Universal layered permutations

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Abstract

We establish an exact formula for the length of the shortest permutation containing all layered permutations of length n, proving a conjecture of Gray.

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We establish the following result, which gives an exact formula for the length of the shortest n-universal permutation for the class of layered permutations, verifying a conjecture of Gray [3]. Definitions follow the statement.

Theorem 1. *For all n, the length of the shortest permutation that is n-universal for the layered permutations is given by the sequence defined by $a(0) = 0$ and*

$$a(n) = n + \min\{a(k) + a(n-k-1) \ : \ 0 \leqslant k \leqslant n-1\} \tag{†}$$

for $n \geqslant 1$.

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Up to shifting indices by 1, the sequence $a(n)$ in Theorem 1 is sequence A001855 in the OEIS [6]. It seems to have first appeared in Knuth's *The Art of Computer Programming, Volume 3* [4, Section 5.3.1, Eq. (3)], where it is related to sorting by binary insertion. Knuth shows there that (in our indexing conventions),

$$a(n) = (n+1)\lceil \log_2(n+1) \rceil - 2^{\lceil \log_2(n+1) \rceil} + 1.$$

This formula also shows that the minimum in (†) is attained when $k = \lfloor n/2 \rfloor$. We refer the reader to the OEIS for further information about this old and interesting sequence.

The permutation π of length n, thought of in one-line or list notation, is said to *contain* the permutation σ of length k if π has a subsequence of length k in the same relative order as σ. Otherwise, π *avoids* σ. For example, $\pi = 432679185$ contains $\sigma = 32514$, as witnessed by the subsequence 32918. We frequently associate the permutation π with its *plot*, which is the set $\{(i, \pi(i))\}$ of points in the plane.

A set of permutations that is closed downward under the containment order is called a *permutation class*. Thus, $\mathcal{C}$ is a class if for all $\pi \in \mathcal{C}$ and all permutations σ contained in π, σ also lies in $\mathcal{C}$. If π has length m and σ has length n, we define the *sum* of π and σ, denoted $\pi \oplus \sigma$, as the permutation of length $m+n$ defined by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{for } 1 \leqslant i \leqslant m, \\ \sigma(i-m)+m & \text{for } m+1 \leqslant i \leqslant m+n. \end{cases}$$

The plot of $\pi \oplus \sigma$ can therefore be represented as below.

$$\pi \oplus \sigma = \begin{array}{cc} & \boxed{\sigma} \\ \boxed{\pi} & \end{array}$$

A permutation is *layered* if it can be expressed as a sum of decreasing permutations, which are themselves called *layers*. For example, the permutation 3214657 is layered, as it can be expressed as $321 \oplus 1 \oplus 21 \oplus 1$ (note that the sum operation is associative and so there is no ambiguity in this expression).

Given a permutation class $\mathcal{C}$, the permutation π is said to be *n-universal for $\mathcal{C}$* if π contains all of the permutations of length n in $\mathcal{C}$ (the alternate term *superpattern* is sometimes used in the literature). A permutation is said to be simply *n-universal* if it contains all permutations of length n.

To date, the best bounds on the length of the shortest n-universal permutation are that it lies between n^2/e^2 (a consequence of Stirling's Formula, because if such a permutation has length m, the inequality $\binom{m}{n} \geqslant n!$ must hold) and $\binom{n+1}{2}$, which was established by Miller [5]. Before Miller's result was established, Eriksson, Eriksson, Linusson, and Wästlund [2] had conjectured that this length is asymptotic to $n^2/2$.

Theorem 1 is concerned with n-universal permutations for the class of layered permutations. Until this work, the best bounds on the length of the shortest such permutations were obtained by Gray [3], who showed that their lengths lie between $n \ln n - n + 2$ and $n\lfloor \log_2 n \rfloor + n$. Similar bounds were obtained independently by Bannister, Cheng, Devanny, and Eppstein [1]. By Theorem 1 we see that this length is asymptotic to $n \log_2 n$.

