Hypertopes with tetrahedral diagram

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Abstract

In this paper we construct an infinite family of hypertopes of rank four having the complete graph K_4 as diagram. Their group of rotational symmetries is isomorphic to $PSL(2, q)$. It turns out some elements of this family are regular hypertopes and some are chiral. Moreover, we show that the chiral ones have both improper and proper correlations simultaneously.

Mathematics Subject Classifications: 51E24, 52B11, 20F05

1 Introduction

Abstract polytopes generalize (the face lattice of) convex polytopes to combinatorial structures. The main interest of the theory of abstract polytopes has been the study of their symmetries. Hence, highly symmetric polytopes (in particular those regular and chiral), together with their automorphism groups, are the most studied ones. A polytope can be regarded as a thin residually connected geometry with linear diagram. The concept of hypertope was introduced in [2] and generalizes the concept of a polytope by dropping the linear condition on the diagram. This generalization was made in such a way that the concept of chirality can be extended to hypertopes. As it is the case with maps and polytopes, we are interested in understanding chiral hypertopes, and finding examples of them can be the first step to achieve this goal.

In the 1970's Branko Grünbaum considered rank 4 polytopes that are locally toroidal, meaning that all their facets and vertex figures are either spherical or toroidal, and not all of them are spherical. In [5] an almost complete answer to Grünbaum's problem, the classification of rank 4 regular locally toroidal polytopes, is given. In this paper we follow [2] and generalize the concept of a polytope to that of a hypertope, and study some interesting locally toroidal ones.

In [3] some examples of chiral hypertopes of rank 4 with certain diagrams are given, satisfying that their residues of rank 3 are either spherical or toroidal. In this paper we continue with the study of locally toroidal 4-hypertopes, in the sense that their residues of rank 3 are toroidal. Using $PSL(2,q)$ as their rotational subgroups, we found an infinite family of hypertopes that contains both regular and chiral hypertopes. Moreover, the chiral ones have the very interesting property that they admit both proper and improper correlations, a feature that is known to be impossible for polytopes (see [4, Lemma 3.1]).

We construct these hypertopes as coset geometries $\Gamma = (G; \{G_0, G_1, G_2, G_3\})$ where G is $PSL(2,q)$ with $q = p$ or $q = p^2$ for certain primes p. In these geometries two maximal parabolic subgroups are alternating groups A_4 and the other two are $E_q : C_3$. Therefore 3 must divide $q - 1$ for our construction to work, and there is a third root of unity e in the field of order q. In the case $q = p^2$ we require that e is not in the subfield of order p, which is equivalent to requiring that $p \equiv 2 \mod 3$. We get an infinite family of regular hypertopes when $q = p^2$ and an infinite family of chiral hypertopes when $q = p$. More precisely in the case $q = p^2$ two residues are hypermaps of type $(3,3,3)_{(2,0)}$ and the other two are hypermaps of type $(3,3,3)_{(p,0)}$; in the case $q = p$ two residues are hypermaps $(3,3,3)_{(2,0)}$ and the other two are both chiral hypermaps of types $(3,3,3)_{(1,e)}$ and $(3,3,3)_{(e,1)}$, respectively.

The paper is organised as follows. In Section 2, we give the definitions and notation needed to understand this paper. In Section 3, we construct $PSL(2,q)$, with $q \in \{p, p^2\}$ and p a prime, as a C^+-group. In Section 4, we show that the C^+-groups obtained in the previous section give hypertopes. In Section 5 we show that the hypertopes obtained are either regular or chiral. In Section 6, we give a geometric description of the hypertopes we constructed, in terms of objects of the projective line $PG(1,q)$.

2 Preliminaries

2.1 Incidence geometries

Following [1], an *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called the *elements* of Γ;

- I is a set whose elements are called the *types* of Γ;
- $t : X \to I$ is a *type function*, associating to each element $x \in X$ a type $t(x) \in I$;
- $*$ is a binary relation on X called *incidence*, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The *incidence graph* of Γ is the graph whose vertex set is X and where two vertices are joined provided the corresponding elements of Γ are incident. A *flag* is a clique of the incidence graph of Γ. The *type* of a flag F is $\{t(x) : x \in F\}$. A *chamber* is a flag of type I.

An element x is *incident* to a flag F and we write $x * F$ for that, when x is incident to all elements of F. An incidence system Γ is a *geometry* or *incidence geometry* if every flag of Γ is contained in a chamber. The *rank* of Γ is the number of types of Γ, namely the cardinality of I.

Let $\Gamma := (X, *, t, I)$ be an incidence geometry and F a flag of Γ. The *residue* of F in Γ is the incidence geometry $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- $X_F := \{x \in X : x * F, x \notin F\}$;
- $I_F := I \setminus t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F.

An incidence system Γ is *connected* if its incidence graph is connected; Γ is *residually connected* when each residue of rank at least two of Γ (including itself) has a connected incidence graph; Γ is called *thin* (resp. *firm*) when every residue of rank one of Γ contains exactly two (resp. at least two) elements. As in [2], we say that a *hypertope* is a thin incidence geometry which is residually connected.

Let $\Gamma := (X, *, t, I)$ be an incidence system. An *automorphism* of Γ is a mapping $\alpha : (X, I) \to (X, I) : (x, t(x)) \mapsto (\alpha(x), t(\alpha(x)))$ where

- α is a bijection on X;
- for each x, $y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each x, $y \in X$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

Note that α induces a bijection on I. When $t(x) = i$ we say that x is an element of *type* i, or equivalently, that x is an i*-element*. The set of automorphisms of Γ is a group denoted by $Aut(\Gamma)$.

An automorphism α of Γ is called *type preserving* when for each $x \in X$, $t(\alpha(x)) = t(x)$ (i.e. α maps each element on an element of the same type). The set of type-preserving automorphisms of Γ is a group denoted by $Aut_I(\Gamma)$ and obviously $Aut_I(\Gamma) \leqslant Aut(\Gamma)$.

A *correlation* is a non-type-preserving automorphism, that is an element of $Aut(\Gamma) \setminus Aut_I(\Gamma)$. A *duality* is correlation that induces an involutory permutation on I.

An incidence geometry Γ is *chamber-transitive* if $Aut_I(\Gamma)$ is transitive on all chambers of Γ. Finally, an incidence geometry Γ is *regular* if $Aut_I(\Gamma)$ acts regularly on the chambers, that is, the action is semi-regular (free) and transitive.

Observe that chamber-transitivity implies flag-transitivity, that is, for each $J \subseteq I$, there is a unique orbits on the flags of type J under the action of $Aut(\Gamma)$.

Sometimes, when $Aut(\Gamma)$ is not transitive on the chambers of Γ, it has two orbits. We are also interested in those hypertopes having two orbits with an extra condition. Two chambers C and C' of a thin incidence geometry of rank r are called *i-adjacent* if C and C' differ only in their i-elements. We then denote C' by C^i. Let $\Gamma(X, *, t, I)$ be a thin incidence geometry. We say that Γ is *chiral* if $Aut_I(\Gamma)$ has two orbits on the chambers of Γ such that any two adjacent chambers lie in distinct orbits. Moreover, if Γ is residually connected, we call Γ a *chiral hypertope.*

When Γ is a chiral hypertope, if $Aut(\Gamma) \neq Aut_I(\Gamma)$, correlations may either interchange the two orbits or preserve them. A correlation that interchanges the two orbits is said to be *improper* and a correlation that preserves them is said to be *proper.*

The following proposition shows how to construct an incident geometry starting from a group.

Proposition 1. *(Tits Algorithm, 1956) [7] Let n be a positive integer and $I := \{1, \dots, n\}$. Let G be a group together with a family of subgroups $(G_i)_{i \in I}$, X the set consisting of all cosets $G_i g$ with $g \in G$ and $i \in I$, and $t : X \to I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by :*

$$G_i g_1 * G_j g_2 \text{ iff } G_i g_1 \cap G_j g_2 \neq \varnothing.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having a chamber. Moreover, the group G acts by right multiplication as an automorphism group on Γ. Finally, the group G is transitive on the flags of rank less than 3.

When a geometry Γ is constructed using the proposition above, we denote it by $\Gamma(G; (G_i)_{i \in I})$ and call it a *coset geometry.* The subgroups $(G_i)_{i \in I}$ are called the *maximal parabolic subgroups.*

2.2 Coset geometries from C^+-groups

As in [2], consider a pair (G^+, R) with G^+ being a group and $R := \{\alpha_1, \dots, \alpha_{r-1}\}$ a set of generators of G^+. Define $\alpha_0 := 1_{G^+}$ and $\alpha_{ij} := \alpha_i^{-1}\alpha_j$ for all $i, j \in I := \{0, \dots, r-1\}$. Let $G^+_J := \langle \alpha_{ij} \mid i, j \in J \rangle$ for $J \subseteq I$.

