ell=0}^{\infty} \mathcal{A}_{\mathcal{R}}^{(\ell)}, \quad \mathcal{A}_{\mathcal{R}}^{(\ell)} = (\mathcal{X} \cdot \mathcal{R}'(\mathcal{A}_{\mathcal{R}}))^\ell \mathcal{A}_{\mathcal{R}}$$

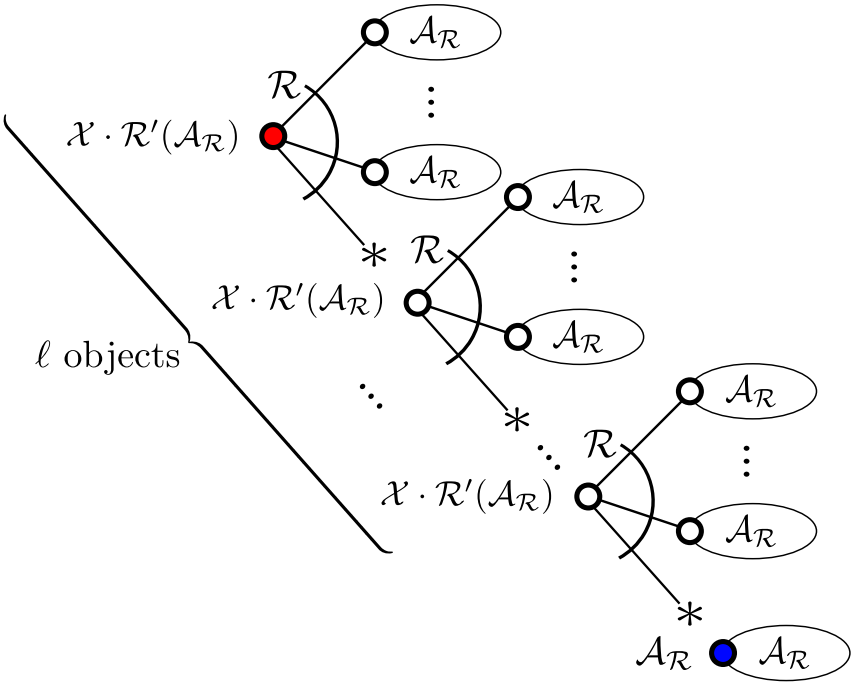

Figure 6: The decomposition of $\mathcal{A}_\mathcal{R}^{(\ell)}$.

corresponding to the pointed enriched trees $((A, \alpha), v)$ in which the outer root v has height ℓ in the rooted tree A. The correspondence is illustrated in Figure 6. Again, this isomorphism is compatible with the weightings, and we may use it to construct the following sampler illustrated in Figure 7.

Lemma 23 (A modified random Sym$(\boldsymbol{\mathcal{R}})$-enriched tree)**.** *For any integer $\ell \geqslant 0$ and parameter $x > 0$ with $(\tilde{\mathcal{A}}_\mathcal{R}^{(1)})^\omega(x) < \infty$ consider the following recursive procedure $\Gamma\mathcal{S}^{(\ell)}(x)$ that samples a random Sym$(\mathcal{R})$-enriched plane tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$ together with a distinguished vertex r which we call the outer root.*

1. *If $\ell = 0$ then return (an independent copy of) the random enriched plane tree $(\mathcal{T}, \beta)$ from Lemma 20 with the outer root being the root-vertex of $\mathcal{T}$. Otherwise, if $\ell \geqslant 1$, then proceed with the following steps.*

2. *Start with a root vertex o and draw a random $\mathcal{R}'$-symmetry (R, σ) from $\bigcup_{k \geqslant 0} Sym(\mathcal{R}')[k]$ with probability proportional to*

$$\frac{\kappa(R)}{|R|!} \tilde{\mathcal{A}}_\mathcal{R}^\omega(x)^{\sigma_1} \tilde{\mathcal{A}}_\mathcal{R}^{\omega^2}(x^2)^{\sigma_2} \cdots .$$

Set $k := |R|$ and make a uniformly at random choice of a bijection f from the set $[k] \cup \{*_k\}$ of labels of the $\mathcal{R}$-structure R to the set of integers $[k+1]$. Relabel the symmetry via the transport function:

$$\beta(o) := (R(o), \sigma(o)) := Sym(\mathcal{R})[f](R, \sigma).$$

Let $b := f(*_k)$ denote the vertex corresponding to $*_k$.

3. *Note that b is a fixed point of the permutation $\sigma(o)$. For each cycle $\tau \neq (b)$ of $\sigma(o)$ draw an independent copy $(\mathcal{T}^\tau, \beta^\tau)$ of the sampler $\Gamma\mathcal{S}(x^{|\tau|})$ with $|\tau| \geqslant 1$ denoting the length of the cycle. For each atom a of the cycle τ make an identical copy $(\mathcal{T}^a, \beta^a)$ of $(\mathcal{T}^\tau, \beta^\tau)$.*

4. *Draw an independent copy $(\mathcal{T}^b, \beta^b)$ of the sampler $\Gamma\mathcal{S}^{(\ell-1)}(x)$.*

5. *For each label $a \in [k+1]$ add an edge between the root vertex o and the root of the plane tree $\mathcal{T}^a$. The ordering of the offspring set is given by the order on the label set $[k+1]$. This defines a plane tree $\mathcal{T}$ with root-vertex v. Moreover, for each $a \in [k+1]$ and each vertex $u \in \mathcal{T}^a$ set $\beta(u) := \beta^a(u)$. This defines an Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$.*

This procedure terminates almost surely. As described in Section 6.2.1, the resulting Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$ corresponds to a symmetry on the vertex set of the tree $\mathcal{T}^{(\ell)}$. Let n denote the number of vertices of the tree $\mathcal{T}^{(\ell)}$ and let $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}(x)$ denote the result of relabelling this symmetry uniformly at random with labels from the set $[n]$. Then $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}(x)$ satisfies a weighted Boltzmann distribution, that is, for any symmetry $((A, r), \sigma)$ from the set $\bigcup_{k=0}^{\infty} Sym(\mathcal{A}_{\mathcal{R}}^{(\ell)})[k]$ we have that

$$\mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}(x) = ((A, r), \sigma)\right) = \omega(A)\frac{x^{|A|}}{|A|!}\tilde{\mathcal{A}}_{\mathcal{R}}^{(\ell)}(x)^{-1} \tag{14}$$

In particular,

$$\mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}(x) = ((A, r), \sigma)\right) = (x\widetilde{\mathcal{R}' \circ \mathcal{A}}_{\mathcal{R}}^{\omega}(x))^{\ell}\mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}(x) = (A, \sigma)\right).$$

Suppose that $\rho > 0$ and consider the Sym($\mathcal{R}$)-enriched tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$ generated by the procedure $\Gamma\mathcal{S}^{(\ell)}(\rho)$. The path from the root to the distinguished vertex in $\mathcal{T}^{(\ell)}$ is its *spine*. If we set $\ell = \infty$, the above construction yields an infinite but locally finite Sym($\mathcal{R}$)-enriched tree $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$ having an infinite spine. We will show that this object, is the local limit weak limit of the random graph $\tilde{\mathsf{A}}_n^{\omega}$, if certain conditions are met.

In order to formalize our notion of local convergence, we require the concept of trimmed $\mathcal{G}$-enriched trees. For any $\mathcal{G}$-enriched plane tree (τ, γ) and any non-negative integer k let $(\tau, \gamma)^{<k>}$ denote the result of trimming at height k. That is,

$$(\tau, \gamma)^{<k>} = (\tau^{[k]}, (\gamma(v))_{v \in \tau^{[k-1]}}),$$

with $\tau^{[k]}$ denoting the plane tree trimmed hat height k. That is, we delete all vertices from τ with height larger than k. In order to simplify notation, we also set $(\mathcal{T}', \beta')^{<k>} := (\tau, \gamma)^{<k>}$ for the Sym($\mathcal{R}$)-enriched plane tree $(\mathcal{T}', \beta')$ corresponding to (τ, γ).

Theorem 24 (Local convergence of random unlabelled $\boldsymbol{\mathcal{R}}$-enriched trees)**.** *Suppose that the ordinary generating series $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(z)$ has radius of convergence $\rho > 0$, and that the series*

$$E(z,u) = zZ_{\mathcal{R}^\kappa}(u, \tilde{\mathcal{A}}^{\omega^2}_{\mathcal{R}}(z^2), \tilde{\mathcal{A}}^{\omega^3}_{\mathcal{R}}(z^3), \ldots)$$

satisfies $E(\rho+\epsilon, \tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(\rho)+\epsilon) < \infty$ for some $\epsilon > 0$. Then for any sequence $k_n = o(\sqrt{n})$ of non-negative integers it holds that

$$d_{\mathrm{TV}}((\mathcal{T}_n, \beta_n)^{<k_n>}, (\mathcal{T}^{(\infty)}, \beta^{(\infty)})^{<k_n>}) \to 0.$$

as n becomes large.

The limit object $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$ admits a more accessible description in terms of $\mathcal{G}$-enriched trees, that we are going to use in the proof of Theorem 24. Let $\hat{\mathsf{G}}$ denote a random variable that is distributed like the $\mathcal{G}$-object corresponding to the root of $\mathcal{T}^{(\infty)}$. Here we do not explicitly distinguish the vertex corresponding the $*$-vertex of the $\mathcal{R}'$-symmetry.

Lemma 25. *Suppose that $\rho > 0$, $\mathbb{E}[\xi] = 1$ and that (ξ, ζ) has a finite covariance matrix.*

1. *The $\mathcal{G}$-object corresponding to the root of $\mathcal{T}^{(\infty)}$ together with the fixed point corresponding to the second spine-vertex is distributed like G with a uniformly at random selected marked fixed point.*

2. *The distribution of the limit enriched tree $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$ may be described as follows. There are normal fixed points and mutant fixed points, and we start with a mutant root. Each normal fixed point receives as $\mathcal{G}$-object an independent copy of G, and each of the fixed points of this $\mathcal{G}$-object is declared normal. Ever mutant fixed point receives an independent copy of $\hat{\mathsf{G}}$, and one of the corresponding fixed points is selected uniformly at random and declared mutant, whereas the remaining fixed points are declared normal.*

3. *Let (τ, γ) denote a $\mathcal{G}$-enriched tree with height at least k. Let $G_1, \ldots, G_t$ denote the depth-first-search ordered list of the $\mathcal{G}$-objects of $\tau^{[k-1]}$. Then*

$$\mathbb{P}\left((\mathcal{T}^{(\infty)}, \beta^{(\infty)})^{<k>} = (\tau, \gamma)^{<k>}\right) = L_k(\tau) \prod_{i=1}^{t} \mathbb{P}\left(\mathsf{G} = G_i\right).$$

4. *For any $\mathcal{G}$-enriched tree (τ, γ) with $\gamma(v) = (\beta_\tau(v), f_\tau(v), F_\tau(v))$ and any integer $k \geqslant 0$ we set*

$$L_k(\tau) = |\{v \in \tau \mid h_\tau(v) = k\}|, \quad L^{\mathcal{G}}_k(\tau) = \sum_{\substack{v \in \tau \\ h_\tau(v) = k}} |F_\tau(v)|, \quad H^{\mathcal{G}}_k(\tau) = \sum_{i=0}^{k} L^{\mathcal{G}}_i(\tau).$$

Let $(\hat{\xi}, \hat{\zeta})$ denote the sizes of the fixed points and non-fixed points of $\hat{\mathsf{G}}$. When (τ, γ) is the $\mathcal{G}$-enriched tree corresponding to $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$, it holds for all $k \geqslant 1$ that

$$\begin{aligned}
\mathbb{E}\left[L_k(\tau)\right] &= k(\mathbb{E}\left[\hat{\xi}\right] - 1) + 1 \\
\mathbb{E}\left[|\tau^{[k]}|\right] &= k(k+1)(\mathbb{E}\left[\hat{\xi}\right] - 1)/2 + k + 1 \\
\mathbb{E}\left[L_k^{\mathcal{G}}(\tau)\right] &= k(\mathbb{E}\left[\hat{\xi}\right] - 1)\mathbb{E}\left[\zeta\right] + \mathbb{E}\left[\hat{\zeta}\right] \\
\mathbb{E}\left[H_k^{\mathcal{G}}(\tau) - |\tau^{[k]}|\mathbb{E}\left[\zeta\right]\right] &= (k+1)(\mathbb{E}\left[\hat{\zeta}\right] - \mathbb{E}\left[\zeta\right]) \\
\mathbb{V}[H_k^{\mathcal{G}}(\tau) - |\tau^{[k]}|\mathbb{E}\left[\zeta\right]] &= k(k+1)(\mathbb{E}\left[\hat{\xi}\right] - 1)\mathbb{V}[\zeta]/2 + (k+1)(\mathbb{V}[\hat{\zeta}] + \mathbb{V}[\zeta])
\end{aligned}$$

6.2.3 Local convergence around a random root

Inspired by Aldous' approach [3] on fringe subtrees of random trees, we may also treat local convergence with respect to a uniformly at random drawn root of the random unlabelled enriched tree $\tilde{\mathsf{A}}_n^{\omega}$ in a similar manner. Let v^* be an uniformly at random drawn vertex of the tree $\mathcal{T}_n$, and let v_0 denote the unique nearest vertex of $\mathcal{T}_n^f$. That is, $v_0 = v^*$ if $v^* \in \mathcal{T}_n^f$, and otherwise v_0 is the unique vertex of the fixed point tree whose $\mathcal{G}$-object contains v^*. For any $i \geqslant 1$, let v_i denote the i'th predecessor of v_0 in the fixed point tree $\mathcal{T}_n^f$, if this predecessor exists. If not, that is, if v_0 has height greater than i in $\mathcal{T}_n^f$, then set $v_i = \diamond$ and $f((\mathcal{T}_n, \beta_n), v_i) = \diamond$ for some symbol $\diamond$ not contained in the set of $\mathcal{G}$-enriched trees. (For example, we could use the empty set.) For any $k \geqslant 0$ we consider the vector of increasing fringe subtrees

$$\mathsf{H}_{[k]}^n := (\mathsf{H}_i^n)_{0 \leqslant i \leqslant k} := (f((\mathcal{T}_n, \beta_n), v_0), \ldots, f((\mathcal{T}_n, \beta_n), v_k)).$$

We are going to establish convergence of these random vectors of enriched trees toward extended enriched fringe subtrees of a limit object, which we introduce in the following lemma.

Lemma 26. *Suppose that $\rho > 0$, $\mathbb{E}\left[\xi\right] = 1$ and that (ξ, ζ) has a finite covariance matrix.*

1. *Let $\bar{\mathsf{G}} = (\bar{\mathsf{S}}, \bar{\mathsf{f}}, \bar{\mathsf{F}})$ denote the random $\mathcal{G}$-object with distribution given by*

$$\mathbb{P}\left(\bar{\mathsf{G}} = (S, f, F)\right) = (1 + |F|)\mathbb{P}\left(\mathsf{G} = (S, f, F)\right)/(1 + \mathbb{E}\left[\zeta\right]).$$

We define an infinite random $\mathcal{G}$-enriched tree $\hat{\mathsf{H}}$ in terms of its sequence $(\hat{\mathsf{H}}_k)_{k \geqslant 0}$ of increasing extended enriched fringe subtrees. The distribution of the fringe subtree tree $\hat{\mathsf{H}}_k$ is given as follows. There are normal fixed points and special fixed points, and we start with a special root. Each normal fixed point receives as $\mathcal{G}$-object an independent copy of G, and all fixed points of this $\mathcal{G}$-object are declared normal. Every special fixed point with height less than ℓ receives an independent copy of $\hat{\mathsf{G}}$, and one of the corresponding fixed points is selected uniformly at random and declared special, whereas the remaining fixed points are declared normal. A special

fixed point with height ℓ receives $\bar{\mathsf{G}}$ and all fixed points of this $\mathcal{G}$-object are declared normal.

Then the special vertices of $\hat{\mathsf{H}}$ form an infinite spine $u_0, u_1, \ldots$ that grows backwards, with

$$\hat{\mathsf{H}}_k = f(\hat{\mathsf{H}}, u_k)$$

for all k. We distinguish a point u^* that is drawn uniformly at random from the set $\{u_0\} \cup \hat{\mathsf{F}}(u_0)$, with $\hat{\mathsf{F}}(u_0)$ denoting the set of non-fixed points of $\mathcal{G}$-object corresponding to the root of $\hat{\mathsf{H}}_0$, and set $\hat{\mathsf{H}}^\bullet = (\hat{\mathsf{H}}, u^*)$.

2. *For any two $\mathcal{G}$-enriched trees A and A', let $Q(A, A')$ denote the number of fixed point children v of the root of A with $f(A, v) = A'$. For any increasing fringe subtree representation $\mathbf{H} = (H_i)_{0 \leqslant i \leqslant k}$ of a $\mathcal{G}$-enriched tree we set*

$$p(\mathbf{H}) = \prod_{i=1}^{k} Q(H_i, H_{i-1}).$$

Then $p(\mathbf{H})$ counts the number of fixed points v at height k in H_k with the property, that the extended enriched fringe subtree representation with respect to v is identical to $\mathbf{H}$.

3. *Let u be either the root of H_0 or a non-fixed point of $\mathcal{G}$-object corresponding to the root of H_0, and let $G_1, \ldots, G_t$ be the $\mathcal{G}$-objects corresponding to H_k. Then*

$$\mathbb{P}\left(\hat{\mathsf{H}}_{[k]} = \mathbf{H}\right) = p(\mathbf{H})(1 + \mathbb{E}\left[\zeta\right])^{-1} \prod_{i=1}^{t} \mathbb{P}\left(\mathsf{G} = G_i\right).$$

4. *For any $\mathcal{G}$-enriched tree (τ, γ) with $\gamma(v) = (\beta_\tau(v), f_\tau(v), F_\tau(v))$ let $\#_f(\tau, \gamma) = |\tau|$ denote its number of fixed points, and $\#(\tau, \gamma) = \sum_{v \in \tau}(1 + F_\tau(v))$ its total size. Then for any sequence $k_n = \sqrt{n}t_n$ of non-negative integers with $t_n = o(1)$ it holds with probability tending to one that*

$$\#_f \hat{\mathsf{H}}_{k_n} \leqslant n t_n \quad \textit{and} \quad |\#\hat{\mathsf{H}}_{k_n} - \mathbb{E}\left[\zeta\right] \#_f \hat{\mathsf{H}}_{k_n}| \leqslant \sqrt{n t_n}.$$

We may now establish convergence of the extended enriched fringe subtrees, that will help us to apply our main theorems to specific examples of random discrete structures, in particular random graphs.

