Quasiregular matriods

S. R. Kingan
Department of Mathematics
Brooklyn College, City University of New York
Brooklyn, NY 11210, U.S.A.
skingan@brooklyn.cuny.edu

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Abstract

Regular matroids are binary matroids with no minors isomorphic to the Fano matroid F_7 or its dual F_7^*. Seymour proved that 3-connected regular matroids are either graphs, cographs, or R_{10}, or else can be decomposed along a non-minimal exact 3-separation induced by R_{12}. Quasiregular matroids are binary matroids with no minor isomorphic to the self-dual binary matroid E_4. The class of quasiregular matroids properly contains the class of regular matroids. We prove that 3-connected quasiregular matroids are either graphs, cographs, or deletion-minors of $PG(3, 2)$, R_{17} or M_{12} or else can be decomposed along a non-minimal exact 3-separation induced by R_{12}, P_9, or P_9^*.

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1 Introduction

Let M be a matroid with ground set E. The *connectivity function* λ is defined as $\lambda(X) = r(X) + r(E - X) - r(M)$ for every subset X of E. Observe that $\lambda(X) = \lambda(E - X)$. A simple matroid is 3-connected if $\lambda(A) \geqslant 2$ for all partitions (A, B) of E with $|A| \geqslant 3$ and $|B| \geqslant 3$.

Let $\mathcal{M}$ be a class of matroids closed under minors. It is sufficient to focus on the 3-connected members of $\mathcal{M}$ since matroids that are not 3-connected can be pieced together from 3-connected matroids using the operations of 1-sum and 2-sum ([8], 8.3.1). A 3-connected matroid is a *splitter* for a minor-closed class if it is in the class, but every 3-connected single-element extension and coextension is not in the class.

We call a partition (A, B) in a matroid a 3-*separation* if $\lambda(A) \leqslant 2$ and $|A| \geqslant 3$ and $|B| \geqslant 3$. If $\lambda(A) = 2$ we call (A, B) an *exact* 3-*separation.* If $\lambda(A) = 2$ and $|A| = 3$ or $|B| = 3$ we call (A, B) a *minimal exact* 3-*separation.* Thus when $\lambda(A) = 2$ and $|A| \geqslant 4$ and $|B| \geqslant 4$, we call (A, B) a *non-minimal exact* 3-*separation.* This is the kind of 3-separation we are interested in finding. Suppose M is a 3-connected matroid in $\mathcal{M}$ with a 3-connected minor N and suppose N has an exact 3-separation (A, B). If M has an exact 3-separation (X, Y) such that $A \subseteq X$ and $B \subseteq Y$, then we say the 3-separation of N is *induced* in M. If, in addition, the exact 3-separation (A, B) is also non-minimal, we call N a 3-*decomposer* for M.

The main theorem of this paper is given below:

Theorem 1. *Suppose M is a binary 3-connected matroid with no E_4-minor. Then either M has a 3-decomposer in $\{R_{12}, P_9, P_9^*\}$ or else M or M^* is a graphic matroid or a deletion-minor of $PG(3,2)$, R_{17}, M_{12}, or the binary spike Z_r, for $r \geqslant 5$.*

Matrix representations for F_7, P_9, E_4, M_{12} and R_{17} are given below. Columns of all matrices in this paper are labeled $\{1, \dots n\}$, where n is the number of columns.

$$F_7 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_3 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array}\right] \quad P_9 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad E_4 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$M_{12} = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_6 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \quad R_{17} = \left[\begin{array}{c|cccccccccccc} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

The matroid R_{17} appears in [3] as a member of the class of binary matroids with no $M^*(K_5 \backslash e)$-minor. It appeared for the first time in [6] as $AG(3,2) \times U_{1,1}$.

Following the terminology in [8], let $EX(M_1 \dots, M_k)$ denote the class of binary matroids with no minors isomorphic to $M_1, \dots, M_k$. Splitters and 3-decomposers were introduced by Seymour in his 1980 result on the decomposition of regular matroids (*i.e.* the class $EX(F_7, F_7^*)$). The significance of Theorem 1 is that $EX(E_4)$ is a superset of $EX(F_7, F_7^*)$, with vastly more members, yet it lends itself to a decomposition just like $EX(F_7, F_7^*)$.