If the pair (G^+, R) satisfies the following condition called the *intersection property* IP^+, we say that (G^+, R) is a *C^+-group.*

$$G^+_J \cap G^+_K = G^+_{J \cap K},$$

for all $J, K \subseteq I$, with $|J|, |K| \geqslant 2$. The following construction produces an incidence system from a C^+-group.

Construction 2.1. *[2] Let $R = \{\alpha_1, \dots, \alpha_{r-1}\}$ be an independent generating set of the group G^+. Define $G_i := \langle \alpha_j | j \neq i \rangle$ for $i = 1, \dots, r-1$ and $G_0 := \langle \alpha_1^{-1}\alpha_j | j \geqslant 2 \rangle$. The coset geometry $\Gamma(G^+, R) := \Gamma(G^+; (G_i)_{i \in \{0, \dots, r-1\}})$ constructed using Tits' algorithm (see Proposition 1) is the incidence system associated to the pair (G^+, R).*

We denote $\Gamma(G^+, R)$ simply by Γ whenever G^+ and R are clear from the context. We set $I := \{0, \dots, r-1\}$, where $r := |R| + 1$.

It is convenient to represent (G^+, R) by a graph $\mathcal{B}$ with r vertices which we call the *B-diagram* of (G^+, R). The vertex set of $\mathcal{B}$ is the set $\{\alpha_0, \dots, \alpha_{r-1}\}$. The edges $\{\alpha_i, \alpha_j\}$ of this graph are labelled by $o(\alpha_i^{-1}\alpha_j) = o(\alpha_j^{-1}\alpha_i)$. We take the convention of dropping an edge if its label is 2 and of not writing the label if it is 3. Vertices of $\mathcal{B}$ are represented by small circles. Finally, we sometimes attach to each vertex α_i the corresponding subgroup G_i defined in Construction 2.1.

Observe that, thanks to Proposition 1 that ensures that G^+ is transitive on the flags of rank less than 3, we not only know the number of elements of type i for every i, that is the index of G_i in G^+, but also the number of elements of type j incident to a given element of type i, that is the index of $G_i \cap G_j$ in G_i.

The residue of the element G_i is the coset geometry $\Gamma(G_i, (G_i \cap G_j)_{j \in I \setminus \{i\}})$. Its diagram is obtained by removing the vertex α_i from the diagram of $\Gamma(G^+; (G_i)_{i \in I})$.

The coset geometry $\Gamma(G^+, R)$ gives an incidence system using Proposition 1. In what follows we prove that any such coset geometry has a connected incidence graph if its rank is at least 3.

Proposition 2. *If $|R| \geqslant 2$, then $\Gamma(G^+, R)$ has a connected incidence graph.*

Proof. As for any $g \in G^+$, $\{G_i g \,|\, i \in I\}$ is a set of mutually incident elements of Γ, it is sufficient to prove that G_i and $G_i g$ are in the same connected component of the incidence graph (of Γ) for every $g \in G^+$ and every $i \in I$. As $G^+ = \langle \alpha_i \,|\, i \in I \rangle$, we can assume $g \in \{\alpha_i : i \in I\}$. If $i \neq 0$, then $G_i \cap G_i\alpha_j = G_i$ for any $j \in I \setminus \{i\}$. Moreover, $G_i * G_k$ and $G_k * G_i\alpha_i$ for any $k \in I \setminus \{0, i\}$, which is a non-empty set since $|I| - 2 = |R| - 1 \geqslant 1$. If $i = 0$, then $G_0 * G_k$ and $G_k * G_0\alpha_j$ for any $k \in I \setminus \{1, j\}$. □

Although the incidence geometry Γ has a connected incidence graph, it need not be residually connected. Moreover, Γ might not be a thin geometry, and hence Γ need not be a hypertope. Furthermore, in general Γ might not be transitive on flags of rank 3, and it might have many orbits of chambers. In the construction we shall give in the next section, the geometry that we obtain will in fact be a hypertope and have only two orbits of chambers. The following theorem will help us to decide if a hypertope is regular of chiral.

Theorem 3. *[2] Let (G^+, R) be a C^+-group. Let $\Gamma := \Gamma(G^+, R)$ be the coset geometry associated to (G^+, R) using Construction 2.1. If Γ is a hypertope and G^+ has two orbits on the set of chambers of Γ, then Γ is chiral if and only if there is no automorphism of G^+ that inverts all the elements of R. Otherwise, there exists an automorphism $\sigma \in Aut(G^+)$ that inverts all the elements of R and the group G^+ extended by σ is regular on Γ.*

2.3 Toroidal hypermaps of type $(3, 3, 3)_{(a,b)}$

The hypertopes we shall construct in Section 4 have residues of rank 3 with a known structure, namely, they are toroidal hypermaps of type $(3, 3, 3)$.

If Γ is a hypertope of rank three, then it is also a hypermap. A toroidal hypertope (or toroidal hypermap) is either a map or a hypermap embeddable on the torus. The toroidal (regular or chiral) hypertopes of rank 3 are divided into the following families: the toroidal maps $\{3,6\}_{(a,b)}$, $\{6,3\}_{(a,b)}$, $\{4,4\}_{(a,b)}$, and the hypermaps $(3,3,3)_{(a,b)}$ with $(a,b)\neq(1,1)$. Note that the hypermap $(3,3,3)_{(a,b)}$ is obtained from the toroidal map $\{6,3\}_{(a,b)}$ by doubling the fundamental region. Indeed as $\{6,3\}_{(a,b)}$ is bipartite it is possible to take one monochromatric set of vertices to be the hyperedges of the hypermap $(3,3,3)_{(a,b)}$ (see [8]).

The rotation subgroup G of the automorphism group of a rank three toroidal hypermap is as follows for some integers a and b.

$$\langle x,y\,|\,x^3,y^3,(x^{-1}y)^3,(xy^{-1}x)^a(xy)^b\rangle \tag{1}$$

This hypermap is denoted by $(3,3,3)_{(a,b)}$. The above presentation readily shows that such a hypermap has a B-diagram that is a triangle with no numbers on the edges.

2.4 $PSL(2,q)$ acting on the projective line

For a prime power $q=p^n$ let $GF(q)^*$ be the multiplicative group of the Galois field $GF(q)$ on q elements.

A *primitive element* of $GF(q)$ is a generator of the multiplicative group $GF(q)^*$.

For any positive integer $k<q$, a solution of $x^k=1$ which is not a solution of $x^j=1$ for any $j\in\{1,\ldots,k-1\}$ is called a *kth primitive root of unity*. If i is a kth primitive root of unity, then k divides $q-1$ and $\sum_{j=0}^{k-1} i^j=0$.

Let V be the 2-dimensional vector space $GF(q)^2$ over $GF(q)$. Consider the relation $\sim$ in $V\setminus\{(0,0)\}$ defined as follows.

$$(x_0,x_1)\sim(y_0,y_1) \text{ if and only if } (y_0,y_1)=\lambda(x_0,x_1), \text{ for some } \lambda\in GF(q)^*.$$

The *projective line* $PG(1,q)$ is the set of equivalence classes $V\setminus\{(0,0)\}/\sim$. The elements of the projective line $[x_0,x_1]$ can be identified with their non-homogeneous coordinates by the following bijection where i is a $(q-1)$th root of unity.

$$\begin{array}{rcl} PG(1,q) & \to & \{0,1,i,i^2,\ldots,i^{q-2},\infty\}\,,\\ [x_0,x_1] & \mapsto & \begin{cases} \frac{x_0}{x_1} & \text{if } x_1\neq 0,\\ \infty & \text{if } x_1=0\,. \end{cases} \end{array}$$

Consider the special linear group $SL(2,q)=\{A\in GL(2,q)\mid \det(A)=1\}$ and denote by Id its identity matrix. As $PSL(2,q)=SL(2,q)/Z$ where $Z=\{\pm Id\}$, the elements of $PSL(2,q)$ can be seen as unordered pairs $\pm A$ with $A\in SL(2,q)$. Observe that in characteristic 2, $|Z|=1$. For convenience, we shall denote $PSL(2,q)$ by G^+. An element of G^+ will be given by one of its two representative elements in $SL(2,q)$. Consequently

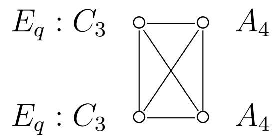

Figure 1: Tetrahedral B-diagram for $PSL(2,q)$.

equalities are to be taken modulo $Z = \{\pm Id\}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,q)$ and $z \in PG(1,q)$, consider the correspondence

$$\varphi :\ G^+ \times PG(1,q) \to PG(1,q),\ (\pm A, z) \mapsto Az$$

defined as follows.

$$Az = \frac{az+b}{cz+d}.$$

As $-Az = Az$, φ is well-defined and gives an action of G^+ on $PG(1,q)$.

