On graphs whose flow polynomials have real roots only

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Abstract

Let G be a bridgeless graph. In 2011 Kung and Royle showed that all roots of the flow polynomial $F(G, \lambda)$ of G are integers if and only if G is the dual of a chordal and plane graph. In this article, we study whether a bridgeless graph G for which $F(G, \lambda)$ has real roots only must be the dual of some chordal and plane graph. We conclude that the answer of this problem for G is positive if and only if $F(G, \lambda)$ does not have any real root in the interval $(1, 2)$. We also prove that for any non-separable and 3-edge connected G, if $G - e$ is also non-separable for each edge e in G and every 3-edge-cut of G consists of edges incident with some vertex of G, then all roots of $P(G, \lambda)$ are real if and only if either $G \in \{L, Z_3, K_4\}$ or $F(G, \lambda)$ contains at least 9 real roots in the interval $(1, 2)$, where L is the graph with one vertex and one loop and Z_3 is the graph with two vertices and three parallel edges joining these two vertices.

Mathematics Subject Classifications: 05C21, 05C31

1 Introduction

The graphs considered in this paper are undirected and finite, and may have loops and parallel edges. For any graph G, let $V(G), E(G), P(G, \lambda)$ and $F(G, \lambda)$ be the set of vertices, the set of edges, the chromatic polynomial and the flow polynomial of G. The roots of $P(G, \lambda)$ and $F(G, \lambda)$ are called *the chromatic roots* and *the flow roots* of G respectively. As $P(G, \lambda) = 0$ (resp. $F(G, \lambda) = 0$) whenever G contains loops (resp. bridges), we will assume that G is loopless (resp. bridgeless) when $P(G, \lambda)$ (resp. $F(G, \lambda)$) is considered.

The chromatic polynomial $P(G, \lambda)$ of G is a function which counts the number of proper λ-colourings whenever λ is a positive integer. A chordal graph G is a graph in which every subgraph of G induced by a subset of $V(G)$ is not isomorphic to any cycle of length larger than 3. It is known that if G is chordal, then all chromatic roots of G are non-negative integers (see [6, 16, 14]). Some non-chordal graphs also have this property (see [2, 6, 7, 5, 15]). Meanwhile, there are graphs which have real chromatic roots only but also have non-integral chromatic roots. For example, when $s \geqslant 7$, the graph H_s obtained from K_s by subdividing a particular edge once is such a graph, as

$$P(H_s, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - s + 2)(\lambda^2 - s\lambda + 2s - 3). \qquad (1)$$

However, it is still unknown if there is a planar graph G with this property, i.e., G *has real chromatic roots only but also contains non-integral chromatic roots.* Due to Tutte [20], $P(G, \lambda) = \lambda F(G^*, \lambda)$ holds for any connected plane graph G, where G^* is the dual of G. Thus, equivalently, it is unknown if there is a planar graph G which has real flow roots only but also has non-integral flow roots. Actually it is also unknown if there is a non-planar graph with this property. It is natural to consider the following problem.

Problem 1. Is there a bridgeless graph which has real flow roots only but also contains non-integral flow roots?

By the following result due to Kung and Royle [13], Problem 1 is equivalent to whether there exists a graph G which is not the dual of any plane and chordal graph but has real flow roots only. If there does not exist any graph asked in Problem 1, then every graph with real flow roots only must be the dual of some chordal and plane graph.

Theorem 2 ([13]). *If G is a bridgeless graph, then its flow roots are integral if and only if G is the dual of a chordal and plane graph.*

In this paper, let $\mathcal{R}$ be the family of bridgeless graphs which have real flow roots only. We will focus on graphs in $\mathcal{R}$ and mainly show that for any graph $G \in \mathcal{R}$, all flow roots of G are integers if and only if G does not contain any real flow roots in the interval $(1, 2)$.

A vertex x in a connected G is called a *cut-vertex* if $G - x$ has more components than G has, where $G - x$ is the graph obtained from G by deleting x and all edges incident with x. A graph $G = (V, E)$ is said to be *non-separable* if either $|E| = |V| = 1$ or G is connected without loops or cut-vertices. An edge-cut S of a graph $G = (V, E)$ is the set of edges joining verteces in V_1 to vertices in V_2 for some partition $\{V_1, V_2\}$ of V. An edge-cut S of G is said to be *proper* if $G - S$ has no isolated vertices.

The definition of the flow polynomial of a graph G is given in (3). By the second equality in (3), $F(G, \lambda) = 0$ holds if G contains bridges. By the second and the fifth equalities in (3), $F(G, \lambda) = F(G/e, \lambda)$ if e is one edge in a 2-edge-cut of G. For this reason, the study of flow polynomials can be restricted to 3-edge connected graphs. By Lemmas 5, 6 and 7, the flow polynomial of any graph can be expressed as the product of flow polynomials of graphs G satisfying the following conditions, divided by $(\lambda-1)^a(\lambda-2)^b$ for some non-negative integers a or b:

(i) G is non-separable and 3-edge connected;

(ii) G does not contain any proper 3-edge-cut; and

(iii) $G - e$ is non-separable for each edge e in G.

