Excluding hooks and their complements

Krzysztof Choromanski
Google Brain Robotics New York
NY 10011, U.S.A.
kchoro@google.com

Dvir Falik*
Commogee Ltd.
Petah Tikva, Israel
dvir.falik@gmail.com

Anita Liebenau†
School of Mathematics and Statistics
University of New South Wales
Sydney NSW 2052, Australia
a.liebenau@unsw.edu.au

Viresh Patel‡
Korteweg-de Vries Institute for Mathematics
University of Amsterdam
Science Park 904, The Netherlands
vpatel@uva.nl

Marcin Pilipczuk§
Institute of Informatics
University of Warsaw, Poland
marcin.pilipczuk@mimuw.edu.pl.

Submitted: Aug 22, 2016; Accepted: Jun 09, 2018; Published: Aug 24, 2018

Abstract

The long-standing Erdős-Hajnal conjecture states that for every n-vertex undirected graph H there exists $\varepsilon(H) > 0$ such that every graph G that does not contain H as an induced subgraph contains a clique or an independent set of size at least $n^{\varepsilon(H)}$. A natural weakening of the conjecture states that the polynomial-size clique/independent set phenomenon occurs if one excludes both H and its complement H^{c}. These conjectures have been shown to hold for only a handful of graphs: it is not even known if they hold for all graphs on 5 vertices.

*Supported by the Warwick-QMUL Alliance in Advances in Discrete Mathematics and its Applications.

†Partially supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007- 2013)/ERC grant agreement no. 259385 and by the Australian Research Council. Previously at Department of Computer Science and DIMAP, University of Warwick, UK; and at School of Mathematical Sciences, Monash University, Melbourne, Australia.

‡Partially supported by the Warwick-QMUL Alliance in Advances in Discrete Mathematics and its Applications and by the Netherlands Organisation for Scientific Research (NWO) through the Gravitation Programme Networks (024.002.003).

§Partially supported by the Centre for Discrete Mathematics and its Applications (DIMAP) at the University of Warwick and by the Warwick-QMUL Alliance in Advances in Discrete Mathematics and its Applications.

In a recent breakthrough, the symmetrized version of the Erdős-Hajnal conjecture was shown to hold for all paths. The goal of this paper is to show that the symmetrized conjecture holds for all trees on 6 (or fewer) vertices. In fact this is a consequence of showing that the symmetrized conjecture holds for any path with a pendant edge at its third vertex; thus we also give a new infinite family of graphs for which the symmetrized conjecture holds.

Mathematics Subject Classifications: 05C35, 05C05, 05C75, 05D99

1 Introduction

The Erdős-Hajnal conjecture is a long-standing conjecture in Ramsey Theory bringing together extremal, structural and probabilistic aspects of graph theory. Informally it says that if a large graph G does not contain a fixed graph H as an induced subgraph then G contains a large clique or independent set.

More formally, a set of vertices in a graph G is *homogeneous* if it induces a clique or independent set, and we denote the largest homogeneous set of G by $\hom(G)$. Given a graph H (resp. a family of graphs $\mathcal{F} = \{H_1, H_2, \ldots\}$), a graph G is said to be H-free (resp. $\mathcal{F}$-free) if G does not contain H (resp. any member of $\mathcal{F}$) as an induced subgraph. The famous Erdős-Hajnal conjecture [21] is the following.

Conjecture 1. For every graph H, there exists $c(H) > 0$ such that if G is an n-vertex H-free graph then $\hom(G) \geqslant n^{c(H)}$.

Bounds on the diagonal Ramsey numbers due to Erdős-Szekeres [20] and Erdős [18] imply that for a general graph G, $\hom(G) > \frac{1}{2}\log_2 n$ and for every sufficiently large n, there exist n-vertex graphs G satisfying $\hom(G) \leqslant 2\log_2 n$. Thus the Erdős-Hajnal conjecture suggests that H-free graphs behave quite differently to general graphs with respect to homogeneous sets. The best known bound for the Erdős-Hajnal conjecture is due to Erdős and Hajnal [22] who showed that the conjecture holds if we replace $n^{c(H)}$ with $e^{c(H)\sqrt{\ln n}}$.

