Countable Menger's theorem with finitary matroid constraints on the ingoing edges

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Abstract

We present a strengthening of the countable Menger's theorem of R. Aharoni. Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$ and let $\mathcal{M} = \bigoplus_{v \in V} \mathcal{M}_v$ be a matroid on A where $\mathcal{M}_v$ is a finitary matroid on the ingoing edges of v. We show that there is a system of edge-disjoint $s \to t$ paths $\mathcal{P}$ such that the united edge set of these paths is $\mathcal{M}$-independent, and there is a $C \subseteq A$ consisting of one edge from each element of $\mathcal{P}$ for which $\mathsf{span}_{\mathcal{M}}(C)$ covers all the $s \to t$ paths in D.

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1 Notation

The variables ξ, ζ denote ordinals and κ stands for an infinite cardinal. We write ω for the smallest limit ordinal (i.e. the set of natural numbers). We apply the abbreviation $H + h$ for the set $H \cup \{h\}$ and $H - h$ for $H \setminus \{h\}$ and we denote by $\triangle$ the symmetric difference (i.e. $H \triangle J := (H \setminus J) \cup (J \setminus H)$).

The digraphs $D = (V, A)$ of this article could be arbitrarily large and may have multiple edges and loops (though the later is irrelevant). For $X \subseteq V$ we denote the ingoing and the outgoing edges of X in D by $\mathsf{in}_D(X)$ and $\mathsf{out}_D(X)$. We write $D[X]$ for the subdigraph induced by the vertex set X. If e is an edge from vertex u to vertex v, then we write $\mathsf{tail}(e) = u$ and $\mathsf{head}(e) = v$. The paths in this paper are assumed to be finite and directed. Repetition of vertices is forbidden in them (we say walk if we want to allow it). For a path P we denote by $\mathsf{start}(P)$ and $\mathsf{end}(P)$ the first and the last vertex of P. If $X, Y \subseteq V$, then P is a $X \to Y$ path if $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$. For singletons we simplify the notation and write $x \to y$ instead of $\{x\} \to \{y\}$. For a path-system (set of paths) $\mathcal{P}$ we denote $\bigcup_{P \in \mathcal{P}} A(P)$ by $A(\mathcal{P})$ and we write for the set of the last edges of the elements of $\mathcal{P}$ simply $A_{\text{last}}(\mathcal{P})$. An s-arborescence is a directed tree in which every vertex is reachable (by a directed path) from its vertex s.

If $\mathcal{M}$ is a matroid and S is a subset of its ground set, then $\mathcal{M}/S$ is the matroid we obtain by the contraction of the set S. We use $\bigoplus$ for the direct sum of matroids. For the rank function we write r and $\mathsf{span}(S)$ is the union of S and the loops (dependent singletons) of $\mathcal{M}/S$. Let us remind that a matroid is called finitary if all of its circuits are finite. One can find a good survey about infinite matroids from the basics in [3].

2 Introduction

In this paper we generalize the countable version of Menger's theorem of Aharoni [1] by applying the results of Lawler and Martel about polymatroidal flows (see [5]).

Let us recall Menger's theorem (directed, edge version).

Theorem 1 (Menger)**.** *Let $D = (V, A)$ be a finite digraph with $s \neq t \in V$. Then the maximum number of the pairwise edge-disjoint $s \to t$ paths is equal to the minimum number of edges that cover all the $s \to t$ paths.*

Erdős observed during his school years that the theorem above remains true for infinite digraphs (by saying cardinalities instead of numbers). He felt that this is not the "right" infinite generalization of the finite theorem and he conjectured the "right" generalization which was known as the Erdős-Menger conjecture. It is based on the observation that in Theorem 1 an optimal cover consists of one edge from each path of an optimal path-system. The Erdős-Menger conjecture states that for arbitrarily large digraphs there are a path-system and a cover that satisfy these complementarity conditions. After a long sequence of partial results the countable case has been settled affirmatively by R. Aharoni:

Theorem 2 (R. Aharoni, [1])**.** *Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$. Then there is a system $\mathcal{P}$ of edge-disjoint $s \to t$ paths such that there is an edge set C which covers all the $s \to t$ paths in D and C consists of one edge from each $P \in \mathcal{P}$.*

It is worth to mention that R. Aharoni and E. Berger proved the Erdős-Menger conjecture in its full generality in 2009 (see [2]) which was one of the greatest achievements in the theory of infinite graphs. We present the following strengthening of the countable Menger's theorem above.

