On the Strong Chromatic Index of Sparse Graphs

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Abstract

The strong chromatic index of a graph G, denoted $\chi'_s(G)$, is the least number of colors needed to edge-color G so that edges at distance at most two receive distinct colors. The strong list chromatic index, denoted $\chi'_{\ell,s}(G)$, is the least integer k such that if arbitrary lists of size k are assigned to each edge then G can be edge-colored from those lists where edges at distance at most two receive distinct colors. We use the discharging method, the Combinatorial Nullstellensatz, and computation to show that if G is a subcubic planar graph with $\text{girth}(G) \geqslant 41$ then $\chi'_{\ell,s}(G) \leqslant 5$, answering a question of Borodin and Ivanova [Precise upper bound for the strong edge chromatic number of sparse planar graphs, *Discuss. Math. Graph Theory*, 33(4), (2014) 759–770]. We further show that if G is a subcubic planar graph and $\text{girth}(G) \geqslant 30$, then $\chi'_s(G) \leqslant 5$, improving a bound from the same paper. Finally, if G is a planar graph with maximum degree at most four and $\text{girth}(G) \geqslant 28$, then $\chi'_s(G) \leqslant 7$, improving a more general bound of Wang and Zhao from [Odd graphs and its application on the strong edge-coloring, *Applied Math. and Computation* 325, (2018), 246–251].

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1 Introduction

A *proper edge-coloring* of a graph G is an assignment of colors to the edges so that incident edges receive distinct colors. A *strong edge-coloring* of a graph G is an assignment of colors to the edges so that edges at distance at most two receive distinct colors. A proper edge-coloring is a decomposition of G into matchings, while a strong edge-coloring is a decomposition of G into *induced* matchings. Fouquet and Jolivet [10, 11] defined the *strong chromatic index* of a graph G, denoted $\chi'_s(G)$, as the minimum integer k such that G has a strong edge-coloring using k colors. Erdős and Nešetřil gave the following conjecture, which is still open, and provided an example to show that it would be sharp, if true.

Conjecture 1 (Erdős and Nešetřil [8])**.** For every graph G, $\chi'_s(G) \leqslant \frac{5}{4}\Delta(G)^2$ when $\Delta(G)$ is even, and $\chi'_s(G) \leqslant \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1)$ when $\Delta(G)$ is odd.

Towards this conjecture, Molloy and Reed [18] bounded $\chi'_s(G)$ away from the trivial upper bound of $2\Delta(G)(\Delta(G) - 1) + 1$ by showing that every graph G with sufficiently large maximum degree satisfies $\chi'_s(G) \leqslant 1.998\Delta(G)^2$. Bruhn and Joos [5] have announced an improvement, claiming $\chi'_s(G) \leqslant 1.93\Delta(G)^2$.

The focus of this paper is the study of strong edge-colorings of *subcubic graphs*, those with maximum degree at most three, and *subquartic graphs*, those with maximum degree at most four. Faudree, Gyárfas, Schelp, and Tuza [9] studied $\chi'_s(G)$ in the class of subcubic graphs, and gave the following conjectures.

Conjecture 2 (Faudree *et al.* [9])**.** Let G be a subcubic graph.

(1) $\chi'_s(G) \leqslant 10$.

(2) If G is bipartite, then $\chi'_s(G) \leqslant 9$.

(3) If G is planar, then $\chi'_s(G) \leqslant 9$.

(4) If G is bipartite and for each edge $xy \in E(G)$, $d(x) + d(y) \leqslant 5$, then $\chi'_s(G) \leqslant 6$.

(5) If G is bipartite and $C_4 \not\subset G$, then $\chi'_s(G) \leqslant 7$.

(6) If G is bipartite and its girth is large, then $\chi'_s(G) \leqslant 5$.

Several of these conjectures have been verified, including (1) by Andersen [2] and (2) by Steger and Yu [20]. Quite recently, Kostochka, Li, Ruksasakchai, Santana, Wang, and Yu [15] announced an affirmative resolution to (3). This result is best possible since the prism, shown in Figure 1, is a subcubic planar graph with $\chi'_s(G) = 9$. Case (4) was also solved [16, 17, 24].

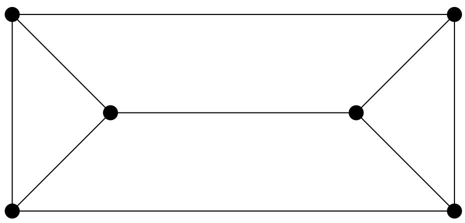


Figure 1: The prism is a subcubic planar graph G with $\chi'_s(G) = 9$.

Several papers prove sharper bounds on the strong chromatic index of planar graphs with additional structure [11, 12, 13, 14], generally by introducing conditions on maximum average degree or girth to ensure that the target graph is sufficiently sparse. For a graph G, the *maximum average degree* of G, denoted $\text{mad}(G)$, is the maximum of average degrees over all subgraphs of G. Hocquard, Montassier, Raspaud, and Valicov [12, 13] proved the following.

Theorem 3 (Hocquard *et al.* [13]). *Let G be a subcubic graph.*

1. If $\text{mad}(G) < \frac{7}{3}$, *then* $\chi'_s(G) \leqslant 6$.

2. If $\text{mad}(G) < \frac{5}{2}$, *then* $\chi'_s(G) \leqslant 7$.

3. If $\text{mad}(G) < \frac{8}{3}$, *then* $\chi'_s(G) \leqslant 8$.

Parts (1) and (2) of Theorem 3 are sharp by the graphs shown in Figures 2 and 3, respectively. An elementary application of Euler's Formula (see [23]) gives the following.

Proposition 4. *If G is a planar graph with girth g then* $\text{mad}(G) < \frac{2g}{g-2}$.

Theorem 3 and Proposition 4 yield the following corollary.

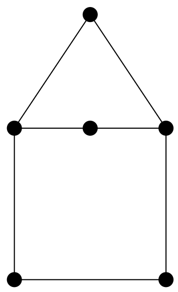


Figure 2: A graph G with $\mathrm{mad}(G) = \frac{7}{3}$ and $\chi'_s(G) > 6$.

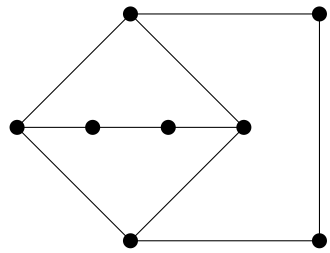


Figure 3: A graph G with $\mathrm{mad}(G) = \frac{5}{2}$ and $\chi'_s(G) > 7$.