Theorem 27 (Local convergence of random unlabelled $\boldsymbol{\mathcal{R}}$-enriched trees around a random root)**.** *Suppose that the ordinary generating series $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(z)$ has radius of convergence $\rho > 0$, and that the series*

$$E(z, u) = z Z_{\mathcal{R}^\kappa}(u, \tilde{\mathcal{A}}^{\omega^2}_{\mathcal{R}}(z^2), \tilde{\mathcal{A}}^{\omega^3}_{\mathcal{R}}(z^3), \ldots)$$

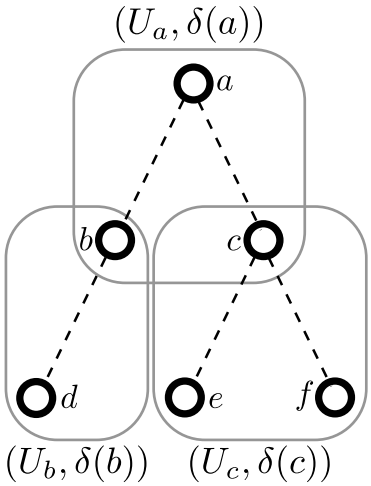

Figure 8: Patching together discrete metric spaces.

satisfies $E(\rho + \epsilon, \tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(\rho) + \epsilon) < \infty$ *for some* $\epsilon > 0$. *Then for any sequence* $k_n = o(\sqrt{n})$ *of non-negative integers the increasing fringe subtree sequence* $\mathsf{H}^n_{[k_n]}$ *of length* k_n, *corresponding to the uniformly at random drawn vertex* v^* *of* $(\mathcal{T}_n, \beta_n)$, *converges in total variation to the fringe subtree sequence of* $\hat{\mathsf{H}}$ *with the same length. That is,*

$$d_{\mathrm{TV}}((\mathsf{H}^n_{[k_n]}, v^*), (\hat{\mathsf{H}}_{[k_n]}, u^*)) \to 0. \tag{15}$$

Here u^* denotes a random vertex of $\hat{\mathsf{H}}$ that we defined in Lemma 26.

6.3 Scaling limits of metric spaces based on $\mathcal{R}$-enriched trees

6.3.1 Patching together discrete metric spaces

We study metric spaces patched together from metrics associated to the vertices of a tree. Let A be a rooted tree. For each vertex $v \in A$ let M_v denote its offspring set. Let δ be a map that assigns to each vertex v of A a metric $\delta(v)$ on the set $U_v := M_v \cup \{v\}$. This induces a metric d on the vertex vertex set $V(A)$ that extends the metrics $\delta(v)$ by patching together as illustrated in Figure 8. Formally, we define this metric as follows. Consider the graph G on the vertex set of A obtained by connecting any two vertices $x \neq y$ if and only if there is some vertex v of the tree A with $x, y \in U_v$ and assigning the weight $\delta(v)(x, y)$ to the edge. The resulting graph is connected and the distance of any two vertices a and b is defined by the minimum of all sums of edge-weights along paths joining a and b in the graph G.

Suppose that for each finite set U and each $\mathcal{R}$-structure $R \in \mathcal{R}[U]$ we are given a random metric δ_R on the set $U \cup \{*_U\}$ with $*_U$ denoting an arbitrary fixed element not contained in U. For example, we could set $*_U := \{U\}$. Let $\tilde{\mathsf{A}}^{\mathcal{R}}_n = (\tilde{\mathsf{A}}_n, \alpha_n)$ denote the random n-sized $\mathcal{R}$-enriched tree drawn with probability proportional to its ω-weight. We construct a random n-element metric space Y_n as follows. For each vertex v of $\tilde{\mathsf{A}}^{\mathcal{R}}_n$ with

offspring set M_v let $\delta_n(v)$ be the metric on the set $M_v \cup \{v\}$ obtained by taking an independent copy of $\delta_{\alpha_n(v)}$ and identifying $*_{M_v}$ with v. Let d_{Y_n} denote the metric patched together from the family $(\delta_n(v))_v$ as described in the preceding paragraph.

In order for this to be a sensible model of a random tree-like structure we require the following two assumptions.

1. We assume that there is a real-valued random variable $\chi \geqslant 0$ such that for any $\mathcal{R}$-structure R the diameter of the metric δ_R is stochastically bounded by the sum of $|R|$ independent copies $\chi_1^R, \ldots, \chi_{|R|}^R$ of χ.

2. For any bijection $\sigma : U \to V$ of finite sets and for any $\mathcal{R}$-structure $R \in \mathcal{R}[U]$ we require that the metric $\delta_{\mathcal{R}[\sigma](R)}$ is identically distributed to the push-forward of the metric δ_R by the bijection $\bar{\sigma} : U \cup \{*_U\} \to V \cup \{*_V\}$ with $\bar{\sigma}|_U = \sigma$.

Recall that by Lemma 22 the radius of convergence ρ of $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(z)$ is finite, and the sum $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(\rho)$ is finite as well.

Theorem 28 (Scaling limits of spaces based on unlabelled enriched trees)**.** *Suppose that the ordinary generating series $\tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(z)$ has radius of convergence $\rho > 0$ and that the series*

$$E(z, u) = zZ_{\mathcal{R}^\kappa}(u, \tilde{\mathcal{A}}^{\omega^2}_{\mathcal{R}}(z^2), \tilde{\mathcal{A}}^{\omega^3}_{\mathcal{R}}(z^3), \ldots)$$

satisfies

$$E(\rho + \epsilon, \tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(\rho) + \epsilon) < \infty$$

for some $\epsilon > 0$. Then the rescaled space $(\mathsf{Y}_n, n^{-1/2} d_{\mathsf{Y}_n})$ converges weakly to a constant multiple of the (Brownian) continuum random tree $\mathcal{T}_{\mathsf{e}}$ with respect to the Gromov–Hausdorff metric as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ tends to infinity.

An explicit expression of the scaling constant in Theorem 28 is given in the corresponding proof in Section 7.2. It is interesting to note that the local weak limit $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$ contains some information on the scaling limit, as it is responsible for one of the factors in the scaling factor. We also provide the following sharp tail-bound for the diameter.

Theorem 29 (Tail bounds for the diameter)**.** *Under the same assumptions of Theorem 28 there are constants $C, c > 0$ such that for all n and $x \geqslant 0$ it holds that*

$$\mathbb{P}\left(\mathrm{D}(\mathsf{Y}_n) \geqslant x\right) \leqslant C(\exp(-cx^2/n) + \exp(-cx)).$$

Again it holds that if χ is bounded, then we have the tail-bound

$$\mathbb{P}\left(\mathrm{D}(\mathsf{Y}_n) \geqslant x\right) \leqslant C \exp(-cx^2/n)$$

for some constants $C, c > 0$.

The main idea of the proofs of Theorems 28 and Theorem 29 is that we may use the random $\mathrm{Sym}(\mathcal{R})$-enriched tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$ of Lemma 23 to relate for any two vertices $x, y \in \mathcal{T}_n^f$ distances $d_{\mathsf{Y}_n}(x, y)$ and $d_{\mathcal{T}_n^f}(x, y)$ by constant factor. The following basic observation then takes care of the rest.

Lemma 30. *Suppose that $\rho > 0$ and that the function*

$$E(z,u) = zZ_{\mathcal{R}^\kappa}(u, \tilde{\mathcal{A}}^{\omega^2}_{\mathcal{R}}(z^2), \tilde{\mathcal{A}}^{\omega^3}_{\mathcal{R}}(z^3), \dots)$$

satisfies $E(\rho+\epsilon, \tilde{\mathcal{A}}^{\omega}_{\mathcal{R}}(\rho)+\epsilon) < \infty$ for some $\epsilon > 0$. Then the following assertions hold:

1. *There are constants $C, c > 0$ such that for all n and $x \geqslant 0$*

$$\mathbb{P}\left(\max_{v\in\mathcal{T}_n^f}(|f_n(v)| + |F_n(v)|) \geqslant x\right) \leqslant Cn^{5/2}\exp(-cx).$$

2. *For any vertex v let D_v denote the d_{Y_n}-diameter of the subspace $\{v\} \cup F_n(v) \subset \mathsf{Y}_n$. Then there are constants $C, c > 0$ such that for all $h \geqslant 0$*

$$\mathbb{P}\left(\max_{v\in\mathcal{T}_n^f} D_v \geqslant h\right) \leqslant Cn^{5/2}\exp(-ch).$$

3. *We have that*

$$\frac{\sqrt{(1+\mathbb{E}\,[\zeta])\mathbb{V}[\xi]}}{2\sqrt{n}}\mathcal{T}_n^f \xrightarrow{d} \mathcal{T}_{\mathsf{e}}$$

in the Gromov–Hausdorff sense as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ tends to infinity.

A result similar to Lemma 30 was used in [61] to provide a combinatorial proof for the scaling limit of uniform random Pólya trees (with possible degree restrictions). Theorem 28 is more general, as it applies for example to scaling limits of models of random graphs with respect to the first-passage-percolation metric, and its proof is deeper and more involved, as it requires the interplay with the Sym($\mathcal{R}$)-enriched tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$ of Lemma 23 that is related to the local weak limit.

6.4 Applications to random unlabelled rooted connected graphs

Let $\mathcal{C}$ denote the class of connected graphs and $\mathcal{B}$ its subclass of graphs that are two-connected or a single edge with its ends. Recall that the rooted class $\mathcal{C}^\bullet$ may be identified with the class of SET $\circ\, \mathcal{B}'$-enriched trees as discussed in Section 6.1.2. Suppose that we have a weighting γ on the class $\mathcal{B}$, that is, for each $\mathcal{B}$-graph B we are given a weight $\gamma(B) \geqslant 0$ such that the weights of isomorphic graphs agree. This induces a weighting κ on the species SET $\circ\, \mathcal{B}'$ by setting the weight of a set of graphs to the product of the individual weights. Hence we also have a weighting ω on $\mathcal{C}$ given by

$$\omega(C) = \prod_B \gamma_B$$

for all $\mathcal{C}$-objects C, with the index B ranging over all blocks of the graph C. In the following, we study random unlabelled rooted graph C_n^ω drawn from the unlabelled $\mathcal{C}^\bullet$-objects of size n with probability proportional to its ω-weight. This corresponds to the

model of random unlabelled enriched trees $\tilde{\mathsf{A}}_n^{\mathcal{R}}$ in Section 6.1 for the special case $\mathcal{R}^\kappa = (\mathrm{SET} \circ \mathcal{B}')^\kappa$.

Under the premise that the cycle index sums related to the random graph C_n^ω satisfy Equation (16), we establish a local weak limit for the vicinity of the fixed root in Theorem 31 and a Benjamini–Schramm limit in Theorem 32. In both cases we actually establish total variational convergence of arbitrary $o(\sqrt{n})$-neighbourhoods, which is best-possible in this setting. We also consider the first-passage-percolation metric on the graph C_n^ω, which is more general than the graph-metric, and establish sharp exponential tail-bounds for the diameter and a Gromov–Hausdorff scaling limit in Theorem 33. As a byproduct, we also obtain a bound for the size of the largest 2-connected component.

As an important special case, these Theorems apply to uniform random unlabelled rooted graph from a subcritical block class. This model was studied by Drmota, Fusy, Kang, Kraus and Rué in [27, Def. 10], and includes uniform random rooted unlabelled cacti graphs, outerplanar graphs, and series-parallel graphs [27, Thm. 15]. The scaling limit Theorem 33 is a strong result in this context and also establishes the correct order of the diameter of this type random graphs.

6.4.1 Local weak limit

The infinite $\mathrm{Sym}(\mathrm{SET} \circ \mathcal{B}')$-enriched tree $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$ from Section 6.2.2 is naturally also a $\mathrm{SET} \circ \mathcal{B}'$-enriched tree, and may hence be interpreted as an infinite locally finite random graph $\hat{\mathsf{C}}$ according to the bijection in Section 6.1.2. Theorem 24 yields local weak convergence of the random graph C_n^ω with respect to neighbourhoods around its fixed root vertex.

Theorem 31 (Local convergence of random unlabelled graphs)**.** *Suppose that the weighted ordinary generating series $(\tilde{\mathcal{C}}^\bullet)^\omega(z)$ has radius of convergence $\rho > 0$, and that*

$$E(z,u) = zZ_{(\mathrm{SET}\circ\mathcal{B}')^\kappa}(u, (\tilde{\mathcal{C}}^\bullet)^{\omega^2}(z^2), (\tilde{\mathcal{C}}^\bullet)^{\omega^3}(z^3), \ldots) \tag{16}$$

satisfies $E(\rho+\epsilon, (\tilde{\mathcal{C}}^\bullet)^\omega(\rho)+\epsilon) < \infty$ for some $\epsilon > 0$. Then for any sequence $k_n = o(\sqrt{n})$ of non-negative integers it holds that

$$d_{\mathrm{TV}}(U_{k_n}(\mathsf{C}_n^\omega), U_{k_n}(\hat{\mathsf{C}})) \to 0, \tag{17}$$

and likewise for the graph-metric neighbourhoods $V_{k_n}(\cdot)$. Thus, the infinite random graph $\hat{\mathsf{C}}$ is the local weak limit of the random graph C_n^ω as n becomes large.

Note that this form of convergence is best-possible, as the diameter of C_n^ω has order $\sqrt{n}$ by Theorem 33, and hence (17) fails if the order of k_n is comparable to $\sqrt{n}$.

6.4.2 Benjamini–Schramm limit and subgraph count asymptotics

The infinite $\mathcal{G}$-enriched tree $\hat{\mathsf{H}}^\bullet$ from Section 6.2.2 may be interpreted as an infinite locally finite random graph $\hat{\mathsf{C}}^\bullet$ according to the bijection in Section 6.1.2. Theorem 27 yields Benjamini–Schramm convergence of the random graph C_n^ω.

Theorem 32 (Benjamini–Schramm convergence of random unlabelled graphs)**.** *Suppose that the weighted ordinary generating series $(\tilde{\mathcal{C}}^\bullet)^\omega(z)$ has radius of convergence $\rho > 0$, and that*

$$E(z,u) = zZ_{(\mathrm{SET}\circ\mathcal{B}')^\kappa}(u,(\tilde{\mathcal{C}}^\bullet)^{\omega^2}(z^2),(\tilde{\mathcal{C}}^\bullet)^{\omega^3}(z^3),\ldots)$$

satisfies $E(\rho+\epsilon,(\tilde{\mathcal{C}}^\bullet)^\omega(\rho)+\epsilon) < \infty$ for some $\epsilon > 0$. Let v^* be a uniformly at random drawn vertex of the random graph C_n^ω. Then for any sequence $k_n = o(\sqrt{n})$ of non-negative integers it holds that

$$d_{\mathrm{TV}}(U_{k_n}(\mathsf{C}_n^\omega, v^*), U_{k_n}(\hat{\mathsf{C}}^\bullet)) \to 0,$$

and likewise for the graph-metric neighbourhoods $V_{k_n}(\cdot)$. Thus, the infinite random graph $\hat{\mathsf{C}}^\bullet$ is the Benjamini–Schramm limit of the random graph C_n^ω as n tends to infinity.

Again this form of convergence is best-possible, as the diameter of C_n^ω has order $\sqrt{n}$ by Theorem 33. Benjamini–Schramm convergent sequences have many nice properties, for example we may apply general results by Kurauskas [48]) and Lyons [56, Thm. 3.2] to deduce laws of large numbers for subgraph count statistics and spanning tree count statistics.

6.4.3 Scaling limit and diameter tail-bounds

We apply our results to first-passage percolation on graphs. Let $\iota > 0$ denote a random variable which has finite exponential moments. Given a connected graph G we may consider the first-passage percolation metric d_{FPP} on G by assigning an independent copy of ι to each edge of G, letting for any two vertices x, y the distance $d_{\mathrm{FPP}}(x,y)$ be given by the minimum of all sums of weights along paths joining x and y. We let $\mathrm{D}_{\mathrm{FPP}}(\cdot)$ and $\mathrm{H}_{\mathrm{FPP}}(\cdot)$ denote the diameter and height with respect to the d_{FPP}-distance Theorems 28 and 29 and the fact, that the diameter and height of the CRT are related by

$$\mathbb{E}\left[\mathrm{D}(\mathcal{T}_\mathsf{e})\right] = \frac{4}{3}\mathbb{E}\left[\mathrm{H}(\mathcal{T}_\mathsf{e})\right] = \frac{4}{3}\sqrt{\frac{\pi}{2}},$$

readily yield the following result.

Theorem 33 (First passage percolation random unlabelled rooted graphs)**.** *Suppose that the weighted ordinary generating series has radius of convergence $\rho > 0$, and that the series*

$$E(z,u) = zZ_{(\mathrm{SET}\circ\mathcal{B}')^\kappa}(u,(\tilde{\mathcal{C}}^\bullet)^{\omega^2}(z^2),(\tilde{\mathcal{C}}^\bullet)^{\omega^3}(z^3),\ldots)$$

is finite at the point $(\rho+\epsilon,\tilde{\mathcal{C}}^{\bullet\omega}(\rho)+\epsilon)$ for some $\epsilon > 0$. Then there exists a constant $a > 0$ such that

$$(\mathsf{C}_n^\omega, an^{-1/2}d_{\mathrm{FPP}}) \xrightarrow{d} (\mathcal{T}_\mathsf{e}, d_{\mathcal{T}_\mathsf{e}})$$

in the Gromov–Hausdorff sense as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ becomes large. Furthermore, if the random edge-length ι is bounded then there are constants $C, c > 0$ with

$$\mathbb{P}\left(\mathrm{D}(\mathsf{C}_n^\omega, d_{\mathrm{FPP}}) \geqslant x\right) \leqslant C\exp(-cx^2/n)$$

for all n and $x \geqslant 0$. In particular, the rescaled height and diameter converge in the space $\mathbb{L}_p$ for all $p \geqslant 1$. We have asymptotically

$$\mathbb{E}\left[\mathrm{D}_{\mathrm{FPP}}(\mathsf{C}_n^\omega)\right] \sim \frac{4}{3}\mathbb{E}\left[\mathrm{H}_{\mathrm{FPP}}(\mathsf{C}_n^\omega)\right] \sim \frac{4}{3a}\sqrt{\frac{\pi n}{2}}.$$

Lemma 30 also yields the following result for the size of the largest 2-connected component of the random graph C_n^ω.