Lemma 4. *[6, Lemma 5.3] Let q be a power of a prime. Then $SL(2,q)$ is generated by the elementary matrices of the form*

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ with } b, c \in GF(q).$$

3 A C$^+$-group for $PSL(2,q)$

Let $q \in \{p, p^2\}$ for a prime number p with 3 being a divisor of $q-1$, and let $e \in GF(q)$ be a third primitive root of unity. In the case $q = p^2$ we also assume that 3 is not a divisor of $p-1$ so that

$$GF(p^2) \cong \{a + eb : a, b \in GF(p)\}.$$

In this section we prove that $PSL(2,q)$ is a C$^+$-group with the B-diagram of Figure 1 (where the labels of the corners are the groups corresponding to each rank 3 residue).

In order to construct this C$^+$-group we consider the following elements of $PSL(2,q)$:

$$\alpha_1 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\ \alpha_2 := \begin{pmatrix} e & e^2 \\ 0 & e^2 \end{pmatrix} \text{ and } \alpha_3 := \begin{pmatrix} e^2 & e \\ 0 & e \end{pmatrix},$$

and we let $G^+ = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, $G_i := \langle \alpha_j | j \neq i \rangle$ for $i = 1,2,3$ and $G_0 := \langle \alpha_1^{-1}\alpha_j | j \geqslant 2 \rangle$. In what follows we show that $G_2 := \langle \alpha_1, \alpha_3 \rangle$ and $G_3 := \langle \alpha_1, \alpha_2 \rangle$ are both isomorphic to A_4, while $G_0 := \langle \alpha_1^{-1}\alpha_2, \alpha_1^{-1}\alpha_3 \rangle$ and $G_1 := \langle \alpha_2, \alpha_3 \rangle$ are both isomorphic to $E_q : C_3$.

Lemma 5. *The groups G_2 and G_3 are both isomorphic to A_4 and their intersection is $\langle \alpha_1 \rangle$. Moreover $(G_2, \{\alpha_1, \alpha_3\})$ and $(G_3, \{\alpha_1, \alpha_2\})$ are C$^+$-groups with triangular B-diagram of type* $(3,3,3)$.

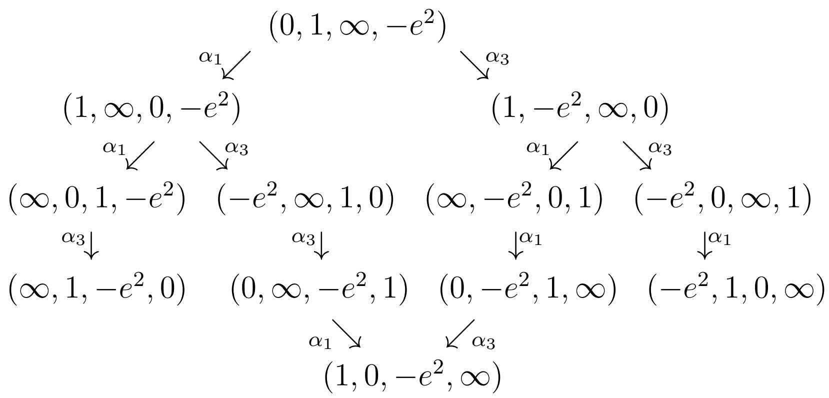

Figure 2: Action of G_2 on the 4-tuple $(0, 1, \infty, -e^2)$.

Proof. We first observe that, for each $i \in \{2, 3\}$, $\{\alpha_1, \alpha_i\}$ is an independent set, which is sufficient to guarantee that it gives a set of generators of a C$^+$-group. Now let us compute the order of the elements that give the B-diagram. It is straightforward to see that α_1 has order 3. To compute the order of the other elements we use the fact that e is a third root of unity and hence $e^2 + e + 1 = 0$. Therefore the following equalities hold for any $c \in GF(q)$.

$$\begin{pmatrix} e & 0 \\ c & e^2 \end{pmatrix}^2 = \begin{pmatrix} e^2 & 0 \\ -c & e \end{pmatrix}, \quad \begin{pmatrix} e & 0 \\ c & e^2 \end{pmatrix}^3 = Id;$$
$$\begin{pmatrix} e & c \\ 0 & e^2 \end{pmatrix}^2 = \begin{pmatrix} e^2 & -c \\ 0 & e \end{pmatrix}, \quad \begin{pmatrix} e & c \\ 0 & e^2 \end{pmatrix}^3 = Id.$$

In particular, this shows that both α_2 and α_3 have order 3. In addition,

$$\alpha_1^{-1}\alpha_2 = \begin{pmatrix} e & 0 \\ e & e^2 \end{pmatrix} \quad \text{and} \quad \alpha_1^{-1}\alpha_3 = \begin{pmatrix} e^2 & 0 \\ e & e \end{pmatrix},$$

hence these elements have also order 3.

To prove that the groups are isomorphic to A_4 we first observe that for $i \in \{2, 3\}$

$$\alpha_1^3 = \alpha_i^3 = (\alpha_1^{-1}\alpha_i)^3 = (\alpha_1\alpha_i^{-1}\alpha_1)^2 = Id.$$

As Id, α_1, α_i, α_1^2, $\alpha_i\alpha_1$, $\alpha_1\alpha_i$, α_i^2, $\alpha_i\alpha_1^2$, $\alpha_i^2\alpha_1$, $\alpha_1^2\alpha_i$, $\alpha_1\alpha_i^2$, $\alpha_1\alpha_i^2\alpha_1$ are distinct elements of $\langle\alpha_1, \alpha_i\rangle$, from 1 we conclude that G_i is isomorphic to A_4, the rotational subgroup of the hypermap of type $(3, 3, 3)_{(2,0)}$. From the given enumeration of the elements of these two groups we conclude that their intersection is $\langle\alpha_1\rangle$. □

Figures 2 and 3 show that G_2 is in the stabilizer of $\{0, 1, \infty, -e^2\}$, while G_3 is in the stabilizer of $\{0, 1, \infty, -e\}$ (and both groups are isomorphic to A_4).

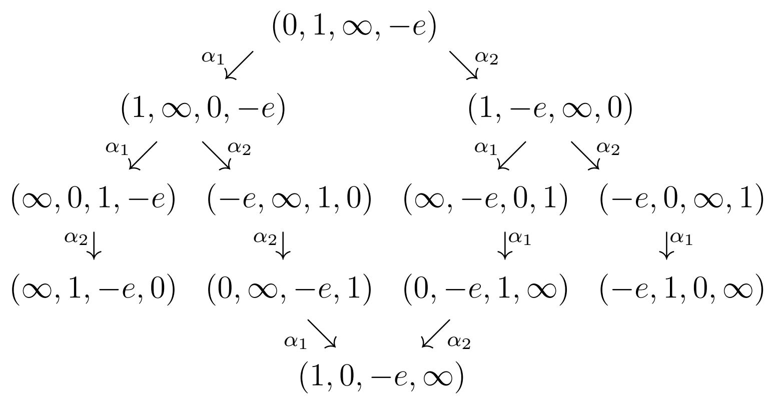

Figure 3: Action of G_3 on the 4-tuple $(0, 1, \infty, -e)$

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Lemma 6. *The groups G_0 and G_1 are both isomorphic to $E_q : C_3$, more precisely, $G_0 = H_0 : \langle \alpha_2^{-1}\alpha_3 \rangle$ and $G_1 = H_1 : \langle \alpha_2^{-1}\alpha_3 \rangle$ where H_0 and H_1 are the following elementary abelian p-groups of order q.*

$$H_0 = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in GF(q) \right\} \text{ and } H_1 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in GF(q) \right\}.$$

Moreover $G_0 \cap G_1 = \langle \alpha_2^{-1}\alpha_3 \rangle$. Finally, $(G_0, \{\alpha_1^{-1}\alpha_2, \alpha_1^{-1}\alpha_3\})$ and $(G_1, \{\alpha_2, \alpha_3\})$ are C^+-groups with triangular B-diagram of type $(3, 3, 3)$.