Corollary 5 (Hocquard *et al.* [13]). *Let G be a subcubic planar graph with girth g.*

1. *If $g \geqslant 14$, then $\chi'_s(G) \leqslant 6$.*
2. *If $g \geqslant 10$, then $\chi'_s(G) \leqslant 7$.*
3. *If $g \geqslant 8$, then $\chi'_s(G) \leqslant 8$.*

Note that no non-trivial sparsity condition on a graph G with maximum degree d will guarantee that $\chi'_s(G) < 2d - 1$ since every graph H with two adjacent vertices of degree d has $\chi'_s(H) \geqslant 2d - 1$. We give sparsity conditions that imply a subcubic planar graph has strong chromatic index at most five and a subquartic planar graph has strong chromatic index at most seven. Previous work in this direction was initiated by Borodin and Ivanova [3], Chang, Montassier, Pěcher, and Raspaud [6], and most recently extended by Wang and Zhao [22]. The current-best bounds are given by the following two results.

Theorem 6 (Borodin and Ivanova [3]). *Let G be a subcubic graph.*

1. *If G has girth at least 9 and $\mathrm{mad}(G) < 2 + \frac{2}{23}$, then $\chi'_s(G) \leqslant 5$.*
2. *If G is planar and has girth at least 41, then $\chi'_s(G) \leqslant 5$.*

Theorem 7 (Wang and Zhao [22]). *Fix $d \geqslant 4$ and let G be a graph with $\Delta(G) \leqslant d$.*

1. *If G has girth at least $2d - 1$ and $\mathrm{mad}(G) < 2 + \frac{2}{6d-7}$, then $\chi'_s(G) \leqslant 2d - 1$.*
2. *If G is planar and has girth at least $10d - 4$, then $\chi'_s(G) \leqslant 2d - 1$.*

One barrier to proving sparsity conditions that imply $\chi'_s(G) \leqslant 5$ is that there exist graphs G with $\mathrm{mad}(G) = 2$ and $\chi'_s(G) = 6$. Let S_3 be a triangle with pendant edges at each vertex, and let S_4 be a 4-cycle with pendant edges at two adjacent vertices. For $k \geqslant 5$, let S_k be a k-cycle with pendant edges at each vertex. Each of S_3, S_4 and S_7 have maximum average degree 2 and strong chromatic index at least 6, see Figure 4. However, these graphs are 6-critical with respect to $\chi'_s(G)$, as the removal of any edge from S_3, S_4 or S_7 results in a graph that has a strong edge-coloring using five colors.

Our main theorem demonstrates that if these few graphs are avoided, and the maximum average degree is not too large, then we can find a strong 5-edge-coloring, improving Theorem 6.

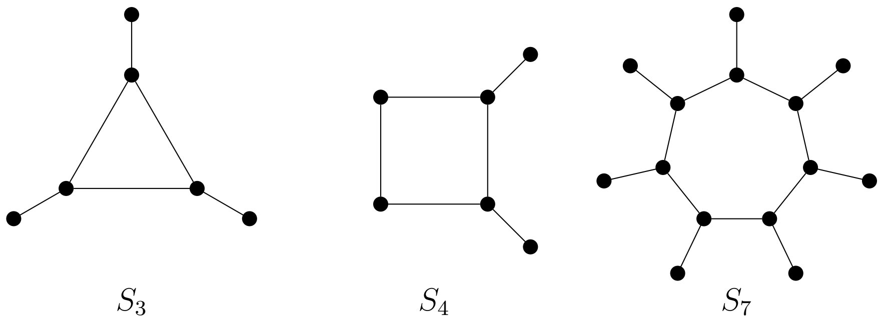


Figure 4: Exceptions in Theorem 8.

Theorem 8. *Let G be a subcubic graph.*

1. *If G does not contain S_3, S_4, or S_7 and* $\mathrm{mad}(G) < 2 + \frac{1}{7}$, *then* $\chi'_s(G) \leqslant 5$.
2. *If G is planar and has girth at least* 30, *then* $\chi'_s(G) \leqslant 5$.

The bound in Theorem 8 is likely not sharp, but is close to optimal. The graph in Figure 5 is subcubic, avoids S_3, S_4, and S_7, and satisfies both $\chi'_s(G) = 6$ and $\mathrm{mad}(G) = 2 + \frac{1}{6}$.

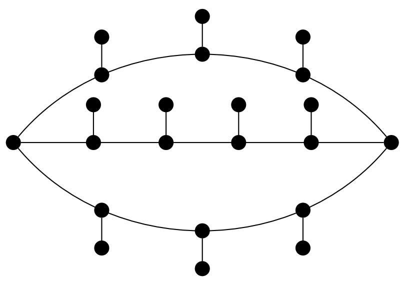


Figure 5: The graph $G = \mathrm{ex}_3(\Theta_{3,3,4})$ with $\mathrm{mad}(G) = 2 + \frac{1}{6}$ and $\chi'_s(G) = 6$.

Using similar methods, we improve the bounds in Theorem 7 when $d = 4$.

Theorem 9. *Let G be a subquartic graph.*

1. *If G has girth at least* 7 *and* $\mathrm{mad}(G) < 2 + \frac{2}{13}$, *then* $\chi'_s(G) \leqslant 7$.
2. *If G is planar and has girth at least* 28, *then* $\chi'_s(G) \leqslant 7$.

We also consider a list variation of the strong chromatic index of G, first introduced by Vu [21]. A *strong list edge-coloring* of a graph G is an assignment of lists to $E(G)$ such that a strong edge-coloring can be chosen from the lists at each edge. The minimum k such that a graph G can be strongly list edge-colored using any lists of size at least k on each edge is the *strong list chromatic index* of G, denoted $\chi'_{\ell,s}(G)$. Borodin and Ivanova [3] asked if there are sparsity conditions that imply $\chi'_{\ell,s}(G) \leqslant 2d-1$ for a planar graph G with maximum degree d. We generalize the bounds in Theorem 6 to apply to list coloring.

Theorem 10. *Let G be a subcubic graph.*

1. *If G has girth at least 9 and $\mathrm{mad}(G) < 2 + \frac{2}{23}$, then $\chi'_{\ell,s}(G) \leqslant 5$.*
2. *If G is planar and has girth at least 41, then $\chi'_{\ell,s}(G) \leqslant 5$.*

The proofs of Theorems 8, 9, and 10 use the discharging method. We begin by proving Theorem 10 in Section 2 as the proof is shorter and the one reducible configuration is used again in the proof of Theorem 8 in Section 3.

1.1 Preliminaries and Notation

Throughout this paper we will only consider simple, finite, undirected graphs. We refer to [23] for any undefined definitions and notation. A graph G has vertex set $V(G)$, edge set $E(G)$, and maximum degree $\Delta(G)$.