Corollary 34. *There is a constant $C > 0$ such that the largest block in the random graph C_n^ω has size at most $C \log n$ with probability tending to 1 as n becomes large. Likewise, the maximum degree admits an $O(\log(n))$ bound with high probability.*

6.5 Applications to random unlabelled front-rooted k-dimensional trees

We consider the species $\mathcal{K}$ of front-rooted k-trees and the subclass $\mathcal{K}^\circ$ where the root-front is required to lie in a single hedron. Let K_n denote a uniform random unlabelled $\mathcal{K}$-object with n hedra and likewise K_n° a uniform random $\mathcal{K}^\circ$-object with n hedra. As discussed in Section 6.1.3, the two species are related by the equations

$$\mathcal{K} \simeq \mathrm{SET} \circ \mathcal{K}^\circ, \qquad \mathcal{K}^\circ \simeq \mathcal{X} \cdot (\mathrm{SEQ}_{\{k\}} \circ \mathrm{SET})(\mathcal{K}^\circ).$$

This identifies the random k-tree K_n° with the random enriched tree $\mathsf{A}_n^{\mathcal{R}}$ for the special case $\mathcal{R} = \mathrm{SEQ}_{\{k\}} \circ \mathrm{SET}$. The random front-rooted k-tree K_n may be interpreted as a random unordered forest of $\mathcal{R}$-enriched trees. We let ρ denote the radius of convergence of $\tilde{\mathcal{K}}^\circ(z) = \tilde{\mathcal{A}}_{\mathcal{R}}^\omega(z)$. Note that

$$Z_{\mathcal{R}}(z_1, z_2, \ldots) = \exp(k \sum_{i=1}^{\infty} z_i/i).$$

Lemmas 21 and 22 readily yield $0 < \rho < 1$ and $\tilde{\mathcal{K}}^\circ(\rho) < \infty$. Hence

$$Z_{\mathcal{R}}(\tilde{\mathcal{K}}^\circ(\rho) + \epsilon, \tilde{\mathcal{K}}^\circ((\rho+\epsilon)^2), \tilde{\mathcal{K}}^\circ((\rho+\epsilon)^3)) < \infty$$

for some $\epsilon > 0$. This parallels some arguments from Drmota and Jin [28], where among other enumerative results the asymptotic number of unlabelled unrooted and rooted k-trees was determined.

6.5.1 Local weak limit

Theorem 24 readily yields a local weak limit of the random graph K_n° with respect to neighbourhoods of the root-front, or any fixed vertex of the root-front. The limit object is the infinite random k-tree $\hat{\mathsf{K}}^\circ$ that corresponds to the limit $\mathrm{Sym}(\mathcal{R})$-enriched tree $(\mathcal{T}^{(\infty)}, \beta^{(\infty)})$ according to the bijection in Section 6.1.3.

The random unlabelled front-rooted k-tree K_n may be viewed as a random unlabelled Gibbs-partition. By Theorem 15 it follows that K_n exhibits a giant component of size $n+$

$O_p(1)$, and the small fragments converge in total variation toward a Boltzmann limit R that follows a $\mathbb{P}_{\widetilde{\mathrm{SET}\circ\mathcal{K}^\circ},\rho}$ distribution. We let $\hat{\mathsf{K}}$ denote the infinite random k-tree obtained by identifying the root-front of $\hat{\mathsf{K}}^\circ$ with the root-front of the front-rooted k-tree corresponding to R according to the bijection in Section 6.1.3.

Theorem 35 (Local convergence of random unlabelled front-rooted $\boldsymbol{k}$-trees)**.** *For any sequence $k_n = o(\sqrt{n})$ it holds that*

$$d_{\mathrm{TV}}(V_{k_n}(\mathsf{K}_n), V_{k_n}(\hat{\mathsf{K}})) \to 0.$$

Thus the infinite random graph $\hat{\mathsf{K}}$ is the local weak limit of the random front-rooted k-tree K_n as n becomes large. Here we may interpret $V_{k_n}(\cdot)$ as the neighbourhood of the root-front, or of any fixed vertex of the root-front. By exchangeability, it does not matter which we choose.

Note that Theorem 35 does not follow directly from the above discussion, as we still need to relate the height of vertices in the $\mathcal{G}$-enriched tree representation with the height in the corresponding k-tree. We provide a detailed justification in Section 7.3.

6.5.2 Benjamini–Schramm limit

The infinite $\mathcal{G}$-enriched tree $\hat{\mathsf{H}}^\bullet$ from Section 6.2.2 may be interpreted as an infinite k-tree $\hat{\mathsf{K}}^\bullet$ according to the bijection in Section 6.1.3.

As we are going to argue in detail in Section 7.3, a random vertex in K_n lies with high probability in the largest $\mathcal{K}^\circ$-component. Theorem 27 together with a large deviation estimate yield Benjamini–Schramm convergence of the random k-tree K_n° toward the random graph $\hat{\mathsf{K}}^\bullet$, and hence also Benjamini–Schramm convergence for the largest $\mathcal{K}^\circ$-component of K_n. Moreover, within this component, a random vertex is unlikely to lie anywhere near the root. Thus $\hat{\mathsf{K}}$ is also the Benjamini–Schramm limit of the random front-rooted k-tree K_n.

Theorem 36 (Benjamini–Schramm convergence of unlabelled front-rooted $\boldsymbol{k}$-trees)**.** *Let v^* denote a uniformly at random selected vertex of the random unlabelled front-rooted k-tree K_n. Then for any sequence of positive integers $k_n = o(\sqrt{n})$ it holds that*

$$d_{\mathrm{TV}}(V_{k_n}(\mathsf{K}_n, v^*), V_{k_n}(\hat{\mathsf{K}}^\bullet)) \to 0.$$

Thus $\hat{\mathsf{K}}^\bullet$ is the Benjamini–Schramm limit of K_n as n becomes large.

6.5.3 Scaling limit

By Theorem 15 we know that K_n exhibits a giant $\mathcal{K}^\circ$-component of size $n + O_p(1)$. Hence in order to establish a scaling limit for K_n it suffices to study the random k-tree K_n° where the root-front lies in a single hedron. In order to establish a scaling limit for K_n° we may not apply Theorem 28 directly, as the metric of K_n° does not fit in the general scheme of random metric spaces considered in Section 6.3. Rather than that, we make direct use of the size-biased $\mathcal{G}$-enriched tree of Lemma 23 and the results of Lemma 30.

Theorem 37. *There is a constant $a_k > 0$ such that*

$$(\mathsf{K}_n, a_k n^{-1/2} d_{\mathsf{K}_n}) \xrightarrow{d} (\mathcal{T}_{\mathsf{e}}, d_{\mathcal{T}_{\mathsf{e}}})$$

in the Gromov–Hausdorff sense as n becomes large.

As a byproduct, we obtain the following properties of the random k-tree K_n°.

Lemma 38. *It holds that*

$$(\mathsf{K}_n^\circ, a_k n^{-1/2} d_{\mathsf{K}_n}) \xrightarrow{d} (\mathcal{T}_{\mathsf{e}}, d_{\mathcal{T}_{\mathsf{e}}}).$$

Moreover, there are constants $C, c > 0$ such that for all $x \geqslant 0$ and $n \geqslant 1$

$$\mathbb{P}\left(\mathrm{D}(\mathsf{K}_n^\circ) \geqslant x\right) \leqslant C \exp(-cx^2/n).$$

6.6 Applications to simply generated Pólya trees

Let $(\kappa_i)_{i\in\mathbb{N}}$ be a sequence of non-negative weights with $\kappa_0 > 0$ and $\kappa_i > 0$ for some $i \geqslant 2$. Hence κ can be seen as a weighting on the species SET. Let d denote the greatest common divisor of the set of all indices i with $\kappa_i > 0$. For $n \equiv 1 \mod d$ large enough we may draw a random Pólya tree τ_n having n vertices with probability $\mathbb{P}\left(\tau_n = \tau\right)$ proportional to $\prod_{v\in\tau} \kappa_{d_\tau^+(v)}$ for any unlabelled unordered tree τ with size n. This corresponds to the random unlabelled enriched tree $\tilde{\mathcal{A}}_n^{\mathcal{R}}$ for $\mathcal{R}^\kappa = \mathrm{SET}^\kappa$. We let ρ denote the radius of convergence of the corresponding generating series $\tilde{\mathcal{A}}(z) := \tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(z)$.

In the following, we only consider the case where

$$\rho > 0 \qquad \text{and} \qquad Z_{\mathrm{SET}^\kappa}(\tilde{\mathcal{A}}(\rho) + \epsilon, \tilde{\mathcal{A}}((\rho+\epsilon)^2), \ldots) < \infty \tag{18}$$

for some $\epsilon > 0$.

6.6.1 Local weak limit

Let $\hat{\tau}$ denote the tree $\mathcal{T}^{(\infty)}$ for the special case $\mathcal{R}^\kappa = \mathrm{SET}^\kappa$. Theorem 24 readily yields the following result.

Theorem 39. *If Condition* (18) *is satisfied, then for any sequence $k_n = o(\sqrt{n})$ of positive integers it holds that*

$$d_{\mathrm{TV}}(V_{k_n}(\tau_n), V_{k_n}(\hat{\tau})) \to 0.$$

Thus, $\hat{\tau}$ is the local weak limit of τ_n as n becomes large.

6.6.2 Benjamini–Schramm limit

Let $\hat{\tau}^\bullet$ denote the pointed plane tree corresponding to the tree $\hat{\mathsf{H}}^\bullet$ for the case $\mathcal{R}^\kappa = \mathrm{SET}^\kappa$. Theorem 27 applies directly and yields a Benjamini–Schramm limit for the simply generated Pólya tree τ_n.

Theorem 40. *Suppose that Condition* (18) *holds. Let v^* denote a uniformly at random selected vertex of the tree τ_n. Then for any sequence $k_n = o(\sqrt{n})$ of positive integers it holds that*

$$d_{\mathrm{TV}}(V_{k_n}(\tau_n, v^*), V_{k_n}(\hat{\tau}^\bullet)) \to 0.$$

In particular, $\hat{\tau}^\bullet$ is the Benjamini–Schramm limit of τ_n as n becomes large.

6.6.3 Scaling limit and diameter tail-bound

Another application of Theorems 28 and 29 is the following scaling limit with a sharp tail-bound for the diameter.

Theorem 41 (Scaling limits of simply generated Pólya trees)**.** *If Condition* (18) *is satisfied, then there is a constant $a > 0$ such that*

$$(\tau_n, an^{-1/2} d_{\tau_n}) \xrightarrow{d} (\mathcal{T}_{\mathsf{e}}, d_{\mathcal{T}_{\mathsf{e}}})$$

with respect to the Gromov–Hausdorff metric as becomes large. Moreover, there are constants $c, C > 0$ such that for all $x \geqslant 0$ and n it holds that

$$\mathbb{P}\left(\mathrm{D}(\tau_n) \geqslant x\right) \leqslant C \exp(-cx^2/n).$$

This is a mild extension of results for uniformly drawn Pólya trees with n vertices and vertex degree restrictions, whose scaling limits were studied in [57, 40, 61].

7 Proofs of the main results

7.1 Proof of the local convergence of unlabelled enriched trees in Section 6.2

Proof of Lemma 20. As discussed in Section 3.5.3, the rules of Lemma 9 concerning the interplay of Boltzmann distributions and operations on species may be used to construct recursive samplers, if the concerning isomorphism satisfies the conditions of Theorem 12. This is the case for isomorphism

$$\mathcal{A}^\omega_{\mathcal{R}} \simeq \mathcal{X} \cdot \mathcal{R}^\kappa(\mathcal{A}^\omega_{\mathcal{R}}),$$

which corresponds to the combinatorial specification $\mathcal{Y} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ with $\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \mathcal{X} \cdot \mathcal{R}^\kappa(\mathcal{Y})$. Indeed, it holds that $\mathcal{H}(0,0) = 0 \cdot \mathcal{R}^\kappa(0) = 0$ and $\partial_2 \mathcal{H}(0,0) = 0 \cdot (\mathcal{R}')^\kappa(0) = 0$.

Thus we may apply the product rule and substitution rule of Lemma 9 to construct a recursive procedure that samples according to the $\mathbb{P}_{\mathrm{Sym}(\mathcal{A}^\omega_{\mathcal{R}}),(x^i)_i}$-Boltzmann distribution. The result is the procedure described in Lemma 20, with one important difference. According to Lemma 9 we would have to apply Lemma 8 for each recursive call to construct an $\mathcal{A}_{\mathcal{R}}$-symmetry out of the $\mathcal{R}$-symmetry and the attached $\mathcal{A}_{\mathcal{R}}$-symmetries, and then relabel uniformly at random. Instead of doing this for each recursive call, we skip this step and keep track of all the $\mathcal{R}$-symmetries, yielding the $\mathrm{Sym}(\mathcal{R})$-enriched tree $(\mathcal{T}, \beta)$. As discussed at the beginning of the Section 6.2.1, we may construct the symmetry that

corresponds to $(\mathcal{T}, \beta)$ in one step by applying Lemma 8 for each of its vertices, starting with the leaves and working our way upwards. Thus $(\mathcal{T}, \beta)$ corresponds to a symmetry on the vertex set of the plane tree $\mathcal{T}$. Moreover, instead of relabelling uniformly at random after each application of Lemma 8, we may postpone this step and just relabel the symmetry corresponding to $(\mathcal{T}, \beta)$ once uniformly at random. The result $\Gamma Z_{\mathcal{A}^\omega_\mathcal{R}}(x)$ then follows $\mathbb{P}_{\mathrm{Sym}(\mathcal{A}^\omega_\mathcal{R}),(x^i)_i}$-Boltzmann distribution, meaning that for each $n \geqslant 0$ and each symmetry $(A, \sigma) \in \mathrm{Sym}(\mathcal{A}_\mathcal{R})[n]$ it holds that

$$\mathbb{P}\left(\Gamma Z_{\mathcal{A}^\omega_\mathcal{R}}(x) = (A, \sigma)\right) = \omega(A)\frac{x^n}{n!} Z_{\mathcal{A}^\omega_\mathcal{R}}(x, x^2, \ldots)^{-1}.$$

By Equation (4) we know that

$$Z_{\mathcal{A}^\omega_\mathcal{R}}(x, x^2, \ldots) = \tilde{\mathcal{A}}^\omega_\mathcal{R}(x).$$

This completes the proof. □

Proof of Lemma 21. As for claim *(1)*, note that the isomorphism $\tilde{\mathcal{A}}^\omega_\mathcal{R} \simeq \mathcal{X} \cdot \mathcal{R}^\kappa(\tilde{\mathcal{A}}^\omega_\mathcal{R})$ implies that

$$\tilde{\mathcal{A}}^\omega_\mathcal{R}(z) = z Z_{\mathcal{R}^\kappa}(\tilde{\mathcal{A}}^\omega_\mathcal{R}(z), \tilde{\mathcal{A}}^\omega_\mathcal{R}(z^2), \ldots).$$

By assumption, there is an $\mathcal{R}$-structure with size zero and one with size at least two such that both have a positive κ-weight. It follows that there are constants $a, b > 0$ and $k \geqslant 2$ with the property that for all $0 \leqslant x < \rho$ it holds that

$$\tilde{\mathcal{A}}^\omega_\mathcal{R}(x) \geqslant x(a + b\tilde{\mathcal{A}}^\omega_\mathcal{R}(x)^k).$$

This implies that $\lim_{x\uparrow\rho} \tilde{\mathcal{A}}^\omega_\mathcal{R}(x) < \infty$ and hence, by non-negativity of coefficients, $\tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho) < \infty$.

Claims *(2)* - *(6)* all follow after a moments consideration from the explicit description of the sampler in Lemma 20. □

Proof of Lemma 22. Claims *(1)* and *(2)* follow from a general enumeration theorem by Bell, Burris and Yeats [12, Thm. 28] which implies the asymptotic behaviour of the coefficients of the power series $\tilde{\mathcal{A}}^\omega_\mathcal{R}(z)$.

Claim *(3)*: The expressions for the moments of ξ and ζ follow from the equations describing the corresponding probability generating functions in Lemma 21. In order to verify that $\mathbb{E}[\xi] = 1$, note that by Pringsheim's theorem the function $\tilde{\mathcal{A}}^\omega_\mathcal{R}(z)$ cannot be analytically continued in a neighbourhood of ρ, and hence by the implicit function theorem it must hold that the function

$$H(z, u) := u - E(z, u)$$

satisfies

$$H_u(\rho, \tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho)) = 0.$$

In other words, $\mathbb{E}[\xi] = E_u(\rho, \tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho)) = 1$. (Compare with the proof of [12, Cor. 12].)

Claim *(4)*: Let Λ denote the lattice spanned by all differences $\mathbf{x}-\mathbf{y}$ of vectors that (ξ,ζ) assumes with positive probability. We assumed that there is an $\mathcal{R}$-structure with size zero and positive κ-weight. Its automorphism is the empty map with no cycles at all. Hence $\mathbb{P}\left((\xi,\zeta)=(0,0)\right)>0$, and Λ is actually generated by the support of (ξ,ζ).

We assumed further that at least one $\mathcal{R}$-structure R with positive κ-weight has a non-trivial automorphism group. Hence there are positive numbers a^*, b^* and c^* such that (ξ,ζ) assumes $(a^*,0)$ and (b^*,c^*) with positive probability, as these points correspond the trivial and a non-trivial automorphism of R. Hence the lattice Λ contains three points that do not lie on any straight line. Consequently, it has rank 2, that is, $\Lambda\simeq\mathbb{Z}^2$ as abelian group.

Let $\mathbf{B}\in\mathbb{Z}^{2\times 2}$ be a basis of the lattice Λ. As the support of (ξ,ζ) is not contained on a straight line, it follows that the covariance matrix Σ is positive-definite. Let $(\xi_i,\zeta_i)_{i\in\mathbb{N}}$ denote a family of independent copies of (ξ,ζ). By Lemma 21 it holds for any positive integer n and ℓ that

$$\mathbb{P}\left(|\mathcal{T}|=n,|\mathcal{T}^f|=\ell\right)=\mathbb{P}\left(\sum_{i=1}^{\ell}(\xi_i,\zeta_i)=(\ell-1,n-\ell),\sum_{i=1}^{m}\xi_i\geqslant m\text{ for all }m<\ell\right).$$

Using rotational symmetry and the cycle lemma 19, it follows that

$$\mathbb{P}\left(|\mathcal{T}|=n,|\mathcal{T}^f|=\ell\right)=\frac{1}{\ell}\mathbb{P}\left(\sum_{i=1}^{\ell}(\xi_i,\zeta_i)=(\ell-1,n-\ell)\right). \tag{19}$$

We know by Claim *(1)* that if n is larger than some fixed constant, then $\mathbb{P}\left(|\mathcal{T}|=n\right)>0$ if and only if $n-1$ is divisible by $\mathsf{span}(\mathbf{w})$. We shall check below that for each such n

$$\{\ell\in\mathbb{Z}\mid(\ell-1,n-\ell)\in\Lambda\}=n+\mathbb{Z}d \tag{20}$$

with $d=|\det\mathbf{B}|/\mathsf{span}(\mathbf{w})$. For now, let us assume that (20) holds. Let $M>0$ be a fixed constant. Then it holds uniformly for all $\ell=(1+\mathbb{E}\left[\zeta\right])n+x\sqrt{n}$ with $|x|\leqslant M$ that

$$\ell\sim n/\sqrt{1+\mathbb{E}\left[\zeta\right]}$$

and

$$\ell^{-1/2}\left((\ell-1,n-\ell)-\ell\mathbb{E}\left[(\xi,\zeta)\right]\right)\sim(0,-x(1+\mathbb{E}\left[\zeta\right])^{3/2}).$$

The central local limit theorem given in Lemma 16 yields that

$$\frac{1}{\ell}\mathbb{P}\left(\sum_{i=1}^{\ell}(\xi_i,\zeta_i)=(\ell-1,n-\ell)\right)\sim\frac{(1+\mathbb{E}\left[\zeta\right])|\det\mathbf{B}|}{n^2\sqrt{2\pi\det\mathbf{\Sigma}}}\exp(-\frac{x^2}{2\sigma^2}). \tag{21}$$

with $\sigma^2=\det\mathbf{\Sigma}/(\mathbb{V}[\zeta](1+\mathbb{E}\left[\zeta\right])^3)$. By Claim *(3)* we know that

$$\mathbb{P}\left(|\mathcal{T}|=n\right)\sim\mathsf{span}(\mathbf{w})n^{-3/2}\sqrt{\frac{1+\mathbb{E}\left[\zeta\right]}{2\pi\mathbb{V}[\xi]}}.$$

Using Equation (19) it follows that

$$\mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \sim \frac{d}{\sigma\sqrt{2\pi n}} \exp(-\frac{x^2}{2\sigma^2}).$$

The central limit theorem now follows from Equation (20), as for any fixed $a < b$

$$\begin{aligned}\mathbb{P}\left(a \leqslant \frac{|\mathcal{T}_n^f| - n/(1+\mathbb{E}\left[\zeta\right])}{\sqrt{n}} \leqslant b\right) &\sim \frac{d}{\sqrt{n}} \sum_x \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2}) \\ &\sim \int_a^b \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})\,\mathrm{d}x,\end{aligned}$$

with the sum index x ranging over $[a,b] \cap ((1-\mu)\sqrt{n} + \mathbb{Z}d/\sqrt{n})$.