Proof. First observe that the sets $\{\alpha_2, \alpha_3\}$ and $\{\alpha_1^{-1}\alpha_2, \alpha_1^{-1}\alpha_3\}$ are independent, thus G_0 and G_1 are C^+ -groups. Now let us compute the order of the elements that give the B-diagram. As noted above, the elements α_2, α_3, $\alpha_1^{-1}\alpha_2$ and $\alpha_1^{-1}\alpha_3$ have order 3. Now $E := \alpha_2^{-1}\alpha_3 = \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix}$ has also order 3 and belongs to $G_0 \cap G_1$.

Consider the following subgroups of $PSL(2, q)$:

$$H_0 = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in GF(q) \right\} \text{ and } H_1 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in GF(q) \right\}$$

When $q = p$, H_0 and H_1 are cyclic groups of order p. More precisely,

$$H_0 = \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle \text{ and } H_1 = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle.$$

When $q = p^2$, H_0 and H_1 are elementary abelian groups of order p^2. More precisely,

$$H_0 = \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \rangle \text{ and } H_1 = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \rangle,$$

since we assumed that 3 does not divide $p-1$ and therefore $GF(q) = \{a + eb : a, b \in GF(p)\}$.

We claim that $G_0 \cong H_0 : \langle E \rangle$ and $G_1 \cong H_1 : \langle E \rangle$. The following equalities prove that H_0 and H_1 are subgroups of G_0 and G_1, respectively.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (\alpha_1^{-1}\alpha_3)(\alpha_1^{-1}\alpha_2)^{-1}(\alpha_1^{-1}\alpha_3) \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \alpha_2\alpha_3^{-1}\alpha_2$$
$$\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} = (\alpha_1^{-1}\alpha_2)^{-1}(\alpha_1^{-1}\alpha_3)^2 \qquad \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \alpha_3^{-1}\alpha_2^{-1} \tag{2}$$

On the other hand, as

$$\alpha_1^{-1}\alpha_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \quad \text{and} \quad \alpha_1^{-1}\alpha_3 = \begin{pmatrix} 1 & 0 \\ e^2 & 1 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix}$$

we have that $\alpha_1^{-1}\alpha_2, \alpha_1^{-1}\alpha_3 \in H_0 : \langle E \rangle$, while

$$\alpha_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \quad \text{and} \quad \alpha_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix}$$

show that $\alpha_2, \alpha_3 \in H_1 : \langle E \rangle$.

With this we have that the groups G_1 and G_0 are both isomorphic to $E_q : \langle E \rangle$.

Since an element of G_0 is equal to $\begin{pmatrix} e^i & 0 \\ c & e^{2i} \end{pmatrix}$ for some $c \in GF(q)$ and $i \in \{0, 1, 2\}$, while an element of G_1 is equal to $\begin{pmatrix} e^j & b \\ 0 & e^{2j} \end{pmatrix}$ for some $b \in GF(q)$ and $j \in \{0, 1, 2\}$, we have that $G_0 \cap G_1 = \langle E \rangle$. □

Lemma 7. *$(G^+, \{\alpha_1, \alpha_2, \alpha_3\})$ is a C^+-group with tetrahedral B-diagram as in Figure 1.*

Proof. By Lemmas 5 and 6 we have that G_i, $i = 0, 1, 2, 3$ are C$^+$-groups, $G_2 \cap G_3 = \langle \alpha_1 \rangle$ and $G_1 \cap G_0 = \langle \alpha_2^{-1}\alpha_3 \rangle$. We have that G_0 is a subgroup of the stabilizer of 0 in G^+ and G_1 is a subgroup of the stabilizer of ∞ in G^+. Figure 2 shows that the only elements of G_2 fixing 0 are the identity, $\alpha_3^2\alpha_1$ and $\alpha_1^2\alpha_3$, hence $G_0 \cap G_2 = \langle \alpha_1^{-1}\alpha_3 \rangle$. The same figure shows that the only elements of G_2 fixing ∞ are the identity, α_3 and α_3^2, thus $G_1 \cap G_2 = \langle \alpha_3 \rangle$. Figure 3 shows that the only elements of G_3 fixing 0 are the identity, $\alpha_2^2\alpha_1$ and $\alpha_1^2\alpha_2$, hence $G_0 \cap G_3 = \langle \alpha_1^{-1}\alpha_2 \rangle$. The same figure shows that the only elements of G_3 fixing ∞ are the identity, α_2 and α_2^2, thus $G_1 \cap G_3 = \langle \alpha_2 \rangle$. From this we conclude that $(G^+, \{\alpha_1, \alpha_2, \alpha_3\})$ is a C$^+$-group. Moreover, as $G_i \cap G_j$ are cyclic groups of order three, for $i, j \in \{0, 1, 2, 3\}$ with $(i \neq j)$, the B-diagram is a complete graph K_4 with edges labelled 3. □

Lemma 8. *The group G^+ is isomorphic to $PSL(2, q)$.*

Proof. This follows from the description of the subgroups H_0 and H_1 in Lemma 6 and Lemma 4. □

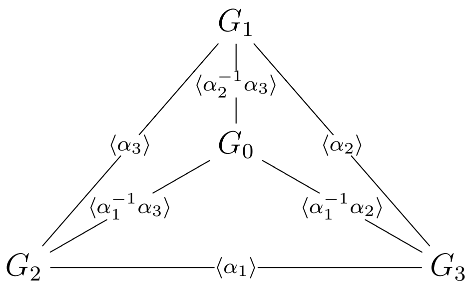

Figure 4: The intersections $G_i \cap G_j$.

4 A hypertope for $PSL(2, q)$

As pointed out in [2], not every coset geometry is a hypertope. In this section we shall see that the coset geometry $\Gamma := \Gamma(G^+, R) := \Gamma(G^+; (G_i)_{i\in I})$ where $I = \{0, 1, 2, 3\}$, G^+ is isomorphic to $PSL(2, q)$ and the G_i's are the groups defined in Section 3, is a hypertope. Since Γ is a coset geometry, then we only have to show that it is thin and residually connected.

We observe that Γ has a lot of symmetry, and we shall use such symmetry to show that it is a hypertope. Note that since G^+ is a C^+-group, then $G_1 \cap G_2 \cap G_3$ is trivial. This implies that the action of G^+ on the flags by right multiplication is free (semi-regular) on the chambers. Moreover, as Γ is a coset geometry, it is transitive on the flags of rank less than 3. We shall show that in fact Γ is transitive on the flags of rank 3. To show this, in the following two lemmas we analyze some incidences among elements of Γ, and some incidences among elements and flags of Γ.

From the B-diagram of G^+, since G_i with $i \in \{2, 3\}$ is isomorphic to A_4 and $G_i \cap G_j \cong C_3$, we know that each element of type $i \in \{2, 3\}$ is incident to exactly four elements of type $j \neq i$. Similarly, since G_i with $i \in \{0, 1\}$ is isomorphic to $E_q : C_3$, we have that each element of type $i \in \{0, 1\}$ is incident to exactly q elements of type $j \neq i$. Remark also that, if $G_i g * G_j$ then $G_i g = G_i h$ for some $h \in G_j$. Moreover, if T is a transversal for $G_i \cap G_j$ in G_j, then $G_i g = G_i h = G_i t$ for some $t \in T$. More precisely we have the following lemma.

Lemma 9. *Let $i, j \in I$ with $i \neq j$ and let $g \in G^+$. Then*

$$G_i g * G_j \Leftrightarrow \begin{cases} G_i g \in \{G_i h \mid h \in H_j\} & \text{if } j \in \{0,1\}; \\ G_i g \in \{G_i, G_i\alpha_i, G_i\alpha_i\alpha_k, G_i\alpha_i\alpha_k^{-1}\} & \text{if } j \in \{2,3\} \text{ and } \{i,j,k\} = \{1,2,3\}; \\ G_i g \in \{G_0, G_0\alpha_1, G_0\alpha_1^{-1}, G_0\alpha_k\} & \text{if } j,k \in \{2,3\} \text{ and } \{i,j,k\} = \{0,2,3\}. \end{cases}$$

Proof. It can be easily checked that all of the cosets in the set on the right side of the equivalences are incident to the respective G_j.

Suppose that $j \in \{0, 1\}$. By Lemma 6, H_j is a subgroup of G_j of order q. Thus $g \in H_j$ implies that $G_i g \cap G_j \neq \varnothing$. As $G_i \cap G_j$ has order 3, if $G_i g * G_j$, then $|G_i g \cap G_j| = 3$. It remains to prove that the cardinality of $\{G_i g \mid g \in H_j\}$ is q. Suppose that $g, h \in H_j$ and $G_i g = G_i h$. Then $gh^{-1} \in G_i \cap H_j = (G_i \cap G_j) \cap H_j$. In all the cases $G_i \cap G_j$ is generated by an element of order 3 which does not belong to H_j, therefore all the intersections $(G_i \cap G_j) \cap H_j$ are trivial, and so $g = h$.