Let $\mathcal{R}_0$ be the family of those graphs in $\mathcal{R}$ which satisfying conditions (i), (ii) and (iii) above. By Lemmas 5, 6 and 7, there exists a graph asked in Problem 1 belonging to $\mathcal{R}$ if and only if there exists a graph asked in Problem 1 belonging to $\mathcal{R}_0$. Thus the study of Problem 1 can be focused on graphs in $\mathcal{R}_0$.

Let $W(G)$ be the set of vertices in a graph G of degrees larger than 3 and let $\bar{d}(G)$ be the mean of degrees of vertices in $W(G)$. Let L denote the graph with one vertex and one loop and let Z_k denote the graph with two vertices and k parallel edges joining these two vertices. Our main result in this paper is the following one.

Theorem 3. *Let $G = (V, E)$ be any graph in $\mathcal{R}$.*

(i) *If some flow roots of G are not in the set $\{1, 2, 3\}$, then $|E| \geqslant |V| + 17$ and G has at least 9 flow roots in the interval $(1, 2)$.*

(ii) *If $G \in \mathcal{R}_0$, then either $G \in \{L, Z_3, K_4\}$ or G has the following properties:*

    (1) $3 \leqslant |W(G)| < \frac{11}{27}|V| + \frac{5}{27}$;

    (2) G contains at least $\lceil \frac{27|W(G)|}{11} - \frac{27}{22} \rceil + 2\mu(6 - |W(G)|) \geqslant 9$ flow roots in $(1, 2)$, where $\mu(x)$ is the function defined by $\mu(x) = 1$ when $x > 0$ and $\mu(x) = 0$ otherwise;

    (3) $\bar{d}(G) > 14.656 - 11.656/|W(G)| > 10.770$;

    (4) $|V| + 8|W(G)| - 7 \leqslant |E| < (32|V| - 49)/5$.

Note that Theorem 3(i) is from Theorem 18, Theorem 3 (ii.1) is from Theorem 16(i) and Lemma 15(vii), Theorem 3(ii.2) are in Theorem 16(ii), while Theorem 3 (ii.3) and (ii.4) are from Lemma 15 (iii), (iv) and (vi).

Remark: Theorem 3(ii) implies that Theorem 2 holds for all graphs in $\mathcal{R}_0$.

Interestingly, Theorems 2 and 3 imply three equivalent statements on a bridgeless graph which has real flow roots only.

Corollary 4. *Let $G \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *G is the dual of some chordal and plane graph;*

(ii) *G does not have any flow root in the interval $(1, 2)$;*

(iii) *each flow root of G is in the set $\{1, 2, 3\}$.*

2 Basic results on flow polynomials

Let $G = (V, E)$ be a finite graph with vertex set V and edge set E and let D be an orientation of G. For any finite additive Abelian group Γ, a Γ-*flow* on D is a mapping $\phi : E \to \Gamma$ such that

$$\sum_{e \in A^+(v)} \phi(e) = \sum_{e \in A^-(v)} \phi(e) \tag{2}$$

holds for every vertex v in G, where $A^+(v)$ (resp. $A^-(v)$) is the set of loopless arcs in D with tail v (resp. with head v). If $\phi(e) \neq 0$ for all $e \in E$, then a Γ-flow ϕ on D is called a *nowhere-zero* Γ-flow on D. For any integer $q \geqslant 2$, a *nowhere-zero q-flow* of G is defined to be a nowhere-zero $\mathbb{Z}$-flow ψ such that $|\psi(e)| \leqslant q - 1$ for all $e \in E$, where $\mathbb{Z}$ is the additive group consisting of all integers. Tutte [21] showed that G has a nowhere-zero q-flow if and only if it has a nowhere-zero Γ-flow, where q is the order of Γ.

The *flow polynomial* $F(G, \lambda)$ of a graph G is a function in λ which counts the number of nowhere-zero Γ-flows on D whenever λ is equal to the order of Γ. Note that the definition of $F(G, \lambda)$ does not depend on the selection of D and the additive Abelian group Γ but on G and the order of Γ. The function $F(G, \lambda)$ can also be obtained recursively by the following rules (see Tutte [22]):

$$F(G, \lambda) = \begin{cases} 1, & \text{if } E = \varnothing; \\ 0, & \text{if } G \text{ has a bridge}; \\ F(G_1, \lambda)F(G_2, \lambda), & \text{if } G = G_1 \cup G_2; \\ (\lambda - 1)F(G - e, \lambda), & \text{if } e \text{ is a loop}; \\ F(G/e, \lambda) - F(G - e, \lambda), & \text{if } e \text{ is not a loop nor a bridge}, \end{cases} \tag{3}$$

where G/e and $G - e$ are the graphs obtained from G by contracting e and deleting e respectively and $G_1 \cup G_2$ is the disjoint union of graphs G_1 and G_2.

A *block* of G is a maximal subgraph of G with the property that it is non-separable. By (3), the following result can be obtained.