Seymour proved that if M is a 3-connected binary matroid in $EX(F_7, F_7^*)$, then either R_{12} is a 3-decomposer for M or else M or M^* is a graphic matroid or R_{10} [9]. The matroid R_{10} is a 10-element rank-5 self-dual matroid. It is a splitter for 3-connected regular matroids. The matroid R_{12} is a 12-element rank-6 self-dual matroid. Observe that $EX(F_7, F_7^*)$ has one 3-decomposer R_{12} (which is self-dual), whereas the non-regular members in $EX(E_4)$ have either P_9 or P_9^* as a 3-decomposer. Besides matroids that can be decomposed by P_9 or P_9^*, the only non-regular members of $EX(E_4)$ are $PG(3,2)$, R_{17} and M_{12}. The matroid M_{12} is a splitter for the class and R_{17} is a maximal 3-connected member of the class. This class is one of few excluded minor classes whose members are described. See Table 1 in [5] for a list of known excluded minor classes. Moreover, after the decomposition of $EX(F_7, F_7^*)$, the matroid F_7^* became the starting point for the analysis of binary non-graphic and non-cographic matroids. As a consequence of Theorem 1, the starting point for the analysis of binary non-graphic and non-cographic matroids is E_4.

The techniques are explained in Section 2. The proof of Theorem 1 is in Section 3. Section 4 contains a discussion of how the computations are done and next steps.

2 Techniques

Theorem 1 builds on Seymour's 1980 result as presented in ([8], 13.1.2 and 13.1.3). While he did not use the term 3-decomposer in his paper, the concept orginated in his paper. It was named so in [3] and [4] for convenience of notation.

Theorem 2. *Suppose M is a 3-connected binary matroid in $EX(F_7, F_7^*)$. Then either R_{12} is a 3-decomposer for M or else M or M^* is a graphic matroid or R_{10}.*

Binary matrix representations for R_{10} and R_{12} are shown below. Note that R_{12} has a non-minimal exact 3-separation (A, B), where $A = \{1, 2, 5, 6, 9, 10\}$.

$$R_{10} = \left[\begin{array}{c|ccccc} & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \end{array} \right] R_{12} = \left[\begin{array}{c|cccccc} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ I_6 & 1 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

Theorem 2 used the Splitter Theorem ([8], 12.2.1) and a Decomposition Theorem ([9], Theorem 9.1). Theorem 1 requires the Strong Splitter Theorem ([2], Theorem 1.4) and a modification of the Decomposition Theorem ([4], Theorem 1.3).

Theorem 3. (Strong Splitter Theorem) *Suppose N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or a whirl then M has no larger wheel or whirl-minor, respectively. Further, suppose $m = r(M) - r(N)$. Then there is a sequence of 3-connected matroids $M_0, M_1, \ldots, M_n$, for some integer $n \geqslant m$, such that:*

(i) $M_0 \cong N$;

(ii) $M_n = M$;

(iii) for $k \in \{1, 2, \ldots, m\}$, $r(M_k) - r(M_{k-1}) = 1$ *and* $|E(M_k) - E(M_{k-1})| \leqslant 3$; *and*

(iv) for $m < k \leqslant n$, $r(M_k) = r(M)$ *and* $|E(M_k) - E(M_{k-1})| = 1$.

Moreover, when $|E(M_k) - E(M_{k-1})| = 3$, for some $1 \leqslant k \leqslant m$, $E(M_k) - E(M_{k-1})$ is a triad of M_k.

We can obtain up to isomorphism M by starting with N and at each step doing a 3-connected single-element extension or coextension, such that at most two consecutive single-element extensions may occur in the sequence before a single-element coextension must occur, unless the rank of the minors involved are the same as the rank of M, as noted in (iv). Moreover, as the last line indicates, if two consecutive single-element extensions by elements $\{e_1, e_2\}$ are followed by a coextension by element f, then $\{e_1, e_2, f\}$ forms a triad in the resulting matroid. Furthermore, the proof indicates that for binary matroids, when coextending a single-element extension only rows in series with existing elements must be considered.