It remains to verify Equation (20), which requires careful reasoning, as we have to relate $\mathsf{span}(\mathbf{w})$ with the involved lattice. In order to simplify our calculations, we would like to pick a "nice" basis $\mathbf{B}$ of Λ. Note that it does not matter for (20) which basis of Λ we choose, as for any two bases $\mathbf{B}_1$ and $\mathbf{B}_2$ there is a matrix $\mathbf{M} \in \mathrm{GL}_2(\mathbb{Z})$ with $\mathbf{B}_1 = \mathbf{M}\mathbf{B}_2$, and as $\det \mathbf{M} \in \{1,-1\}$ it follows that $|\det \mathbf{B}_1| = |\det \mathbf{B}_2|$. The inconvenient part is that, contrary to vector spaces, not every linear independent subset of a lattice may be extended to a basis. However, a classical algebraic result states that for any free $\mathbb{Z}$-module M with rank $r(M)$ and for any submodule $N \subset M$ with rank $r(N)$ there is a $\mathbb{Z}$-basis $v_1, \ldots, v_{r(M)}$ of M and integers $\lambda_1, \ldots, \lambda_{r(N)}$ such that $\lambda_1 v_1, \ldots, \lambda_k v_k$ is a basis of N. See for example Roman's book [65, Thm. 6.7], which states this in the more general context of modules over principal ideal domains. If $\mathbf{B} = (\mathbf{b}_1, \mathbf{b_2})$ is such a basis of Λ for the submodule $N := \mathbb{Z}(a^*, 0)^\intercal \subset \Lambda$ (recall that we defined a^* at the beginning of the proof, when we showed that Λ has rank 2), then there is an integer λ with $\lambda\mathbf{b}_1 \in \mathbb{Z}(a^*,0)^\intercal$. But this implies that the second coordinate of $\mathbf{b}_1$ must be zero, and hence

$$\mathbf{B} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \tag{22}$$

with $a, b, c > 0$ is an upper-triangular matrix. For any ℓ, n it holds that $(\ell-1, (n-1) - (\ell-1))^\intercal \in \Lambda$ if and only if $(\ell-1, n-1)^\intercal \in \Lambda' := \mathbf{C}\mathbb{Z}^2$ with

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ a & b+c \end{pmatrix}.$$

We may easily calculate that

$$(\ell-1, n-1) \in \mathbf{C}\mathbb{Z}^2 \quad \text{if and only if} \quad n-1 \in \mathbb{Z}\gcd(a, b+c),\ \ell \in n + \mathbb{Z}ac/\gcd(a, b+c).$$

Indeed, it necessarily holds that $n-1 \in \mathbb{Z}a + \mathbb{Z}(b+c) = \mathbb{Z}\gcd(a, b+c)$. Conversely, if $n-1 = \lambda_0 a + \mu_0 b$, then any pair $(\lambda, \mu) \in \mathbb{Z}^2$ satisfies $\lambda a + \mu b = n-1$ precisely if there is an arbitrary integer t with $\lambda - \lambda_0 = t(b+c)/\gcd(a, b+c)$ and $\mu - \mu_0 = ta/\gcd(a, b+c)$. Hence $(\ell-1, n-1)^\intercal = \mathbf{C}(\lambda, \mu)^\intercal$ reduces to $\ell = n + act/\gcd(a, b+c)$ with $t \in \mathbb{Z}$.

Since $\det(\mathbf{B}) = ac$, it remains to check that $\mathsf{span}(\mathbf{w}) = \gcd(a, b+c)$. That is, we show that $\mathsf{span}(\mathbf{w})$ is the span of the projection of Λ' on the y-axis. The support S of $(|\mathcal{T}^f| - 1, |\mathcal{T}| - 1)$ is a subset of the lattice Λ'. Hence $\mathsf{span}(\mathbf{w})$, which is the span of the projection of S to the y-axis, is a multiple of $\gcd(a, b+c)$. In order to show equality we need to verify that indeed two successive multiples of $\gcd(a, b+c)$ exist that $|\mathcal{T}| - 1$ assumes with positive probability. We may do this probabilistically. Let $C > 0$ be a given constant. By Equation 19 and the central local limit theorem given in Lemma 16 it follows that there is a positive integer L such that for all $\ell \geqslant L$ and all n with $(l-1, n-1)^\intercal \in \Lambda'$ and

$$\ell^{-1/2} \left\| ((\ell - 1, n - \ell) - \ell \mathbb{E}\left[(\xi, \zeta)\right]) \right\| \leqslant C \tag{23}$$

it holds that $\mathbb{P}\left(|\mathcal{T}| = n, |\mathcal{T}^f| = \ell\right) > 0$. There is a constant $c > 0$ such that for every n at least $c\sqrt{n}$ consecutive integers $\ell \geqslant L$ exist that satisfy Inequality (23). If $n-1$ is additionally a constant multiple of $\gcd(a, b+c)$, then the intersection Γ_n of the lattice Λ' and the affine space $(0, n-1)^\intercal + \mathbb{Z}(1, 0)$ is non-empty, and hence an affine subspace of rank one, whose span does not depend on n. In particular, Γ_n hits the cone (23) about a constant fraction of $\sqrt{n}$ many times. This proves that $\mathsf{span}(\mathbf{w}) = \gcd(a, b+c)$ and concludes the proof of Claim *(4)*.

□

Proof of Lemma 23. The task is to provide a sampler for the species $\mathcal{A}_{\mathcal{R}}^{(\ell)}$ using the isomorphism

$$\mathcal{A}_{\mathcal{R}}^{(\ell)} \simeq (\mathcal{X} \cdot (\mathcal{R}')^{\kappa}(\mathcal{A}_{\mathcal{R}}^{\omega}))^{\ell} \mathcal{A}_{\mathcal{R}}^{\omega}.$$

Note that for $\ell = 0$ we have $\mathcal{A}_{\mathcal{R}}^{(0)} = \mathcal{A}_{\mathcal{R}}^{\omega}$ and for $\ell \geqslant 1$ it holds that

$$\mathcal{A}_{\mathcal{R}}^{(\ell)} \simeq \mathcal{X} \cdot (\mathcal{R}')^{\kappa}(\mathcal{A}_{\mathcal{R}}^{\omega}) \cdot \mathcal{A}_{\mathcal{R}}^{(\ell-1)}.$$

Thus for $\ell \geqslant 1$ we may apply the product rule and substitution rule of Lemma 9 to construct a procedure that samples according to the $\mathbb{P}_{\mathrm{Sym}(\mathcal{A}_{\mathcal{R}}^{(\ell)}),(x^i)_i}$-Boltzmann distribution, employing a $\mathbb{P}_{\mathrm{Sym}(\mathcal{A}_{\mathcal{R}}^{(\ell-1)}),(x^i)_i}$-distributed symmetry, which we may sample by a recursive call to the sampler for $\ell - 1$.

The result is essentially the procedure described in Lemma 23, but we make some modifications. According to Lemma 9 we would have to apply Lemma 8 to construct an $\mathcal{A}_{\mathcal{R}}$-symmetry out of the $\mathcal{R}'$-symmetry, the attached $\mathcal{A}_{\mathcal{R}}$-symmetries, and the $\mathcal{A}_{\mathcal{R}}^{(\ell-1)}$-symmetry, and relabel uniformly at random afterwards. Instead of doing this for each call, we skip this step and keep track of all the $\mathcal{R}'$-symmetries. Also, instead of taking $\mathcal{A}_{\mathcal{R}}$-symmetries directly, we use the $\mathrm{Sym}(\mathcal{R})$-enriched tree $(\mathcal{T}, \beta)$ from Lemma 23. This yields the $\mathrm{Sym}(\mathcal{R})$-enriched tree $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$.

As discussed at the beginning of the Section 6.2.1, we may construct the symmetry that corresponds to $(\mathcal{T}^{(\ell)}, \beta^{(\ell)})$ in one step by applying Lemma 8 for each of its vertices, starting with the leaves and working our way upwards. If we additionally relabel uniformly at random, the resulting symmetry $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}$ follows a $\mathbb{P}_{\mathrm{Sym}(\mathcal{A}_{\mathcal{R}}^{(\ell)},(x^i)_i}$-distribution.

We defined $\mathcal{A}_{\mathcal{R}}^{(\ell)}$ as a subspecies of the species of pointed $\mathcal{R}$-enriched trees. In particular, for all $n \geqslant 0$ and any symmetry $((A,r),\sigma) \in \mathrm{Sym}(\mathcal{A}_{\mathcal{R}}^{(\ell)})$ we have that (A,σ) is an $\mathcal{A}_{\mathcal{R}}^{\omega}$-symmetry and r is a fixed point of the automorphism σ. It follows from the definition of the Boltzmann distributions and

$$\tilde{\mathcal{A}}_{\mathcal{R}}^{(\ell)}(x) = (x\widetilde{(\mathcal{R}')^{\kappa} \circ \mathcal{A}_{\mathcal{R}}^{\omega}}(x))^{\ell}\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(x)$$

that

$$\begin{aligned}\mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}(x) = ((A,r),\sigma)\right) &= \omega(A)\frac{x^{|A|}}{|A|!}\tilde{\mathcal{A}}_{\mathcal{R}}^{(\ell)}(x)^{-1} \\ &= (x\widetilde{(\mathcal{R}')^{\kappa} \circ \mathcal{A}_{\mathcal{R}}^{\omega}}(x))^{-\ell}\mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}(x) = (A,\sigma)\right). \qquad \square\end{aligned}$$

Proof of Lemma 25. Claim *(1)*: By Lemma 23, the $\mathcal{R}$-symmetry $(\hat{\mathsf{R}},\hat{\sigma})$ of $\hat{\mathsf{G}}$ follows, up to relabeling, a weighted Pólya-Boltzmann distribution for the species $(\mathcal{R}')^{\kappa}$ with parameter $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho)$. It makes no difference whether we distinguish the vertex corresponding to the $*$-fixed point, or a uniformly at random drawn fixed point, as the results are identically distributed. By the discussion in Section 3.2 it holds that

$$Z_{(\mathcal{R}')^{\kappa}}(s_1,s_2,\ldots) = \frac{\partial}{\partial s_1}Z_{\mathcal{R}^{\kappa}}(s_1,s_2,\ldots).$$

Hence, for any $\mathcal{R}$-symmetry (R,σ) with a marked fixed point u, the probability that $(\hat{\mathsf{R}},\hat{\sigma}) = (R,\sigma)$ and that precisely the fixed point u gets marked, is simply given by the probability that a Pólya-Boltzmann distributed $\mathcal{R}$-symmetry with parameter $\tilde{\mathcal{A}}_{\mathcal{R}}^{\omega}(\rho)$ equals (R,σ). Consequently, for any $\mathcal{G}$-object G with a marked fixed point u the probability that $\hat{\mathsf{G}}$ assumes G and that precisely the vertex u is marked is given by $\mathbb{P}(\mathsf{G} = G)$.

Claim *(2)*: This is a reformulation of Lemma 23 in terms of $\mathcal{G}$-enriched trees.

Claim *(3)*: The event $(\mathcal{T}^{(\infty)},\beta^{(\infty)})^{<k>} = (\tau,\gamma)^{<k>}$ corresponds to precisely $L_\tau(k)$ different outcomes of the first $k-1$ levels, depending on which leaf of $\tau^{[k]}$ the spine of the fixed point tree $(\mathcal{T}^{(\infty)})^f$ is supposed to pass through. By Claim *(1)*, each of these events is equally likely with probability $\prod_{i=1}^{t}\mathbb{P}(\mathsf{G} = G_i)$.

Claim *(4)*: The $k+1$-th level of $(\mathcal{T}^{(\infty)},\beta^{(\infty)})$ interpreted as $\mathcal{G}$-enriched tree is obtained by taking for each fixed point of the k-th level an independent copy of G, except for the unique distinguished fixed point, which receives an independent copy of $\hat{\mathsf{G}}$. This yields

$$\begin{aligned}\mathbb{E}[L_k(\tau)] &= (\mathbb{E}[L_{k-1}(\tau)] - 1)\mathbb{E}[\xi] + \mathbb{E}\left[\hat{\xi}\right] = \ldots = k(\mathbb{E}\left[\hat{\xi}\right] - 1) + 1, \\ \mathbb{E}\left[|\tau^{[k]}|\right] &= \mathbb{E}[L_0(\tau) + \ldots + L_k(\tau)] = k(k+1)(\mathbb{E}\left[\hat{\xi}\right] - 1)/2 + k + 1, \\ \mathbb{E}\left[L_k^{\mathcal{G}}(\tau)\right] &= (\mathbb{E}[L_k] - 1)\mathbb{E}[\zeta] + \mathbb{E}\left[\hat{\zeta}\right] = \ldots = k(\mathbb{E}\left[\hat{\xi}\right] - 1)\mathbb{E}[\zeta] + \mathbb{E}\left[\hat{\zeta}\right], \\ \mathbb{E}\left[H_k^{\mathcal{G}}(\tau) - |\tau^{[k]}|\mathbb{E}[\zeta]\right] &= \sum_{i=0}^{k}(\mathbb{E}\left[L_i^{\mathcal{G}}(\tau)\right] - \mathbb{E}[L_i(\tau)]\,\mathbb{E}[\zeta]) = (k+1)(\mathbb{E}\left[\hat{\zeta}\right] - \mathbb{E}[\zeta]).\end{aligned}$$

For the variance, note that

$$\mathbb{V}[H_k^{\mathcal{G}}(\tau) - |\tau^{[k]}|\mathbb{E}[\zeta]] = \mathbb{E}\left[(D_0 + \dots D_k)^2)\right]$$

with

$$D_i := L_i^{\mathcal{G}}(\tau) - L_i(\tau)\mathbb{E}[\zeta] - \mathbb{E}\left[\hat{\zeta}\right] + \mathbb{E}[\zeta]$$

satisfying $\mathbb{E}[D_i \mid L_i(\tau) = \ell] = 0$ for all ℓ. In particular, for all $i < j$,

$$\mathbb{E}\left[D_i D_j \mid (\tau^{[k]}, (\gamma(v))_{v\in\tau^{[k-1]}})\right] = D_i\mathbb{E}[D_j \mid L_j(\tau)] = 0.$$

It also holds that

$$\mathbb{E}\left[D_i^2 \mid L_i(\tau)\right] = L_i(\tau)\mathbb{V}[\zeta] + \mathbb{V}[\hat{\zeta}],$$

and consequently

$$\begin{aligned}\mathbb{V}[H_k^{\mathcal{G}}(\tau) - |\tau^{[k]}|\mathbb{E}[\zeta]] &= \sum_{i=0}^{k}\mathbb{E}\left[D_i^2\right] = \sum_{i=0}^{k}(\mathbb{E}[L_i(\tau)]\,\mathbb{V}[\zeta] + \mathbb{V}[\hat{\zeta}]) \\ &= k(k+1)(\mathbb{E}\left[\hat{\xi}\right] - 1)\mathbb{V}[\zeta]/2 + (k+1)(\mathbb{V}[\hat{\zeta}] + \mathbb{V}[\zeta]). \qquad \square\end{aligned}$$

Proof of Theorem 24. Let $\mathcal{E}$ denote the countably infinite set of all $\mathcal{G}$-enriched plane trees and set

$$\mathcal{E}_k = \{A^{<k>} \mid A \in \mathcal{E}\}.$$

We have to show that

$$\lim_{n\to\infty}\sup_{\mathcal{H}\subset\mathcal{E}_{k_n}} |\mathbb{P}\left((\mathcal{T}_n, \beta_n)^{<k_n>} \in \mathcal{H}\right) - \mathbb{P}\left((\mathcal{T}^{(\infty)}, \beta^{(\infty)})^{<k_n>} \in \mathcal{H}\right)| = 0. \tag{24}$$

Recall that for any $\mathcal{G}$-enriched tree $(\tau, (G_\tau(v))_v)$ with $G_\tau(v) = (\beta_\tau(v), f_\tau(v), F_\tau(v))$ and any integer $k \geqslant 0$ we set

$$L_k(\tau) = |\{v \in \tau \mid \mathrm{h}_\tau(v) = k\}|, \quad L_k^{\mathcal{G}}(\tau) = \sum_{\substack{v\in\tau\\ \mathrm{h}_\tau(v)=k}} |F_\tau(v)|, \quad \text{and} \quad H_k^{\mathcal{G}}(\tau) = \sum_{i=0}^{k} L_i^{\mathcal{G}}(\tau).$$

By assumption, there is a sequence $t_n \to 0$ with $k_n = \sqrt{n}t_n$. We may without loss of generality assume that k_n tends to infinity. For any $C > 0$ and all k and n we define with foresight the set $\mathcal{E}_{C,k,n}$ of all $(\tau, \gamma) \in \mathcal{E}_k$ satisfying $\mathbb{P}\left((\mathcal{T}^{(\infty)}, \beta^{(\infty)})^{<k>} = (\tau, \gamma)\right) > 0$ and

$$1 \leqslant L_k(\tau) \leqslant C(nt_n)^{1/2}, \quad |H_{k-1}^{\mathcal{G}}(\tau) - |\tau^{[k-1]}|\mathbb{E}[\zeta]| \leqslant C(nt_n)^{1/2}, \quad |\tau^{[k-1]}| \leqslant nt_n.$$

Using Markov's and Chebyshev's inequalities and the expressions of the moments in Lemma 25, it follows that there is a constant C such that

$$\lim_{n\to\infty}\mathbb{P}\left((\mathcal{T}^{(\infty)}, \beta^{(\infty)})^{<k_n>} \in \mathcal{E}_{C,k_n,n}\right) = 1. \tag{25}$$