If a vertex v has degree j we refer to it as a *j-vertex*, and if v has a neighbor that is a j-vertex, we say it is a *j-neighbor* of v. When G is planar let $F(G)$ denote the set of faces of G, and $\ell(f)$ denote the length of a face f. The *girth* of a graph G is length of its shortest cycle. A graph G is $\{a, b\}$-regular if for every v in G, the degree of v is either a or b. Every graph G with maximum degree d is contained in a prescribed $\{1, d\}$-regular graph, denoted $\mathrm{ex}_d(G)$, the *d-expansion* of G. To construct $\mathrm{ex}_d(G)$, add $d - d(v)$ pendant edges to each vertex v in G where $d(v) \in \{2, \ldots, d\}$. Additionally, let the *contracted graph* of G, denoted by $\mathrm{ct}(G)$, be the graph obtained by deleting all 1-vertices of G. A vertex v in G is a *$2^\perp$-vertex* if v is a 2-vertex in $\mathrm{ct}(G)$. Thus, for the remainder of the paper a vertex v is a *k^+-vertex* in G if it has degree at least k in $\mathrm{ct}(G)$.

We will make use of the discharging method for some of our results. For an introduction to this method, see the survey by Cranston and West [7]. We will directly use two standard results that can be proven using this method. Both of Theorems 6 and 7 rely on Lemmas 11 and 12.

Let G be a graph and $\mathrm{ct}(G)$ be the contracted graph. An *ℓ-thread* is a path $v_1 \ldots v_\ell$ in $\mathrm{ct}(G)$ where each v_i is a $2^\perp$-vertex.

Lemma 11 (Cranston and West [7])**.** *If G is a graph with girth at least $\ell+1$ and $\mathrm{mad}(G) < 2 + \frac{2}{3\ell-1}$, then $\mathrm{ct}(G)$ contains a 1-vertex or an ℓ-thread.*

Lemma 12 (Nešetřil, Raspaud, and Sopena [19])**.** *If G is a planar graph with girth at least $5\ell + 1$, then $\mathrm{ct}(G)$ contains a 1-vertex or an ℓ-thread.*

2 Strong List Edge-Coloring of Subcubic Graphs

In this section, we prove Theorem 10. Our proof uses the discharging method, wherein we assign an initial charge to the vertices and faces of a theoretical minimal counterexample. This initial charge is then disbursed according to a set of discharging rules in order to draw a contradiction to the existence of such a minimal counterexample. We will often make use of the following, which is another simple and well known application of Euler's Formula.

Proposition 13. *In a planar graph G,*

$$\sum_{f\in F(G)} (\ell(f)-6) + \sum_{v\in V(G)} (2d(v)-6) = -12.$$

We will also use the Combinatorial Nullstellensatz, which will be applied to show we can extend certain list colorings.

Theorem 14 (Combinatorial Nullstellensatz [1])**.** *Let f be a polynomial of degree t in m variables over a field $\mathbb{F}$. If there is a monomial $\prod x_i^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero in $\mathbb{F}$, then f is nonzero at some point of $\prod S_i$, where each S_i is a set of t_i+1 distinct values in $\mathbb{F}$.*

Theorem 10(2) follows from the following strengthened theorem.

Theorem 15. *Let G be a planar $\{1,3\}$-regular graph of girth at least 41, and let $p \in V(G)$. Assign distinct colors to the edges incident to p and let L be a 5-list-assignment to the remaining edges of G. There exists a strong edge-coloring c where $c(e) \in L(e)$ for all $e \in E(G)$.*

Proof. For the sake of contradiction, select G, p, c, and L as in the theorem statement, and assume there does not exist a strong edge-coloring of $E(G)$ using colors from L. In this selection, minimize $n(G)$. Note that G is connected and $e(G) > 5$. We can further assume that $d(p) > 1$, since if $d(p) = 1$ and $\{p'\} = N(p)$ then we can instead color the edges incident to p'.

Lemma 16. *There does not exist a cut-edge uv such that $d(u) = d(v) = 3$.*

Proof. Suppose that G contains a cut-edge uv with $d(u) = d(v) = 3$. There are exactly two components in $G-uv$, call them G_1 and G_2, with $u \in V(G_1)$ and $v \in V(G_2)$. Without loss of generality, $p \in V(G_1)$. For each $i \in \{1,2\}$, let $G'_i = G_i + uv$.

Since $d(v) = 3$, $n(G'_1) < n(G)$. Thus there is a strong edge-coloring of G'_1 using the 5-list-assignment L. Next, color the other two edges incident to v using colors distinct from those on the edges incident to u. Now, G'_2 is a subcubic planar graph of girth at least 41 with distinctly colored edges about the vertex v and $n(G'_2) < n(G)$. Thus, there is an extension of the coloring to G'_2.

The colorings of G'_1 and G'_2 form a strong edge-coloring of G, a contradiction. □

Define a *k-caterpillar* to be a k-thread $v_1, \dots, v_k$ in G where $p \notin \{v_1, \dots, v_k\}$. Figure 6 is an 8-caterpillar.

Lemma 17. *$G-p$ does not contain an 8-caterpillar.*

Proof. We will show that if $G-p$ contains an 8-caterpillar, then G has a strong edge L-coloring. If $v_1, \dots, v_8$ form an 8-caterpillar, then let v'_i be the 1-vertex adjacent to v_i, v_0 and v_9 be the other neighbors of v_1 and v_8. For $i \in \{0,9\}$, let v'_i and u'_i be the neighbors of v_i other than v_1 or v_8.

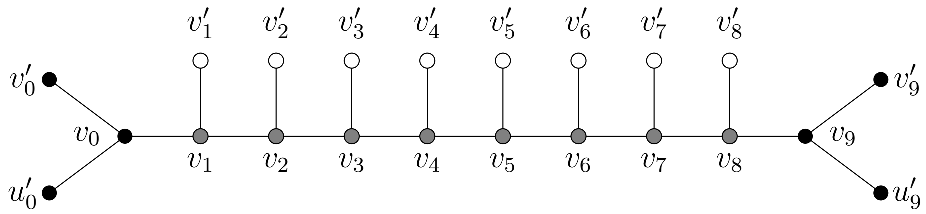


Figure 6: An 8-caterpillar.

By removing all edges incident to $v_2, \ldots, v_7$ and $v'_1, \ldots, v'_8$, as well as any isolated vertices that are produced, we obtain a graph G' with fewer vertices than G, so we can strongly edge-color G' with 5 colors. We fix such a coloring of G' and generate a contradiction by extending this coloring to a strong edge-coloring of G. Suppose that $c_1, \ldots, c_6$ are the colors of the edges incident to the vertices v_0 and v_9, and assign variables $y_1, \ldots, y_8$ to the pendant edges, and variables $x_1, \ldots, x_7$ to the interior edges as shown in Figure 7.

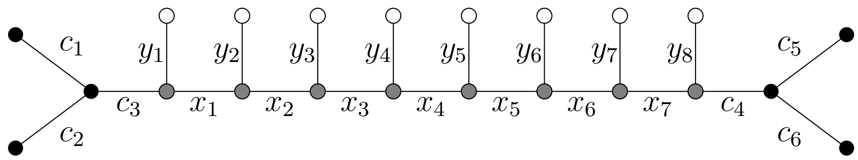


Figure 7: The assignment of colors and variables to the 8-caterpillar.