For the cases when $j \in \{2, 3\}$, by Lemma 7 (see also Figure 4 we have the following:

- If $i \neq 0$, then $G_i \cap G_j = \langle \alpha_k \rangle$ where $\{i, j, k\} = \{1, 2, 3\}$. Hence

$$T = \{Id,\ \alpha_i,\ \alpha_i\alpha_k,\ \alpha_i\alpha_k^{-1}\}$$

  is a transversal for $G_i \cap G_j$ in G_j and $G_i g = G_i t$ for some $t \in T$.

- If $i = 0$, then $G_0 \cap G_j = \langle \alpha_1^{-1}\alpha_k \rangle$ where $\{j, k\} = \{2, 3\}$. Hence

$$T = \{Id,\ \alpha_1,\ \alpha_1^{-1},\ \alpha_k\}$$

  is a transversal for $G_0 \cap G_j$ in G_j and $G_0 g = G_0 t$ for some $t \in T$. □

Proposition 10. *Let $i, j, k \in I$ be distinct and let $g \in G^+$. Then*

$$G_i g * \{G_j, G_k\} \Leftrightarrow G_i g = G_i h \ \textit{ for some } \ h \in G_j \cap G_k\,.$$

Proof. Obviously the right hand side of the equivalence implies the left one. The proof of the converse will be divided in cases covering all the possibilities for the set $\{j, k\}$. Assume that $G_i g * \{G_j, G_k\}$.

Case 1. If $\{j, k\} = \{0, 1\}$, then $i \in \{2, 3\}$ and

$$G_i g \in \{G_i h_0 \mid h_0 \in H_0\} \cap \{G_i h_1 \mid h_1 \in H_1\}$$

by means of Lemma 9. Looking at the 12 elements of G_i, we see that for $h_0 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H_0$ and $h_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H_1$,

$$h_0 h_1^{-1} = \begin{pmatrix} 1 & -b \\ c & 1 - bc \end{pmatrix} \in G_i \Leftrightarrow (b, c) \in \{(0, 0), (1, 1), (-e^{1-i}, -e^{i-1})\}\,.$$

As before, we set $E := \alpha_2^{-1}\alpha_3$. We have to prove that $G_i g = G_i h$ for some $h \in \langle E \rangle = G_0 \cap G_1$ (Figure 4). If $(b, c) = (0, 0)$, then $G_i g = G_i h_0 = G_i h_1 = G_i$. If $(b, c) = (1, 1)$, then $h_1 = \alpha_2\alpha_3^{-1}\alpha_2$ by 2. Hence

$$G_i g = G_i h_0 = G_i h_1 = G_i \alpha_2\alpha_3^{-1}\alpha_2 = G_i E^{i-1}$$

since $G_2\alpha_2\alpha_3^{-1}\alpha_2 = G_2(\alpha_3^{-1}\alpha_2)^2 = G_2 E^{-2} = G_2 E$
and $G_3\alpha_2\alpha_3^{-1}\alpha_2 = G_3\alpha_3^{-1}\alpha_2 = G_2 E^{-1} = G_3 E^2$.

If $(b, c) = (-e^{1-i}, -e^{i-1})$, then

$$G_i g = G_i h_0 = G_i h_1 = G_i h_1 E^{i-1} E^{1-i} = G_i E^{1-i}$$

since $h_1 E^{i-1} = \begin{pmatrix} e^{1-i} & e^{i-1} \\ 0 & e^{i-1} \end{pmatrix}^{-1} \in G_i$. Thus $G_i g = G_i h$ for some $h \in \langle E \rangle = G_0 \cap G_1$.

Case 2. If $\{j,k\} = \{0,2\}$ or $\{0,3\}$, then $i \neq 0$ and we can assume $k = 0$, $j \in \{2,3\}$. By means of Lemma 9 there is $h_0 \in H_0$ such that

$$G_i g = G_i h_0 \in \{G_i, G_i\alpha_i, G_i\alpha_i\alpha_l, G_i\alpha_i\alpha_l^{-1}\}$$

where $l \in I$ is such that $\{i,j,k,l\} = I$. In other words, $G_i g$ must belong to the set

$$S := \{G_i t \mid t \in \{Id, \alpha_i, \alpha_i\alpha_l, \alpha_i\alpha_l^{-1}\},\ H_0 \cap G_i t \neq \varnothing\}.$$

Obviously $G_i \in S$. To see that $G_i\alpha_i \in S$, consider the following elements:

$$\beta_1 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in G_1, \quad \beta_2 := \alpha_1\alpha_3^{-1}\alpha_1 \in G_2 \quad \text{and} \quad \beta_3 := \alpha_1\alpha_2^{-1}\alpha_1 \in G_3.$$

Then $\beta_i\alpha_i \in H_0 \cap G_i\alpha_i$ for any $i \in \{1,2,3\}$. To show that $G_i\alpha_i\alpha_l \notin S$ we remark that

$$\alpha_i\alpha_l(0) = \alpha_i(1) = \begin{cases} \infty & \text{if } i = 1, \\ -e & \text{if } i = 2, \\ -e^2 & \text{if } i = 3, \end{cases}$$

from which it follows $H_0 \cap G_i\alpha_i\alpha_l = \varnothing$, since there is no element in G_i sending $\alpha_i(1)$ to 0. Finally, considering the following elements

$$\gamma_1 := E^{j-1} \in G_1, \quad \gamma_2 := \alpha_1\alpha_3 \in G_2 \quad \text{and} \quad \gamma_3 := \alpha_1\alpha_2 \in G_3,$$

we have that $\gamma_i\alpha_i\alpha_l^{-1} \in H_0$ for any $i \in \{1,2,3\}$. Hence $G_i\alpha_i\alpha_l^{-1} \in S$ and so

$$S = \{G_i, G_i\alpha_i, G_i\alpha_i\alpha_l^{-1}\}.$$

It is now straightforward to check that $S = \{G_i h \mid h \in \langle \alpha_1^{-1}\alpha_l \rangle\}$, from which we conclude that $G_i g = G_i h$ for some $h \in \langle \alpha_1^{-1}\alpha_l \rangle = G_0 \cap G_j$ (Figure 4).

Case 3. If $\{j,k\} = \{1,2\}$ or $\{1,3\}$ we assume $k = 1$ and $j \in \{2,3\}$. Let $l \in I$ such that $\{j,l\} = \{2,3\}$. According to Lemma 9 we have the following:

- If $i = 0$ then $G_i g = G_0 g$ belongs to

  $$\{G_0 t \mid t \in \{Id, \alpha_1, \alpha_1^{-1}, \alpha_l\},\ H_1 \cap G_0 t \neq \varnothing\}$$

  which is equal to $\{G_0, G_0\alpha_1^{-1}, G_0\alpha_l\}$. Namely, E ,E^2 and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ are elements of G_0 such that

  $$E^2\alpha_2, E\alpha_3, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\alpha_1^{-1} \in H_1,$$

  while $\alpha_1(\infty) = 0$ gives $H_1 \cap G_0\alpha_1 = \varnothing$. As $G_0\alpha_1^{-1} = G_0\alpha_l^{-1}$, we conclude that $G_0 g = G_0 h$ for some $h \in \langle \alpha_l \rangle = G_j \cap G_1 = G_j \cap G_k$ (Figure 4).

- If $i \neq 0$ then $G_i g$ belongs to

$$\{G_i t \mid t \in \{Id, \alpha_i, \alpha_i\alpha_1, \alpha_i\alpha_1^{-1}\},\ H_1 \cap G_i t \neq \varnothing\}$$

which is equal to $\{G_i, G_i\alpha_i, G_i\alpha_i\alpha_1\}$. Namely, $\alpha_j\alpha_i, \alpha_i\alpha_j \in H_1$, $G_i\alpha_i\alpha_1 = G_i\alpha_j^{-1}\alpha_i^{-1} = G_i\alpha_i^{-1}$, while $\alpha_i\alpha_1^{-1}(\infty) = \alpha_i(1) = -e^{i-1}$ gives $H_1 \cap G_i\alpha_i\alpha_1^{-1} = \varnothing$. We conclude so that $G_i g = G_i h$ for some $h \in \langle \alpha_i \rangle = G_j \cap G_1 = G_j \cap G_k$ (Figure 4).