Hence, if we verify (24) with the index $\mathcal{H}$ ranging only over all subsets of $\mathcal{E}_{C,k_n,n}$, then (25) implies that

$$\lim_{n\to\infty} \mathbb{P}\left((\mathcal{T}_n,\beta_n)^{<k_n>} \notin \mathcal{E}_{C,k_n,n}\right) = 1 \tag{26}$$

and consequently (24) already holds with the index ranging over all subsets of $\mathcal{E}_{k_n}$. So the only thing that is left to show is that (24) holds for $\mathcal{H} \subset \mathcal{E}_{C,k_n,n}$. That is, we have to verify that for any $\epsilon > 0$ it holds for large enough n that

$$\sup_{\mathcal{H}\subset\mathcal{E}_{C,k_n,n}} |\mathbb{P}\left((\mathcal{T}_n,\beta_n)^{<k_n>} \in \mathcal{H}\right) - \mathbb{P}\left((\mathcal{T}^{(\infty)},\beta^{(\infty)})^{<k_n>} \in \mathcal{H}\right)| \leqslant \epsilon. \tag{27}$$

Let $\mathbf{B}, \mathbf{\Sigma}, \mu, \sigma$ and $d = |\det \mathbf{B}|/\mathsf{span}(\mathbf{w})$ be as in Lemma 22. We have shown in this Lemma that, as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ tends to infinity,

$$\sqrt{n}\mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \sim \frac{d}{\sigma\sqrt{2\pi}} \exp(-\frac{x_\ell^2}{2\sigma^2}) \tag{28}$$

uniformly for all bounded x_ℓ with

$$\ell := \mu n + x_\ell\sqrt{n} \in n + d\mathbb{Z}.$$

Moreover,

$$\frac{|\mathcal{T}_n^f| - n\mu}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\sigma^2)$$

and for any $\epsilon_1 > 0$ there is a constant $M > 0$ such that for all n

$$\mathbb{P}\left(|\mathcal{T}_n^f| \notin I_n\right) \leqslant \epsilon_1 \quad \text{with} \quad I_n := (n + d\mathbb{Z}) \cap [n/(1+\mathbb{E}\,[\zeta]) - M\sqrt{n}, n/(1+\mathbb{E}\,[\zeta]) + M\sqrt{n}].$$

Hence the expression in (27) may be bounded by

$$\epsilon_1 + \sup_{(\tau,\gamma)\in\mathcal{E}_{C,k_n,n}} \left| \frac{\mathbb{P}\left((\mathcal{T}_n,\beta_n)^{<k_n>} = (\tau,\gamma), |\mathcal{T}_n^f| \in I_n\right)}{\mathbb{P}\left((\mathcal{T}^{(\infty)},\beta^{(\infty)})^{<k_n>} = (\tau,\gamma)\right)} - 1 \right|. \tag{29}$$

Let $(\tau,\gamma) \in \mathcal{E}_{C,k_n,n}$ and let $G_i = (R_i, f_i, F_i)$, $1 \leqslant i \leqslant t$ be the depth-first-search ordered list of its $\mathcal{G}$-objects. We set $(h,H) := \sum_{i=1}^t (|f_i|, |F_i|)$. Moreover, let $\mathsf{G}_i = (\mathsf{R}_i, \mathsf{f}_i, \mathsf{F}_i)$ denote a family of independent copies of the random $\mathcal{G}$-enriched tree G. For each $\mathcal{G}$-enriched plane tree G set $\pi_G = \mathbb{P}\,(\mathsf{G} = G)$. By Lemma 21 it follows that for all ℓ the probability

$$\mathbb{P}\left((\mathcal{T},\beta)^{<k_n>} = (\tau,\gamma), |\mathcal{T}^f| = \ell\right) = \pi_{G_1}\cdots\pi_{G_t}\cdot$$
$$\mathbb{P}\left(\sum_{i=t+1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell-1-h, n-\ell-H), \sum_{i=t+1}^{m} \mathsf{f}_i \geqslant m-h \text{ for all } 1 \leqslant m < \ell\right).$$

Using rotational symmetry and the Cycle Lemma 19, this may be simplified further to

$$\pi_{G_1}\cdots\pi_{G_t}\frac{h-t+1}{\ell-t}\mathbb{P}\left(\sum_{i=t+1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell-1-h, n-\ell-H)\right).$$

The tree τ has precisely $h-t+1$ many leaves with height k and thus Lemma 25 implies that

$$\mathbb{P}\left((\mathcal{T}^{(\infty)},\beta^{(\infty)})^{<k_n>}=(\tau,\gamma)\right)=\pi_{G_1}\cdots\pi_{G_t}(h-t+1).$$

It holds uniformly for $(\tau,\gamma)\in\mathcal{E}_{C,k_n,n}$ and $\ell=\mu n+x_\ell\sqrt{n}\in I_n$ that

$$\ell^{-1/2}\left|(\ell-1-h,n-\ell-H)-(\ell-t)\mathbb{E}\left[(\xi,\zeta)\right]\right|\sim(0,-x_\ell(1+\mathbb{E}\left[\zeta\right])^{3/2})$$

as n becomes large. Hence, as $\ell-t\sim\ell$, we may apply the bivariate Central Local Limit Theorem 16 to obtain

$$\frac{1}{\ell-t}\mathbb{P}\left(\sum_{i=t+1}^{\ell}(|\mathsf{f}_i|,|\mathsf{F}_i|)=(\ell-1-h,n-\ell-H)\right)\sim\frac{(1+\mathbb{E}\left[\zeta\right])|\det\mathbf{B}|}{n^2\sqrt{2\pi\det\boldsymbol{\Sigma}}}\exp(-\frac{x_\ell^2}{2\sigma^2}).$$

By Lemma 22 we know that

$$\mathbb{P}\left(|\mathcal{T}|=n\right)\sim\mathsf{span}(\mathbf{w})n^{-3/2}\sqrt{\frac{1+\mathbb{E}\left[\zeta\right]}{2\pi\mathbb{V}[\xi]}}.$$

It follows that uniformly for all $(\tau,\gamma)\in\mathcal{E}_{C,k_n,n}$

$$\begin{aligned}\frac{\mathbb{P}\left((\mathcal{T}_n,\beta_n)^{<k_n>}=(\tau,\gamma),|\mathcal{T}_n^f|\in I_n\right)}{\mathbb{P}\left((\mathcal{T}^{(\infty)},\beta^{(\infty)})^{<k_n>}=(\tau,\gamma)\right)}&\sim\frac{d}{\sqrt{n}}\sum_{\ell\in I_n}\frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x_\ell^2}{2\sigma^2})\\&\sim\int_{-M}^{M}\frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})\,\mathrm{d}x.\end{aligned}$$

Taking $\epsilon_1=\epsilon/2$ and M sufficiently large it follows that the bound (29) is smaller than ϵ for sufficiently large n. This completes the proof. □

Proof of Lemma 26. Claims *(1)* and *(2)* are straight-forward. For Claim *(3)*, note that the event $\hat{\mathsf{H}}_{[k]}=\mathbf{H}_{[k]}$ means that $\hat{\mathsf{H}}_k=H_k$, and that u_0 lies at one of the $p(\mathbf{H})$ many locations, and that u^* lies in precisely the location in $\{u_0\}\cup F(u_0)$ that corresponds to u, with $F(u_0)$ denoting the forest of non-fixed points of the $\mathcal{G}$-object corresponding to u_0. Let $G_1,\ldots,G_t$ denote the depth-first-search ordered list of the $\mathcal{G}$-objects $G_i=(S_i,f_i,F_i)$ of H_k and let $G_{\ell_1},\ldots,G_{\ell_2}$ be the segment that corresponds to H_0. For any $\mathcal{G}$-object $G=(S,f,F)$ with a marked fixed point v from f, the probability for $\hat{\mathsf{G}}=(\hat{\mathsf{S}},\hat{f},\hat{F})$ to assume G and that a uniformly at random drawn fixed point of $\hat{f}$ equals v is given by

$$\mathbb{P}\left(\hat{\mathsf{G}}=G\right)|f|^{-1}=\mathbb{P}\left(\mathsf{G}=G\right).$$

Likewise, if we distinguish a vertex v from $\{*\}\sqcup F$, then the probability for $\bar{\mathsf{G}}=(\bar{\mathsf{S}},\bar{\mathsf{f}},\bar{\mathsf{F}})$ to assume G and that a uniformly at random drawn vertex from $\{*\}\sqcup\bar{\mathsf{F}}$ equals v is given by

$$\mathbb{P}\left(\bar{\mathsf{G}}=G\right)(1+|F|)^{-1}=\mathbb{P}\left(\mathsf{G}=G\right)(1+\mathbb{E}\left[\zeta\right])^{-1}.$$

Thus

$$\mathbb{P}\left((\hat{\mathsf{H}}_{[k]}, u^*) = (\mathbf{H}, u)\right) = p(\mathbf{H})\mathbb{P}\left(\mathsf{G} = G_{\ell_1}\right)(1 + \mathbb{E}\left[\zeta\right])^{-1} \prod_{i \neq \ell_1} \mathbb{P}\left(\mathsf{G}_i = G_i\right).$$

For Claim *(4)*, let us start with the fixed points. By assumption, there is a sequence $t_n \to 0$ such that $k_n = t_n\sqrt{n} \to \infty$. For any integer $m \geqslant 0$ let S_m denote the sum of m independent copies of the size of the ξ-Galton–Watson tree $\mathcal{T}^f$. The number of fixed points $\#_f\hat{\mathsf{H}}_k$ is given by the sum of

$$\#_f\hat{\mathsf{H}}_0 \stackrel{(d)}{=} 1 + S_{|\bar{\mathsf{f}}|}.$$

and the independent differences

$$\#_f\hat{\mathsf{H}}_i - \#_f\hat{\mathsf{H}}_{i-1} \stackrel{(d)}{=} S_{|\hat{\mathsf{f}}|-1}.$$

Consequently,

$$\#_f\hat{\mathsf{H}}_k \stackrel{(d)}{=} 1 + S_{M_k}$$

with $M_k = |\bar{\mathsf{f}}| + \sum_{i=1}^{k}(\hat{\xi}_i - 1)$ and $(\hat{\xi}_i)_{i\geqslant 1}$ denoting a family of independent copies of $|\hat{\mathsf{f}}|$. By a general result for the size of Galton–Watson forests, there is a constant $C > 0$ such that

$$\mathbb{P}\left(S_m \geqslant x\right) \leqslant Cmx^{-1/2}$$

for all m and $x > 0$. See Devroye and Janson [24, Lem. 2.3] and Janson [43, Lem. 2.1]. As $\hat{\xi}$ and $|\bar{\mathsf{f}}|$ have finite first moments, it follows that

$$\mathbb{P}\left(S_{M_k} \geqslant x\right) \leqslant C\mathbb{E}\left[M_k\right]x^{-1/2} = C(\mathbb{E}\left[|\bar{\mathsf{f}}|\right] + k(\mathbb{E}\left[\hat{\xi}\right] - 1))x^{-1/2}.$$

Setting $x = nt_n - 1$ and $k = k_n$, it follows that $\#_f\hat{\mathsf{H}}_{k_n} \leqslant nt_n$ with probability tending to one as n becomes large.

For the second statement, let R_m denote the sum of m independent copies of $|\mathcal{T}| - (1 + \mathbb{E}\left[\zeta\right])|\mathcal{T}^f|$. The difference $\#\hat{\mathsf{H}}_k - (1 + \mathbb{E}\left[\zeta\right])\#_f\hat{\mathsf{H}}_k$ is given by the sum of

$$\#\hat{\mathsf{H}}_0 - (1 + \mathbb{E}\left[\zeta\right])\#_f\hat{\mathsf{H}}_0 \stackrel{(d)}{=} |\bar{\mathsf{F}}| - \mathbb{E}\left[\zeta\right] + R_{|\bar{\mathsf{f}}|}$$

and the independent differences

$$\#\hat{\mathsf{H}}_i - \#\hat{\mathsf{H}}_{i-1} - (1 + \mathbb{E}\left[\zeta\right])(\#_f\hat{\mathsf{H}}_i - \#_f\hat{\mathsf{H}}_{i-1}) \stackrel{(d)}{=} |\hat{\mathsf{F}}| - \mathbb{E}\left[\zeta\right] + R_{|\hat{\mathsf{f}}|-1}, \quad i = 1 \ldots k.$$

Consequently,

$$\#\hat{\mathsf{H}}_k - (1 + \mathbb{E}\left[\zeta\right])\#_f\hat{\mathsf{H}}_k \stackrel{(d)}{=} |\bar{\mathsf{F}}| - \mathbb{E}\left[\zeta\right] + \sum_{i=1}^{k}(\hat{\zeta}_i - \mathbb{E}\left[\zeta\right]) + R_{M_k} \tag{30}$$

with $\hat{\zeta}_i$ denoting independent copies of $|\hat{\mathsf{F}}|$. Markov's inequality implies that for $k = k_n$

$$||\bar{\mathsf{F}}| - \mathbb{E}[\zeta] + \sum_{i=1}^{k_n} (\hat{\zeta}_i - \mathbb{E}[\zeta])| \leqslant \sqrt{nt_n}/2 \tag{31}$$

with probability tending to one as n becomes large.

It remains to show that $|R_{M_{k_n}}| \leqslant \sqrt{nt_n}/2$ with probability tending to one. If ξ and ζ were independent, then this would be rather simple. But, as this is not necessarily the case, it requires a bit of effort. We may write

$$|\mathcal{T}| - (1 + \mathbb{E}[\zeta])|\mathcal{T}^f| = \sum_{i=0}^{\infty} D_i \quad \text{with} \quad D_i = L_i^{\mathcal{G}}(\mathcal{T}^f) - L_i(\mathcal{T}^f)\mathbb{E}[\zeta]\,. \tag{32}$$

The sum is finite, as $D_i = 0$ for $i > \mathrm{H}(\mathcal{T}^f)$. Since

$$\mathbb{P}\left(\mathrm{H}(\mathcal{T}^f) \geqslant h\right) \sim 2/(\mathbb{V}[\xi]h)$$

as h becomes large, it follows that the probability for the maximum height of m independent copies of $\mathcal{T}^f$ to be less than $k_n/\sqrt{t_n}$ is given by

$$\left(1 - \frac{(1 + o(1))2\sqrt{t_n}}{\mathbb{V}[\xi]k_n}\right)^m, \tag{33}$$

with the $o(1)$ term not depending on m. As $\mathbb{E}[M_{k_n}] \sim k_n(\mathbb{E}\left[\hat{\xi}\right] - 1)$ and $\mathbb{V}[M_{k_n}] = k_n\mathbb{V}[\hat{\xi}]$, it follows by Chebyshev's inequality that

$$|M_{k_n} - k_n| \leqslant k_n^{3/4}$$

with probability tending to one. Hence Expression (33) implies that the probability for the maximum height of M_{k_n} independent copies of $\mathcal{T}^f$ to be smaller than $k_n/\sqrt{t_n}$ tends to one as n becomes large. Equation (32) hence implies for all x that

$$\mathbb{P}\left(|R_{M_{k_n}}| \geqslant x\right) \leqslant o(1) + \mathbb{P}\left(|U_1 + \ldots + U_{k_n}| \geqslant x\right) \tag{34}$$

with the U_j denoting independent copies of $\sum_{i=0}^{\lfloor k_n/\sqrt{t_n}\rfloor} D_i$. We aim to apply Chebyshev's inequality again, and hence compute the expected value and variance of the U_j. It holds for all i and ℓ that

$$\mathbb{E}\left[D_i \mid L_i(\mathcal{T}^f) = \ell\right] = \mathbb{E}\left[\sum_{j=1}^{\ell} \zeta_j - \ell\mathbb{E}[\zeta]\right] = 0,$$

with $(\zeta_j)_j$ denoting a family of independent copies of ζ. Consequently, for all k

$$\mathbb{E}\left[\sum_{i=0}^{k} D_i\right] = 0 \quad \text{and} \quad \mathbb{V}[\sum_{i=0}^{k} D_i] = \sum_{i=0}^{k} \mathbb{E}\left[D_i^2\right] + 2\sum_{0 \leqslant i < j \leqslant k} \mathbb{E}[D_i D_j]\,.$$

For all $0 \leqslant i < j$ it holds that

$$\mathbb{E}\left[D_i D_j \mid ((\mathcal{T}^f)^{[j]}, (\beta(v))_{v\in(\mathcal{T}^f)^{[j-1]}}\right] = D_i \mathbb{E}\left[D_j \mid L_j(\mathcal{T}^f)\right] = 0,$$

because for all ℓ

$$\mathbb{E}\left[D_j \mid L_j(\mathcal{T}^f) = \ell\right] = \mathbb{E}\left[\sum_{r=1}^{\ell} \zeta_i - \ell\mathbb{E}\left[\zeta\right]\right] = 0.$$

Clearly it also holds that

$$\mathbb{E}\left[D_j^2 \mid L_j(\mathcal{T}^f) = \ell\right] = \ell\mathbb{V}[\zeta].$$

Hence,

$$\mathbb{V}[\sum_{i=0}^{k} D_i] = \mathbb{V}[\zeta]\mathbb{E}\left[\sum_{i=0}^{k} L_i(\mathcal{T}^f)\right] = (k+1)\mathbb{V}[\zeta].$$

Setting $k = \lfloor k_n/\sqrt{t_n}\rfloor$, we may thus apply Chebyshev's inequality to (34) and obtain

$$\mathbb{P}\left(|R_{M_{k_n}}| \geqslant \sqrt{nt_n}/2\right) \leqslant o(1) + \frac{k_n(k_n/\sqrt{t_n}+1)\mathbb{V}[\zeta]}{nt_n/4} = o(1).$$

Together with (30) and (31) this implies that

$$|\#\hat{\mathsf{H}}_{k_n} - (1+\mathbb{E}\left[\zeta\right])\#_f\hat{\mathsf{H}}_{k_n}| \leqslant \sqrt{nt_n}$$

with probability tending to one as n becomes large. □

Proof of Theorem 27. For any k, let $\mathcal{E}_k$ denote the set of pairs $(\mathbf{H}, u)$ where $\mathbf{H} = (H_i)_{0\leqslant i\leqslant k}$ is a representation of a $\mathcal{G}$-enriched tree as sequence of increasing enriched fringe subtrees, and u is either the root of H_0 or an element of the non-fixed points of the $\mathcal{G}$-object corresponding to the root of H_0. We need to show that

$$\lim_{n\to\infty} \sup_{\mathcal{H}\subset\mathcal{E}_{k_n}} |\mathbb{P}\left((\mathsf{H}^n_{[k_n]}, v^*) \in \mathcal{H}\right) - \mathbb{P}\left((\hat{\mathsf{H}}_{[k_n]}, u^*) \in \mathcal{H}\right)| = 0. \tag{35}$$

By assumption, there is a sequence $t_n \to 0$ such that $k_n = \sqrt{n}t_n$. We define the subset

$$\mathcal{E}_{k,n} = \{((H_i)_{0\leqslant i\leqslant k}, u) \in \mathcal{E}_k \mid \#_f H_k \leqslant nt_n, |\#H_k - (1+\mathbb{E}\left[\zeta\right])\#_f H_k| \leqslant \sqrt{nt_n}\}.$$

Lemma 26 implies that $(\hat{\mathsf{H}}_{[k_n]}, u^*)$ lies in $\mathcal{E}_{k_n,n}$ with probability tending to one as n becomes large. Hence, if we verify Equation (35) with the index $\mathcal{H}$ only ranging over the subsets of $\mathcal{E}_{k_n,n}$, then it follows that $(\mathsf{H}^n_{[k_n]}, v^*)$ also lies with probability tending to one in $\mathcal{E}_{k_n,n}$. But this already verifies (35) when $\mathcal{H}$ ranges over all subsets of $\mathcal{E}_{k_n}$, and we are done.