Identifying the conflicts between variables and colors produces the following polynomial,

$$\begin{aligned} f(y_1, \ldots, y_8, x_1, \ldots, x_7) = {} & (y_2 - c_3)(x_2 - c_3)(y_7 - c_4)(x_6 - c_4) \\ & \cdot \prod_{i=1}^{3}(x_1 - c_i) \prod_{i=1}^{3}(y_1 - c_i) \prod_{i=4}^{6}(x_7 - c_i) \prod_{i=4}^{6}(y_8 - c_i) \\ & \cdot \prod_{j-i\in\{1,2\}} (x_i - x_j) \prod_{j-i=1} (y_i - y_j) \prod_{i-j\in\{-1,0,1,2\}} (y_i - x_j). \end{aligned}$$

We will use the Combinatorial Nullstellensatz to show that there is an assignment of colors $\hat{c}_1, \ldots, \hat{c}_8$ and $c'_1, \ldots, c'_7$ such that $f(\hat{c}_1, \ldots, \hat{c}_8, c'_1, \ldots, c'_7) \neq 0$. Such an assignment of colors would extend the inductive coloring of $G - p$ to a strong edge-coloring of G. If the coefficient of

$$(x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7\ y_8)^4$$

is nonzero, then there are values from L for $x_1, \ldots, x_7, y_1, \ldots, y_8$ such that f is nonzero by Theorem 14. Using the Magma algebra system [4], this monomial has coefficient -2, and thus there is a strong edge-coloring using the 5-list assignment[1]. Thus, the 8-caterpillar does not exist in a vertex minimal counterexample. □

[1]All source code and data is available at http://www.math.iastate.edu/dstolee/r/scindex.htm.

Note that the proof in Lemma 17 cannot be extended to exclude a 7-caterpillar in G, as there exists a 5-coloring of the external edges that does not extend to the caterpillar, even when the lists are all the same.

To complete the proof, we apply a discharging argument[2] to $\mathrm{ct}(G)$. First, observe that by Lemma 16, $\mathrm{ct}(G)$ is 2-connected and so every face is a simple cycle of length at least 41. Also observe that by Lemma 17, $\mathrm{ct}(G)$ does not contain a path of length 8 where every vertex is of degree 2, unless one of those vertices is p.

Assign charge $2d(v) - 6$ to every vertex $v \neq p$, charge $\ell(f) - 6$ to every face f, and charge $2d(p) + 5$ to p. By Proposition 13, the total amount of charge on $\mathrm{ct}(G)$ is -1. Apply the following discharging rules.

(R1) For every $v \in G - p$, if v is a 2–vertex, v pulls 1 from each incident face.

(R2) If p is a 2–vertex, then p gives $\frac{9}{2}$ to each incident face.

Observe that every vertex has nonnegative final charge after this discharging process. It remains to show that every face has nonnegative charge.

Let f be a face, and let r_2 be the number of 2–vertices on the boundary of f, not counting p, and consider two cases.

Case 1: $d(p) = 3$ or p is not adjacent to f.

In this case, p does not give charge to f, and therefore f has charge $\ell(f) - r_2 - 6$ after discharging. Also, the boundary of f does not contain a path of length 8 containing only vertices of degree 2, thus $r_2 \leqslant \left\lfloor \frac{7}{8}\ell(f) \right\rfloor$. Since $\ell(f) \geqslant 41$, we have

$$\ell(f) - r_2 - 6 \geqslant \ell(f) - \left\lfloor \frac{7}{8}\ell(f) \right\rfloor - 6 \geqslant 0.$$

Case 2: $d(p) = 2$ and p is adjacent to f.

By (R2), p gives charge $\frac{9}{2}$ to f, so that f has charge $\ell(f) - r_2 - \frac{3}{2}$ after discharging. The boundary of f does not contain a path of length 8 containing only vertices of degree 2, except when using p, so, $r_2 \leqslant \left\lfloor \frac{7}{8}\ell(f) \right\rfloor$. Since $\ell(f) \geqslant 41$, we have

$$\ell(f) - r_2 - \frac{3}{2} \geqslant \ell(f) - \left\lfloor \frac{7}{8}\ell(f) \right\rfloor - \frac{3}{2} \geqslant 0.$$

Thus, all vertices and faces have nonnegative charge, contradicting Proposition 13. □

Theorem 10(1) follows quickly from Lemmas 11 and 17.

Proof of Theorem 10 (1). Suppose that G is a minimal counterexample, and L is a 5-list-assignment on the edges of G that cannot be strongly colored. For $\ell = 8$, we have $\mathrm{girth}(G) \geqslant \ell + 1$ and $\mathrm{mad}(G) < 2 + \frac{2}{3\ell - 1}$, so Lemma 11 implies that $ct(G)$ contains a 1-vertex or an 8-thread.

[2] Our discharging approach is similar to the proof of Lemma 12 where $\ell = 8$, but some care is needed due to the precolored vertex p.

If $\mathrm{ct}(G)$ contains a 1-vertex v, then let S be the set of 1-vertices adjacent to v; there is at least one or v would be pruned from $\mathrm{ct}(G)$. Note also that $|S| \in \{1, 2\}$. By minimality of G, there is an L-coloring of $E(G - S)$. For each $u \in S$, the edge uv has five possible colors and at most three edges at distance at most two that have assigned colors. Thus, there are at most two uncolored edges and each have at least two available colors, giving a strong L-coloring of $E(G)$.

Therefore, $\mathrm{ct}(G)$ has an 8-thread, which gives an 8-caterpillar in $\mathrm{ex}_2(G)$. Observe that the proof of Lemma 17 demonstrates that a strong L-coloring of $E(G)$ exists, a contradiction. $\square$

3 Strong Edge-Coloring of Sparse Graphs

In this section, we prove Theorems 8 and 9.

Let G be a graph with maximum degree $\Delta(G) \leqslant d$. For a vertex v in $\mathrm{ct}(G)$ denote by $N_3(v)$ the set of 3^+-vertices u where $\mathrm{ct}(G)$ contains a path P from u to v where all internal vertices of P are $2^\perp$-vertices. For $u \in N_3(v)$, let $\mu(v, u)$ be the number of paths from v to u whose internal vertices have degree 2 in $\mathrm{ct}(G)$. For a 3-vertex v, let the *responsibility set*, denoted $\mathrm{Resp}(v)$, be the set of $2^\perp$-vertices that appear on the paths between v and the vertices in $N_3(v)$.

Let D be a subgraph of G. We call D a *k-reducible configuration* if there exists a subgraph D' of D such that any strong k-edge-coloring of $G - D'$ can be extended to a strong k-edge-coloring of G. One necessary property for the selection of D' is that no two edges that remain in $G - D'$ can have distance at most two in G but distance strictly larger than two in $G - D'$. In the next subsection we describe several reducible configurations.