One very useful property that layered permutations have, and which is used throughout our proofs, is that when determining whether the layered permutation $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_\ell$ (where each λ_i is a decreasing permutation) is contained in the layered permutation π, it suffices to take a greedy approach: embed the first layer λ_1 as far to the left as possible, then embed the second layer λ_2 as far left as possible, and so on.

Our first result establishes that among the shortest n-universal permutations for the class of layered permutations, there is one that is layered. This is a key component in our proof of Theorem 1.

Proposition 2. *Given any permutation π of length m, there is a layered permutation of length m that contains every layered permutation contained in π.*

Proof. We prove the claim by induction on m. Note that the base case is trivial. Let D denote a decreasing subsequence of π of maximum possible length. Because D is a maximal decreasing subsequence, every entry of π that is not in D must either lie to the southwest of an entry of D or to the northeast of such an entry, but not both. Let D^- denote the set of entries that lie to the southwest of an entry of D and let D^+ denote the set of entries that lie to the northeast of such an entry, so that D, D^-, and D^+ together constitute a partition of the entries of π. An example of this decomposition is shown on the leftmost panel of Figure 1.

Define π^- (resp., π^+) to be the permutation in the same relative order as the entries of D^- (resp., D^+), let $\delta = |D| \cdots 21$, and set $\pi^* = \pi^- \oplus \delta \oplus \pi^+$. Thus in some sense π^* is a "straightened-out" version of π; an example is shown in the central panel of Figure 1.

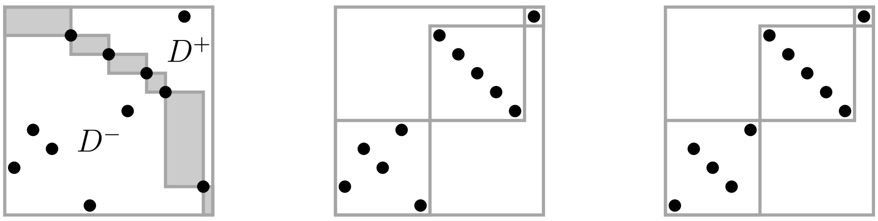

Figure 1: The steps in the proof of Proposition 2. From left to right, the drawings show an example of π, of π^*, and of the layered permutation $\tau^- \oplus \delta \oplus \tau^+$.

We claim that every layered permutation contained in π is also contained in π^*. Suppose $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_\ell$ is a layered permutation contained in π, where each λ_i is a decreasing permutation, and fix an embedding of λ into π. Choose j maximally so that in this embedding of λ into π, the layers $\lambda_1 \oplus \cdots \oplus \lambda_{j-1}$ are embedded entirely using entries in D^-. It follows that the entries of $\lambda_{j+1} \oplus \cdots \oplus \lambda_\ell$ are embedded entirely using entries of D^+. Since λ_j certainly embeds into D and consequently into δ, we have that $\lambda \leqslant \pi^*$.

Finally, by induction we see that there are layered permutations μ^- and μ^+ which contain all of the layered permutations contained in π^- and π^+, respectively. It follows that $\mu^- \oplus \delta \oplus \mu^+$ is layered and contains all of the layered permutations contained in π^*, which in turn contains all of the layered permutations contained in π, proving the proposition. An example of this final construction is shown in the rightmost panel of Figure 1. $\square$

We can now prove our main result.

Proof of Theorem 1. We prove the theorem by induction on n. As the base case is trivial, suppose that the statement is true for all values less than n.