Theorem 3. *Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$. Assume that there is a finitary matroid $\mathcal{M}_v$ on the ingoing edges of v for any $v \in V$. Let $\mathcal{M}$ be the direct sum of the matroids $\mathcal{M}_v$. Then there is a system of edge-disjoint $s \to t$ paths $\mathcal{P}$ such that the united edge set of the paths is $\mathcal{M}$-independent, and there is an edge set C consisting of one edge from each element of $\mathcal{P}$ for which $\mathsf{span}_{\mathcal{M}}(C)$ covers all the $s \to t$ paths in D.*

Note that on the one hand, if $\mathcal{M}$ is a free matroid in Theorem 3, then we get back the edge version of the countable Menger's theorem. On the other hand, if exactly the singletons are independent in $\mathcal{M}_v$ for $v \in V \setminus \{t\}$ and $\mathcal{M}_t$ is a free matroid, then $\mathcal{P}$ is an internally vertex-disjoint system of $s \to t$ paths. Furthermore, if there is no edge from s to t, then the vertex set

$$C' := \{\mathsf{tail}(e) : e \in C \cap \mathsf{in}_D(t)\} \cup \{\mathsf{head}(e) : e \in C \setminus \mathsf{in}_D(t)\} \subseteq V \setminus \{s, t\}$$

covers all the $s \to t$ paths in D and consists of one internal vertex from each path in $\mathcal{P}$. Thus we obtained both the edge and the vertex version of the countable Menger's theorem as a special case of Theorem 3.

In the proof of Theorem 3 it will be more convenient focusing on $t - s$ cuts ($X \subseteq V$ is a $t - s$ cut if $t \in X \subseteq V \setminus \{s\}$) instead of dealing with covers directly. Let us call a path-system $\mathcal{P}$ independent if $A(\mathcal{P})$ is independent in $\mathcal{M}$. Suppose that an independent system $\mathcal{P}$ of edge-disjoint $s \to t$ paths and a $t - s$ cut X satisfy the **complementarity conditions**:

Condition 4.

1. $A(\mathcal{P}) \cap \mathsf{out}_D(X) = \varnothing$,
2. $A(\mathcal{P}) \cap \mathsf{in}_D(X)$ spans $\mathsf{in}_D(X)$ in $\mathcal{M}$.

Then clearly $\mathcal{P}$ and $C := A(\mathcal{P}) \cap \mathsf{in}_D(X)$ satisfy the demands of Theorem 3. Therefore it is enough to prove the following reformulation of the theorem.

3 Main result

Theorem 5. *Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$ and suppose that there is a finitary matroid $\mathcal{M}_v$ on $\mathsf{in}_D(v)$ for each $v \in V$ and let $\mathcal{M} = \bigoplus_{v \in V} \mathcal{M}_v$. Then there are a system $\mathcal{P}$ of edge-disjoint $s \to t$ paths where $A(\mathcal{P})$ is independent in $\mathcal{M}$ and a $t - s$ cut X such that $\mathcal{P}$ and X satisfy the complementarity conditions (Condition 4).*

Proof. Without loss of generality we may assume that $\mathcal{M}$ does not contain loops. A pair $(\mathcal{W}, X)$ is called a **wave** if X is a $t - s$ cut and $\mathcal{W}$ is an independent system of edge-disjoint $s \to X$ paths such that the second complementarity condition holds for $\mathcal{W}$ and X (i.e. $A_{\text{last}}(\mathcal{W})$ spans $\mathsf{in}_D(X)$ in $\mathcal{M}$).

Remark 6. By picking an arbitrary base B of $\mathsf{out}(s)$ and taking $\mathcal{W} := B$ as a set of single-edge paths and $X := V \setminus \{s\}$ we obtain a wave $(\mathcal{W}, X)$ thus there always exists some wave.

We say that the wave $(\mathcal{W}_1, X_1)$ extends the wave $(\mathcal{W}_0, X_0)$ and write $(\mathcal{W}_0, X_0) \leqslant (\mathcal{W}_1, X_1)$ if

1. $X_0 \supseteq X_1$,
2. $\mathcal{W}_1$ consists of the forward-continuations of some of the paths in $\mathcal{W}_0$ (i.e. every path in $\mathcal{W}_1$ has an initial segment which is in $\mathcal{W}_0$) such that the continuations lie in X_0,
3. $\mathcal{W}_1$ contains all of those paths of $\mathcal{W}_0$ that meet X_1.