Case 4. If $\{j,k\} = \{2,3\}$, then by Lemma 9 we have the following:

- If $i = 0$, then $G_i g = G_0 g$ belongs to

$$\{G_0, G_0\alpha_1, G_0\alpha_1^{-1}, G_0\alpha_2\} \cap \{G_0, G_0\alpha_1, G_0\alpha_1^{-1}, G_0\alpha_3\}$$

which is equal to $\{G_0, G_0\alpha_1, G_0\alpha_1^{-1}\}$ since $\alpha_2\alpha_3^{-1} \notin G_0$.

- If $i = 1$, then $G_i g = G_1 g$ belongs to

$$\{G_1, G_1\alpha_1, G_1\alpha_1\alpha_2, G_1\alpha_1\alpha_2^{-1}\} \cap \{G_1, G_1\alpha_1, G_1\alpha_1\alpha_3, G_1\alpha_1\alpha_3^{-1}\}.$$

Remark that $G_1\alpha_1\alpha_k = G_1\alpha_k^{-1}\alpha_1^{-1} = G_1\alpha_1^{-1}$ by means of the fact that $\alpha_1\alpha_k$ is an involution for $k = 2,3$. Moreover $(\alpha_1\alpha_2^{-1})(\alpha_1\alpha_3^{-1})^{-1} \notin G_1$ because it does not fix ∞. Hence $G_1 g$ belongs to $\{G_1, G_1\alpha_1, G_1\alpha_1^{-1}\}$. □

We are now ready to show that G^+ is transitive on the flags of rank 3.

Proposition 11. *Let $J = \{i,j,k\} \subset \{0,1,2,3\}$ with i,j and k all distinct. The action of G^+ on the flags of type J is transitive.*

Proof. Let $F = \{G_i, G_j, G_k\}$ and $F' = \{G_i a_i, G_j a_j, G_k a_k\}$. By Proposition 1, G^+ is transitive on the flags of rank 2. Hence, there exists a flag F'' and $a \in G^+$ such that $F' = F''a$ with $F'' = \{G_i g, G_j, G_k\}$. Without loss of generality we may assume that $(i,j,k) \in \{(0,2,3),(1,2,3),(0,1,3),(0,1,2)\}$. Then by Proposition 10, there exists $h \in G_i g \cap G_j \cap G_k$. Hence $F'' = Fh$ and therefore $F' = Fha$. □

To show that Γ is a thin geometry, we need to show that every residue of rank 1 has exactly two elements. A residue of rank 1 is the set of elements of Γ that are incident to a flag of type $J \subset \{0,1,2,3\}$, with $|J| = 3$. By Proposition 11, all residues of rank 1 corresponding to flags of a given type J are isomorphic. Hence, it is enough to show that, for $\{i,j,k,l\} = \{0,1,2,3\}$, the sets $\{G_j g \mid G_j g * \{G_i, G_k, G_l\}\}$ all have cardinality two.

Proposition 12. *The coset geometry Γ is a thin geometry.*

Proof. Following the above discussion, we need to show that four sets have cardinality two. To do this, we shall show that:

$$\begin{array}{lcl} G_0 g * \{G_1, G_2, G_3\} & \Leftrightarrow & G_0 g \in \{G_0, G_0\alpha_1^{-1}\}; \\ G_1 g * \{G_0, G_2, G_3\} & \Leftrightarrow & G_1 g \in \{G_1, G_1\alpha_1\}; \\ G_2 g * \{G_0, G_1, G_3\} & \Leftrightarrow & G_2 g \in \{G_2, G_2\alpha_2\}; \\ G_3 g * \{G_0, G_1, G_2\} & \Leftrightarrow & G_3 g \in \{G_3, G_3\alpha_3\}. \end{array} \tag{3}$$

It is straightforward to see that the elements on the sets of the right are incident to the flags on the left.

We start by assuming that $G_ig*\{G_j,G_2,G_3\}$ where $\{i,j\}=\{0,1\}$. As $G_ig*\{G_2,G_3\}$, by Proposition 10, $G_ig\in\{G_i,G_i\alpha_1,G_i\alpha_1^{-1}\}$. As we also have G_ig*G_j we observe that

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\alpha_1^{-1}\in G_0\alpha_1^{-1}\cap G_1$$

while $G_0\alpha_1\cap G_1=\varnothing$ since any element of $G_0\alpha_1$ sends ∞ to 0 and so it does not belong to G_1 which is in the stabilizer of ∞. As $G_0g*G_1\Leftrightarrow G_0*G_1g^{-1}$, this proves the first two equivalences in 3.

Now let $G_ig*\{G_0,G_1,G_j\}$ where $\{i,j\}=\{2,3\}$. As G_ig*G_0 and G_ig*G_1, there exists an element of G_ig fixing 0 and an element fixing ∞. In addition, as G_ig*G_j, by Lemma 9, $G_ig\in\{G_i,G_i\alpha_i,G_i\alpha_i\alpha_1,G_i\alpha_i\alpha_1^{-1}\}$.

Let $i=2$. Looking at Figure 2 and Figure 3, we have that $\alpha_2\alpha_1(0)=-e$ and $\alpha_2\alpha_1^{-1}(\infty)=-e$. As G_2 stabilizes $\{0,1,\infty,-e^2\}$, there is no element in $G_2\alpha_2\alpha_1$ fixing 0 and there is no element in $G_2\alpha_2\alpha_1^{-1}$ fixing ∞. Thus $G_2g\in\{G_2,G_2\alpha_2\}$, as $\alpha_3^{-1}\alpha_2\in G_2\alpha_2$ fixes both 0 and ∞.

Now consider $i=3$. We have that $\alpha_3\alpha_1(0)=-e^2$ and $\alpha_3\alpha_1^{-1}(\infty)=-e^2$. As G_2 stabilizes $\{0,1,\infty,-e\}$, there is no element in $G_3\alpha_3\alpha_1$ fixing 0 and there is no element in $G_3\alpha_3\alpha_1^{-1}$ fixing ∞. Thus $G_3g\in\{G_3,G_3\alpha_3\}$, as $\alpha_2^{-1}\alpha_3\in G_3\alpha_3$ fixes both 0 and ∞.

Therefore all the equivalences in (3) are satisfied and so Γ is a thin geometry. □

In order to show that the geometry Γ is a hypertope, we need to show that it is residually connected. That is, we need to show that each residue of rank at least two has a connected incidence graph. We already showed that all the residues of rank 3 are hypermaps of type $(3,3,3)$ (Lemmas 5 and 6). Since we know that all such hypermaps are residually connected, as observed in [2], that will conclude the proof. More precisely, we have the following theorem.

Theorem 13. *Γ is a hypertope.*

Proof. As pointed out before, Γ is a geometry, and by Proposition 12, it is thin. By Lemmas 5 and 6, its rank 3 redidues Γ_i are toroidal hypertopes isomorphic to the coset geometries $\Gamma(G_i,(G_i\cap G_j)_{j\in I\setminus\{i\}})$. This, together with Proposition 2, implies that Γ is residually connected. Therefore Γ is a hypertope. □

4.1 The residues of rank 3

The residues of rank three of Γ are hypermaps of type $(3,3,3)_{(a,b)}$ for some a and b that depend on the type of the residue. To determine the vector (a,b) of the rank 3 residues of Γ we now fix an order for the generators of each G_i. Let the first generator of G_i be the one with minimal label when $i\neq 0$ and let the first generator of G_0 be $\alpha_1^{-1}\alpha_2$.

Lemma 14. *The residues Γ_i with $i\in\{2,3\}$ are regular hypermaps of type $(3,3,3)_{(0,2)}$. When $q=p^2$ the residues Γ_i with $i\in\{0,1\}$ are regular hypermaps of type $(3,3,3)_{(0,p)}$.*

When $q = p$ the residue Γ_0 is the chiral hypermap of type $(3,3,3)_{(1,e)}$ while the residue Γ_1 is the chiral hypermap of type $(3,3,3)_{(e,1)}$, where e is a third root of unity (which is an integer, as $q = p$).

Proof. For the following see 1. As $(\alpha_1\alpha_2)^2 = (\alpha_1\alpha_3)^2 = 1_{G^+}$, the residues Γ_2 and Γ_3 are regular hypermaps of type $(3,3,3)_{(0,2)}$.

Consider first the case $q = p^2$. In this case the order of $(\alpha_1^{-1}\alpha_2)(\alpha_1^{-1}\alpha_3)$ is p. Hence Γ_0 and Γ_1 are regular hypermaps of type $(3,3,3)_{(0,p)}$.