Theorem 4. (Decomposition Theorem) *Let N be a simple and cosimple matroid in $\mathcal{M}$ with an exact 3-separation (A, B), such that A is a union of circuits and a union of cocircuits. If $\lambda_M(A) = 2$ for every simple single-element extension and cosimple single-element coextension of N in $\mathcal{M}$, then the 3-separation (A, B) of N is induced in M for every $M \in \mathcal{M}$ with N as a minor.*

Since the result in this paper uses both the Strong Splitter Theorem and a decomposition theorem, it is important to point out (as did Seymour in [9]) that in a decomposition theorem, unlike in the Splitter Theorem, no isomorphism is involved.

We end this section by describing our method for calculating extensions and coextensions which is an alternative to using grafts. This technique for finding single-element extensions and coextensions, the Strong Splitter Theorem, and the Decomposition Theorem lead to short proofs.

Let N be a $GF(q)$-representable n-element rank-r matroid represented by the matrix $A = [I_r|D]$ over $GF(q)$. The columns of A may be viewed as a subset of the columns of the matrix that represents the projective geometry $PG(r-1, q)$. Let M be a simple single-element extension of N over $GF(q)$. Then $N = M\backslash e$ and M may be represented by $[I_r|D']$, where D' is the same as D, but with one additional column corresponding to the element e. The new column is distinct from the existing columns and has at least two non-zero elements. If the existing columns are labeled $\{1, \ldots, r, \ldots, n\}$, then the new column is labeled $(n+1)$.

Suppose M is a cosimple single-element coextension of N over $GF(q)$. Then $N = M/f$ and M may be represented by the matrix $[I_{r+1}|D'']$, where D'' is the same as D, but with one additional row. The new row is distinct from the existing rows and has at least two non-zero elements. The columns of $[I_{r+1}|D'']$ are labeled $\{1, \ldots, r+1, r+2, \ldots, n, n+1\}$. The coextension element f corresponds to column $r+1$. The coextension row is selected from $PG(n-r, q)$. We can visualize the new element f as appearing in the new dimension and lifting several points into the higher dimension. Observe that f forms a cocircuit with the elements corresponding to the non-zero elements in the new row. In $[I_{r+1}|D'']$ the labels of columns beyond r are increased by 1 to accomodate the new column $r+1$.

We refer to the simple single-element extensions of N as Type (i) matroids and the cosimple single-element coextensions of N as Type (ii) matroids. The structure of Type (i) and Type (ii) matroids are shown in Figure 1. Note that for fields of order $q \geqslant 4$, if the representation of N is not unique, Type (i) and (ii) matroids must be calculated for each inequivalent representation of N.

Once the simple single-element extensions (Type (i) matroids) and cosimple single-element coextensions (Type (ii) matroids) are determined, the number of permissable rows

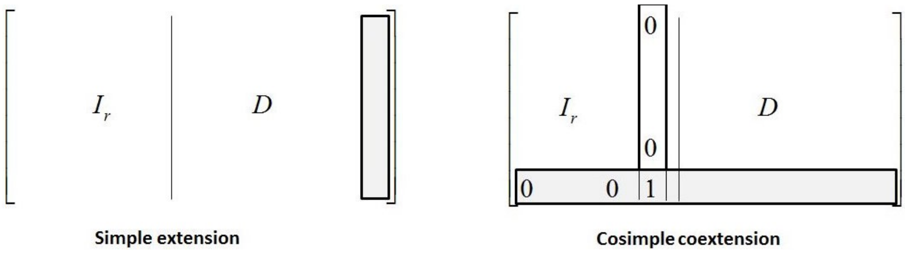

Figure 1: Structure of Type (i) and Type (ii) matroids

and columns give a bound on the choices for the cosimple single-element extensions of the Type (i) matroids and the simple single-element extensions of the Type (ii) matroids, respectively. The structure of the cosimple single-element coextensions of a Type (i) matroid and the simple single-element extensions of a Type (ii) matroid are shown in Figure 2.

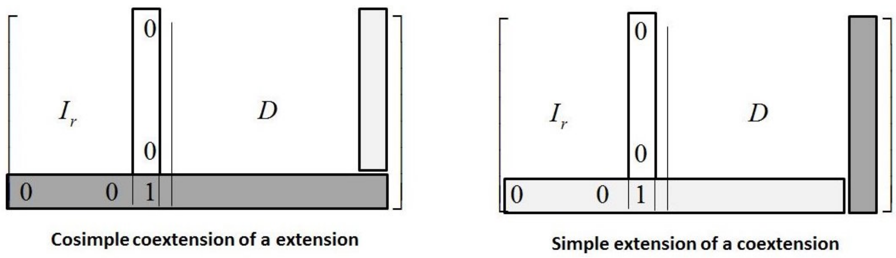

Figure 2: Structure of M, where $|E(M) - E(N)| = 2$

When computing a cosimple single-element coextension of a Type (i) matroid, there are three types of rows that may be inserted into the last row. (This paper is only on binary matroids, so we will talk only of zeros and ones.)