If in addition $\mathcal{W}_1$ contains a forward-continuation of *all* the elements of $\mathcal{W}_0$, then the extension is called **complete**. Note that $\leqslant$ is a partial order on the waves and if $(\mathcal{W}_0, X_0) \leqslant (\mathcal{W}_1, X_1)$ holds, then the extension is proper (i.e. $(\mathcal{W}_0, X_0) < (\mathcal{W}_1, X_1)$) iff $X_1 \subsetneq X_0$.

Observation 7. *If $(\mathcal{W}_1, X_1)$ is an incomplete extension of $(\mathcal{W}_0, X_0)$, then it is a proper extension thus $X_1 \subsetneq X_0$. Furthermore, $\mathcal{W}_1$ and X_0 do not satisfy the second complementarity condition (Condition 4/2).*

Lemma 8. *If a nonempty set $\mathcal{X}$ of waves is linearly ordered by $\leqslant$, then $\mathcal{X}$ has a unique smallest upper bound $\sup(\mathcal{X})$.*

Proof. We may suppose that $\mathcal{X}$ has no maximal element. Let $\langle(\mathcal{W}_\xi, X_\xi) : \xi < \kappa\rangle$ be a cofinal increasing sequence of $(\mathcal{X}, \leqslant)$. We define $X := \bigcap_{\xi<\kappa} X_\xi$ and

$$\mathcal{W} := \bigcup_{\zeta<\kappa} \bigcap_{\zeta<\xi} \mathcal{W}_\xi.$$

For $P \in \mathcal{W}$ we have $V(P) \cap X_\xi = \{\mathsf{end}(P)\}$ for all large enough $\xi < \kappa$ hence $V(P) \cap X = \{\mathsf{end}(P)\}$. The paths in $\mathcal{W}$ are pairwise edge-disjoint since $P_1, P_2 \in \mathcal{W}$ implies that $P_1, P_2 \in \mathcal{W}_\xi$ for all large enough ξ. Since the matroid $\mathcal{M}$ is finitary the same argument shows that $\mathcal{W}$ is independent.

Suppose that $e \in \mathsf{in}_D(X) \setminus A(\mathcal{W})$. For a large enough $\xi < \kappa$ we have $e \in \mathsf{in}_D(X_\xi)$. Then the last edges of those elements of $\mathcal{W}_\xi$ that terminate in $\mathsf{head}(e)$ span e in $\mathcal{M}$. These paths have to be elements of all the further waves of the sequence (because of the definition of $\leqslant$) and thus of $\mathcal{W}$ as well. Therefore $(\mathcal{W}, X)$ is a wave and clearly an upper bound.

Suppose that $(\mathcal{Q}, Y)$ is another upper bound for $\mathcal{X}$. Then $X_\xi \supseteq Y$ for all $\xi < \kappa$ and hence $X \supseteq Y$. Let $Q \in \mathcal{Q}$ be arbitrary. We know that $\mathcal{W}_\xi$ contains an initial segment Q_ξ of Q for all $\xi < \kappa$ because $(\mathcal{Q}, Y)$ is an upper bound (see the definition of $\leqslant$). For $\xi < \zeta < \kappa$ the path Q_ζ is a (not necessarily proper) forward-continuation of Q_ξ. From some index the sequence $\langle Q_\xi : \xi < \kappa\rangle$ need to be constant, say Q^*, since Q is a finite path. But then $Q^* \in \mathcal{W}$. Thus any $Q \in \mathcal{Q}$ is a forward-continuation of a path in $\mathcal{W}$. Finally assume that some $P \in \mathcal{W}$ meets Y. Pick a $\xi < \kappa$ for which $P \in \mathcal{W}_\xi$. Then $(\mathcal{W}_\xi, X_\xi) \leqslant (\mathcal{Q}, Y)$ guarantees $P \in \mathcal{Q}$. Therefore $(\mathcal{W}, X) \leqslant (\mathcal{Q}, Y)$. □

Remark 6 and Lemma 8 imply via Zorn's Lemma the following.

Corollary 9. *There exists a maximal wave. Furthermore, there is a maximal wave which is greater or equal to an arbitrary prescribed wave.*

Let $(\mathcal{W}, X)$ be a maximal wave. To prove Theorem 5 it is enough to show that there is an independent system of edge-disjoint $s \to t$ paths $\mathcal{P}$ that consists of forward-continuations of all the paths in $\mathcal{W}$. Indeed, condition $A(\mathcal{P}) \cap \mathsf{out}_D(X) = \varnothing$ will be true automatically (otherwise $\mathcal{P}$ would violate independence, when the violating path "comes back" to X) and hence $\mathcal{P}$ and X will satisfy the complementarity conditions.