Now let $q = p$. We have that $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^e$ which gives the following relation in G^+,

$$(\alpha_2\alpha_3^{-1}\alpha_2)^e(\alpha_2\alpha_3) = 1_{G^+}.$$

Hence Γ_1 is a chiral hypermap of type $(3,3,3)_{(e,1)}$. We also have that $(\alpha_1^{-1}\alpha_2)(\alpha_1^{-1}\alpha_3) = \begin{pmatrix} 1 & 0 \\ 1+e & 1 \end{pmatrix}$ and $(\alpha_1^{-1}\alpha_2)(\alpha_1^{-1}\alpha_3)^{-1}(\alpha_1^{-1}\alpha_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. As $\begin{pmatrix} 1 & 0 \\ 1+e & 1 \end{pmatrix}^e = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,

$$(\alpha_1^{-1}\alpha_2)(\alpha_1^{-1}\alpha_3)^{-1}(\alpha_1^{-1}\alpha_2)[(\alpha_1^{-1}\alpha_2)(\alpha_1^{-1}\alpha_3)]^e = 1_{G^+}.$$

Hence Γ_0 is a chiral hypermap of type $(3,3,3)_{(1,e)}$. □

5 The symmetries of Γ

As pointed out in the introduction, the hypertopes we have constructed in this paper are highly symmetric. Recall that we have two families: a family of hypertopes $\Gamma^{(p)} = \Gamma(G^+, R)$ with $G^+ = PSL(2,p)$ for any prime number p satisfying $p \equiv 1 \bmod 3$, and a family of hypertopes $\Gamma^{(p^2)} = \Gamma(G^+, R)$ with $G^+ = PSL(2,p^2)$ for any prime p satisfying $p \equiv 2 \bmod 3$, where $R = \{\alpha_1, \alpha_2, \alpha_3\}$ for both families. In this section we shall show that all hypertopes $\Gamma^{(p)}$ are chiral, while all hypertopes $\Gamma^{(p^2)}$ are regular. Furthermore, we study the correlations of the hypertopes and find that all the chiral hypertopes $\Gamma^{(p)}$ have both proper and improper correlations. For any Γ in one of these two families we have the following lemma.

Lemma 15. *G^+ acts with two orbits on chambers of Γ with adjacent chambers in different orbits.*

Proof. This is a consequence of Propositions 11 and 12. □

Lemma 16. *Any hypertope $\Gamma^{(p)}$ is chiral while any hypertope $\Gamma^{(p^2)}$ is regular.*

Proof. By Lemma 14, one residue of $\Gamma^{(p)}$ is chiral. Hence $\Gamma^{(p)}$ is itself chiral, indeed by Theorem 3 the residues of a regular hypertope must all be regular. For $\Gamma^{(p^2)}$ we have that $G^+ = PSL(2, p^2)$. Then, the Frobenius automorphism of $GF(p^2)$ gives rise to the involutory automorphism

$$\varphi : G^+ \to G^+, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

which fixes α_1 and interchanges α_2 and α_3 (since $p \equiv 2 \bmod 3$ and therefore $e^p = e^2$ in that case). Let ψ be the automorphism of G^+ given by conjugation with

$$A := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2, p^2).$$

Then $\psi\varphi$ is an automorphism of G^+ inverting α_1, α_2 and α_3. Hence, by Theorem 3 $\Gamma^{(p^2)}$ is a regular hypertope. □

For the chiral hypertopes $\Gamma^{(p)}$ we will now exhibit proper and improper correlations. In order to give a proper correlation of $\Gamma^{(p)}$, we remark the following general fact. Given a coset geometry $\Gamma := \Gamma(G, (G_i)_{i\in I})$, any $\varphi \in Aut(G)$ satisfying $\{\varphi(G_i) \mid i \in I\} = \{G_i \mid i \in I\}$ gives rise to an automorphism of Γ mapping $G_i g$ to $\varphi(G_i g) = \varphi(G_i)\varphi(g)$. In particular, we have the following lemma.

Lemma 17. *The automorphism η of $G^+ = PSL(2, p)$ defined by conjugation with the involution*

$$B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, p)$$

gives rise to a proper duality of $\Gamma^{(p)}$.

Proof. As η is an involution satisfying

$$\eta(\alpha_1) = \alpha_1^{-1}, \quad \eta(\alpha_2) = \alpha_1^{-1}\alpha_3, \quad \eta(\alpha_3) = \alpha_1^{-1}\alpha_2,$$

it interchanges G_0 with G_1 and G_2 with G_3. □

To give an improper correlation of $\Gamma^{(p)}$, we extend the assignment

$$\mu : G_0 \mapsto G_0\alpha_1^{-1}, \; G_1 \mapsto G_1, \; G_2 \mapsto G_3, \; G_3 \mapsto G_2$$

to a bijection of the set of elements of $\Gamma^{(p)}$ by setting

$$\mu(G_i g) := \mu(G_i) g^A, \qquad \text{where} \quad A := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2, p).$$

Note that A is an involution and that $\alpha_1^A = \alpha_1^{-1}$, $\alpha_2^A = \alpha_3^{-1}$, $\alpha_3^A = \alpha_2^{-1}$. Therefore $\mu(G_0)^A = \alpha_1 G_0$ and $\mu(G_i)^A = G_i$ for any $i \in \{1, 2, 3\}$. Then, for any $j \in I$ we have

$$\mu(G_j g) = \mu(G_j h) \Leftrightarrow \mu(G_j) g^A = \mu(G_j) h^A \Leftrightarrow \mu(G_j)^A g = \mu(G_j)^A h \Leftrightarrow G_j g = G_j h,$$

showing that μ is well-defined and a bijection. To prove that μ is a correlation we need the following lemma.

Lemma 18. $\{G_i g^A \alpha_1 \mid g \in H_0\} = \{G_i h \mid h \in H_0\}$ *for any* $i \in \{1,2,3\}$.

Proof. Let $g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H_0$. Then

$$\begin{aligned} g^A \alpha_1 = \begin{pmatrix} -c & 1 \\ -c-1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c-1 & 1 \end{pmatrix} \in \{G_1 h \mid h \in H_0\} \\ &= \alpha_1 \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \in \{G_2 h \mid h \in H_0\} \cap \{G_3 h \mid h \in H_0\} . \end{aligned}$$

Hence $G_i g^A \alpha_1 \in \{G_i h \mid h \in H_0\}$. □

Lemma 19. *The bijection* μ *is an improper duality of* $\Gamma^{(p)}$.

Proof. By definition μ sends the chamber $\{G_0, G_1, G_2, G_3\}$ to $\{G_0\alpha_1^{-1}, G_1, G_2, G_3\}$, the adjacent chamber. Hence we have only to prove that μ is a correlation. It is enough to prove that for any $g \in G^+$ and any $i, j \in I$ with $i > j$

$$\mu(G_i g) * \mu(G_j) \Leftrightarrow G_i g * G_j .$$

According to the definition of μ and Lemma 9, we have

$$\begin{aligned} \mu(G_i g) * \mu(G_j) &\Leftrightarrow \mu(G_i) g^A * \mu(G_j) \\ &\Leftrightarrow \begin{cases} \mu(G_i) g^A \alpha_1 \in \{\mu(G_i) h \mid h \in H_0\} & \text{if } j = 0 , \\ \mu(G_i) g^A \in \{\mu(G_i) h \mid h \in H_1\} & \text{if } j = 1 , \\ G_3 g^A \in \{G_3, G_3\alpha_3, G_3\alpha_3\alpha_1, G_3\alpha_3\alpha_1^{-1}\} & \text{if } j = 2 \ (\Rightarrow i = 3). \end{cases} \\ &\Leftrightarrow \begin{cases} G_i g \alpha_1^{-1} \in \{G_i h^A \mid h \in H_0\} & \text{if } j = 0 , \\ G_i g \in \{G_i h^A \mid h \in H_1\} & \text{if } j = 1 , \\ G_3 g \in \{G_3, G_3\alpha_3^{-1}, G_3\alpha_3^{-1}\alpha_1^{-1}, G_3\alpha_3^{-1}\alpha_1\} & \text{if } j = 2 . \end{cases} \\ &\Leftrightarrow \begin{cases} G_i g \in \{G_i h^A \alpha_1 \mid h \in H_0\} & \text{if } j = 0 , \\ G_i g \in \{G_i h^A \mid h \in H_1\} & \text{if } j = 1 , \\ G_3 g \in \{G_3, G_3\alpha_3^{-1}, G_3\alpha_3^{-1}\alpha_1^{-1}, G_3\alpha_3^{-1}\alpha_1\} & \text{if } j = 2 . \end{cases} \end{aligned}$$

Using Lemma 18 ($j = 0$), checking that A normalizes H_1 ($j = 1$) and verifying that

$$\{G_3, G_3\alpha_3^{-1}, G_3\alpha_3^{-1}\alpha_1^{-1}, G_3\alpha_3^{-1}\alpha_1\} = \{G_3, G_3\alpha_3\alpha_1, G_3\alpha_3, G_3\alpha_3\alpha_1^{-1}\} \quad (j = 2) ,$$

we conclude that $\mu(G_i g) * \mu(G_j) \Leftrightarrow G_i g * G_j$ for any $i, j \in I$ with $i > j$ using again Lemma 9. □

As it has been shown in Lemmas 17 and 19, the chiral hypertopes $\Gamma^{(p)}$ have both proper and improper correlations simultaneously. We therefore have the following theorem.