So it remains to show that for any $\epsilon > 0$ it holds for sufficiently large n that

$$\sup_{\mathcal{H}\subset\mathcal{E}_{k_n,n}} |\mathbb{P}\left((\mathsf{H}^n_{[k_n]}, v^*) \in \mathcal{H}\right) - \mathbb{P}\left((\hat{\mathsf{H}}_{[k_n]}, u^*) \in \mathcal{H}\right)| \leqslant \epsilon. \tag{36}$$

In order to show this, we first exert some control over the number of fixed points in $(\mathcal{T}_n, \beta_n)$. Let $\mathbf{B}, \mathbf{\Sigma}, \mu$ and σ be as in Lemma 22. If follows from this Lemma that, as $n \equiv 1$ mod $\mathsf{span}(\mathbf{w})$ tends to infinity,

$$\sqrt{n}\mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \sim \frac{d}{\sigma\sqrt{2\pi}} \exp(-\frac{x_\ell^2}{2\sigma^2})$$

uniformly for all bounded $x_\ell \in \mathbb{R}$ with

$$\ell := \mu n + x_\ell \sqrt{n} \in n + d\mathbb{Z}.$$

Furthermore,

$$\frac{|\mathcal{T}_n^f| - n\mu}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

and for any $\epsilon_1 > 0$ there is a constant $M > 0$ such that for all n

$$\mathbb{P}\left(|\mathcal{T}_n^f| \notin I_n\right) \leqslant \epsilon_1 \quad \text{with} \quad I_n := (n + d\mathbb{Z}) \cap [n/(1 + \mathbb{E}\left[\zeta\right]) - M\sqrt{n}, n/(1 + \mathbb{E}\left[\zeta\right]) + M\sqrt{n}].$$

Hence we may bound the expression in (36) by

$$\epsilon_1 + \sup_{(\mathbf{H},u)\in\subset\mathcal{E}_{k_n,n}} \left| \frac{\mathbb{P}\left((\mathsf{H}^n_{[k_n]}, v^*) = (\mathbf{H}, u), |\mathcal{T}_n^f| \in I_n\right)}{\mathbb{P}\left((\hat{\mathsf{H}}_{[k_n]}, u^*) = (\mathbf{H}, u)\right)} - 1 \right|. \tag{37}$$

Throughout the rest of the proof, we set $k = k_n$. Let $(\mathbf{H}, u) \in \mathcal{E}_{k,n}$ with $\mathbf{H} = (H_i)_{0\leqslant i\leqslant k}$ be given. We set $L = L_k(H_k) + L_k^{\mathcal{G}}(H_k)$ and let F_0 denote the size of the forest of the $\mathcal{G}$-object corresponding to the root of H_0. Let A denote a $\mathcal{G}$-enriched tree with a total number of vertices $\#A = n$. Given $(\mathcal{T}_n, \beta_n) = A$, the vertex v^* is drawn uniformly at random from A. Given $(\mathcal{T}_n, \beta_n) = A$ and $\mathsf{H}_k^n = H_k$, the vertex v^* is drawn uniformly at random from L possible locations. Out of these, exactly $p(\mathbf{H})(1 + F_0)$ many correspond to the event $\mathsf{H}^n_{[k]} = \mathbf{H}$, since the number $p(\mathbf{H})$ defined in Lemma 26 counts the number of fixed points v at height k in H_k with the property, that the extended enriched fringe subtree representation with respect to v is identical to $\mathbf{H}$. Moreover, given additionally $\mathsf{H}^n_{[k]} = \mathbf{H}$, there is precisely one out of $1 + F_0$ possible locations such that $v^* = u$. Hence

$$\mathbb{P}\left(\mathsf{H}^n_{[k]} = \mathbf{H} \mid (\mathcal{T}_n, \beta_n) = A\right) = \mathbb{P}\left(\mathsf{H}_n^k = H_k \mid (\mathcal{T}_n, \beta_n) = A\right) p(\mathbf{H})/L. \tag{38}$$

There is a 1 to L correspondence between the fixed points v of A with $f(A, v) = H_k$, and the possible locations for v^* such that $\mathsf{H}_k^n = H_k$. Hence

$$\mathbb{P}\left(\mathsf{H}_n^k = H_k \mid (\mathcal{T}_n, \beta_n) = A\right) = \mathbb{E}\left[\sum_{v\in\mathcal{T}_n^f} \mathbb{1}_{f((\mathcal{T}_n,\beta_n),v)=H_k} \mid (\mathcal{T}_n, \beta_n) = A\right] L/n \tag{39}$$

Let $(G_1, \ldots, G_t)$ and G_i^n, $i = 1, \ldots, |\mathcal{T}_n^f|$ denote the depth-first-search ordered lists of $\mathcal{G}$-objects of the enriched trees H_k and $\mathcal{T}_n^f$. The occurrences of $(G_1, \ldots, G_t)$ as substrings

of $(\mathsf{G}_i^n)_i$ correspond precisely to the vertices of $\mathcal{T}_n^f$ where the fringe-subtree equals H_k. As Equations (38) and (39) hold uniformly for all $\mathcal{G}$-enriched trees with n vertices, it follows that

$$\mathbb{P}\left(\mathsf{H}_{[k]}^n = \mathbf{H}, |\mathcal{T}_n^f| \in I_n\right) = \sum_{\ell \in I_n} \mathbb{E}\left[\sum_{j=1}^{\ell} J_j^n, |\mathcal{T}_n^f| = \ell\right] p(\mathbf{H})/n \tag{40}$$

with J_j^n denoting the indicator variable for the event $(\mathsf{G}_j^n, \dots, \mathsf{G}_{j+t-1}^n) = (G_1, \dots, G_t)$. Here we set $\mathsf{G}_i^n := \mathsf{G}_{i-\ell}^n$ whenever $i > \ell$. Note that $J_j = 0$ whenever $j + t - 1 > \ell$, as then the sequence $(\mathsf{G}_i^n)_{j \leqslant i \leqslant j+t-1}$ does not correspond to any $\mathcal{G}$-enriched tree at all. Hence the sum of the J_j really counts the number of occurrences of $(G_i)_{1 \leqslant i \leqslant t}$ in $(\mathsf{G}_i)_{1 \leqslant i \leqslant \ell}$. Recall that by Lemma 21 there is a natural coupling of the unconditioned $\mathcal{G}$-enriched tree $(\mathcal{T}, \beta)$ with a family $(\mathsf{G}_i)_{i \in \mathbb{N}}$ of independent copies $\mathsf{G}_i = (\mathsf{S}_i, \mathsf{f}_i, \mathsf{F}_i)$ of the random $\mathcal{G}$-object G, conditioned on the event that there is an initial segment of $(\mathsf{G}_i)_{i \in \mathbb{N}}$ that corresponds to an $\mathcal{G}$-enriched tree. Let $(J_j)_{j=1,\dots,\ell}$ denote the unconditioned pendants of J_j^n for the sequence $(\mathsf{G}_i)_{1 \leqslant i \leqslant \ell}$. Lemma 21 implies that

$$\mathbb{E}\left[\sum_{j=1}^{\ell} J_j^n, |\mathcal{T}_n^f| = \ell\right] = \mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \mathbb{E}\left[\sum_{j=1}^{\ell} J_j \mid \sum_{i=1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell - 1, n - \ell), \sum_{i=1}^{m} |\mathsf{f}_i| \geqslant m \text{ for all } m < \ell\right].$$

The sum $\sum_{j_1}^{\ell} J_j$ is invariant under cyclic permutations of the list $(\mathsf{G}_1, \dots, \mathsf{G}_\ell)$. Hence the Cycle Lemma 19 yields

$$\mathbb{E}\left[\sum_{j=1}^{\ell} J_j^n, |\mathcal{T}_n^f| = \ell\right] = \mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \mathbb{E}\left[\sum_{j=1}^{\ell} J_j \mid \sum_{i=1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell - 1, n - \ell)\right].$$

Conditioned on this simpler event, the J_j are all identically distributed. Hence

$$\mathbb{E}\left[\sum_{j=1}^{\ell} J_j^n, |\mathcal{T}_n^f| = \ell\right] = \ell \mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \mathbb{E}\left[J_1 \mid \sum_{i=1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell - 1, n - \ell)\right].$$

Setting $\pi_G = \mathbb{P}(\mathsf{G} = G)$ for all $\mathcal{G}$-objects G, it follows that

$$\mathbb{E}\left[J_1, \sum_{i=1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell - 1, n - \ell)\right] = \pi_{G_1} \cdots \pi_{G_t} \mathbb{P}\left(\sum_{i=t+1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_i|) = (\ell - 1 - t, n - \#H_k)\right).$$

It holds uniformly for $(\mathbf{H}, u) \in \mathcal{E}_{k_n,n}$ and $\ell = \mu n + x_\ell \sqrt{n} \in I_n$ that

$$\ell^{-1/2} \left| (\ell - 1 - t, n - \#H_k) - (\ell - t)\mathbb{E}\left[(\xi, \zeta)\right] \right| \sim (0, -x_\ell (1 + \mathbb{E}\left[\zeta\right])^{3/2})$$

as n becomes large. Hence, as $\ell - t \sim \ell$, we may apply the multivariate local central limit theorem (Lemma 16) to obtain

$$\mathbb{P}\left(\sum_{i=t+1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_\mathsf{i}|) = (\ell - 1 - t, n - \#H_k) \right) \sim \frac{|\det \mathbf{B}|}{n\sqrt{2\pi \det \mathbf{\Sigma}}} \exp(-\frac{x_\ell^2}{2\sigma^2}).$$

Likewise, the probability $\mathbb{P}\left(\sum_{i=1}^{\ell} (|\mathsf{f}_i|, |\mathsf{F}_\mathsf{i}|) = (\ell - 1, n - \ell) \right)$ has precisely the same asymptotic order. It follows from Equation (40) that

$$\mathbb{P}\left(\mathsf{H}^n_{[k]} = \mathbf{H}, |\mathcal{T}^f_n| \in I_n \right) \sim p(\mathbf{H}) \pi_{G_1} \cdots \pi_{G_t} \sum_{\ell \in I_n} \frac{\ell}{n} \mathbb{P}\left(|\mathcal{T}^f_n| = \ell \right).$$

Since $\ell/n \sim (1 + \mathbb{E}\left[\zeta\right])^{-1}$ and $\mathbb{P}\left(|\mathcal{T}^f_n| \notin I_n \right) \leqslant \epsilon_1$ it follows that uniformly for $(\mathbf{H}, u) \in \mathcal{E}_{k_n,n}$

$$\left| \frac{\mathbb{P}\left(\mathsf{H}^n_{[k]} = \mathbf{H}, |\mathcal{T}^f_n| \in I_n \right)}{p(\mathbf{H})(1 + \mathbb{E}\left[\zeta\right])^{-1} \pi_{G_1} \cdots \pi_{G_t}} - 1 \right| \leqslant \epsilon_1 \tag{41}$$

for n large enough. Setting $\epsilon_1 = \epsilon/2$, Inequality (41) and Lemma 26 imply that the bound in (37) is smaller than ϵ for large enough n. This verifies (36) and hence the proof is complete. □

7.2 Proofs of the scaling limits and diameter tail bounds in Section 6.3

Proof of Lemma 30. Claim *(1)*: Let $x \geqslant 0$ be arbitrary and let $\mathcal{E}$ denote the event that there is a vertex $v \in \mathcal{T}^f$ with $|f(v)| + |F(v)| \geqslant x$. It follows from Claim *(1)* that

$$\mathbb{P}\left(|\mathcal{T}| = n \right) = [z^n] \tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho z) / \tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho) = \Theta(n^{-3/2})$$

and hence

$$\mathbb{P}\left(\mathcal{E} \mid |\mathcal{T}| = n \right) = O(n^{3/2}) \mathbb{P}\left(\mathcal{E}, |\mathcal{T}| = n \right).$$

By Lemma 21, the probability $\mathbb{P}\left(\mathcal{E}, |\mathcal{T}| = n \right)$ is given by

$$\mathbb{P}(\exists \ell \leqslant n : \max(\xi_1 + \zeta_1, \ldots, \xi_\ell + \zeta_\ell) \geqslant x, \sum_{i=1}^{\ell} \xi_i = \ell - 1, \sum_{i=1}^{\ell} (1 + \zeta_i) = n, \forall m < \ell : \sum_{i=1}^{m} \xi_i \geqslant m),$$

with $(\xi_i, \zeta_i)_{i \in \mathbb{N}}$ denoting a list of independent copies of (ξ, ζ). We are not interested in precise asymptotics here and hence bound this very roughly by

$$\mathbb{P}\left(\mathcal{E}, |\mathcal{T}| = n \right) \leqslant \mathbb{P}\left(\max(\xi_1 + \zeta_1, \ldots, \xi_n + \zeta_n) \geqslant x \right) \leqslant n \mathbb{P}\left(\xi + \zeta \geqslant x \right).$$

As (ξ, ζ) has finite exponential moments, it follows that

$$\mathbb{P}(\mathcal{E}, |\mathcal{T}| = n) \leqslant Cn^{5/2} \exp(-cn)$$

for some constants $C, c > 0$ that do not depend on n.

Claim *(2)*: We may form a random metric space Y by constructing a metric d_{Y} on the vertex set of $\Gamma\mathcal{S}(\rho)$ by patching together independent copies of the metrics δ_R just as in the construction of the metric space Y_n. Hence Y_n is distributed like the space Y conditioned on having size n. For any vertex v of the fixed point tree $\mathcal{T}^f$ let D_v denote the d_{Y}-diameter of the subspace $\{v\} \cup F(v) \subset \mathsf{Y}$. Given $h \geqslant 0$ let $\mathcal{E}'$ denote the event that $D_v \geqslant h$ for at least one vertex $v \in \mathcal{T}^f$.

Using Claim *(1)* it follows that

$$\mathbb{P}(\mathcal{E}' \mid |\mathsf{Y}| = n) = O(n^{3/2})\mathbb{P}(\mathcal{E}', |\mathsf{Y}| = n) = O(n^{5/2})\mathbb{P}(D_o \geqslant h) \qquad (*)$$

with o denoting the root of the fixed point tree $\mathcal{T}^f$.

By assumption, for any vertex $u \in F(o)$, the distance $d_{\mathsf{Y}}(o, u)$ is bounded by the sum of $\sum_e d^+_{\mathcal{T}}(e)$ many independent copies of a real-valued random variable $\chi \geqslant 0$ having finite exponential moments, with the sum index e ranging over all ancestors of the vertex u in the tree $\mathcal{T}$. Clearly we have that

$$\sum_e d^+_{\mathcal{T}}(e) \leqslant |F(o)|$$

for all $u \in F(o)$. Since $D_o \leqslant 2 \sup_{u \in F(o)} d_{\mathsf{Y}}(o, u)$, it follows that

$$\mathbb{P}(D_o \geqslant h) \leqslant \sum_{k=0}^{\infty} \mathbb{P}(|F(o)| = k)\, k\mathbb{P}(\chi_1 + \ldots + \chi_k \geqslant h/2)$$

with $(\chi_i)_{i \in \mathbb{N}}$ a family independent copies of χ. Moreover,

$$\mathbb{P}(|F(o)| = k) = \mathbb{P}(\zeta = k) = O(\gamma_1^k)$$

for some constant $0 < \gamma_1 < 1$. By the deviation inequality given in Lemma 18 it follows that there are constants $a, b > 0$ such that

$$\gamma_1^k \mathbb{P}(\chi_1 + \ldots + \chi_k \geqslant h/2) \leqslant 2e^{-ak-bh}$$

for all k and h. Hence

$$\mathbb{P}(D_o \geqslant h) = O(\gamma_2^h)$$

for some constant $0 < \gamma_2 < 1$. Hence Equality $(*)$ implies that

$$\mathbb{P}(\mathcal{E}' \mid |\mathsf{Y}| = n) = O(n^{5/2})\gamma_2^h$$

and we are done.

Claim *(3)*: Set $t := \sqrt{(1+\mathbb{E}[\zeta])}\sigma/2$ with $\sigma^2 = \mathbb{V}[\xi]$. Let $g : \mathbb{K}^\bullet \to \mathbb{R}$ denote a Lipschitz-continuous, bounded function defined on the space of isometry classes of pointed compact metric spaces. Note that $\mathcal{T}_n^f$ conditioned on having size ℓ is distributed like $\mathcal{T}^f$ conditioned on having size ℓ. Hence it is distributed like a ξ-Galton–Watson tree conditioned on having ℓ vertices, which we denote by $\mathcal{T}'_\ell$. It follows from claim *ii*) that

$$\mathbb{E}\left[g(t\mathcal{T}_n^f/\sqrt{n})\right] = o(1) + \sum_\ell \mathbb{E}\left[g(t\mathcal{T}'_\ell/\sqrt{n})\right]\mathbb{P}\left(|\mathcal{T}^f| = \ell\right)$$

with the sum index ℓ ranging over all integers $\ell \equiv 1 \mod \mathsf{span}(\mathbf{w})$ contained in the interval $(1 \pm n^{-s})\frac{n}{1+\mathbb{E}[\zeta]}$. Since g is Lipschitz-continuous, we have that

$$|\mathbb{E}\left[g(t\mathcal{T}'_\ell/\sqrt{n})\right] - \mathbb{E}\left[g(\sigma\mathcal{T}'_\ell/(2\sqrt{\ell}))\right]| \leqslant a_{n,\ell}\mathbb{E}\left[\mathrm{D}(\mathcal{T}'_\ell)/\sqrt{\ell}\right]$$

for some constants $a_{n,\ell}$ with $\sup_\ell(a_{n,\ell}) \to 0$ as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ tends to infinity. Moreover, the average rescaled diameter $\mathbb{E}\left[\mathrm{D}(\mathcal{T}'_\ell)/\sqrt{\ell}\right]$ converges as ℓ becomes large to a multiple of the expected diameter of the CRT $\mathcal{T}_\mathsf{e}$. In particular, it is a bounded sequence. Since $\mathbb{E}\left[g(\frac{\sigma\mathcal{T}'_\ell}{2\sqrt{\ell}})\right]$ converges to $\mathbb{E}[g(\mathcal{T}_\mathsf{e})]$ as ℓ becomes large, it follows that $\mathbb{E}\left[g(\frac{t\mathcal{T}_n^f}{\sqrt{n}})\right]$ converges to $\mathbb{E}[g(\mathcal{T}_\mathsf{e})]$ as n becomes large. This concludes the proof. □

Proof of Theorem 28. By Lemma 30 it follows that with high probability all vertices $v \in \mathcal{T}_n^f$ have the property that the d_{Y_n}-diameter of the subspace $\{v\} \cup F(v)$ is at most $O(\log(n))$. This implies that the Gromov–Hausdorff distance between the metric spaces $(\mathsf{Y}_n, d_{\mathsf{Y}_n}/\sqrt{n})$ and $(\mathcal{T}_n^f, d_{\mathsf{Y}_n}/\sqrt{n})$ converges in probability to zero. Moreover, by Lemma 30 we know that $(\mathcal{T}_n^f, cd_{\mathcal{T}_n^f}/\sqrt{n})$ with $c = \sqrt{(1+\mathbb{E}[\zeta])\mathbb{V}[\xi]}/2$ converges weakly to the CRT $\mathcal{T}_\mathsf{e}$. It remains to show that there is a constant c' such that the Gromov–Hausdorff distance between $(\mathcal{T}_n^f, d_{\mathsf{Y}_n}/\sqrt{n})$ and $(\mathcal{T}_n^f, c'd_{\mathcal{T}_n^f}/\sqrt{n})$ converges in probability to zero.