Despite much attention, the conjecture is known to hold for only a limited choice of H. We give some brief background here and refer the interested reader to the survey [14] of Chudnovsky. The upper bounds on Ramsey numbers due to Erdős and Szekeres [20] immediately imply that every clique satisfies the Erdős-Hajnal conjecture. In [2], Alon, Pach and Solymosi show that if the conjecture is true for two graphs H_1 and H_2, then it is also true for the graph formed by blowing up a vertex of H_1 and inducing a copy of H_2 amongst the new vertices. It is known that the Erdős-Hajnal conjecture holds for all graphs on at most four vertices and using the result in [2] immediately implies that the Erdős-Hajnal conjecture holds for all graphs on five vertices except the four-edge path, its complement, the cycle on five vertices, and a graph commonly called the *bull* (a triangle with two pendant edges). Chudnovsky and Safra [15] settled the conjecture for the bull, but the question remains unsolved for the other three graphs on five vertices.

The following slightly weaker (symmetric) form of the Erdős-Hajnal conjecture can be found e.g. in Chudnovsky [14].

Conjecture 2. For every graph H, there exists a constant $c(H) > 0$ such that if G is an n-vertex $\{H, H^c\}$-free graph then $\hom(G) \geqslant n^{c(H)}$.

Compared to the Erdős-Hajnal conjecture, this conjecture is known to hold for only a few additional choices of H. Let P_k be the path on k vertices. Chudnovsky and Seymour [16] proved Conjecture 2 when H is P_6. In a recent breakthrough Bousquet, Lagoutte and Thomassé [8] proved the conjecture for all paths. A natural next step is to consider the conjecture for trees. Our main contribution is to generalise their result from k-paths to what we call k-hooks, giving a new infinite family of H for which Conjecture 2 holds. In particular, this allows us to prove Conjecture 2 for all trees on at most 6 vertices as we discuss below.

A *k-hook*, denoted by H_k, is the graph on $k+4$ vertices $\{v_1, \dots, v_{k+4}\}$, where $\{v_1, \dots, v_{k+3}\}$ form a $(k+3)$-vertex path, and $v_{k+1}v_{k+4}$ is a pendant edge. An illustration can be found in Figure 1a. We call this graph a k-hook (rather than, say, a $(k+3)$-hook), since it is more convenient to treat it like a k-vertex path with a *hook*, i.e. a four-vertex path, attached to it.

Theorem 3. *For every $k \geqslant 1$ there exists c_k such that if G is an n-vertex $\{H_k, H_k^c\}$-free graph then* $\hom(G) > n^{c_k}$.

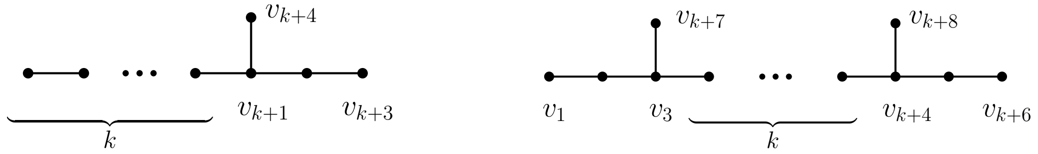
From Theorem 3, we deduce the following.

Theorem 4. *If H is a tree on at most 6 vertices, then H satisfies Conjecture 2.*

Let us check that Theorem 4 follows from Theorem 3. The Erdős-Hajnal conjecture, and hence Conjecture 2, is known to hold for all trees on at most five vertices except P_5, for which Conjecture 2 was shown to hold in [16]. There are six non-isomorphic trees on six vertices, which are depicted in Figure 2. The trees T_1, T_2, and T_3 can be obtained by the substitution operation discussed earlier by substituting independent sets into P_3 or P_4, so it follows (from [2]) that T_1, T_2, and T_3 satisfy the Erdős-Hajnal conjecture. The path P_6 satisfies Conjecture 2 by [8]. The tree H_2 satisfies Conjecture 2 by Theorem 3. Finally since T_4 is an induced subgraph of H_3 (which satisfies Conjecture 2 by Theorem 3), then T_4 also satisfies Conjecture 2, so establishing Theorem 4.

1.1 Further results

We now describe some results used to prove Theorem 3, which may be of independent interest. A class $\mathcal{G}$ of graphs is said to have the *(weak) Erdős-Hajnal property* if there



(a) A k-hook.