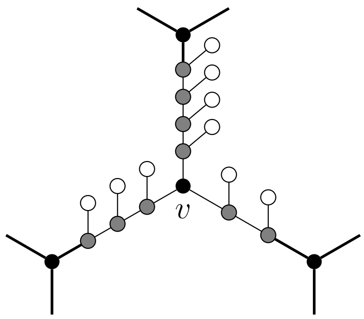
3.1 Reducible Configurations

This subsection contains description of four types of reducible configurations. Each configuration is described in terms of how it appears within $\mathrm{ct}(G)$ where G is a graph with maximum degree $\Delta(G) \leqslant d$ for some $d \geqslant 4$.

Let t be a positive integer. The *t-caterpillar* is formed by two 3^+-vertices v_0 and v_{t+1} with a path $v_0 v_1 \dots v_t v_{t+1}$ where each v_i is a $2^\perp$-vertex for every $i \in \{1, \dots, t\}$.

Let $t_1, \dots, t_k$ be nonnegative integers. A configuration $Y(t_1, \dots, t_k)$ is formed by a k^+-vertex v and k internally disjoint paths of lengths $t_1 + 1, \dots, t_k + 1$ with v as a common endpoint, where the internal vertices of the paths are $2^\perp$-vertices. We call such configuration a *Y-type configuration about v*, see Figure 8.

A configuration $H(t_1, t_2; r; s_1, s_2)$ is formed by two 3-vertices u and v and 5 internally disjoint paths of lengths $t_1 + 1$, $t_2 + 1$, $r + 1$, $s_1 + 1$, and $s_2 + 1$, where the internal vertices of the paths are $2^\perp$-vertices. The paths of lengths $t_1 + 1$ and $t_2 + 1$ have v as an endpoint, the path of length $r + 1$ has u and v as endpoints and the paths of lengths $s_1 + 1$ and $s_2 + 1$ have u as an endpoint. We call such configuration an *H-type configuration about v and u*, see Figure 9.



$Y(2,3,4)$

Figure 8: The configuration $Y(t_1, t_2, t_3)$.

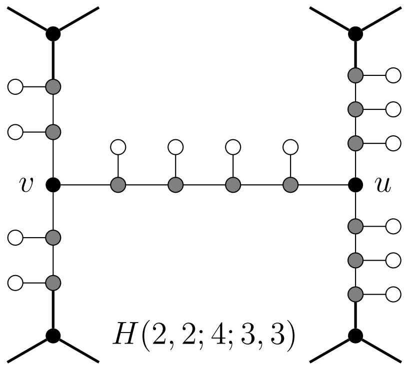
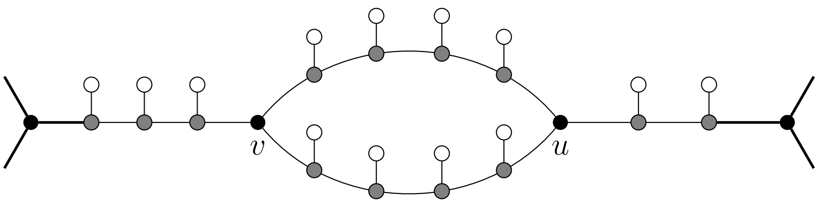


Figure 9: The configuration $H(t_1, t_2; r; s_1, s_2)$.

A configuration $\Phi(t, a_1, a_2, s)$ is formed by two 3-vertices u and v and 4 internally disjoint paths of lengths $t+1$, a_1+1, a_2+1, and $s+1$, where the internal vertices of the paths are $2^{\perp}$-vertices. The path of length $t+1$ has v as an endpoint, the paths of lengths a_1+1 and a_2+1 have u and v as endpoints and the path of length $s+1$ has u as an endpoint. We call such configuration a Φ-*type configuration about* v *and* u, see Figure 10.



$\Phi(3,4,4,2)$

Figure 10: The configuration $\Phi(t, a_1, a_2, s)$.

The reducibility of these configurations was verified using computer[3], and in addition

[3]All source code and data is available at http://www.math.iastate.edu/dstolee/r/scindex.htm and http://www.combinatorics.org/ojs/index.php/eljc/article/view/v25i3p18/html.

the 8-caterpillar is addressed in Lemma 17. Given the definition of a $2^{\perp}$-vertex, the vertices of degree two in these configurations may, or may not, be adjacent to some 1-vertices in G. We demonstrate the reducibility of the instances of these configurations wherein each vertex of degree 2 is adjacent to $d-2$ 1-vertices, as depicted in Figures 8–10. This suffices to address all other instances of these configurations that may occur.

Claim 18. *The following caterpillars with maximum degree d are reducible:*

1. *(Borodin and Ivanova [3]) For $d = 3$, the 8-caterpillar is 5-reducible.*
2. *(Wang and Zhao [22]) For $d \geqslant 4$, the $(2d-2)$-caterpillar is $(2d-1)$-reducible.*

These caterpillars are likely the smallest that are reducible for each degree d. Thus, the bounds in Theorems 6 and 7 are best possible using only Lemma 12. To improve these bounds, we demonstrate larger reducible configurations and use a more complicated discharging argument.

Claim 19. *The following configurations with maximum degree 3 are 5-reducible:*

1. $Y(1,6,7)$, $Y(2,5,6)$ *and* $Y(3,4,5)$.
2. $H(7,7;0;3,7)$, $H(7,7;0;4,6)$, $H(7,7;0;5,5)$, $H(6,7;0;3,7)$, $H(6,7;0;4,6)$, $H(6,7;0;5,5)$, $H(6,6;1;2,7)$, $H(6,6;1;3,6)$, $H(6,6;1;4,5)$, $H(5,7;1;2,7)$, $H(5,7;1;3,6)$, $H(5,7;1;4,5)$, $H(4,7;2;1,7)$, $H(4,7;2;2,6)$, $H(4,7;2;3,5)$, $H(4,7;2;4,4)$, $H(3,7;3;1,6)$, $H(3,7;3;2,5)$ *and* $H(3,7;3;3,4)$.
3. $\Phi(7,0,7,1)$, $\Phi(7,0,6,1)$, $\Phi(6,0,7,1)$, $\Phi(6,1,6,1)$, $\Phi(7,1,5,1)$, $\Phi(5,1,7,1)$, $\Phi(7,2,4,1)$, $\Phi(4,2,7,1)$, $\Phi(7,3,3,1)$, $\Phi(3,3,7,1)$ *and* $\Phi(3,7,0,7)$.