We need a method developed by Lawler and Martel in [5] for the augmentation of polymatroidal flows in finite networks which works in the infinite case as well.

Lemma 10. *Let $\mathcal{P}$ be an independent system of edge-disjoint $s \to t$ paths. Then there is either an independent system of edge-disjoint $s \to t$ paths $\mathcal{P}'$ with $\mathsf{span}_{\mathcal{M}_t}(A_{last}(\mathcal{P})) \subsetneq \mathsf{span}_{\mathcal{M}_t}(A_{last}(\mathcal{P}'))$ or there is a $t-s$ cut X such that the complementarity conditions (Condition 4) hold for $\mathcal{P}$ and X.*

Proof. Call W an augmenting walk if

1. W is a directed walk with respect to the digraph that we obtain from D by changing the direction of edges in $A(\mathcal{P})$,
2. $\mathsf{start}(W) = s$ and W meets s no more,
3. $A(W) \triangle A(\mathcal{P})$ is independent,
4. if for some initial segment W' of W the set $A(W')\triangle A(\mathcal{P})$ is not independent, then for the one edge longer initial segment $\mathcal{W}'' = \mathcal{W}'e$ the set $A(W'')\triangle A(\mathcal{P})$ is independent again.

If there is an augmenting walk terminating in t, then let W be a shortest such a walk. Build $\mathcal{P}'$ from the edges $A(W)\triangle A(\mathcal{P})$ in the following way. Keep untouched those $P \in \mathcal{P}$ for which $A(W) \cap A(P) = \varnothing$ and replace the remaining finitely many paths, say $\mathcal{Q} \subseteq \mathcal{P}$ where $|\mathcal{Q}| = k$, by $k+1$ new $s \to t$ paths constructed from the edges $A(W)\triangle A(\mathcal{Q})$ by the greedy method. Obviously $\mathcal{P}'$ is an independent system of edge-disjoint $s \to t$ paths. We need to show that

$$\mathsf{span}_{\mathcal{M}_t}(A_{\mathrm{last}}(\mathcal{P})) \subsetneq \mathsf{span}_{\mathcal{M}_t}(A_{\mathrm{last}}(\mathcal{P}')).$$

If only the last vertex of W is t, then it is clear. Let $f_1, e_1, \dots, f_n, e_n, f_{n+1}$ be the ingoing-outgoing edge pairs of t in W with respect to the direction of W (enumerated with respect to the direction of W). The initial segments of W up to the inner appearances of t cannot be augmenting walks (since W is a shortest that terminates in t) hence by condition 4 the one edge longer and the one edge shorter segments are. It follows that for any $1 \leqslant i \leqslant n$ there exists a $\mathcal{M}_t$-circuit C_i in

$$A_i := A(\mathcal{P}) \cap \mathsf{in}_D(t) + f_1 - e_1 + f_2 - e_2 + \dots + f_i$$

and $f_i \notin A(\mathcal{P})$ and $e_i \in C_i \cap A(\mathcal{P})$. It implies by induction that $A_i \setminus \{e_i\}$ spans the same set in $\mathcal{M}_t$ as $A(\mathcal{P}) \cap \mathsf{in}_D(t)$ whenever $1 \leqslant i \leqslant n$ and hence $A_n \cup \{f_{n+1}\}$ spans a strictly larger set.

Suppose now that none of the augmenting walks terminates in t. Let us denote the set of the last vertices of the augmenting walks by Y. We show that $\mathcal{P}$ and $X := V \setminus Y$ satisfy the complementarity conditions. Obviously X is a $t-s$ cut. Suppose, to the contrary, that $e \in A(\mathcal{P}) \cap \mathsf{out}_D(X)$. Pick an augmenting walk W terminating in $\mathsf{head}(e)$. Necessarily $e \in A(W)$, otherwise We would be an augmenting walk contradicting the definition of X. Consider the initial segment W' of W for which the following edge is e. Then $W'e$ is an augmenting walk (if W' itself is not, then it is because of condition 4) which leads to the same contradiction.