Lemma 5. *If $G_1, G_2, \cdots, G_k$ are the components of G, or $G_1, G_2, \cdots, G_k$ are the blocks of a connected graph G, then*

$$F(G, \lambda) = \prod_{1 \leqslant i \leqslant k} F(G_i, \lambda). \tag{4}$$

If G is non-separable, $F(G, \lambda)$ can also be factorized when $G - e$ is separable for some edge e or G has a proper 3-edge-cut S. The results have been given in [11] (see [4, 10, 12] also).

Lemma 6 ([11]). *Let G be a bridgeless connected graph, v be a vertex of G, $e = u_1 u_2$ be an edge of G, and H_1 and H_2 be edge-disjoint subgraphs of G such that $E(H_1) \cup E(H_2) = E(G - e)$, $V(H_1) \cap V(H_2) = \{v\}$, $V(H_1) \cup V(H_2) = V(G)$, $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$, as shown in Figure 1. Then*

$$F(G, \lambda) = \frac{F(G_1, \lambda)F(G_2, \lambda)}{\lambda - 1}. \tag{5}$$
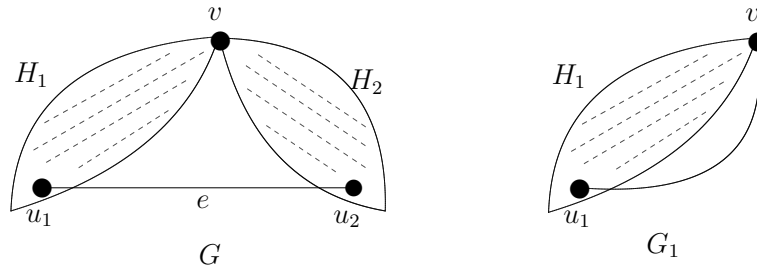
where $G_i = H_i + vu_i$ for $i \in \{1, 2\}$.

Figure 1: $G - e$ is separable.

If G has an edge-cut S with $2 \leqslant |S| \leqslant 3$, then $F(G, \lambda)$ also has a factorization [11].

Lemma 7 ([11]). *Let G be a bridgeless connected graph, S be an edge-cut of G and H_1 and H_2 be the sides of S, as shown in Figure 2 when $|S| = 3$. Let G_i be obtained from G by contracting $E(H_{3-i})$, for $i \in \{1, 2\}$. Then, for $2 \leqslant |S| \leqslant 3$,*

$$F(G, \lambda) = \frac{F(G_1, \lambda) F(G_2, \lambda)}{(\lambda - 1)_{|S|-1}}, \tag{6}$$

where $(x)_k$ is the polynomial $x(x - 1) \cdots (x - k + 1)$.
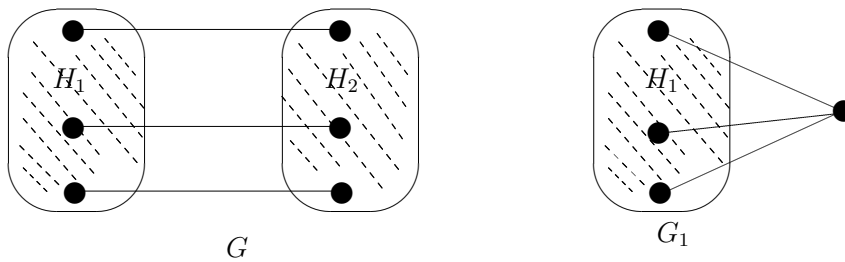

Figure 2: G has a 3-edge-cut.

It is not difficult to prove that for any bridgeless graph G, $F(G, \lambda)$ has no zero in $(-\infty, 1)$. But 1 is a zero of $F(G, \lambda)$ whenever G is not an empty graph. These conclusions can be obtained by equalities in (3). The next zero-free interval for flow polynomials is $(1, 32/27]$, due to Wakelin [17].

Theorem 8 ([17]). *Let $G = (V, E)$ be a bridgeless connected graph. Then*

(i) *$F(G, \lambda)$ is non-zero with sign $(-1)^{|E|-|V|+1}$ for $\lambda \in (-\infty, 1)$;*

(ii) *$F(G, \lambda)$ has a zero of multiplicity 1 at $\lambda = 1$ if G is non-separable;*

(iii) *$F(G, \lambda)$ is non-zero for $\lambda \in (1, 32/27]$.*

For any integer $k \geqslant 0$, let ξ_k be the supremum in $(1, 2]$ such that $F(G, \lambda)$ is non-zero in the interval $(1, \xi_k)$ for all bridgeless graphs G with at most k vertices of degrees larger than 3 (i.e., $|W(G)| \leqslant k$). Clearly that $\xi_0 \geqslant \xi_1 \geqslant \xi_2 \geqslant \cdots$. It is shown in [4] that each ξ_k can be determined by the flow roots of graphs from a finite set.