Claim 20. *The following configurations with maximum degree 4 are 7-reducible:*

$$Y(2,4,4),\ Y(1,5,5),\ Y(2,4,5),\ Y(3,4,4),\ \text{and } Y(2,5,5).$$

3.2 Proof of Theorem 8

Proof. Among graphs G with $\mathrm{mad}(G) < 2+\frac{1}{7}$ not containing S_3, S_4, or S_7, with $\chi'_s(G) > 5$, select G while minimizing the number of vertices in $\mathrm{ct}(G)$. Note that $e(G) > 5$ since $\chi'_s(G) > 5$, and let n be the number of vertices in $\mathrm{ct}(G)$. By using the discharging method, we will show that $\mathrm{mad}(\mathrm{ct}(G)) \geqslant 2 + \frac{1}{7}$, which is a contradiction, so no such minimal counterexample exists.

Observe that G does not contain any of the reducible configurations addressed in Claim 19. We also have the following additional structure on $\mathrm{ct}(G)$.

Lemma 21. $\mathrm{ct}(G)$ *is 2-connected.*

Proof. Suppose that $\mathrm{ct}(G)$ contains a cut-edge uv. In G, the vertices u and v have degree at least two. There are exactly two components, G_1 and G_2, in $G - uv$, with $u \in V(G_1)$ and $v \in V(G_2)$. Let u_1, u_2 be neighbors of u in G_1 and v_1, v_2 be neighbors of v in G_2; let $u_1 = u_2$ only when u has a unique neighbor in G_1, and $v_1 = v_2$ only when v has a unique neighbor in G_2. Let $G'_1 = G_1 + \{uv, vv_1, vv_2\}$ and $G'_2 = G_2 + \{uv, uu_1, uu_2\}$.

If $G'_1 = G$, then consider $G' = G - v_1 - v_2$. Since $n(G') < n(G)$ and $\mathrm{mad}(G') \leqslant \mathrm{mad}(G)$, there is a strong 5-edge-coloring c of G'. Extend the coloring c to color $c(vv_1)$ and $c(vv_2)$ from the colors not in $\{c(uv), c(uu_1), c(uu_2)\}$, a contradiction. We similarly reach a contradiction when $G'_2 = G$.

Therefore, $n(G'_i) < n(G)$ and $\mathrm{mad}(G'_i) \leqslant \mathrm{mad}(G)$ for each $i \in \{1, 2\}$. Thus, there exist strong 5-edge-colorings c_1 and c_2 of G'_1 and G'_2, respectively. For each coloring, the colors on the edges $uv, uu_1, uu_2, vv_1, vv_2$ are distinct. Let π be a permutation of the five colors satisfying $\pi(c_2(e)) = c_1(e)$ for each edge $e \in \{uv, uu_1, uu_2, vv_1, vv_2\}$. Then, we extend the coloring c_1 of G'_1 to all of G by assigning $c_1(e) = \pi(c_2(e))$ for all edges $e \in E(G'_2)$. The coloring c_1 is a strong 5-edge-coloring of G, a contradiction. □

If $\mathrm{ct}(G)$ does not have any 3-vertices, then $\mathrm{ct}(G)$ must be isomorphic to cycle C_n. If $n \geqslant 9$, then $\mathrm{ex}_3(G)$ contains an 8-caterpillar. If $n \in \{5, 6, 8\}$, then G is a subgraph of S_5, S_6, or S_8, which each has a strong edge-coloring using five colors, discovered by computer. When $n \in \{3, 4, 7\}$, G does not contain S_3, S_4, or S_7, and any proper subgraph of these graphs is 5 strong edge-colorable, discovered by computer. Therefore, $\mathrm{ct}(G)$ is not isomorphic to a cycle, and hence for every $2^{\perp}$-vertex u in G, $|N_3(u)| \geqslant 1$.

If G has some vertex v such that $|N_3(v)| = 1$, then G must be a subgraph of $\Theta(t_1, t_2, t_3)$, which is the graph consisting of three internally disjoint $x - y$ paths of length $t_1 + 1, t_2 + 1$ and $t_3 + 1$, for some $0 \leqslant t_1 \leqslant t_2 \leqslant t_3$.

If $t_3 \geqslant 8$, then $\mathrm{ex}_3(G)$ contains an 8-caterpillar, so we assume that $t_3 < 8$. Observe that if $\mathrm{mad}(\Theta(t_1, t_2, t_3)) < 2 + \frac{1}{7}$, then $t_1 + t_2 + t_3 \geqslant 13$. However, if $\Theta(t_1, t_2, t_3)$ does not contain a reducible Y-type configuration, then by Claim 19 the sequence (t_1, t_2, t_3) is one of $(0, 7, 7)$, $(0, 6, 7)$, $(1, 6, 6)$, $(1, 5, 7)$, $(2, 5, 6)$, $(2, 4, 7)$, or $(3, 3, 7)$. In each of these cases, we have verified by computer that $\Theta(t_1, t_2, t_3)$ has a strong edge-coloring using five colors. The colorings are depicted in Figure 11.

Therefore, $|N_3(v)| \geqslant 2$ for every $v \in \mathrm{ct}(G)$. We proceed using discharging. Assign each vertex initial charge $d(v)$. Note that the total charge on the graph is $2e(\mathrm{ct}(G))$, which is at most $\mathrm{mad}(G)n < (2 + \frac{1}{7})n$. We shall distribute charge among the vertices of $\mathrm{ct}(G)$ and result with charge at least $2 + \frac{1}{7}$ on every vertex, giving a contradiction.

Distribute charge among the vertices according to the following discharging rules, applied to each pair of vertices $u, v \in V(\mathrm{ct}(G))$:

(R1) If u is a 2-vertex and $v \in N_3(u)$, then v sends $\frac{1}{14}$ to u.

(R2) If v is a 3-vertex with $|\mathrm{Resp}(v)| \leqslant 10$ and $u \in N_3(v)$, then

- (a) if $d(u, v) = 1$ and $|\mathrm{Resp}(u)| = 14$, then v sends $\frac{1}{7}$ to u;
- (b) otherwise, if $d(u, v) \leqslant 4$, then v sends $\frac{1}{14}$ to u.