To show the second complementarity condition assume that $f \in \mathsf{in}_D(X) \setminus A(\mathcal{P})$. Choose an augmenting walk W that terminates in $\mathsf{tail}(f)$. We may suppose that $f \notin A(W)$ otherwise we consider the initial segment W' of W for which the following edge is f (it is an augmenting walk, otherwise $W'f$ would be by applying condition 4 contradicting the definition of X). The initial segments of Wf that terminate in $\mathsf{head}(f)$ cannot be augmenting walks. Let $f_1, e_1, \dots, f_n, e_n$ be the ingoing-outgoing edge pairs of $\mathsf{head}(f)$ in W with respect to the direction of W (enumerated with respect to the direction of W) and let $f_{n+1} := f$. Then for any $1 \leqslant i \leqslant n+1$ there is a unique $\mathcal{M}$-circuit C_i in

$$A(\mathcal{P}) \cap \mathsf{in}_D(\mathsf{head}(f)) + f_1 - e_1 + f_2 - e_2 + \cdots + f_i.$$

It follows by using condition 4 and the definition of X that for $1 \leqslant i \leqslant n$

1. $f_i \notin A(\mathcal{P})$ and $e_i \in C_i \cap A(\mathcal{P})$,
2. $\mathsf{tail}(e_i), \mathsf{tail}(f_i) \in Y$ (tail with respect to the original direction),
3. $C_i \subseteq \mathsf{in}_D(X)$.

Assume that we already know for some $1 \leqslant i \leqslant n$ that f_j is spanned by $F := A(\mathcal{P}) \cap \mathsf{in}_D(X)$ in $\mathcal{M}$ whenever $j < i$. Any element of $C_i \setminus \{f_i\}$ which is not in F has a form f_j for some $j < i$ thus by the induction hypothesis it is spanned by F and hence we obtain that $f_i \in \mathsf{span}_{\mathcal{M}}(F)$ as well. By induction it is true for $i = n+1$. □

Proposition 11. *Assume that $(\mathcal{W}, X)$ and $(\mathcal{Q}, Y)$ are waves where $X \supseteq Y$ and $\mathcal{Q}$ consists of the forward-continuation of some of the paths in $\mathcal{W}$ where the new terminal segments lie in X. Let $\mathcal{W}_Y := \{P \in \mathcal{W} : \mathsf{end}(P) \in Y\}$. Then for an appropriate $\mathcal{Q}' \subseteq \mathcal{Q}$ the pair $(\mathcal{W}_Y \cup \mathcal{Q}', Y)$ is a wave with $(\mathcal{W}, X) \leqslant (\mathcal{W}_Y \cup \mathcal{Q}', Y)$.*

Proof. The path-system $\mathcal{W}_Y \cup \mathcal{Q}$ (not necessarily disjoint union) is edge-disjoint since the edges in $A(\mathcal{Q}) \setminus A(\mathcal{W})$ lie in X. For the same reason it may violate independence only at the vertices $\{\mathsf{end}(P) : P \in \mathcal{W}_Y\} \subseteq Y$. Pick a base B of $\mathsf{in}_D(Y)$ for which

$$A_{\text{last}}(\mathcal{W}_Y) \subseteq B \subseteq A_{\text{last}}(\mathcal{W}_Y) \cup A_{\text{last}}(\mathcal{Q}).$$

It is routine to check that the choice $\mathcal{Q}' = \{P \in \mathcal{Q} : A(P) \cap B \neq \varnothing\}$ is suitable. □

For $A_0 \subseteq A$ let us denote $(D - \mathsf{span}_{\mathcal{M}}(A_0), \mathcal{M}/\mathsf{span}_{\mathcal{M}}(A_0))$ by $\boldsymbol{\mathfrak{D}(A_0)}$. We may iterate this operation i.e. for $A_1 \subseteq A - \mathsf{span}_{\mathcal{M}}(A_0)$ we define $\mathfrak{D}(A_0)(A_1)$ similarly (which is of course $\mathfrak{D}(A_0 \cup A_1)$). Note that for any A_0 the matroid corresponding to $\mathfrak{D}(A_0)$ has no loops and $(D, \mathcal{M}) = \mathfrak{D}(\varnothing) =: \boldsymbol{\mathfrak{D}}$.

Observation 12. *If $(\mathcal{W}, X)$ is a wave and for some $A_0 \subseteq A \setminus A(\mathcal{W})$ the set $A_0 \cup A(\mathcal{W})$ is independent, then $(\mathcal{W}, X)$ is a $\mathfrak{D}(A_0)$-wave as well.*

Lemma 13. *If $(\mathcal{W}, X)$ is a maximal $\mathfrak{D}$-wave and $e \in A \setminus A(\mathcal{W})$ for which $A(\mathcal{W}) \cup \{e\}$ is independent, then all the extensions of the $\mathfrak{D}(e)$-wave $(\mathcal{W}, X)$ in $\mathfrak{D}(e)$ are complete.*

Proof. Assume that we have an incomplete extension $(\mathcal{Q}, Y)$ of $(\mathcal{W}, X)$ with respect to $\mathfrak{D}(e)$. Observe that necessarily $e \in \mathsf{in}_D(Y)$ and $r_{\mathcal{M}}(\mathsf{in}_D(Y)/A_{\mathrm{last}}(\mathcal{Q})) = 1$. Furthermore, $Y \subsetneq X$ by Observation 7.