(I) Rows that can be added to the original matroid N to obtain a coextension, augmented by a 0 or 1 as the last entry;

(II) The identity rows augmented by a 1 in the last position; and

(III) Rows "in-series" to the right-hand side of the matrix with the last entry reversed.

When computing a simple single-element extension of a Type (ii) matroid, there are three types of columns that may be inserted into the last column.

(I) Columns that can be added to the original matroid N to obtain an extension augmented by a 0 or 1 as the last entry;

(II) The identity columns augmented by a 1 in the last position; and

(III) Columns "in-parallel" to the right-hand side of matrix with the last entry reversed.

Suppose N' is a simple double-element extension of N formed by adding columns e_1 and e_2 and M is a cosimple single-element coextension of N' by element f. Then, by Theorem 3 $M\backslash e_1$ or $M\backslash e_2$ is 3-connected except when $\{e_1, e_2, f\}$ is a triad. Thus the only 3-connected coextension of N' we must check is the one formed by adding row $[00\ldots011]$ to D. Moreover, no further calculations are necessary.

3 Proof of Theorem 1

Observe that $F_7 = PG(2,2)$ and therefore has no 3-connected extensions in the class of binary matroids. Coextensions of F_7 are duals of extensions of F_7^*. Thus we may focus on the 3-connected extensions of F_7^*. The two non-isomorphic 3-connected single-element extensions of F_7^* are $AG(3,2)$ and S_8 [9]. Since they are self-dual, they are also coextensions of F_7. The matroid S_8 has two non-isomorphic 3-connected single-element extensions P_9 and Z_4 and $AG(3,2)$ has one 3-connected single-element extension Z_4 [7]. Matrix representations for S_8, $AG(3,2)$, Z_4, P_9^*, and the rank-4 binary projective geometry $PG(3,2)$ are given below. The simple extensions and cosimple single-element coextensions of P_9 are given in Table 1a and 1b in the Appendix.

$$S_8 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \end{array}\right] AG(3,2) = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{array}\right] Z_4 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$P_9^* = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 \end{array}\right] PG(3,2) = \left[\begin{array}{c|ccccccccccc} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

Oxley proved that a 3-connected binary non-regular matroid M has no minor isomorphic to P_9 or P_9^* if and only if M or M^* is a deletion-minor of Z_r, for $r \geqslant 4$ [7].
The main theorem of this section is the decomposition of 3-connected binary non-regular matroids with no E_4-minor.

Theorem 5. *Suppose M is a binary 3-connected non-regular matroid with no E_4-minor. Then either M has a 3-decomposer in $\{P_9, P_9^*\}$ or else M or M^* is a deletion-minor of $PG(3,2)$, R_{17}, M_{12}, or Z_r, for $r \geqslant 5$.*

Proof. Suppose M is a 3-connected binary non-regular matroid with no E_4-minor. If M has no P_9 or P_9^*-minor, then M or M^* is a deletion-minor of a rank-r binary spike Z_r for $r \geqslant 4$ [7]. Therefore we may assume M has a P_9 or P_9^*-minor. From Tables 1a and 1b we see that P_9 has three non-isomorphic simple single-element extensions D_1, D_2, and D_3, and eight non-isomorphic cosimple single-element coextensions E_1, E_2, E_3, E_4, E_5, E_6, E_6^*, and E_7.

$$D_1 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] D_2 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right] D_3 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$E_1 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 1&1&0&0&0 \end{array}\right] E_2 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 1&1&0&1&1 \end{array}\right] E_3 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 1&1&0&0&1 \end{array}\right]$$

$$E_4 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 0&1&0&0&1 \end{array}\right] E_5 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 1&0&1&0&0 \end{array}\right] E_6 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 0&0&1&0&1 \end{array}\right]$$

$$E_6^* = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 0&0&1&1&1 \end{array}\right] E_7 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 0&0&0&1&1 \end{array}\right]$$