Theorem 20. *There exist chiral hypertopes of rank* 4 *having both proper and improper correlations simultaneously.*

We finish this section with the following two remarks.

1. All residues of rank 3 of the regular hypertope $\Gamma^{(4)}$ are regular hypermaps of type $(3,3,3)_{(0,2)}$ (Lemma 14). Consequently a correspondence fixing a generator of G^+ and interchanging the other two, can be extended to an automorphism of G^+. Thus $\Gamma^{(4)}$ has all possible kinds of correlations. In other words, $Aut(\Gamma^{(4)}) \cong S_5 \times S_4 \cong PSL(2,4) \times S_4$.

2. We observe that the tetrahedral groups G^+ arise naturally as subgroups of index 4 in the symmetry group $[6,3,6] = \langle r_0, r_1, r_2, r_3 \rangle$ of the hyperbolic tessellation $\{6,3,6\}$ given in [6, Section 11] (generated by r_1, r_2, $(r_1)^{r_0}$ and $(r_2)^{r_3}$).

6 A geometric construction of Γ

We now give a geometric description of the rank 4 hypertopes $\Gamma = (X, *, t, I)$ constructed in the previous sections. The set $I = \{0,1,2,3\}$ of types gives a partition of the set of elements of Γ into four sets X_0, X_1, X_2 and X_3, where $x \in X_j$ if and only if $t(x) = j$. The elements of X_0, X_1, X_2 and X_3 can be described from a certain action in the projective line $PG(1,q)$, where $q = p$ if $\Gamma = \Gamma^{(p)}$ and $q = p^2$ if $\Gamma = \Gamma^{(p^2)}$. In what follows we define each of the X_i and the incidence between them.

Let G^+ be the group $PSL(2,q)$ acting on the projective line $PG(1,q) = GF(q) \cup \{\infty\}$ as follows.

$$PG(1,q) \times G^+ \to PG(1,q),\ (z,g) \mapsto z^g := g^{-1}(z),$$

where $g^{-1}(z) = Az$ if $g^{-1} = \pm A$ with $A \in SL(2,q)$, as in Section 2.4.

Consider a transversal U for the stabilizer G^+_∞ of ∞. The index of G^+_∞ in G^+ is $q+1$, since G^+_∞ is the semi-direct product of the elementary abelian p-group H_1 of order q (given in Lemma 6) with the cyclic group D of diagonal matrices in G^+, which has order $\frac{q-1}{d}$ with $d = \gcd(2, q-1)$. As $q+1 = |PG(1,q)|$, we can identify any $u \in U$ with the element $z := \infty^u$ of $PG(1,q)$ and write $u = u_z$.

By Lemma 6 the group $G_1 = H_1 : (G_0 \cap G_1)$ is a subgroup of $G^+_\infty = H_1 : D$. Consider a transversal T for G_1 in G^+_∞. Let X_0 and X_1 be two disjoint copies of $T \times PG(1,q)$. For $x_0 = (t,z) \in X_0$ and $x_1 = (s,w) \in X_1$ we say that

$$x_0 * x_1 \text{ whenever } t = s \text{ and } z \neq w. \tag{4}$$

It is straightforward to see that an element $x_0 \in X_0$ is incident with exactly q elements of X_1 and vice-versa.

Let X_2 and X_3 be the orbits of the sets $\{1, e, e^2, \infty\}$ and $\{1, e, e^2, 0\}$ under the action of G^+, respectively. Remarking that all possible cross ratio of four distinct points in $\{1, e, e^2, \infty\}$ or in $\{1, e, e^2, 0\}$ take only two values, we conclude that both sets have a stabilizer isomorphic to A_4. Hence $|X_2| = |X_3| = \frac{(q+1)q(q-1)}{12d}$.

Given $x_2 \in X_2$ and $x_3 \in X_3$, we shall say that

$$x_2 * x_3 \text{ whenever } |x_2 \cap x_3| = 3. \tag{5}$$

Then, each element $x_2 \in X_2$ is incident with exactly 4 elements of X_3 and vice-versa.

Before defining the incidences between any other two elements, we shall first define the incidence between $\{1, e, e^2, \infty\} \in X_2$ and an element (t, z) of X_0 or X_1. We say that

$$\{1, e, e^2, \infty\} * (t, z) \text{ if and only if } t = 1_{G^+} \text{ and } z \in \{1, e, e^2, \infty\}. \quad (6)$$

Having 6 in mind, the transversal U can be choosen in such a way that $u_z \in G^+_{\{1,e,e^2,\infty\}}$ for $z \in \{1, e, e^2, \infty\}$. This is possible and motivated by the following. If the coset $G_1 t u_z$ intersects $G^+_{\{1,e,e^2,\infty\}}$, then $z \in \{1, e, e^2, \infty\}$ and therefore $u_z \in G^+_{\{1,e,e^2,\infty\}}$ by our choice of U. It follows then that $G_1 t$ intersects $G^+_{\{1,e,e^2,\infty\}}$. As $G_1 t$ is in G^+_∞ and $G^+_\infty \cap G^+_{\{1,e,e^2,\infty\}} < G_1$, we have that $G_1 t \cap G_1 \neq \varnothing$, that is $t = 1_{G^+}$. Now we can say that an element $\{1, e, e^2, \infty\}^g$ in X_2 will be incident to those (t, z) (in X_0 or X_1) such that $G^+_{\{1,e,e^2,\infty\}} \cap G_1 t u_z g^{-1} \neq \varnothing$. Note that, if $g \in G^+_{\{1,e,e^2,\infty\}}$, then $\{1, e, e^2, \infty\}^g = \{1, e, e^2, \infty\}$ will be incident to those (t, z) such that $G^+_{\{1,e,e^2,\infty\}}$ is incident with $G_1 t u_z g^{-1} = G_1 t u_{z^{g^{-1}}}$ (by the choice of U). As we saw, these are the pairs (t, z) with $t = 1_{G^+}$ and $z \in \{1, e, e^2, \infty\}$. Thus incidence is well-defined.

Finally, an element $\{1, e, e^2, 0\}^g \in X_3$ will be incident with an element (t, z) in X_0 or X_1 if the corresponding element $\{1, e, e^2, \infty\}^g \in X_2$ is incident with (t, z).

By definition, $\Gamma = (X, *, t, I)$ is an incidence system, where $X = \cup_{j=0}^3 X_j$, $I = \{0, 1, 2, 3\}$, $t : X \to I$ is such that $t(x) = j$ whenever $x \in X_j$ and the incidence $*$ is given by (4), (5), (6) and the last paragraph above. Furthermore, it is straightforward to see that every flag is in a chamber, implying that Γ is in fact an incidence geometry. By construction, Γ is isomorphic to the hypertope given in section 4.

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References

[1] Francis Buekenhout and Arjeh M. Cohen. *Diagram Geometry. Related to classical groups and buildings.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A

Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 57. Springer, Heidelberg, 2013. xiv+592 pp.

[2] Maria Elisa Fernandes, Dimitri Leemans and Asia Ivić Weiss. *Highly symmetric hypertopes*. Aeq. Math. 90:1045–1067, 2016.

[3] Maria Elisa Fernandes, Dimitri Leemans and Asia Ivić Weiss. *Hexagonal extensions of toroidal maps and hypermaps*, Springer Contributed Volume dedicated to Karoly Bezdek and Egon Schulte on the occasion of their 60th birthdays, 21 pages. To appear.

[4] Isabel Hubard and Asia Ivič Weiss. *Self-duality of chiral polytopes*, J. Combin. Theory Ser. A, 111:128–136, 2005.

[5] Peter McMullen and Egon Schulte. *Abstract regular polytopes*, volume 92 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2002.

[6] Egon Schulte and Asia Ivić Weiss. *Chirality and projective linear groups* . Discrete Math. 131:221–261, 1994.

[7] Jacques Tits. Sur les analogues algébriques des groupes semi-simples complexes. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, Centre Belge de Recherches Mathématiques, 261–289, Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris, 1957.

[8] T. R. S. Walsh. *Hypermaps versus bipartite maps.* J. Combinatorial Theory Ser. B, 18(2):155–163, 1975.