We show that $(\mathcal{W}, X)$ has a proper extension with respect to $\mathfrak{D}$ as well contradicting its maximality. Without loss of generality we may assume that $\mathsf{in}_D(X) = A_{\mathrm{last}}(\mathcal{W})$. Indeed, otherwise we delete the edges $\mathsf{in}_D(X) \setminus A(\mathcal{W})$ from D and from $\mathcal{M}$. It is routine to check that after the deletion $(\mathcal{W}, X)$ is still a wave and a proper extension of it remains a proper extension after putting back these edges.

Contract $V \setminus X$ to s and contract Y to t in D and keep $\mathcal{M}$ unchanged. Apply the augmenting walk method (Lemma 10) in the resulting system with the $V \setminus X \to Y$ terminal segments of the paths in $\mathcal{Q}$. If the augmentation is possible, then the assumption $\mathsf{in}_D(X) = A_{\mathrm{last}}(\mathcal{W})$ ensures that the first edge of any element of the resulting path-system $\mathcal{R}$ is the last edge of some path in $\mathcal{W}$. By uniting the elements of $\mathcal{R}$ with the corresponding paths from $\mathcal{W}$ we can get a new independent system of edge-disjoint $s \to Y$ paths $\mathcal{Q}'$ (with respect to $\mathfrak{D}$). Furthermore, $r_{\mathcal{M}}(\mathsf{in}_D(Y)/A_{\mathrm{last}}(\mathcal{Q})) = 1$ guarantees that $A_{\mathrm{last}}(\mathcal{Q}')$ spans $\mathsf{in}_D(Y)$ in $\mathcal{M}$ and hence $(\mathcal{Q}', Y)$ is a wave. Thus by Proposition 11 we get an extension of $(\mathcal{W}, X)$ and it is proper because $Y \subsetneq X$ which is impossible.

Thus the augmentation must be unsuccessful which implies by Lemma 10 that there is some Z with $Y \subseteq Z \subseteq X$ such that Z and $\mathcal{Q}$ satisfy the complementarity conditions. By Observation 7 we know that $Z \subsetneq X$. For the initial segments $\mathcal{Q}_Z$ of the paths in $\mathcal{Q}$ up to Z the pair $(\mathcal{Q}_Z, Z)$ forms a wave. Thus by applying Proposition 11 with $(\mathcal{W}, X)$ and $(\mathcal{Q}_Z, Z)$ we obtain an extension of $(\mathcal{W}, X)$ which is proper because $Z \subsetneq X$ contradicting the maximality of $(\mathcal{W}, X)$. □

Proposition 14. *If $(\mathcal{W}, X)$ is a maximal wave and $v \in X$, then there is a $v \to t$ path Q in $D[X]$ such that $A(\mathcal{W}) \cup A(Q)$ is independent.*

Proof. It is equivalent to show that there exists a $v \to t$ path Q in $D - \mathsf{span}_{\mathcal{M}}(A(W))$ (path Q will necessarily lie in $D[X]$ because $D - \mathsf{span}_{\mathcal{M}}(A(W))$ does not contain any edge entering into X.) Suppose, to the contrary, that it is not the case. Let $X' \subsetneq X$ be the set of those vertices in X that are unreachable from v in $D - \mathsf{span}_{\mathcal{M}}(A(W))$ (note that $v \notin X'$ but $t \in X'$ by the indirect assumption). Let $\mathcal{W}'$ consist of the paths in $\mathcal{W}$ that meet X'. If we prove that $(\mathcal{W}', X')$ is a wave, then we are done since it would be a proper extension of the maximal wave $(\mathcal{W}, X)$. Assume that $f \in \mathsf{in}_D(X') \setminus A(\mathcal{W}')$. Then by the definition of X' we have $\mathsf{tail}(f) \in V \setminus X$ thus $f \in \mathsf{in}_D(X)$. Hence f is spanned by the last edges of the paths in $\mathcal{W}$ terminating in $\mathsf{head}(f)$ and all these paths are in $\mathcal{W}'$ as well. Therefore $(\mathcal{W}', X')$ is a wave. □

Lemma 15. *Let $(\mathcal{W}, X_0)$ be a maximal wave and assume that $P \in \mathcal{W}$ and let $\mathcal{W}_0 = \mathcal{W} \setminus \{P\}$. Then there is an s-arborescence $\mathcal{A}$ such that*

1. $A(P) \subseteq A(\mathcal{A})$,

2. $A(\mathcal{A}) \cap A(\mathcal{W}_0) = \varnothing$,

3. $A(\mathcal{A}) \cup A(\mathcal{W}_0)$ *is independent,*

4. $t \in V(\mathcal{A})$,

5. *there is a maximal wave with respect to* $\mathfrak{D}(A(\mathcal{A}))$ *which is a complete extension of the* $\mathfrak{D}(A(\mathcal{A}))$*-wave* $(\mathcal{W}_0, X_0)$.