Claim 1. *If M has a P_9- or P_9^*-minor, but no D_2, D_2^*, E_4, or E_5-minor, then either P_9 or P_9^* is a 3-decomposer for M.*

Proof. As noted in Section 1, the columns of P_9 are labeled $\{1, \ldots 9\}$. It has a non-minimal exact 3-separation (A, B), where $A = \{1, 2, 5, 6\}$ is both a circuit and a cocircuit. It is easy to check that $\lambda(\{1, 2, 5, 6\}) = 2$ in D_1 and D_3 (every column is checked), whereas D_2 is internally 4-connected. The set $A = \{1, 2, 5, 6\}$ corresponds to $A' = \{1, 2, 6, 7\}$ in the coextension since the fifth column is the coextended element. It is easy to check that $\lambda(\{1, 2, 6, 7\}) = 2$ in every single-element coextension (every row is checked). Further note that E_4 and E_5 are self-dual. The claim follows from Theorem 4. ☐

Next, we must consider matroids that have an E_5, D_2, or D_2^*-minor, but no E_4-minor. Consider the 3-connected single-element extensions and coextensions of E_5 shown in Tables 2a and 2b in the Appendix. Observe that E_5 has seven non-isomorphic simple single-element extensions all of which have an E_4-minor except A, B and C. Matrix representations for A, B, and C are given below.

$$A = \left[\begin{array}{c|cccccc} & 0&1&1&1&1&0\\ & 1&0&1&1&0&0\\ I_5 & 1&1&0&1&1&1\\ & 1&1&1&1&0&0\\ & 1&1&0&0&0&1 \end{array}\right] B = \left[\begin{array}{c|cccccc} & 0&1&1&1&1&1\\ & 1&0&1&1&0&0\\ I_5 & 1&1&0&1&1&0\\ & 1&1&1&1&0&1\\ & 1&1&0&0&0&1 \end{array}\right] C = \left[\begin{array}{c|cccccc} & 0&1&1&1&1&1\\ & 1&0&1&1&0&1\\ I_5 & 1&1&0&1&1&0\\ & 1&1&1&1&0&0\\ & 1&1&0&0&0&1 \end{array}\right]$$

The next claim is the key step in Theorem 5. We use the following representation of E_5 (dual of the previous representation, since E_5 is self dual):

$$E_5 = \left[\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&0\\ I_5 & 1&1&0&1&1\\ & 1&1&1&1&0\\ & 1&1&0&0&0 \end{array}\right]$$

Claim 2. *Suppose M is a binary 3-connected matroid with an E_5-minor and no E_4-minor. Then M or M^* is isomorphic to a 3-connected deletion-minor of M_{12} or R_{17}.*

Proof. The proof is in three stages. First, we will show that all the cosimple coextensions of A, B, and C have an E_4-minor with the exception of M_{12}. Suppose M is a cosimple single-element coextension of A, B, C. The three types of rows that may be added to A, B and C to obtain M are:

(I) rows that can be added to E_5 to obtain a coextension with no E_4-minor, with a 0 or 1 as the last entry;

(II) the identity rows with a 1 in the last position; and

(III) the rows "in-series" to the right-hand side of matrices A, B, C with the last entry reversed.

Type I rows are [001110], [001111] [010010], [010011], [010100], [010101], [011000], [011001], [100110], [100111], [101010], [101011], [111010], and [111011]. They are obtained from Table 2b. Type II rows are [100001], [010001], [001001], [000101], and [000011]. Type III rows are specific to the matrices A, B, C. For matrix A they are [011111], [101101], [110110], [111101], and [110000]. For matrix B they are [011110], [101101], [110111], [111100], and [110000]. For C they are [011110], [101100], [110111], [111101], and [110000].

Type I, II, and III rows are shown in bold in Table 3 of the Appendix. Most result in matroids that clearly have an E_4-minor. Only a few coextensions must be specifically checked for an E_4-minor. They are the 11th coextension of A, denoted as $(A, coext11)$, the eight coextension of B, denoted as $(B, coext8)$, and five coextensions of C denoted as $(C, coext8)$, $(C, coext9)$, $(C, coext10)$, $(C, coext12)$, and $(C, coext14)$.