Theorem 9 ([4]). *Each ξ_k can be determined by the flow roots of graphs in a finite set Θ_k, and $\xi_k = 2$ for $k = 0, 1, 2$, $\xi_3 = 1.430 \cdots$, $\xi_4 = 1.361 \cdots$ and $\xi_5 = 1.317 \cdots$, where the last three numbers are the real roots of $\lambda^3 - 5\lambda^2 + 10\lambda - 7$, $\lambda^3 - 4\lambda^2 + 8\lambda - 6$ and $\lambda^3 - 6\lambda^2 + 13\lambda - 9$ in $(1, 2)$ respectively.*

Corollary 10. $\xi_k > 32/27$ *for all $k \geqslant 0$.*

Proof. By Theorem 9, ξ_k is determined by the flow roots of graphs from a finite set Θ_k. Thus ξ_k is the flow root of some graph in Θ_k. By Theorem 8, $\xi_k > 32/27$. $\qquad\square$

3   Graphs with real flow roots only

In this section, we assume that $G = (V, E)$ is a connected and bridgeless graph and r, $\gamma, \alpha, k, \mathcal{R}$ and ω are some invariants related to G defined below:

(i) $r = |E| - |V| + 1$;

(ii) $\alpha = \sum\limits_{i \geqslant 3}(i - 3)v_i$, where v_i is the number of vertices in G of degree i;

(iii) γ is the number of 3-edge-cuts of G;

(iv) $k = \sum\limits_{i \geqslant 4} v_i$;

(v) $\mathcal{R}$ is the multiset of real roots of $F(G, \lambda)$ in $(1, 2)$; and

(vi) $\omega = \sum\limits_{u \in \mathcal{R}}(2 - u)$.

If we take another graph H, the above parameters related to H are denoted by $r(H), \alpha(H)$, $\gamma(H), k(H), \mathcal{R}(H)$ and $\omega(H)$ respectively. It is straightforward to verify the following relations on these parameters.

Lemma 11. *The following relations hold:*

(i) $\alpha = 2|E| - 3|V|$;

(ii) $k = W(G) = |V| - v_3$;

(iii) $\gamma \geqslant v_3$, *where the equality holds if and only if G has no proper 3-edge-cut;*

(iv) $|V| = 2r - 2 - \alpha$;

(v) $|E| = 3r - 3 - \alpha$;

(vi) $\omega + \sum_{u \in \mathcal{R}} u = 2|\mathcal{R}|$.

It can be verified by (3) that $F(G, \lambda)$ is a polynomial of order r. Furthermore, if G is 3-edge connected, the coefficients of the three leading terms can be expressed in terms of $r, |E|$ and γ (see [13]).

Lemma 12. *If G is 3-edge connected, $F(G, \lambda)$ can be expressed as $\sum_{0 \leqslant i \leqslant r} b_i \lambda^i$, where $b_r = 1, b_{r-1} = -|E|$ and $b_{r-2} = \binom{|E|}{2} - \gamma$.*

Recall that L is the graph with one vertex and one loop and $\mathcal{R}$ is the family of bridgeless graphs which have real flow roots only. Obviously, we have the following conclusion on r.

Lemma 13. *$r \geqslant 1$. Furthermore, if G is a 3-edge connected graph in $\mathcal{R}$, then $r = 1$ if and only if G is the graph L.*

From now on, we assume that G is a 3-edge connected graph in $\mathcal{R}$. By Lemma 12, we can get a lower bound for γ in terms of $|E|$ and r.

Lemma 14. *Let $G = (V, E)$ be a 3-edge connected graph in $\mathcal{R}$ with $|V| \geqslant 2$. Then*

(i) $\gamma \geqslant (|E| - r)(|E| - 1)/(2r - 2)$, where the inequality is strict if $r - 1$ does not divide $|E| - 1$;

(ii) if $r \geqslant 3$ and G is not an even graph[1], then $\gamma \geqslant ((|E| - r)(|E| - 4) + r - 1)/(2r - 4)$, where the inequality is strict if $r - 2$ does not divide $|E| - 3$.

Proof. (i) It is known that any non-empty graph does not have nowhere-zero 1-flows, i.e., $F(G, \lambda)$ has a root 1. Write

$$F(G, \lambda) = (\lambda - 1)(\lambda^{r-1} - a_1 \lambda^{r-2} + a_2 \lambda^{r-3} - \cdots). \tag{7}$$

By Lemma 12, $a_1 + 1 = |E|$ and $a_2 + a_1 = \binom{|E|}{2} - \gamma$. So $\gamma = \binom{|E|}{2} - a_2 - |E| + 1$. Since all roots of $F(G, \lambda)$ are real, applying Lemma 3.1 in [13] or the Newton Inequality [8] to the coefficients of the three leading terms in the second factor of the right-hand side of (7), we have

$$a_2 \leqslant \binom{r-1}{2} \left(\frac{|E| - 1}{r - 1}\right)^2, \tag{8}$$

where the inequality is strict if $(|E| - 1)/(r - 1)$ is not a root of $F(G, \lambda)$. Note that if $(|E| - 1)/(r - 1)$ is not an integer, it is not a root of $F(G, \lambda)$. Thus

$$\gamma \geqslant \binom{|E|}{2} - \binom{r-1}{2} \left(\frac{|E| - 1}{r - 1}\right)^2 - |E| + 1$$

and (i) follows.