Proof.

Proposition 16. *The pair* $(\mathcal{W}_0, X_0) = (\mathcal{W} \setminus \{P\}, X_0)$ *is a maximal wave with respect to* $\mathfrak{D}(A(P))$.

Proof. It is clearly a wave thus we show just the maximality. Seeking a contradiction, suppose that $(\mathcal{Q}, Y)$ is a proper extension of $(\mathcal{W} \setminus \{P\}, X_0)$ with respect to $\mathfrak{D}(A(P))$. Necessarily $\mathsf{end}(P) \in Y$ otherwise it would be a wave with respect to $\mathfrak{D}$ which properly extends $(\mathcal{W}, X_0)$. Let e be the last edge of P. We know that $A_{\text{last}}(\mathcal{Q})$ spans $\mathsf{in}_D(Y)$ in $\mathcal{M}/e$. Since $A(\mathcal{Q})$ is $[\mathcal{M}/\mathsf{span}_{\mathcal{M}}(A(P))]$-independent it follows that $(\mathcal{Q} \cup \{P\}, Y)$ is a $\mathfrak{D}$-wave. But then it properly extends $(\mathcal{W}, X_0)$ which is a contradiction. □

Fix a well-ordering of A with order type $|A| \leqslant \omega$. We build the arborescence $\mathcal{A}$ by recursion. Let $\mathcal{A}_0 := P$. Assume that $\mathcal{A}_m, \mathcal{W}_m$ and X_m have already been defined for $m \leqslant n$ in such a way that

1. $A(\mathcal{A}_m) \cap A(\mathcal{W}_m) = \varnothing$,

2. $A(\mathcal{A}_m) \cup A(\mathcal{W}_m)$ is independent,

3. $(\mathcal{W}_m, X_m)$ is a maximal wave with respect to $\mathfrak{D}_m := \mathfrak{D}(A(\mathcal{A}_m))$ and a complete extension of the $\mathfrak{D}_m$-wave $(\mathcal{W}_k, X_k)$ whenever $k < m$,

4. for $0 \leqslant k < n$ we have $\mathcal{A}_{k+1} = \mathcal{A}_k + e_k$ for some $e_k \in \mathsf{out}_D(V(\mathcal{A}_k))$.

If $t \in V(\mathcal{A}_n)$, then $\mathcal{A}_n$ satisfies the requirements of Lemma 15 thus we are done. Hence we may assume that $t \notin V(\mathcal{A}_n)$.

Proposition 17. $\mathsf{out}_{D-\mathsf{span}_{\mathcal{M}}(A(\mathcal{W}_n))}(V(\mathcal{A}_n)) \neq \varnothing$.

Proof. We claim that the $\mathfrak{D}_n$-wave $(\mathcal{W}_n, X_n)$ is not a $\mathfrak{D}$-wave. Indeed, suppose it is, then $\mathsf{end}(P) \notin X_n$ (since $A_{\text{last}}(\mathcal{W}_n)$ does not span the last edge e of P) and therefore $X_n \subsetneq X_0$ thus it extends $(\mathcal{W}, X_0)$ properly with respect to $\mathfrak{D}$ contradicting the maximality of $(\mathcal{W}, X_0)$. Hence the s-arborescence $\mathcal{A}_n$ needs to have an edge $e \in \mathsf{in}_D(X_n)$. Let Q be a path that we obtain by applying Proposition 14 with $(\mathcal{W}_n, X_n)$ and $\mathsf{head}(e)$ in the system $\mathfrak{D}_n$. Consider the last vertex v of Q which is in $V(\mathcal{A}_n)$. Since $v \neq t$ there is an outgoing edge f of v in Q and hence $f \in \mathsf{out}_{D-\mathsf{span}_{\mathcal{M}}(A(\mathcal{W}_n))}(V(\mathcal{A}_n))$. □

Pick the smallest element e_n of $\mathsf{out}_{D-\mathsf{span}_{\mathcal{M}}(A(\mathcal{W}_n))}(V(\mathcal{A}_n))$ and let $\mathcal{A}_{n+1} := \mathcal{A}_n + e_n$. Let $(\mathcal{W}_{n+1}, X_{n+1})$ be a maximal wave with respect to $\mathfrak{D}_{n+1}$ which extends $(\mathcal{W}_n, X_n)$ (exists by Corollary 9). Lemma 13 ensures that it is a complete extension. The recursion is done.