Observe that $(A, coext11)/11\backslash 3 \cong E_4$, $(C, coext8)/12\backslash 2 \cong E_4$, $(C, coext9)/12\backslash 1 \cong E_4$, $(C, coext10)/12\backslash 10 \cong E_4$, and $(C, coext14)/12\backslash 6 \cong E_4$. Further $(B, coext8) \cong (C, coext12)$ and this matroid does not have an E_4-minor. This is the matroid M_{12}.

Second, we must establish that M_{12} is a splitter for $EX[E_4]$. By the Splitter Theorem and the fact that M_{12} is self-dual, we only need to check the single-element coextensions of M_{12}. From Table 3 observe that M_{12}, as a coextension of C, may be obtained by adding exactly one row. Thus there are no further rows that may be added to form coextensions without an E_4-minor. It follows that M_{12} is a splitter for the class of binary matroids with no E_4-minor.

Third, we must show that either $M \cong M_{12}$ or $r(M) \leqslant 5$. To show this we compute the simple single-element extensions of A, B, and C with no E_4-minor. From Table 2a the only columns that can be added to E_5 to obtain a matroid with no E_4-minor are [00101], [00110], [01011], [01100] [10011], [11001], and [11101]. They give the matroids D, E, F, and G shown below.

$$D = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] E = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$F = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] G = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Specifically, adding to A column [00110], [01100], or [10011] gives D; adding column [01011] gives E; adding [11001] gives F; and adding [11101] gives G. Similarly, we can check that B extends to D and F, and C extends to F and G. Observe that adding all seven columns to E_5 gives the 17-element matroid shown below which is isomorphic to the representation of R_{17} shown in the introduction.

$$R_{17} = \left[\begin{array}{c|cccccccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

By Theorem 3 the only cosimple single-element coextensions of D, E, and F we must consider are the ones with [0000011] as the new row. Let us call them D', E', F', and G', respectively. In each case we can find an E_4 minor. In particular, $D'/1\backslash\{3,11\} \cong E_4$, $E'/1\backslash\{7,11\} \cong E_4$, $F'/1\backslash\{3,11\} \cong E_4$, and $G'/1\backslash\{7,11\} \cong E_4$. This concludes the proof of Claim 2. □

Returning to the proof of Theorem 5 it remains to show that if M has a D_2-minor and no E_4-minor, then we do not get any new matroids other than those already found in Claim 2. Suppose M is a cosimple single-element coextension of D_2. From Appendix Table 4 we see that M is isomorphic to A, B, C, or Z. A matrix representation for Z is shown below:

$$Z = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Since Z is formed by adding only one row to D_2 (namely [000111]) any coextension of Z will also be a coextension of A, B, and C.

Suppose M is a single-element extension of D_2. From Table 1a we see that that D_2 has two single-element extensions X_1 and X_3 shown below:

$$X_1 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}\right] X_3 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{array}\right]$$

By Theorem 3 the only coextensions of X_1 and X_3 we must check are the ones formed with [00000011] as the new row. Both these matroids have an E_4-minor.

Lastly, suppose M is a simple single-element extension of Z. It is straightforward to compute the three non-isomorphic simple single-element extensions which are D, F and Y (obtained by adding one of columns [00111], [01011], [01101], [10101], or [11100]). The result follows again by Claim 2 and the fact that when we add the above five columns to Z we get the sixteen element matroid shown below which is isomorphic to $R_{17}\backslash\{17\} = R_{16}$.

$$R_{16} = \left[\begin{array}{c|ccccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

Thus Z does not contribute any new matroids to $EX(E_4)$ other than those found in Claim 2. This completes the proof of Theorem 5. □

The proof of Theorem 1 follows from Theorems 2 and 5. Once R_{17} is shown to be the largest rank-5 member of the class it is straightforward to compute its 3-connected deletion-minors. See [3] for details.

4 Further research

The next step is to make a more precise identification of the members of $EX(E_4)$ that have P_9 or P_9^* as a 3-decomposer. This seems quite difficult. In the case of regular matroids, R_{12} is a 3-sum of a graphic matroid and a cographic matroid. It follows that any regular matroid having R_{12} as a 3-decomposer is the 3-sum of a graphic matroid and cographic matroid. The class of quasiregular matroids is very large. Especially problematic are the quasiregular matroids with minors isomorphic to all three matroids: P_9, P_9^* and $AG(3,2)$. However, it seems to be an exercise in completeness, since for most practical purposes knowing that in theory the matroids can be written as 3-sums appears to be enough.