(ii). It can be obtained similarly. As G is not even, both 1 and 2 are flow roots of G. Write

$$F(G, \lambda) = (\lambda - 1)(\lambda - 2)(\lambda^{r-2} - c_1 \lambda^{r-3} + c_2 \lambda^{r-4} - \cdots). \tag{9}$$

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[1]It is known that G has a nowhere-zero 2-flow if and only if every vertex of G has an even degree.

Applying the idea used in the proof of (i), we have $c_1 = |E| - 3$, $\binom{|E|}{2} - \gamma = c_2 + 3c_1 + 2$ and

$$c_2 \leqslant \binom{r-2}{2} \left(\frac{|E| - 3}{r - 2} \right)^2,$$

where the inequality is strict whenever $\frac{|E|-3}{r-2}$ is not an integer. Thus

$$\gamma \geqslant \binom{|E|}{2} - \binom{r-2}{2} \left(\frac{|E| - 3}{r - 2} \right)^2 - 3(|E| - 3) - 2$$

and (ii) follows. □

Recall that $\bar{d}(G)$ is the average value of degrees of all vertices $x \in W(G)$ in G, i.e.,

$$\bar{d}(G) = \frac{2|E| - 3(|V| - k)}{k}.$$

Lemma 15. *Let $G = (V, E)$ be a 3-edge connected graph in $\mathcal{R}$ with block number b. If $|V| \geqslant 3$ and G does not contain any proper 3-edge-cut, then*

(i) *$r \geqslant 3$ and $|V| \geqslant 2k + 1$;*

(ii) *$|E| \geqslant 2|V| + 2k - 3 + \frac{4(k-1)^2}{|V|-2k}$, where the inequality is strict if $r - 2$ does not divide $|E| - 3$;*

(iii) *$|E| \geqslant |V| + 8k - 7$, where the inequality is strict if $|V| \neq 4k - 2$;*

(iv) *$\bar{d}(G) \geqslant 9 + 4\sqrt{2} - 2(\sqrt{2} + 1)^2/k > 14.656 - 11.656/k$;*

(v) *$\omega \geqslant |E| - 2|V| + 2 - b$, where the inequality is strict if $F(G, \lambda)$ has some real roots in $(2, \infty)$;*

(vi) *$|E| \leqslant \frac{\xi_k}{\xi_k - 1}|V| + b - \frac{2}{\xi_k - 1} < b + \frac{32|V|-54}{5}$;*

(vii) *$k < \frac{11}{27}|V| + \frac{b+9}{54}$.*

Proof. (i) and (ii). As G is 3-edge connected, $d(u) \geqslant 3$ for each vertex u in G, implying that

$$r = |E| - |V| + 1 \geqslant \lceil 3|V|/2 \rceil - |V| + 1 = \lceil |V|/2 + 1 \rceil \geqslant 3.$$

Since G does not contain any proper 3-edge-cut, $\gamma = v_3 = |V| - k$ by Lemma 11. As $r = |E| - |V| + 1$, by Lemma 14 (i), we have

$$|V| - k \geqslant (|V| - 1)(r + |V| - 2)/(2(r - 1)), \tag{10}$$

which is equivalent to

$$(|V| - 2k + 1)r \geqslant |V|^2 - |V| - 2k + 2.$$

As $|V| \geqslant k$ and $|V| \geqslant 3$, we have $|V|^2 - |V| - 2k + 2 > 0$. Thus $|V| \geqslant 2k$, implying that $v_3 = |V| - k \geqslant \frac{|V|}{2} > 0$. Thus G is not an even graph. As $r = |E| - |V| + 1$, Lemma 14 (ii) implies that

$$|V| - k \geqslant \frac{(|V| - 1)(r + |V| - 5) + r - 1}{2(r - 2)}, \qquad (11)$$

where the inequality is strict if $r - 2$ does not divide $|E| - 3$. Observe that (11) is equivalent to

$$r(|V| - 2k) \geqslant |V|^2 - 4k - 2|V| + 4 \geqslant 4(k - 1)^2 \geqslant 0. \qquad (12)$$

But $|V|^2 - 4k - 2|V| + 4 = 0$ implies that $k = 1$ and $|V| = 2k = 2$, a contradiction. Thus $|V| \geqslant 2k + 1$ and (i) holds. The above inequality (12) also implies that

$$r \geqslant \frac{|V|^2 - 4k - 2|V| + 4}{|V| - 2k} = |V| + 2k - 2 + \frac{4(k - 1)^2}{|V| - 2k}. \qquad (13)$$

If $r - 2$ does not divide $|E| - 3$, then, by Lemma 14 (ii), the inequalities in (11) and (13) are strict. As $r = |E| - |V| + 1$, the above inequality (13) implies (ii) directly.

(iii). By (13),

$$r \geqslant 4k - 2 + |V| - 2k + \frac{4(k - 1)^2}{|V| - 2k} \geqslant 4k - 2 + 2 \times \sqrt{4(k - 1)^2} = 8k - 6, \qquad (14)$$

where the last inequality is strict if and only if $|V| \neq 4k - 2$. As $r = |E| - |V| + 1$, (iii) follows.