Let $a(n)$ denote the length of the shortest permutation which is n-universal for the layered permutations. First we show that

$$a(n) \leqslant n + \min\{a(k) + a(n-k-1) \,:\, 0 \leqslant k \leqslant n-1\}.$$

Choose k so that $a(k) + a(n-k-1)$ is minimized. Then choose a permutation σ of length $a(k)$ containing all layered permutations of length k and a permutation τ of length $a(n-k-1)$ containing all layered permutations of length $n-k-1$. We claim that the permutation $\pi = \sigma \oplus (n \cdots 21) \oplus \tau$ is n-universal for the layered permutations. Consider any layered permutation $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_\ell$, where each λ_i is a decreasing permutation, of length $n = |\lambda_1| + \cdots + |\lambda_\ell|$, and choose j so that

$$\begin{array}{ccccccccc} |\lambda_1| & + & \cdots & + & |\lambda_{j-1}| & & & \leqslant & k, \\ |\lambda_1| & + & \cdots & + & |\lambda_{j-1}| & + & |\lambda_j| & > & k. \end{array}$$

Note that this implies that $|\lambda_{j+1}| + \cdots + |\lambda_\ell| \leqslant n-k-1$. Since σ is k-universal for the layered permutations, it contains $\lambda_1 \oplus \cdots \oplus \lambda_{j-1}$. Clearly λ_j is contained in $n \cdots 21$. Finally, as τ is $(n-k-1)$-universal for the layered permutations, it contains $\lambda_{j+1} \oplus \cdots \oplus \lambda_\ell$. This verifies that π contains λ, as desired.

To establish the reverse inequality, suppose that the permutation π is n-universal for the class of layered permutations and is as short as possible subject to this constraint. Proposition 2 allows us to assume that π is itself layered, so we may decompose π as $\pi_1 \oplus \cdots \oplus \pi_\ell$ where each π_i is a decreasing permutation.

Because π contains all layered permutations of length n, it contains $n \cdots 21$. Moreover, as every embedding of $n \cdots 21$ into π may use entries from only one layer, there must be some index j such that $|\pi_j| \geqslant n$. Now choose k so that

$$a(k) \leqslant |\pi_1 \oplus \cdots \oplus \pi_{j-1}| < a(k+1).$$

Thus there is at least one layered permutation λ of length $k+1$ that does not embed into $\pi_1 \oplus \cdots \oplus \pi_{j-1}$. Therefore the earliest that λ may embed into π is if it embeds into $\pi_1 \oplus \cdots \oplus \pi_j$. Now let μ denote an arbitrary layered permutation of length $n-k-1$. Because $\lambda \oplus \mu$ has length n, it must embed into π, and by our observations about where λ may embed into π we see that μ must embed into $\pi_{j+1} \oplus \cdots \oplus \pi_\ell$. As μ was an arbitrary layered permutation of length $n-k-1$, $\pi_{j+1} \oplus \cdots \oplus \pi_\ell$ is therefore $(n-k-1)$-universal for the class of layered permutations, so we have

$$\begin{aligned} |\pi| &= |\pi_1 \oplus \cdots \oplus \pi_{j-1}| + |\pi_j| + |\pi_{j+1} \oplus \cdots \oplus \pi_\ell| \\ &\geqslant a(k) + n + a(n-k-1) \\ &\geqslant a(n), \end{aligned}$$

completing the proof of the theorem. □

Proposition 2 implies that among the shortest permutations which are n-universal for the layered permutations, there is at least one that is itself layered. Computation shows that this property is not shared by all permutation classes. For example, one of the 5-universal permutations of minimum length for the class of 231-avoiding permutations is

$$1\ 5\ 11\ 9\ 3\ 2\ 8\ 4\ 7\ 6\ 10,$$

However, this permutation contains 231, and there is no 231-avoiding permutation of length 11 which is 5-universal for the class of 231-avoiding permutations. The shortest 231-*avoiding* permutations which are 5-universal for the class of 231-avoiding permutations instead have length 12; one such permutation is

$$1\ 11\ 3\ 2\ 10\ 7\ 5\ 4\ 6\ 9\ 8\ 12.$$

On the other hand, computational evidence leads us to suspect that the class of 321-avoiding permutations does possess this property:

Conjecture 3. For all n, among the shortest permutations that are n-universal for the class of 321-avoiding permutations there is at least one which avoids 321 itself.

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