We define the random number η as follows. Choose a random $\mathcal{R}'$-symmetry (R, σ) from $\bigcup_{k\geqslant 0}\mathrm{Sym}(\mathcal{R}')[k]$ with probability proportional to the weight

$$\frac{\kappa(\mathsf{R})}{|\mathsf{R}|!}\tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho)^{\sigma_1}\tilde{\mathcal{A}}^\omega_\mathcal{R}(\rho^2)^{\sigma_2}\dots$$

and let η denote the δ_R-distance of the two distinct $*$-labels. Note that by our assumptions on the cycle index sum $Z_{\mathcal{R}^\kappa}$ we have that $|\mathsf{R}|$ has finite exponential moments. Moreover, the diameter of the metric δ_R is bounded by $|\mathsf{R}|$ many independent copies of a real-valued random variable $\chi \geqslant 0$ with finite exponential moments. Hence η has finite exponential moments. We are going to show that the Gromov–Hausdorff distance of $(\mathcal{T}_n^f, d_{\mathsf{Y}_n}/\sqrt{n})$ and $(\mathcal{T}_n^f, \mathbb{E}[\eta]\, d_{\mathcal{T}_n^f}/\sqrt{n})$ converges in probability to zero. By the discussion in the preceding paragraph this implies that

$$\frac{\sqrt{(1+\mathbb{E}[\zeta])\mathbb{V}[\xi]}}{2\mathbb{E}[\eta]\sqrt{n}}\mathsf{Y}_n \xrightarrow{d} \mathcal{T}_\mathsf{e}$$

and we are done.

Let $s > 1$ and $t > 0$ be arbitrary constants and set $s_n = \log(n)^s$ and $t_n = n^t$. Let $\epsilon > 0$ be given and let $\mathcal{E}_1$ denote the event that there exists a fixed point $v \in \mathcal{T}_n^f$ and an ancestor u of v with the property that

$$d_{\mathcal{T}_n^f}(u,v) \geqslant s_n \quad \text{and} \quad d_{\mathsf{Y}_n}(u,v) \notin (1 \pm \epsilon)\mathbb{E}\left[\eta\right] d_{\mathsf{T}_n}(u,v).$$

Likewise, let $\mathcal{E}_2$ denote the event that there exists a vertex v and an ancestor u of v with

$$d_{\mathcal{T}_n}(u,v) \leqslant s_n \quad \text{and} \quad d_{\mathsf{Y}_n}(u,v) \geqslant t_n.$$

We are going to show that with high probability none of the events $\mathcal{E}_1$ and $\mathcal{E}_2$ takes place

This suffices to show the claim: Take $s = 2$ and $t = 1/4$ and suppose that the complementary events $\mathcal{E}_1^c$ and $\mathcal{E}_2^c$ hold. Given vertices $a \neq b$ in the tree $\mathcal{T}_n^f$ let x denote their lowest common ancestor. If $x \in \{a, b\}$ then we have

$$d_{\mathsf{Y}_n}(a,b) = d_{\mathsf{Y}_n}(a,x) + d_{\mathsf{Y}_n}(b,x).$$

If $x \neq a, b$, then let a' denote the offspring of x that lies on the $\mathcal{T}_n^f$-path joining a and x and likewise b' the offspring of x lying on the path joining x and b. Hence we have that

$$d_{\mathsf{Y}_n}(a,b) = d_{\mathsf{Y}_n}(a,x) + d_{\mathsf{Y}_n}(b,x) + R \quad \text{with} \quad R = d_{\mathsf{Y}_n}(a,a') - d_{\mathsf{Y}_n}(a',x) - d_{\mathsf{Y}_n}(b',x).$$

By property $\mathcal{E}_2^c$ and the triangle inequality it follows that $|R| = -R \leqslant 2n^{1/4}$. Thus, regardless whether $x \in \{a, b\}$, it holds that

$$d_{\mathsf{Y}_n}(a,b) = d_{\mathsf{Y}_n}(a,x) + d_{\mathsf{Y}_n}(b,x) + O(n^{1/4}).$$

Moreover, if $d_{\mathcal{T}_n^f}(a,x) \geqslant \log(n)^2$, then it follows by property $\mathcal{E}_1^c$ that

$$d_{\mathsf{Y}_n}(a,x) \in (1 \pm \epsilon)\mathbb{E}\left[\eta\right] d_{\mathcal{T}_n^f}(a,x).$$

Otherwise, if $d_{\mathcal{T}_n^f}(a,x) < \log(n)^2$ then it follows by property $\mathcal{E}_2^c$ that $d_{\mathsf{Y}_n}(a,x) \leqslant n^{1/4}$ and thus

$$|d_{\mathsf{Y}_n}(a,x) - \mathbb{E}\left[\eta\right] d_{\mathcal{T}_n^f}(a,x)| \leqslant Cn^{1/4}$$

for a fixed constant C that does not depend on n or the points a and x. It follows that

$$|d_{\mathsf{Y}_n}(a,b)/\sqrt{n} - \mathbb{E}\left[\eta\right] d_{\mathcal{T}_n^f}(a,b)/\sqrt{n}| \leqslant \epsilon \mathrm{D}(\mathcal{T}_n^f)/\sqrt{n} + o(1),$$

with $\mathrm{D}(\mathcal{T}_n^f)$ denoting the diameter. Thus

$$d_{\mathrm{GH}}(\mathsf{Y}_n, \mathbb{E}\left[\eta\right] \mathcal{T}_n^f) \leqslant \epsilon \mathrm{D}(\mathcal{T}_n^f)/\sqrt{n} + o(1)$$

holds with high probability. Since we may choose ϵ arbitrarily small, and $\mathrm{D}(\mathcal{T}_n^f)/\sqrt{n}$ converges in distribution (to a multiple of the diameter of the CRT), it follows that $d_{\mathrm{GH}}(\mathsf{Y}_n, \mathbb{E}\left[\eta\right] \mathcal{T}_n^f) \to 0$ in probability and we are done.

For each finite subset $U \in \mathbb{N}$ and each $\mathcal{R}$-structure $R \in \mathcal{R}[U]$ let $(\delta_R^i)_{i\in\mathbb{N}_0}$ be a family of independent copies of the metric δ_R. Given a $\mathcal{A}_\mathcal{R}$-symmetry $S = ((T,\alpha),\sigma)$ with label set $[k]$ for some $k \geqslant 0$ we may form the family $(\delta^S(v))_{v\in T}$ of random metrics by traversing bijectively the vertices of T in ascending order $1, 2, \ldots k$ and assigning to each vertex v the "leftmost" unused copy from the list $(\delta^1_{\alpha(v)}, \delta^2_{\alpha(v)}, \ldots)$. The metrics can be patched together to a metric d^S on the vertex set $[k]$ of the tree T just as described in Section 6.3.

We may assume that all random variables considered so far are defined on the same probability space and that the metric d_{Y_n} of Y_n coincides with the metric d^{Z_n} with Z_n denoting the sampler $\Gamma Z_{\mathcal{A}^\omega_\mathcal{R}}(\rho)$ conditioned on having size n. Given the family $(\delta^i_R)_{R,i}$ let $\mathcal{H} \subset \bigcup_{k=0}^\infty \mathrm{Sym}(\mathcal{A}_\mathcal{R})[k]$ denote the finite set symmetries of size n such that the event $\mathcal{E}_1$ takes place if and only if $\mathsf{Z}_n \in \mathcal{H}$. By the definition of the event $\mathcal{E}_1$ for any symmetry $S = ((T,\alpha),\sigma) \in \mathcal{H}$ we may choose a fixed point v_S of σ having the property that there exists an ancestor u in the tree T with

$$d_T(u, v_S) \geqslant s_n \quad \text{and} \quad d^S(u, v_S) \notin (1 \pm \epsilon)\mathbb{E}\,[\eta]\, d_T(u, v).$$

Let ℓ_S denote the height $h_T(v_S)$. Note that since v_S is a fixed point, the tupel (T, α, v_S, σ) is a $\mathcal{A}^{(\ell_S)}_\mathcal{R}$-symmetry. By Lemma 22 the probability for the sampler $\Gamma Z_{\mathcal{A}^\omega_\mathcal{R}}(\rho)$ to have size n is $\Theta(n^{-3/2})$ and we have that $\rho\widetilde{\mathcal{R}' \circ \mathcal{A}^\omega_\mathcal{R}}(\rho) = 1$. Hence Equation (14) implies that the conditional distribution of the event $\mathcal{E}_1$ given $(\delta^i_R)_{R,i}$ equals

$$\sum_{S\in\mathcal{H}} \mathbb{P}\left(\mathsf{Z}_n = S \mid (\delta^i_R)_{R,i}\right) = \Theta(n^{3/2}) \sum_{S\in\mathcal{H}} \mathbb{P}\left(\Gamma Z_{\mathcal{A}^{(\ell_S)}_\mathcal{R}}(\rho) = (S, v_S) \mid (\delta^i_R)_{R,i}\right).$$

Let $v_0, \ldots, v_\ell$ denote the spine of $\Gamma Z_{\mathcal{A}^{(\ell)}_\mathcal{R}}(\rho)$, that is, v_ℓ is the outer root, v_0 is the inner root, and $(v_0, \ldots, v_\ell)$ is the directed path connecting the roots. It follows that the probability for the event $\mathcal{E}_1$ is bounded by

$$\Theta(n^{3/2}) \sum_{\ell=1}^{n} \sum_{k=0}^{\ell-s_n} \mathbb{P}\left(d^{\left(\Gamma Z_{\mathcal{A}^{(\ell)}_\mathcal{R}}(\rho)\right)}(v_k, v_\ell) \notin (1 \pm \epsilon)\mathbb{E}\,[\eta]\,(\ell - k)\right) \qquad (*)$$

But the $d^{\left(\Gamma Z_{\mathcal{A}^{(\ell)}_\mathcal{R}}(\rho)\right)}$-distance between spine vertices v_i and v_j is distributed like the sum $\eta_1 + \ldots + \eta_{|i-j|}$ of independent copies $(\eta_i)_i$ of η. We know that η has finite exponential moments and hence by the deviation inequality in Lemma 18 the bound $(*)$ converges to zero as $n \equiv 1 \mod \mathsf{span}(\mathbf{w})$ tends to infinity. Thus with high probability $\mathcal{E}_1$ does not hold. By the same arguments we may bound the probability for the event $\mathcal{E}_2$ by

$$\Theta(n^{3/2}) \sum_{\ell=1}^{n} \sum_{k=1}^{\min(s_n,\ell)} \mathbb{P}\left(\eta_1 + \ldots + \eta_k \geqslant t_n\right)$$

which also converges to zero. This concludes the proof. □

Proof of Theorem 29. It suffices to show that there are constants $C, c, N > 0$ such that for all $n \geqslant N$ and $h \geqslant \sqrt{n}$ we have that

$$\mathbb{P}\left(\mathrm{H}(\mathsf{X}_n) \geqslant h\right) \leqslant C(\exp(-ch^2/n) + \exp(-ch)).$$

For any fixed point $v \in \mathcal{T}_n^f$ set

$$\ell(v) = \sum_u d^+_{\mathcal{T}_n}(u)$$

with the sum index u ranging over all ancestors of the vertex v in the plane tree $\mathcal{T}_n^f$. Note that we are summing up the outdegrees in the tree $\mathcal{T}_n$ and not in the tree $\mathcal{T}_n^f$. Moreover, for any vertex $y \in \mathcal{T}_n$ let v_y denote its closest fixed point, that is, $v_y = y$ if y is a fixed point and otherwise v_y is the unique vertex with $y \in F_n(v_y)$. If y has height $\mathrm{h}_{\mathsf{Y}_n}(y) \geqslant 2h$ then $\mathrm{h}_{\mathsf{Y}_n}(v_y) \geqslant h$ or $d_{\mathsf{Y}_n}(u, v_y) \geqslant h$. Thus either $\mathrm{H}_{\mathsf{Y}_n}(\mathcal{T}_n^f) \geqslant h$ or there exists a fixed point $v \in \mathcal{T}_n^f$ such that the d_{Y_n}-diameter D_v of the subspace $\{v\} \cup F(v)$ is greater than or equal to h.

Let $s > r > 0$ be constants. Given $h \geqslant \sqrt{n}$ let $\mathcal{E}^{2h}$ denote the event that $\mathrm{H}(\mathsf{X}_n) \geqslant h$. It follows that

$$\mathcal{E}^{2h} \subset \mathcal{E}_0^h \cup \mathcal{E}_1^h \cup \mathcal{E}_2^h \cup \mathcal{E}_3^h$$

with the events $\mathcal{E}_i^h$ given as follows. $\mathcal{E}_0^h$ is the event that there exists a fixed point $v \in \mathcal{T}_n^f$ with $D_v \geqslant h$. $\mathcal{E}_1^h$ is the event that $\mathrm{H}(\mathcal{T}_n^f) \geqslant rh$. $\mathcal{E}_2^h$ is the event that $\mathrm{H}(\mathcal{T}_n^f) \leqslant rh$ and $\ell(v) \geqslant sh$ for some fixed point $v \in \mathcal{T}_n^f$. $\mathcal{E}_3^h$ is the event that $\ell(v) \leqslant sh$ for all fixed points $v \in \mathcal{T}_n^f$ and $\mathrm{H}_{\mathsf{Y}_n}(\mathcal{T}_n^f) \geqslant h$.

We are going to show that if we choose r and s sufficiently small, then each of these events is sufficiently unlikely. By Lemma 30 we have that

$$\mathbb{P}\left(\mathcal{E}_0^h\right) = O(n^{5/2})\gamma^h$$

for some $0 < \gamma < 1$. Hence there are constants $C_0, c_0 > 0$ such that

$$\mathbb{P}\left(\mathcal{E}_0^h\right) \leqslant C_0 \exp(-c_0 h^2/n)$$

if $h \leqslant n$ and

$$\mathbb{P}\left(\mathcal{E}_0^h\right) \leqslant C_0 \exp(-c_0 h)$$

if $h \geqslant n$.

In order to bound the probability for the event $\mathcal{E}_1^h$ note that the tree $\mathcal{T}_n^f$ conditioned on having size ℓ is distributed like $\mathcal{T}^f$ conditioned on having size ℓ. That is, it is identically distributed to a ξ-Galton–Watson tree conditioned on having size ℓ which we denote by $\mathcal{T}'_\ell$. Hence

$$\mathbb{P}\left(\mathcal{E}_1^h\right) \leqslant \sum_{\ell=1}^{n} \mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \mathbb{P}\left(\mathrm{H}(\mathcal{T}'_\ell) \geqslant rh\right).$$

By Inequality (2) there exist constants $C_1, c_1 > 0$ that do not depend on n or h such that for all $1 \leqslant \ell \leqslant n$ we have the tail bound

$$\begin{aligned}\mathbb{P}\left(\mathrm{H}(\mathcal{T}'_\ell) \geqslant rh\right) &\leqslant C_1 \exp(-c_1 r^2 h^2/\ell) \\ &\leqslant C_1 \exp(-c_1 r^2 h^2/n).\end{aligned}$$

In particular, it holds that

$$\mathbb{P}\left(\mathcal{E}_1^h\right) \leqslant C_1 \exp(-c_1 r^2 h^2/n)$$

for all n and h.

We proceed to bound the probability for the event $\mathcal{E}_2^h$. Let Z_n denote the sampler $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}(\rho)$ conditioned on having size n and let

$$\mathcal{H} \subset \bigcup_{k=0}^{\infty} \mathrm{Sym}(\mathcal{A}_{\mathcal{R}})[k]$$

denote the set of $\mathcal{A}_{\mathcal{R}}$-symmetries $S = ((T, \alpha), \sigma)$ having the property that there exists a fixed point vertex v_S in T with the property that

$$\ell_S := \mathrm{h}_T(v_S) \leqslant rh$$

and

$$\sum_u d_T^+(u) \geqslant sh$$

with the sum-index u ranging over all ancestors of the vertex v in the tree T. By Equation 14 we may bound the probability for the event $\mathcal{E}_2^h$ by

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_2^h\right) &= \mathbb{P}\left(\mathsf{Z}_n \in \mathcal{H}\right) \\ &= O(n^{3/2}) \sum_{S \in \mathcal{H}} \mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell_S)}}(\rho) = (S, v_S)\right).\end{aligned}$$

Let η denote the outdegree of the root in the sampler $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(1)}}(\rho)$. By the assumptions on the cycle index sum $Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}$ it follows that η has finite exponential moments. Note that the outdegrees along the spine of $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell)}}$ are distributed like independent copies of η. It follows that

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_2^h\right) &= O(n^{3/2}) \sum_{\ell=1}^{\min(n, rh)} \mathbb{P}\left(\eta_1 + \ldots + \eta_\ell \geqslant sh\right) \\ &= O(n^{5/2}) \mathbb{P}\left(\eta_1 + \ldots + \eta_{\lfloor rh \rfloor} \geqslant sh\right).\end{aligned}$$

By the deviation inequality in Lemma 18 it follows that there are constants $c, \lambda > 0$ such that the above quantity is bounded by a constant multiple of $\exp(5/2 \log(n) + crh - \lambda sh)$.