(b) A double k-hook.

Figure 1

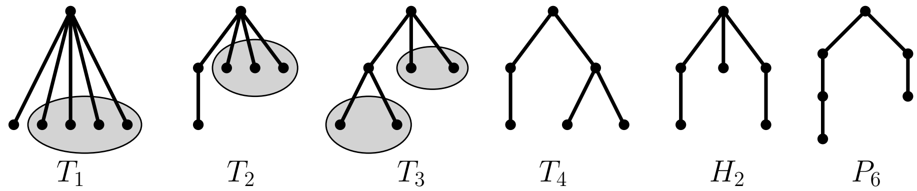


Figure 2: All trees on 6 vertices.

exists a constant $c > 0$ such that every graph $G \in \mathcal{G}$ satisfies $\hom(G) \geqslant n^c$ where n is the number of vertices in G. Clearly, Conjecture 1 is equivalent to the statement that, for every graph H, the class of H-free graphs satisfies the Erdős-Hajnal property. Instead of asking for homogeneous sets, one can ask for homogeneous pairs: for a graph G and two disjoint subsets of its vertices P and Q, we say that (P, Q) is a *homogeneous pair* if every edge between P and Q is present or if every edge between P and Q is absent. In the former case, we call (P, Q) an *adjacent pair* and in the latter case an *anti-adjacent pair*. A graph class $\mathcal{G}$ has *the strong Erdős-Hajnal property* if there exists a constant $\delta > 0$ such that every $G \in \mathcal{G}$ with at least two vertices has a homogeneous pair (P, Q) with $|P|, |Q| \geqslant \delta|V(G)|$. It is not hard to show (see e.g. [1], [24]) that if a graph class $\mathcal{G}$ has the strong Erdős-Hajnal property, then it also has the (weak) Erdős-Hajnal property.

We shall prove Theorem 3 by proving that a more general graph class has the strong Erdős-Hajnal property. A *double k-hook*, denoted by H_k^2, is the graph on $k+8$ vertices $\{v_1, \ldots, v_{k+8}\}$, where the vertices $\{v_1, \ldots, v_{k+6}\}$ form a $(k+6)$-vertex path, and v_3v_{k+7} and v_{k+4}, v_{k+8} are pendant edges. Again, we prefer to view this graph as a k-vertex path, with a hook attached to each end of the path: hence the name. For an illustration, see Figure 1b. Let $\mathcal{H}^2_{\geqslant k} := \{H_\ell^2, (H_\ell^2)^{\mathrm{c}} : \ell \geqslant k\}$ i.e. the set of double ℓ-hooks and their complements for all $\ell \geqslant k$. Since the class of $\{H_k, H_k^{\mathrm{c}}\}$-free graphs is a subclass of $\mathcal{H}^2_{\geqslant k}$-free graphs, Theorem 3 is implied by the following.

Theorem 5. *For every $k \geqslant 1$, the class of $\mathcal{H}^2_{\geqslant k}$-free graphs has the strong Erdős-Hajnal property.*

Note that the result in [8] mentioned earlier, that the class of $\{P_k, P_k^{\mathrm{c}}\}$-free graphs has the Erdős-Hajnal property, is in fact proved by showing that $\{P_k, P_k^{\mathrm{c}}\}$-free graphs have the strong Erdős-Hajnal property. Furthermore, Bonamy, Bousquet and Thomassé [5] show that $\mathcal{G}_k$ has the strong Erdős-Hajnal property, where $\mathcal{G}_k$ is the class of graphs that do not contain the cycle C_ℓ on ℓ vertices or its complement C_ℓ^{c} as an induced subgraph for all $\ell \geqslant k$.

In the course of the paper, we shall prove that two further hereditary graph classes have the strong Erdős-Hajnal property. We believe these results may be of independent interest. A *hole* in a graph is an induced cycle of length at least 4 and an *antihole* is the complement of such a graph. A *Berge graph* is a graph that does not contain any odd hole or odd antihole. It follows easily from the Strong Perfect Graph Theorem that the class of Berge graphs satisfies the (weak) Erdős-Hajnal property, but a certain random poset construction [23] shows that it does not satisfy the strong Erdős-Hajnal property. However if we also forbid the claw, i.e. the star on four vertices then the strong Erdős-Hajnal property holds.