(iv). By (ii) and the definition of $\bar{d}(G)$,

$$4|V| + 4k - 6 + \frac{8(k - 1)^2}{|V| - 2k} \leqslant 2|E| = 3(|V| - k) + k\bar{d}(G),$$

implying that

$$\bar{d}(G) \geqslant 7 + \frac{|V| - 6 + \frac{8(k-1)^2}{|V|-2k}}{k} = 7 + \frac{2k - 6 + |V| - 2k + \frac{8(k-1)^2}{|V|-2k}}{k}$$

$$\geqslant 7 + \frac{2k - 6 + 4\sqrt{2}(k - 1)}{k} = 9 + 4\sqrt{2} - 2(\sqrt{2} + 1)^2/k > 14.656 - 11.656/k.$$

(v). Let $t = |\mathcal{R}(G)|$, i.e., t is the number of real roots of $F(G, \lambda)$ in the interval $(1, 2)$. Thus t is the sum of the multiplicities of all flow roots of G in $(1, 2)$.

By Theorem 8, one root of $F(G, \lambda)$ is 1 with multiplicity b, exactly t of its roots are in $(1, 2)$ and $(r - t - b)$ of its roots are at least 2. As $|E|$ is the sum of all flow roots of G, we have

$$|E| \geqslant b + \sum_{u \in \mathcal{R}} u + 2(r - b - t) = b + 2t - \omega + 2(r - b - t) = 2r - b - \omega, \qquad (15)$$

implying that $\omega \geqslant |E| - 2|V| + 1 - b$ as $r = |E| - |V| + 1$, where the inequality is strict if $F(G, \lambda)$ has some real roots in $(2, \infty)$.

(vi). Since $|V| \geqslant 2k+1$ by i, G has some vertices of degree 3 and thus 2 is a root of $F(G, \lambda)$. By Lemma 5 and Theorem 8 (ii), $F(G, \lambda)$ has a root of multiplicity b at $\lambda = 1$. Thus $|\mathcal{R}| \leqslant r - 1 - b$ and

$$|E| - |V| - b = r - b - 1 \geqslant |\mathcal{R}|. \tag{16}$$

On the other hand,

$$|\mathcal{R}|(2 - \xi_k) \geqslant \omega \geqslant |E| - 2|V| + 2 - b, \tag{17}$$

where the last inequality is from v. So (16) and (17) imply that

$$(|E| - |V| - b)(2 - \xi_k) \geqslant |E| - 2|V| + 2 - b. \tag{18}$$

Then it follows that

$$|E| \leqslant \frac{(|V| + b)\xi_k - b - 2}{\xi_k - 1} = \frac{\xi_k}{\xi_k - 1}|V| + \frac{b\xi_k - b - 2}{\xi_k - 1} < b + \frac{32|V| - 54}{5}, \tag{19}$$

where the last inequality follows from the fact that $\xi_k > 32/27$ by Corollary 10.

(vii). By ii and vi, we have

$$2|V| + 2k - 3 + \frac{4(k-1)^2}{|V| - 2k} < b + (32|V| - 54)/5.$$

Solving this inequality gives that $k < \frac{11}{27}|V| + \frac{b+9}{54}$. So (vii) holds. $\qquad\square$

We are now going to establish the following important result. Recall that $\mathcal{R}_0$ is the family of non-separable and 3-edge connected graphs G in $\mathcal{R}$ such that G does not contain any proper 3-edge-cut and $G - e$ is non-separable for each edge e in G.

Theorem 16. *Let $G = (V, E) \in \mathcal{R}_0$. If $G \notin \{L, Z_3, K_4\}$, then*

(i) $k \geqslant 3$;

(ii) $|\mathcal{R}| \geqslant \lceil \frac{27k}{11} - \frac{27}{22} \rceil + 2\mu(6 - k) \geqslant 9$, where $\mu(x)$ is the function defined in Theorem 3 (ii).

Proof. Suppose that $G \notin \{L, Z_3, K_4\}$. Clearly, $|V| \geqslant 2$, as $|V| = 1$ and $G \in \mathcal{R}_0$ imply that $G = L$. It is easy to verify that all flow roots of Z_s are real if and only if $s = 3$. As $G \neq Z_3$, $|V| \neq 2$. Hence $|V| \geqslant 3$.

(i) We first prove that $k \neq 0$. Suppose that $k = 0$. Then G is a cubic graph and so $|E| = \frac{3}{2}|V|$, implying that $|V|$ is even. Thus $|V| \geqslant 4$. By Lemma 15 (ii), $|E| \geqslant 2|V| - 2$. Thus $3|V|/2 \geqslant 2|V| - 2$, implying that $|V| \leqslant 4$. Thus $|V| = 4$. Since G is cubic and 3-edge connected, it is not difficult to verify that $G \cong K_4$, contradicting to the assumption.