Suppose, to the contrary, that the recursion does not stop after finitely many steps (i.e. the arborescence never reaches t). Let

$$\mathcal{A}_\infty := \left(\bigcup_{n=0}^{\infty} V(\mathcal{A}_n), \bigcup_{n=0}^{\infty} A(\mathcal{A}_n) \right).$$

Note that $A(\mathcal{A}_\infty)$ is independent. Furthermore, $\langle (\mathcal{W}_n, X_n) : n < \omega \rangle$ is an $\leqslant$-increasing sequence of $\mathfrak{D}(A(\mathcal{A}_\infty))$-waves. Let $(\mathcal{W}_\infty, X_\infty)$ be a maximal $\mathfrak{D}(A(\mathcal{A}_\infty))$-wave which extends $\sup_n (\mathcal{W}_n, X_n)$ (see Lemma 8).

It cannot be a wave with respect to $\mathfrak{D}$ (the reason is similar as in the first part of the proof of Proposition 17). Hence the s-arborescence $\mathcal{A}_\infty$ contains an edge $e \in \mathsf{in}_D(X_\infty)$. Apply Proposition 14 with $(\mathcal{W}_\infty, X_\infty)$ and $\mathsf{head}(e)$ in the system $\mathfrak{D}(A(\mathcal{A}_\infty))$. Consider the last vertex v of the resulting Q which is in $V(\mathcal{A}_\infty)$. Since $v \neq t$ by assumption there is an outgoing edge f of v in Q. Then $f \in \mathsf{out}_{D-\mathsf{span}_{\mathcal{M}}(A(\mathcal{W}_\infty))}(V(\mathcal{A}_\infty))$ which implies that for some $n_0 < \omega$ we have $f \in \mathsf{out}_{D-\mathsf{span}_{\mathcal{M}}(A(\mathcal{W}_n))}(V(\mathcal{A}_n))$ whenever $n > n_0$. But then the infinitely many pairwise distinct edges $\{e_n : n_0 < n < \omega\}$ are all smaller than f in our fixed well-ordering of A which contradicts the fact that the type of this well-ordering is at most ω. $\square$

Theorem 3 follows easily from Lemma 15. Indeed, pick a maximal wave $(\mathcal{W}_0, X_0)$ with respect to $\mathfrak{D}_0 := \mathfrak{D}$ where $\mathcal{W}_0 = \{P_n\}_{n<\omega}$. Apply Lemma 15 with $P_0 \in \mathcal{W}_0$. The resulting arborescence $\mathcal{A}_0$ contains a unique $s \to t$ path P_0^* which is necessarily a forward-continuation of P_0 (usage of a new edge from $\mathsf{in}_D(X_0)$ would lead to dependence). Then by Lemma 15 we have a maximal wave $(\mathcal{W}_1, X_1)$ (where $X_1 \subseteq X_0$) with respect to $\mathfrak{D}_1 := \mathfrak{D}_0(A(\mathcal{A}_0))$ such that $\mathcal{W}_1 = \{P_n^1\}_{1 \leqslant n < \omega}$ where P_n^1 is a forward-continuation of P_n. Then we apply Lemma 15 with the $\mathfrak{D}_1$-wave $(\mathcal{W}_1, X_1)$ and $P_1^1 \in \mathcal{W}_1$ and continue the process recursively. By the construction $\bigcup_{n<m} A(P_n^*)$ is independent for each $m < \omega$. Since $\mathcal{M}$ is finitary $\bigcup_{n<\infty} A(P_n^*)$ is independent as well thus $\mathcal{P} := \{P_n^*\}_{n<\omega}$ is a desired paths-system that satisfies the complementarity conditions with X_0. $\square$

4 Open problems

We suspect that one can omit the countability condition for D in Theorem 5 by analysing the famous infinite Menger's theorem [2] of Aharoni and Berger. We also think that it is possible to put matroid constraints on the outgoing edges of each vertex as well but this generalization contains the Matroid intersection conjecture for finitary matroids as a special case, which problem is hard enough itself. The finitarity of the matroids is used several times in the proof; we do not know yet if one can omit this condition.

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