We assumed that $h \geqslant \sqrt{n}$, hence if we choose r sufficiently small depending only on s, c and λ, it follows that there are constants $C_2, c_2 > 0$ such that

$$\mathbb{P}\left(\mathcal{E}_2^h\right) \leqslant C_2 \exp(-c_2 h).$$

It remains to treat the event $\mathcal{E}_3^h$. By assumption, for any fixed point $v \in \mathcal{T}_n^f$ we have that the height $\mathrm{h}_{\Upsilon_n}(v)$ is bounded by $\ell(v)$ many independent copies of a random variable χ having finite exponential moments. Thus

$$\mathbb{P}\left(\mathcal{E}_3^h\right) \leqslant n\mathbb{P}\left(\chi_1 + \ldots + \chi_{\lfloor sh \rfloor} \geqslant h\right)$$

with $(\chi_i)_{i\in\mathbb{N}}$ a family of independent copies of χ. By the deviation inequality in Lemma 18 there are constants $c, \lambda > 0$ such that this quantity is bounded by a constant multiple of

$$\exp(\log(n) + \lfloor sh \rfloor c - \lambda h).$$

We assumed that $h \geqslant \sqrt{n}$, hence we may bound this by

$$\exp(h(\log(n)/\sqrt{n} + sc - \lambda)).$$

If s is sufficiently small, then it follows that there are constants $C_3, c_3 > 0$ such that

$$\mathbb{P}\left(\mathcal{E}_3^h\right) \leqslant C_3 \exp(-c_3 h)$$

for all n and $h \geqslant \sqrt{n}$.

Thus there exist constants $C, c > 0$ with

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}^{2h}\right) &\leqslant \sum_i \mathbb{P}\left(\mathcal{E}_i^h\right) \\ &\leqslant C(\exp(-ch^2/n) + \exp(-ch))\end{aligned}$$

for all n and $h \geqslant \sqrt{n}$. This concludes the proof. □

7.3 Proofs concerning the applications

7.3.1 Random weighted graphs

As argued in Section 6.4, Theorems 31, 32 and 33 are special cases of the more general results we established in Sections 6.2 and 6.3.

7.3.2 Random front-rooted k-dimensional trees

We start with global geometric properties, as some intermediate results in there will also be useful in the study of the local properties.

Proof of Theorem 37. By Theorem 15 it follows that the largest $\mathcal{K}^\circ$-component of the random front-rooted k-tree K_n has size $n + O_p(1)$. Hence Lemma 38 readily implies that K_n converges toward the CRT after rescaling by the same factor as for K_n°. □

Proof of Lemma 38. The random front-rooted k-tree K_n° corresponds to the random enriched tree $\tilde{\mathsf{A}}_n^{\mathcal{R}}$ for $\mathcal{R} = \mathrm{SEQ}_{\{k\}} \circ \mathrm{SET}$. Hence our framework applies.

We first show a tail-bound for the diameter. By the bijection discussed in Section 6.1.3, we know that the distance between any vertex in $(\mathcal{T}_n, \beta_n)$ and its offspring is always 1, as the two vertices are also joined by an edge in the corresponding k-tree. Let $\sqrt{n} \leqslant x \leqslant n$ be given. If $\mathrm{H}(\mathsf{K}_n^\circ) \geqslant x$, then it follows that $\mathrm{H}(\mathcal{T}_n \geqslant x)$. (Here we define the height with respect to the vertex that corresponds to the root of $\mathcal{T}_n$.) Hence $\mathrm{H}(\mathcal{T}_n^f) \geqslant x/2$ or $|F(v)| \geqslant x/2$ for some $v \in \mathcal{T}_n^f$. Let us denote these events by $\mathcal{E}_1$ and $\mathcal{E}_2$. Lemma 30 states that there are constants $C_1, c_1 > 0$ such that uniformly for all n and non-negative x

$$\mathbb{P}\left(\max_{v \in \mathcal{T}_n^f}(|f_n(v)| + |F_n(v)|) \geqslant x\right) \leqslant C_1 n^{5/2} \exp(-c_1 x). \tag{42}$$

As we assumed that $\sqrt{n} \leqslant x \leqslant n$, it follows that

$$\mathbb{P}(\mathcal{E}_2) \leqslant C_1 n^{5/2} \exp(-c_1 x/2) \leqslant C_2 \exp(-c_2 x^2/n)$$

with the constants $C_2, c_2 > 0$ not depending on n or x. As for the event $\mathcal{E}_1$, Lemma 2 implies that there are constants $C_3, c_3 > 0$ that do not depend on n or x such that

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_1) &= \sum_{\ell=1}^{n} \mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \mathbb{P}\left(\mathrm{H}(\mathcal{T}_n^f) \geqslant x/2 \mid |\mathcal{T}_n^f| = \ell\right) \\
&\leqslant C_3 \sum_{\ell=1}^{n} \mathbb{P}\left(|\mathcal{T}_n^f| = \ell\right) \exp(-c_3 x^2/\ell) \\
&\leqslant C_3 \exp(-c_3 x^2/\ell).
\end{aligned}$$

Thus, there are constants $C_4, c_4 > 0$ such that

$$\begin{aligned}
\mathbb{P}(\mathrm{H}(\mathsf{K}_n^\circ) \geqslant x) &\leqslant \mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2) \\
&\leqslant C_4 \exp(-c_4 x^2/n)
\end{aligned}$$

for all n and all $\sqrt{n} \leqslant x \leqslant n$. It is clear that, by possibly adjusting the constants involved, such an inequality also holds for all $x \geqslant 0$. This verifies the exponential tail-bound for the diameter of the k-tree K_n°.

It remains to establish the scaling limit. Inequality (42) implies that with high probability all vertices $v \in \mathcal{T}_n^f$ have the property that the number of vertices in the forest $F(v)$ is at most $O(\log(n))$. This implies that the (pointed) Gromov–Hausdorff distance between the k-tree K_n° and the subspace corresponding to the vertices of the fixed point tree $\mathcal{T}_n^f$ is with high probability at most $O(\log n)$. Consequently, it suffices to show that there is a constant $a_k > 0$ such that this subspace rescaled by $a_k/\sqrt{n}$ converges toward the continuum random tree.

Let

$$b_k = \left(k \sum_{i=1}^{k} \frac{1}{i}\right)^{-1}. \tag{43}$$

We are going to show that there are exponents $t > 0$ and $1/2 < s < 1$ such that with high probability

$$|d_{\mathsf{K}_n^\circ}(u,v) - b_k d_{\mathcal{T}_n^f}(u,v)| \leqslant d_{\mathcal{T}_n^f}(u,v)^s + \log^t n \tag{44}$$

for all $u, v \in \mathcal{T}_n^f$. This suffices to complete the proof. Indeed, it follows that with high probability

$$d_{\mathrm{GH}}((\mathcal{T}_n^f, d_{\mathsf{K}_n^\circ}), (\mathcal{T}_n^f, b_k d_{\mathcal{T}_n^f})) \leqslant \mathrm{D}(\mathcal{T}_n^f)^s + \log^t n \tag{45}$$

By Lemma 30 we know that

$$\left(\mathcal{T}_n^f, \frac{\sqrt{(1+\mathbb{E}\left[\zeta\right])\mathbb{V}[\xi]}}{2\sqrt{n}} d_{\mathcal{T}_n^f}\right) \xrightarrow{d} (\mathcal{T}_\mathsf{e}, d_{\mathcal{T}_\mathsf{e}}). \tag{46}$$

in the (pointed) Gromov–Hausdorff sense. In particular, $\mathrm{D}(\mathcal{T}_n^f) = O_p(\sqrt{n})$, and hence it follows from Equation (45) that

$$d_{\mathrm{GH}}((\mathcal{T}_n^f, n^{-1/2} d_{\mathsf{K}_n^\circ}), (\mathcal{T}_n^f, b_k n^{-1/2} d_{\mathcal{T}_n^f})) \xrightarrow{p} 0. \tag{47}$$

Equations (46) and (47) then readily imply that the subspace of the k-tree K_n°, that corresponds to $\mathcal{T}_n^f$, converges toward the CRT after rescaling the metric by $c_k/\sqrt{n}$ with

$$c_k = \frac{\sqrt{(1+\mathbb{E}\left[\zeta\right])\mathbb{V}[\xi]}}{2b_k}.$$

It follows that

$$(\mathsf{K}_n^\circ, c_k n^{-1/2} d_{\mathsf{K}_n^\circ}) \xrightarrow{d} (\mathcal{T}_\mathsf{e}, d_{\mathcal{T}_\mathsf{e}}).$$

Hence Inequality (44) is sufficient to complete the proof.

Let $u, v \in \mathcal{T}_n^f$ be arbitrary vertices and let $x \in \mathcal{T}_n^f$ denote their youngest common ancestor. Let o denote the root of $\mathcal{T}_n^f$. Then any shortest path in K_n° from o to u, or o to v, or u to v contains at least one vertex with $d_{\mathsf{K}_n^\circ}$-distance at most 1 from x. Thus the expression

$$|d_{\mathsf{K}_n^\circ}(u,v) - (d_{\mathsf{K}_n^\circ}(o,u) + d_{\mathsf{K}_n^\circ}(o,v) - 2d_{\mathsf{K}_n^\circ}(o,x))|$$

is bounded by a fixed constant that does not depend on u, v or x. Thus, in order to show Inequality (44), it suffices to show that for a sufficiently small but fixed constant $c > 0$ and it holds with high probability that

$$|d_{\mathsf{K}_n^\circ}(o,v) - b_k d_{\mathcal{T}_n^f}(o,v)| \leqslant c(d_{\mathcal{T}_n^f}(o,v)^s + \log^t n) \tag{48}$$

for all $v \in \mathcal{T}_n^f$.

To this end, let Z_n denote the sampler $\Gamma Z_{\mathcal{A}_\mathcal{R}^\omega}(\rho)$ conditioned on having size n. That is, Z_n is the symmetry corresponding to the $\mathrm{Sym}(\mathcal{R})$-enriched tree $(\mathcal{T}_n, \beta_n)$. Consider the set $\mathcal{H} \subset \bigcup_{k=0}^\infty \mathrm{Sym}(\mathcal{A}_\mathcal{R})[k]$ of $\mathcal{A}_\mathcal{R}$-symmetries $S = ((T,\alpha),\sigma)$ having the property that the

there exists a fixed point vertex v_S in T with the property that $\ell_S := \mathrm{h}_T(v_S) \geqslant \log^t(n)$ but the corresponding distance $\mathrm{h}_*(v_S)$ in the k-tree corresponding to (T, α) satisfies

$$|\mathrm{h}_*(v_S) - b_k \ell_S| > c\ell_S^s.$$

By Lemma 22 the probability for the sampler $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{\omega}}(\rho)$ to have size n is $\Theta(n^{-3/2})$. It follows from Equation 14 that

$$\begin{aligned}\mathbb{P}\left(\mathsf{Z}_n \in \mathcal{H}\right) &= O(n^{3/2}) \sum_{S \in \mathcal{H}} \mathbb{P}\left(\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell_S)}}(\rho) = (S, v_S)\right) \\ &\leqslant O(n^{3/2}) \sum_{\log^t n \leqslant \ell \leqslant n} p_\ell, \end{aligned} \tag{49}$$

with p_ℓ denoting the probability that the k-tree distance d_ℓ between the root vertex v_0 and the tip v_ℓ of the spine $v_0, \ldots, v_\ell$ in $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell_S)}}(\rho)$ satisfies

$$|d_\ell - b_k \ell| > c\ell^s.$$

We are going to bound the p_ℓ to show that the bound in (49) converges to zero. In order to simplify the calculations, let u denote any fixed vertex of the $*$-place-holder root-front of the k-tree corresponding to $\Gamma Z_{\mathcal{A}_{\mathcal{R}}^{(\ell_S)}}(\rho)$. Let d'_ℓ denote the k-tree distance from u to v_ℓ. Thus

$$|d_\ell - d'_\ell| \leqslant 1$$

for all ℓ and it suffices to study the deviation of d'_ℓ from $b_k \ell$. We are going to exploit properties of the bijection in Section 6.1.3 to take a Markov chain approach. Consider the set M_0 consisting of all k unlabelled vertices. The vertex v_1 is incident to v_0 and to a $k-1$-element subset $S_0 \subset M_0$. By the construction in Lemma 23 each $(k-1)$-element subset of M_0 is equally likely. The distance of v_1 to u is given by

$$d'_1 = 1 + \min_{v \in S_0} d(u, v)$$

with $d(\cdot, \cdot)$ denoting the k-tree distance. This follows from the fact that $M_0 \cup \{v_0\}$ is a $(k+1)$-clique. Setting $M_1 = \{v_0\} \cup S_0$, the distance of v_2 to u is again given by

$$d'_2 = 1 + \min_{v \in S_1} d(u, v)$$

with $S_1 \subset M_1$ denoting the $(k-1)$-element subset of M_1 that is incident with v_2. Here each $(k-1)$-element subset of M_1 is (conditionally) equally likely. We may continue this construction, yielding sequences $S_0, \ldots, S_{\ell-1}$ and $M_0, \ldots, M_\ell$, such that for all $0 \leqslant i \leqslant \ell - 1$ it holds that

$$d'_{i+1} = 1 + \min_{v \in S_i} d(u, v) \qquad \text{and} \qquad M_{i+1} = \{v_i\} \cup S_i, \tag{50}$$

and such that, conditioned on M_i, the subset S_i gets drawn uniformly at random among the $(k-1)$-element subsets of M_i. Note that for all $v \in M_i$ it holds that $d(u, v) = d'_i$ or

$d(u,v) = d'_i - 1$. For all $0 \leqslant i \leqslant \ell$ we let $1 \leqslant X_i \leqslant k$ denote the number of vertices in M_i with $d(u,v) = d'_i - 1$. Equation (50) implies that for all $0 \leqslant i \leqslant \ell - 1$

$$d'_{i+1} = \begin{cases} d'_i, & X_{i+1} < k \\ d'_i + 1, & X_{i+1} = k \end{cases}.$$

As $d'_0 = 1$ and $X_0 = 1$, it holds that

$$d'_\ell = 1 + \sum_{i=1}^{\ell} \mathbb{1}_{X_i = k}. \tag{51}$$

Recall that given M_i, the set S_i gets drawn uniformly at random from the $(k-1)$-element subsets of M_i. Thus, $(X_i)_i$ is Markov chain with the transition probabilities $p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$ given by the matrix

$$\mathbf{P} = (p_{ij})_{i,j} = \begin{pmatrix} \frac{k-1}{k} & & & & & \frac{1}{k} \\ \frac{2}{k} & \frac{k-2}{k} & & & & \\ & \frac{3}{k} & \frac{k-3}{k} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{k-1}{k} & \frac{1}{k} & \\ & & & & 1 & \end{pmatrix} \in \mathbb{R}^{k \times k}.$$

Here we use the convention, that empty spaces in a matrix denote zero entries. The stationary distribution is given by

$$\pi = (\pi_i)_{1 \leqslant i \leqslant k} = \frac{1}{\sum_{i=1}^{k} \frac{1}{i}} \left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right).$$

This chain is clearly irreducible. However, unless $k = 2$, it is not reversible. As the multiplicative symmetrization $\mathbf{P}^\intercal \mathbf{P}$ is irreducible, we may apply Lemma 17 to obtain that there are constants $a, b > 0$ such that for all sufficiently small $\epsilon > 0$ and all $\ell \geqslant 1$ it holds that

$$\mathbb{P}(|d'_\ell - 1 - \pi_k \ell| \geqslant \epsilon \ell) \leqslant a \exp(-b\epsilon^2 \ell).$$

As $\pi_k = b_k$, follows that for any $1/2 < s < 1$ we may choose $t > 0$ large enough such that the bound in (49) tends to zero as n becomes large.

Thus, it holds with high probability that all vertices $v \in \mathcal{T}_n^f$ with $d_{\mathcal{T}_n^f}(o,v) \geqslant \log^t n$ satisfy

$$|d_{\mathsf{K}_n^\circ}(o,v) - b_k d_{\mathcal{T}_n^f}(o,v)| \leqslant c d_{\mathcal{T}_n^f}(o,v)^s. \tag{52}$$

This readily verifies Equation (48) and hence completes the proof. □

Proof of Theorem 35. Let $k_n = o(\sqrt{n})$ be a given sequence. The random front-rooted unlabelled k-tree K_n may be viewed as a Gibbs partition. Theorem 15 ensures that K_n

exhibits a giant component, and that the small fragments converge in total variation toward a Boltzmann limit. Thus, it suffices to show that

$$d_{\mathrm{TV}}(V_{k_n}(\mathsf{K}_n^\circ), V_{k_n}(\hat{\mathsf{K}}^\circ)) \to 0 \tag{53}$$

as n becomes large. Set $h_n = 2b_k k_n + n^{1/4} = o(\sqrt{n})$ with b_k defined in Equation (43). Theorem 24 ensures that

$$d_{\mathrm{TV}}((\mathcal{T}_n, \beta_n)^{<h_n>}, (\mathcal{T}^{(\infty)}, \beta^{(\infty)})^{<h_n>}) \to 0.$$

as n becomes large.

By Equation (52) we know that with high probability the trimmed tree $(\mathcal{T}_n, \beta_n)^{<h_n>}$ already contains all information required to determine the k_n-neighbourhood $V_{k_n}(\mathsf{K}_n^\circ)$. This verifies Equation (53) and hence completes the proof. □

Proof of Theorem 36. Let $k_n = o(\sqrt{n})$ be a given sequence. Theorem 15 ensures that K_n exhibits a giant component, which is distributed like $\mathsf{K}_(r_n)^\circ$ for some random size $r_n = n + O_p(1)$. A uniformly at random selected vertex of K_n lies with high probability in this giant component and outside of its root-front. Conditioned on this event, the random vertex is uniformly distributed among the non-root vertices of the large component. Note that every path from a vertex of the giant component to a vertex of a smaller component must pass through the root-front. Thus, it suffices to show that if v^* is a random vertex of $(\mathcal{T}_n, \beta_n)$, then

$$d_{\mathrm{TV}}(V_{k_n}(\mathsf{K}_n^\circ, v^*), V_{k_n}(\hat{\mathsf{K}}^\circ)) \to 0,$$

and with high probability $V_{k_n}(\mathsf{K}_n^\circ, v^*)$ contains no vertex of the root-front of K_n°. Set $h_n = 2b_k k_n + n^{1/4} = o(\sqrt{n})$ with b_k defined in Equation (43). By Theorem 27 it follows that

$$d_{\mathrm{TV}}((\mathsf{H}^n_{[h_n]}, v^*), (\hat{\mathsf{H}}_{[h_n]}, u^*)) \to 0.$$

and that with high probability the vertex v^* has height $\mathrm{h}_{\mathcal{T}_n}(v^*) > h_n$. By Equation (52) it follows that with high probability it holds that $\mathrm{h}_{\mathsf{K}_n^\circ}(v^*) > k_n$ and that $(\mathsf{H}^n_{[k_n]}, v^*)$ already contains all information necessary to determine $V_{k_n}(\mathsf{K}_n^\circ, v^*)$. This completes the proof. □

7.3.3 Simply generated Pólya trees

We argued in Section 6.6 how Theorems 39, 40 and 41 follow from the results on random $\mathcal{R}$-enriched trees of Sections 6.2 and 6.3.

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