We will end with a discussion of our method for finding the binary simple single-element extensions. Let M be a binary 3-connected n-element rank r matroid represented over $GF(2)$ by the matrix A. Note that since M is 3-connected, simple extensions of M will also be 3-connected. Each of $2^r - 1 - n$ columns $\{x_1, \ldots, x_{2^r-1-n}\}$ from $PG(r-1, 2)$ when adjoined to the matrix A gives a single-element extension represented by the matrix $A \cup x_i$. We must then check whether $M(A \cup x_i) \cong M(A \cup x_j)$, for some $i, j \in \{1, 2, \ldots, 2^r - 1 - n\}$. To do this first various invariants are calculated for $M(A \cup x_i)$ and $M(A \cup x_j)$. If the invariants all match, then a list of candidate mappings preserving element-wise independent sets, circuits, cocircuits, bases, and spanning sets is generated, and the mappings are tested sequentially for basis preservation. (The matroid software program Oid finds isomorphisms among other features [1].) When an isomorphism is obtained, checking that it is indeed an isomorphism is straightforward.

Once isomorphism is determined, equivalence classes are created among the extension columns. For example, Table 1a shows the three non-isomorphic single-element extensions of P_9: D_1 is obtained by adjoining just one column [1110]; D_2 is obtained by adjoining any one of columns [1001] [0101] [0110], [1010]; and D_3 is obtained by adjoining column [0011]. When there is a choice of columns the bolded column is selected for further processing.

Before Oid was created, we found by hand automorphisms of P_9, P_9^*, and E_5, and expressed them in terms of the row operations that induced them. Consider for example P_9 using the representation in Section 1. Swapping row 1 with row 2 induces an automorphism on P_9. Pivoting on element a_{16} and swapping row 3 with row 4 also induces an automorphism. The corresponding maps on $(x_1, x_2, x_3, x_4)^T$ are shown below:

$$\alpha : (x_1, x_2, x_3, x_4)^T \longrightarrow (x_2, x_1, x_3, x_4)^T$$
$$\beta : (x_1, x_2, x_3, x_4)^T \longrightarrow (x_1, x_2, x_4 + x_1, x_3 + x_1)^T$$

Thus $\alpha(1010) = (0110)$, $\beta(0110) = (0101)$ and $\alpha(0101) = (1001)$.

Tables 1, 2, and 3 appear in the author's dissertation. Tables 3 and 4 were created by Oid and verified by hand since, once Oid gives the exact ismorphism between two matroids, the isomorphism is easy to verify by hand.

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Appendix

Matroid	Extension Columns	Name
P_9	[**1110**]	D_1
	[**1001**] [0101] [0110], [1010]	D_2
	[**0011**]	D_3
D_1	[0101] [0110] [1001] [1010]	X_1
	[**0011**]	X_2
D_2	[**1010**] [1110]	X_1
	[**0011**] [0101] [0110]	X_3
D_3	[1110]	X_2
	[0101] [0110] [1001] [1010]	X_3
X_1	[**0011**] [0101] [0110]	Y_1
	[**1110**]	Y_2
X_2	[0101] [0110] [1001] [1010]	Y_1
X_3	[0101] [0110] [1010] [1110]	Y_1

Table 1a: Rank 4 extensions of P_9

Coextension Rows	Name
[11000] [11111]	E_1
[11011] [11100]	E_2
[11001] [11101]	E_3
[01001] [01010] [01101] [01110] [10001] [10010] [10101] [10110]	E_4
[01011] [01100] [10011] [10100]	E_5
[00101] [00110]	E_6
[00111]	E_6^*
[00011]	E_7

Table 1b: Single-element coextensions of P_9

Extension Columns	Name	E_4-minor
[00101] [00110] [01011] [01100]	A	No
[10011]	B	No
[11001] [11101]	C	No
[00011] [00111] [01001] [01101]		Yes
[01010] [01110]		Yes
[10001] [10010] [11011] [11100]		Yes
[10101] [10110] [11000] [11111]		Yes