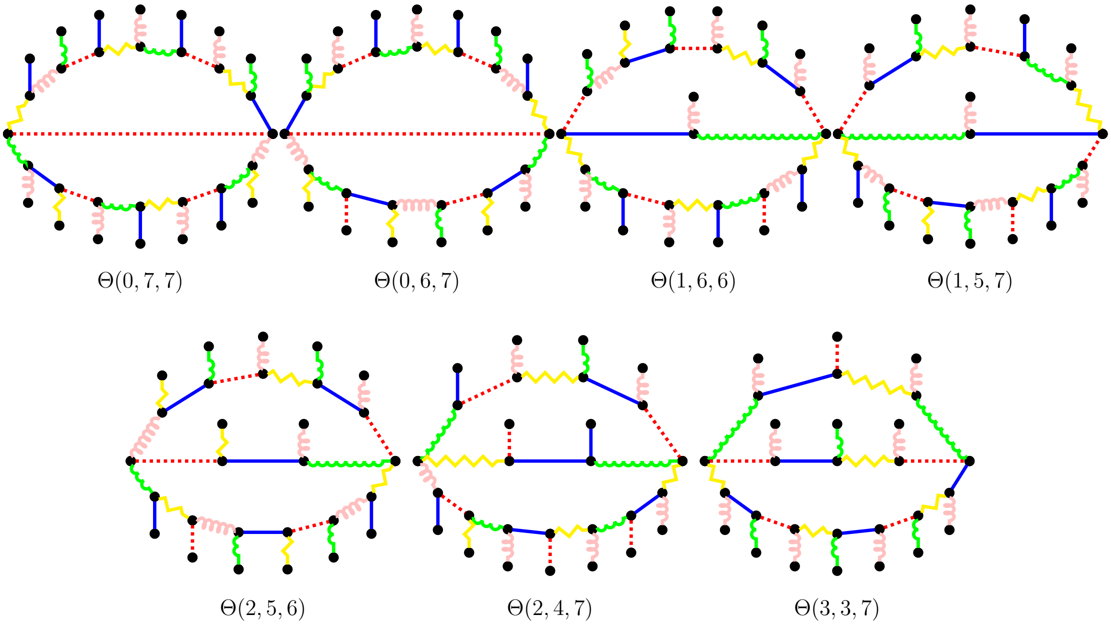


Figure 11: Strong edge-colorings of $\Theta(t_1, t_2, t_3)$. Style of line corresponds to color to make colors visible on a black and white print.

We will now verify the assertion that each vertex has final charge at least $2 + \frac{1}{7}$. If v is a 2-vertex, then since $|N_3(v)| = 2$ the final charge on v is $2 + \frac{1}{7}$ after by (R1). Let v be a 3-vertex. If $u \in N_3(v)$, then $d(u, v) \leqslant 8$ by Lemma 17. Claim 19 implies that $|\operatorname{Resp}(v)| \leqslant 14$.

Case 1: $|\operatorname{Resp}(v)| \in \{11, 12\}$. In this case, v only loses charge by (R1), so the final charge is at least $3 - \frac{12}{14} = 2 + \frac{1}{7}$.

Case 2: $|\operatorname{Resp}(v)| = 14$. By Claim 19, the Y-type configuration about v is $Y(0, 7, 7)$. Thus, some vertex $u_1 \in N_3(v)$ is at distance one from v. If $\mu(v, u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(7, 7; 0; s_1, s_2)$; by Claim 19 $s_1 + s_2 \leqslant 9$, $|\operatorname{Resp}(u_1)| \leqslant 9$, and u_1 sends $\frac{1}{7}$ to v by (R2a). If $\mu(v, u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form $\Phi(7, 0, 7, s)$; by Claim 19 $s = 0$, $|\operatorname{Resp}(u_1)| \leqslant 7$, and u_1 sends $\frac{1}{7}$ to v by (R2a).

Case 3: $|\operatorname{Resp}(v)| = 13$. By Claim 19, the Y-type configuration $Y(t_1, t_2, t_3)$ about v is one of $Y(0, 6, 7)$, $Y(1, 6, 6)$, $Y(1, 5, 7)$, $Y(2, 4, 7)$, or $Y(3, 3, 7)$. We consider each case separately.

Case 3.i: $(t_1, t_2, t_3) = (0, 6, 7)$. Let u_1 be the vertex in $N_3(v)$ at distance 1 from v. If $\mu(v, u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(6, 7; 0; s_1, s_2)$; by Claim 19 $s_1 + s_2 \leqslant 9$, $|\operatorname{Resp}(u_1)| \leqslant 9$, and u_1 sends $\frac{1}{14}$ to v by (R2b). If $\mu(v, u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form

$\Phi(6,0,7,s)$ or $\Phi(7,0,6,s)$; by Claim 19 $s=0$, $|\operatorname{Resp}(u_1)| \leqslant 7$, and u_1 sends $\frac{1}{14}$ to v by (R2b).

Case 3.ii: $(t_1,t_2,t_3) = (1,6,6)$. Let u_1 be the vertex in $N_3(v)$ at distance 2 from v. If $\mu(v,u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(6,6;1;s_1,s_2)$; by Claim 19 $s_1+s_2 \leqslant 8$, $|\operatorname{Resp}(u_1)| \leqslant 9$, and u_1 sends $\frac{1}{14}$ to v by (R2b). If $\mu(v,u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form $\Phi(6,1,7,s)$ or $\Phi(7,1,6,s)$; by Claim 19 $s=0$, $|\operatorname{Resp}(u_1)| \leqslant 8$, and u_1 sends $\frac{1}{14}$ to v by (R2b).

Case 3.iii: $(t_1,t_2,t_3) = (1,5,7)$. Let u_1 be the vertex in $N_3(v)$ at distance 2 from v. If $\mu(v,u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(5,7;1;s_1,s_2)$; by Claim 19 $s_1+s_2 \leqslant 8$, $|\operatorname{Resp}(u_1)| \leqslant 9$, and u_1 sends $\frac{1}{14}$ to v by (R2b). If $\mu(v,u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form $\Phi(5,1,7,s)$ or $\Phi(7,1,5,s)$; by Claim 19 $s=0$, $|\operatorname{Resp}(u_1)| \leqslant 8$, and u_1 sends $\frac{1}{14}$ to v by (R2b).

Case 3.iv: $(t_1,t_2,t_3) = (2,4,7)$. Let u_1 be the vertex in $N_3(v)$ at distance 3 from v. If $\mu(v,u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(4,7;2;s_1,s_2)$; by Claim 19 $s_1+s_2 \leqslant 7$, $|\operatorname{Resp}(u_1)| \leqslant 9$, and u_1 sends $\frac{1}{14}$ to v by (R2b). If $\mu(v,u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form $\Phi(4,2,7,s)$ or $\Phi(7,2,4,s)$; by Claim 19 $s=0$, $|\operatorname{Resp}(u_1)| \leqslant 8$, and u_1 sends $\frac{1}{14}$ to v by (R2b).

Case 3.v: $(t_1,t_2,t_3) = (3,3,7)$. Let u_1 be the vertex in $N_3(v)$ at distance 4 from v. If $\mu(v,u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(3,7;3;s_1,s_2)$; by Claim 19 $s_1+s_2 \leqslant 7$, $|\operatorname{Resp}(u_1)| \leqslant 10$, and u_1 sends $\frac{1}{14}$ to v by (R2b). If $\mu(v,u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form $\Phi(3,3,7,s)$ or $\Phi(7,3,3,s)$; by Claim 19 $s=0$, $|\operatorname{Resp}(u_1)| \leqslant 10$, and u_1 sends $\frac{1}{14}$ to v by (R2b).

Case 4: $|\operatorname{Resp}(v)| \leqslant 10$. In this case, v loses charge at most $\frac{10}{14}$ by (R1), so if it sends at most $\frac{1}{7}$ by (R2), then the final charge on v is at least $2+\frac{1}{7}$. Consider how much is sent by (R2).