Now suppose that $1 \leqslant k \leqslant 2$. By Theorem 9, we have $\mathcal{R} = \varnothing$ and so $\omega = 0$. By Lemma 15 v, $|E| \leqslant 2|V| - 2 + b = 2|V| - 1$ as $b = 1$ (i.e., G is non-separable). By Lemma 15 ii,

$$|E| \geqslant \begin{cases} 2|V| - 1, & \text{if } k = 1; \\ 2|V| + 2, & \text{if } k = 2. \end{cases}$$

Thus $k = 1$ and $2|V| - 1 \leqslant |E| \leqslant 2|V| - 1$, implying that $|E| = 2|V| - 1$. As $k = 1$, $|E| = 2|V| - 1$ and $d(v) \geqslant 3$ for each vertex v in G, it can be verified that G has a vertex u of degree $|V| + 1$ and $|V| - 1$ vertices of degree 3. So $G - u$ is a graph of order $|V| - 1 \geqslant 2$ and size $|V| - 2$. Since G is non-separable, $G - u$ is connected. Thus $G - u$ is a tree of order at least 2, implying that $G - e$ is separable for each edge e in $G - u$, contradicting the given condition that $G \in \mathcal{R}_0$.

Thus (i) holds.

(ii). By the definitions of ξ_k, ω and $\mathcal{R}$, we have $\omega \leqslant (2 - \xi_k)|\mathcal{R}|$. As $k \geqslant 3$ and G is non-separable, v and ii in Lemma 15 imply that $\omega \geqslant 2k - 1$. Thus $|\mathcal{R}| \geqslant (2k - 1)/(2 - \xi_k)$ by the definition of ω. By Theorem 9, $\xi_3 \geqslant 1.430$, $\xi_4 \geqslant 1.361$ and $\xi_5 \geqslant 1.317$. By Corollary 10, $\xi_k > 32/27$ for all $k \geqslant 0$. As $k \geqslant 3$ by (i), it is trivial to verify the result in (ii). $\qquad\square$

Note that for $k = 3, 4, 5, 6, 7, 8, 9, 10$, the values of the function $\lceil \frac{27k}{11} - \frac{27}{22} \rceil + 2\mu(6 - k)$ are respectively $9, 11, 14, 14, 16, 19, 21, 24$. By Theorem 16 ii, the following conclusion is obtained.

Corollary 17. *Let $G = (V, E) \in \mathcal{R}_0$. Then all flow roots of G are integers if and only if $G \in \{L, Z_3, K_4\}$.*

Assume that $G = (V, E) \in \mathcal{R}_0$. If some flow root of G is not in the set $\{1, 2, 3\}$, then Theorem 16 i and Lemma 15 iii imply that $|E| \geqslant |V| + 17$, and Theorem 16 ii implies that $|\mathcal{R}| \geqslant 9$. In fact, these conclusions still hold even if the condition "$G \in \mathcal{R}_0$" is replaced by "$G \in \mathcal{R}$".

Theorem 18. *Let $G = (V, E) \in \mathcal{R}$. If some flow root of G is not in the set $\{1, 2, 3\}$, then $|E| \geqslant |V| + 17$ and $|\mathcal{R}| \geqslant 9$.*

Proof. Let $\mathcal{Z}$ be the set of graphs in $\mathcal{R}$ which contain flow roots not in the set $\{1, 2, 3\}$. Suppose that the result fails and G is a graph in $\mathcal{Z}$ with the minimum value of $|E(G)|$ such that $|E| < |V| + 17$ or $|\mathcal{R}| < 9$. We first prove the following claims.
Claim 1: G is non-separable.

Suppose that G is separable. By Lemma 5, some block B of G is contained in $\mathcal{Z}$. By the minimality of $|E(G)|$, $|E(B)| \geqslant |V(B)| + 17$ and $\mathcal{R}(B) \geqslant 9$ hold. As G is bridgeless, $|E(B')| \geqslant |V(B')|$ holds for each block of G. Thus $|E(G)| \geqslant |V(G)| + 17$ also holds. By Lemma 5 again, $\mathcal{R}(B) \geqslant 9$ implies that $\mathcal{R}(G) \geqslant 9$, a contradiction. Thus this claim holds.
Claim 2: $|V(G)| \geqslant 3$.

It is easy to verify that for any non-separable graph H of order at most 2, if all flow roots of H are real, then each flow root of G is in $\{1, 2, 3\}$. As $G \in \mathcal{Z}$, this claim holds.
Claim 3: G is 3-edge connected.

Assume that e is an edge contained in a 2-edge-cut of G. By (3), $F(G, \lambda) = F(G/e, \lambda)$. Thus $\mathcal{R}(G) = \mathcal{R}(G/e)$, and $G \in \mathcal{Z}$ implies that $G/e \in \mathcal{Z}$. Also note that $|E(G)| - |V(G)| = |E(G/e)| - |V(G/e)|$, implying that G/e is also a counter-example to the result, contradicting the assumption of G. Hence Claim 3 holds.
Claim 4: G does not have any proper 3-edge cut.