Table 2a: Simple single-element extensions of E_5

Coextension Rows	Name
[00111] [01001] [01010] [01100]	A^*
[10011]	B^*
[10101] [11101]	C^*
[00011] [00101] [01011] [01101]	
[00110] [01110]	
[10001] [10010] [10111] [11100]	
[10100] [11001] [11010] [11111]	

Table 2b: Cosimple single-element coextensions of E_5

Matroid	Name	**Coextension Row**
A	coext 1	[**000011**] [**000101**] [001010] [011010] [101111] [111001]
	coext 2	[000110] [110011] [110101]
	coext 3	[000111] [**101011**] [**111011**]
	coext 4	[**001001**] [010110] [**011111**]
	coext 5	[001011] [011011] [**100111**]
	coext 6	[001100] [011100] [**110000**]
	coext 7	[001101] [**010010**] [**010100**] [011101] [101110] [111000]
	coext 8	[**001110**] [**011000**] [**101101**] [110010] [110100] [**111101**]
	coext 9	[**001111**] [**011001**] [100011] [100101] [**101010**] [**111010**]
	coext 10	[**010001**] [**100010**] [100100]
	coext 11	[**010011**] [**010101**] [**100110**]
	coext 12	[010111]
	coext 13	[**100001**] [101000] [111110]
	coext 14	[101001] [**110110**] [111111]
B	coext 1	[**000011**] [**000101**] [000110] [**001001**] [001010] [**001111**] [**010010**] [**010100**] [010111] [**011000**] [011011] [**011110**]
	coext 2	[000111] [001011] [010110] [011010]
	coext 3	[001100] [**010001**] [011101]
	coext 4	[001101] [**001110**] [**010011**] [**010101**] [**011001**] [011100]
	coext 5	[**100001**] [100010] [100100] [101000] [**101101**] [101110] [**110000**] [110011] [110101] [111001] [**111100**] [111111]
	coext 6	[100011] [100101] [**101010**] [101111] [111000] [**111011**]
	coext 7	[**100110**] [101001] [110010] [110100] [**110111**] [111110]
	coext 8	[**100111**] [**101011**] [**111010**]
C	coext 1	[**000011**] [**000101**] [**001001**] [**001111**] [**010010**] [**010100**] [**011000**] [**011110**] [100010] [100100] [101000] [101110] [110011] [110101] [111001] [111111]
	coext 2	[000110] [010111]
	coext 3	[000111] [010110] [**100110**] [**110111**]
	coext 4	[001010] [011011]
	coext 5	[001011] [011010] [**101010**] [**111011**]
	coext 6	[001100] [011101]
	coext 7	[001101] [011100] [**101100**] [**111101**]
	coext 8	[**001110**] [**010011**] [**010101**] [**011001**]
	coext 9	[**010001**]
	coext 10	[**100001**] [**110000**]
	coext 11	[100011] [100101] [101111] [111000]
	coext 12	[**100111**]
	coext 13	[101001] [110010] [110100] [111110]
	coext 14	[**101011**] [**111010**]

Table 3: Cosimple single-element coextensions of A B and C

Matroid	**Coextension Rows**	**Name**	Relevant minors
D_2	[000011] [000101] [000110] [001111] [100111] [101000]	A_{26} **A**	E_5, E_6^*, E_7
	[000111]	A_{31} **Z**	E_7, R_{10}
	[001001] [010100] [011101]	A_{23}	E_4, E_5
	[001010] [001100] [010001] [010010] [011011] [011110]	A_{20}	E_4, E_6
	[001011] [001101] [010101] [010110] [011001] [011100]	A_{21}	E_4, E_5
	[001110] [010011] [011010]	A_{24}	E_4
	[100001] [101000] [101011] [101101] [110110] [111001]	A_{15}	E_2, E_5
	[100010] [100100] [110000] [110101] [111100] [111111]	A_6	E_1, E_4
	[100011] [100101] [110010] [110111] [111000] [111011]	A_{16}	E_2, E_3, E_4, E_6^*
	[100110] [101010] [101100] [101111] [110001] [111110]	A_7	E_4, E_5
	[100111] [110011] [111010]	A_{18} **C**	E_3, E_5, E_6^*, E_7
	[101001]	A_{27} **B**	E_5

Table 4: Cosimple single-element coextensions of D_2