Case 4.i: v sends $\frac{3}{14}$ by (R2). If $|\operatorname{Resp}(v)| \leqslant 9$, then the final charge on v is at least $2+\frac{1}{7}$, so assume that $|\operatorname{Resp}(v)| = 10$. If v sends $\frac{1}{14}$ to each of three vertices in $N_3(v)$, then $d(v,u) \leqslant 4$ for each $u \in N_3(v)$ and hence $|\operatorname{Resp}(v)| < 10$. Thus, v sends $\frac{1}{7}$ to some $u_1 \in N_3(v)$ and $\frac{1}{14}$ to some $u_2 \in N_3(v)$. Since $|\operatorname{Resp}(u_1)| = 14$, Claim 19 implies that the Y-type configuration about u_1 is of the form $Y(0,7,7)$. Since v is adjacent to u_1, $d(v,u_2) \leqslant 4$, and $|\operatorname{Resp}(v)| = 10$, the Y-type configuration about v is of the form $Y(0,3,7)$. If $\mu(v,u_1) = 1$, then the H-type configuration about v and u_1 is of the form $H(3,7;0;7;7)$ which is reducible by Claim 19. If $\mu(v,u_1) = 2$, then the Φ-type configuration about v and u_1 is of the form $\Phi(3,7,0,7)$ which is reducible by Claim 19.

Case 4.ii: v sends $\frac{2}{7}$ by (R2). In this case, v must send charge $\frac{1}{7}$ to at least one vertex u_1 in $N_3(v)$. If v sends $\frac{1}{7}$ to another vertex u_2 in $N_3(v)$, then, as G contains no 8-caterpillar, $|\operatorname{Resp}(v)| \leqslant 7$ and hence the final charge on v is at least $2 + \frac{3}{14}$. If v sends $\frac{1}{14}$ to the other two vertices u_2 and u_3 in $N_3(v)$, then $|\operatorname{Resp}(v)| \leqslant 6$ and hence the final charge on v is at least $2 + \frac{5}{14}$.

Case 4.iii: v either sends $\frac{5}{14}$ or $\frac{3}{7}$ by (R2). Suppose that v sends $\frac{5}{14}$ by (R2). Thus, v must send charge $\frac{1}{7}$ to two of three vertices in $N_3(v)$, and $\frac{1}{14}$ to the third vertex. This implies that $|\operatorname{Resp}(v)| \leqslant 3$ and hence the final charge on v is at least $2 + \frac{3}{7}$. Similarly, if v sends $\frac{3}{7}$ by (R2), then $|Resp(v)| = 0$. Thus, the final charge on v is $2 + \frac{4}{7}$.

In all cases, we verified that the final charge is at least $2 + \frac{1}{7}$, contradicting that the average degree of $\operatorname{ct}(G)$ is strictly less than $2 + \frac{1}{7}$. □

We note that it is possible to improve the bound $\operatorname{mad}(G) < 2 + \frac{1}{7}$ by a small amount. In particular, the discharging method used above essentially states that the average size of a responsibility set in $\operatorname{ct}(G)$ is at most 12. By careful analysis, we can find that a 3-vertex v with $|\operatorname{Resp}(v)| \leqslant 11$ has some excess charge after the discharging argument that could be used to increase the charge on nearby vertices by a small fraction. We have verified using computation that for every 3-vertex v, there is at least one vertex $u \in N_3(v)$ where $|\operatorname{Resp}(u)| < 12$. Thus, it is impossible to have a minimal counterexample where all responsibility sets have size 12, and it is feasible to construct a discharging argument that will improve on the bound $\operatorname{mad}(G) < 2 + \frac{1}{7}$ by a small fraction. We do not do this explicitly as it requires significant detail without significant gain.

In order to prove that $\operatorname{mad}(G) < 2 + \frac{1}{6}$ implies that G can be strongly 5-edge-colored, then the proof will imply that the average size of a responsibility set is at most 10. This will require sending charge to all of the vertices with 11 or 12 vertices in the responsibility set, and also making sure that the charge comes from vertices with responsibility sets much smaller. Likely, larger reducible configurations will grant some improvement in this direction, but our algoritheorem is insufficient to effectively test reducibility for larger configurations.

3.3 Proof of Theorem 9

Proof. Note that the second item of Theorem 9 follows from the first by Proposition 4. For the first item, we follow a similar discharging argument as in Theorem 8. The argument will be simpler as we will only discharge from 3^+-vertices to $2^\perp$-vertices. Select a graph G that satisfies the hypotheses and minimizes $n(G)$. Observe that $\operatorname{ct}(G)$ is 2-connected by an argument similar to Lemma 21.

Since the 6-caterpillar is 7-reducible by Claim 18, $\operatorname{ct}(G)$ does not contain a path of six $2^\perp$-vertices. Since G has girth at least 7, $\operatorname{ct}(G)$ is not a cycle, so it contains at least one 3^+-vertex.

If v is a 3^+-vertex, then let $\mathrm{Resp}(v)$ be the set of $2^\perp$-vertices reachable from v using only $2^\perp$-vertices. We consider $\mathrm{Resp}(v)$ to be a multiset, where the multiplicity of a vertex $u \in \mathrm{Resp}(v)$ is given by the number of paths from v to u using only $2^\perp$-vertices. Note that the multiplicity is either 1 or 2.

Assign charge $d_{\mathrm{ct}(G)}(v)$ to each vertex $v \in V(\mathrm{ct}(G))$. Note that the average charge on each vertex is equal to the average degree of G. To discharge, let $\varepsilon = \frac{1}{13}$ and each 3^+-vertex v sends εm to each $2^\perp$-vertex in $\mathrm{Resp}(v)$ with multiplicity m. Thus, every $2^\perp$-vertex ends with charge $2 + \frac{2}{13}$.

Suppose $d_{\mathrm{ct}(G)}(v) = 3$. Since $\mathrm{ct}(G)$ is 2-connected, all vertices in $\mathrm{Resp}(v)$ appear with multiplicity one. By Claim 20, $|\mathrm{Resp}(v)| \leqslant 11$. Thus each 3-vertex ends with final charge at least $3 - \frac{11}{13} = 2 + \frac{2}{13}$.

Suppose $d_{\mathrm{ct}(G)}(v) = 4$. Since the $(6, 4)$-caterpillar is reducible, each path of $2^\perp$-vertices has length at most five, and hence $|\mathrm{Resp}(v)| \leqslant 20$, including multiplicity. Thus each 4-vertex ends with final charge at least $4 - \frac{20}{13} = 2 + \frac{6}{13} > 2 + \frac{2}{13}$.

Therefore, every vertex ends with final charge at least $2 + \frac{2}{13}$ and thus the average degree of G is at least $2 + \frac{2}{13}$, a contradiction. □

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