Suppose that S is a proper 3-edge-cut of G, as shown in Figure 2. By Lemma 7,

$$F(G, \lambda) = \frac{F(G_1, \lambda)F(G_2, \lambda)}{(\lambda - 1)(\lambda - 3)}, \tag{20}$$

where G_1 and G_2 are the graphs stated in Lemma 7. By (20), $G \in \mathcal{Z}$ implies that $G_i \in \mathcal{Z}$ for some i. Say $i = 1$. By the minimality of $|E(G)|$, $|E(G_1)| \geqslant |V(G_1)| + 17$ and $|\mathcal{R}(G_1)| \geqslant 9$ hold. By (20) again, $|\mathcal{R}(G_1)| \geqslant 9$ implies that $|\mathcal{R}(G)| \geqslant 9$. As G is bridgeless, it is not difficult to verify that $|E(G_1)| - |V(G_1)| \leqslant |E(G)| - |V(G)|$. Thus $|E(G_1)| \geqslant |V(G_1)| + 17$ implies that $|E(G)| \geqslant |V(G)| + 17$, a contradiction. Hence Claim 4 holds.

Claim 5: $G - e$ is non-separable for each edge e in G.

Suppose that $G - e$ is separable for some edge $e = u_1 u_2$ as shown in Figure 1. By Lemma 6,

$$F(G, \lambda) = \frac{F(G_1, \lambda)F(G_2, \lambda)}{\lambda - 1}, \tag{21}$$

where G_1 and G_2 are the graphs stated in Lemma 6. Then this claim can be proved similarly as the previous claim.

By the above claims, we have $G \in \mathcal{R}_0 \cap \mathcal{Z}$. But, by Theorem 16 and Lemma 15 iii, we have $k = |W(G)| \geqslant 3$, $|E| \geqslant |V| + 8k - 7 \geqslant |V| + 17$ and $\mathcal{R}(G) \geqslant 9$, contradicting the assumption of G.

Hence the result holds. $\qquad \square$

By Theorems 18 and 2, we have the following result on plane graphs which have real chromatic roots only.

Corollary 19. *Let H be a connected planar graph of order n and size m. Assume that H has real chromatic roots only but H is not a chordal graph. Then $n \geqslant 19$ and H has at least 9 chromatic roots in $(1, 2)$ (counting multiplicity for each root). Furthermore, if every vertex-cut of H does not induce a clique of H, then $32n/27 - 5/9 < m \leqslant 2n - 8$.*

Proof. Assume that H is a connected plane graph and H^* is its dual. By the equality $P(H, \lambda) = \lambda F(H^*, \lambda)$ due to Tutte [20], the given conditions implies that H^* has real flow roots only. As H is not chordal, $P(H, \lambda)$ has non-integral roots by the result in [5] that planar graphs with integral chromatic roots are chordal. Thus H^* has real flow roots only but also contains non-integral flow roots. By Theorem 18, H^* has at least 9 flow roots in $(1, 2)$, implying that H has at least 9 chromatic roots in $(1, 2)$. Notice that H^* has m edges and n faces. By Euler's polyhedron formula, the order of H^* is

$$|V(H^*)| = m - n + 2. \tag{22}$$

By Theorem 18 again, $|E(H^*)| \geqslant |V(H^*)| + 17$, implying that $m \geqslant (m - n + 2) + 17$, i.e., $n \geqslant 19$.

Now assume that every vertex-cut-set of H does not induce a clique of H. Then, it is trivial to verify that H^* is a graph contained in $\mathcal{R}_0$. By Theorem 16 i, $k(H^*) \geqslant 3$. Then, by Lemma 15 ii and vi, we have

$$2|V(H^*)| + 4 \leqslant |E(H^*)| < (32|V(H^*)| - 49)/5. \tag{23}$$

By (22),
$$2(m - n + 2) + 4 \leqslant m < (32(m - n + 2) - 49)/5. \tag{24}$$
Thus $32n/27 - 5/9 < m \leqslant 2n - 8$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We end this article with the following remark.

Remark: By Lemmas 5, 6 and 7, the study of Problem 1 can be restricted to those graphs in the family $\mathcal{R}_0$. Thus, by Theorem 16, there exist graphs asked in Problem 1 if and only if $\mathcal{R}_0 - \{L, Z_3, K_4\} \neq \varnothing$. By Theorem 16 again, for any $G \in \mathcal{R}_0 - \{L, Z_3, K_4\}$, G contains at least at least $\lceil \frac{27k}{11} - \frac{27}{22} \rceil + 2\mu(6 - k) \geqslant 9$ flow roots in the interval $(1, 2)$, where $k = |W(G)| \geqslant 3$. However, as I know, no much research is conducted on counting the number of real flow roots of a graph in the interval $(1, 2)$, except some study which confirms certain families of graphs having no real flow roots in the interval $(1, 2)$ (see [3, 4, 10, 11, 12]). It is unknown if there exists a graph H with at least $\lceil \frac{27|W(H)|}{11} - \frac{27}{22} \rceil + 2\mu(6 - |W(H)|)$ flow roots in $(1, 2)$.

Problem 20. Is there a graph H with $|W(H)| = k \geqslant 3$ and at least $\lceil \frac{27k}{11} - \frac{27}{22} \rceil + 2\mu(6 - k)$ flow roots in $(1, 2)$?

Acknowledgements

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