Theorem 6. *The class of claw-free Berge graphs has the strong Erdős-Hajnal property.*

In [29], Lagoutte and Trunck show that another subclass of Berge graphs has the strong Erdős-Hajnal property. This class of graphs is incomparable to the class of claw-free Berge graphs.

The *line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$ where ef is an edge in $L(G)$ if and only if e and f share a vertex in G. While the class of line graphs is a proper subclass of the class of claw-free graphs, it is incomparable to the class of claw-free Berge graphs so the result below gives another hereditary class for which the strong Erdős-Hajnal property holds.

Theorem 7. *The class of line graphs has the strong Erdős-Hajnal property.*

In fact we shall require weighted versions of Theorem 6 and Theorem 7, which we state and prove in Section 3.

We remark that although the strong Erdős-Hajnal property implies the (weak) Erdős-Hajnal property, there are graphs H such that the class $\mathcal{G}$ of H-free graphs satisfies the Erdős-Hajnal property, yet the class of $\{H, H^c\}$-free graphs (and thus $\mathcal{G}$) fails to satisfy the *strong* Erdős-Hajnal property. The bull, a self-complementary graph, is such an example, as implied by [15] and the following.

Theorem 8. *Let H be a graph.*

(*a*) *The class of H-free graphs has the strong Erdős-Hajnal property if and only if H is an induced subgraph of the four-vertex path P_4.*

(*b*) *If both H and its complement H^c contain a cycle, then the class of $\{H, H^c\}$-free graphs does not have the strong Erdős-Hajnal property.*

We expect that the result above, which is proved by a simple random construction, is probably known, but we cannot find it recorded anywhere. We give the details in Section 7.

While the strong Erdős-Hajnal property requires homogeneous pairs of linear size, if we require homogeneous pairs of only polynomial size, then some strong results are known. In [19], Erdős, Hajnal and Pach improve results from [21] and show that for every graph H there exists $c > 0$ such that every H-free graph G on n vertices admits a homogeneous pair (P, Q) with $|P|, |Q| \geqslant n^c$. Fox and Sudakov [26] showed that in fact, every H-free graph G contains either a clique of size n^c or an anti-adjacent pair (P, Q) with $|P|, |Q| \geqslant n^c$.

Finally, we remark that, as shown by Bousquet, Lagoutte, and Thomassé [7], if a hereditary graph class satisfies the strong Erdős-Hajnal property, then it also admits a clique-independent set separation family of polynomial size (for precise definitions we refer the reader to [7]). Consequently, the latter conclusion holds for the family for $\mathcal{H}^2_{\geqslant k}$-free graphs for every fixed $k \geqslant 1$. We point out that a conjecture of Yannakakis [35], stemming from communication complexity and asserting that *every* graph admits a clique-independent set separation family of polynomial size, was very recently disproved by Göös [27].

Methods. Suppose H is a fixed graph and $\mathcal{G}$ is the class of $\{H, H^c\}$-free graphs. Our first observation, which is essentially expressed in Lemma 19 but also requires a result from [25], is the following. If (for a contradiction) the strong Erdős-Hajnal property does not hold for $\mathcal{G}$, then we may assume that each (connected) n-vertex graph $G \in \mathcal{G}$ has maximum degree $o(n)$ and a minimal separator (an inclusion-wise minimal set of vertices whose deletion leaves the graph disconnected) of linear size. It immediately follows that G has three disjoint subsets of vertices A, B, S where S has linear size, where every vertex of S has at least one neighbour in A and B, and where there are no edges between A and B.

The next step is to use this additional structure of G to form a hook i.e. an induced path on four vertices and to grow a k-vertex induced path from the third vertex of the hook. This gives an induced copy of H_k and the desired contradiction. In order to obtain the k-vertex induced path, we apply the simple but ingenious argument used in Bousquet, Lagoutte, and Thomassé [8] that allows one to grow an arbitrarily long (but constant size) induced path in a connected graph with sublinear maximum degree. The main work in our proof is to set up the hook so that it will not interfere with the path we wish to grow. If such a hook does not exist, then an involved analysis of vertices in S and how their neighbourhoods interact reveals that S has quite a restricted structure: in particular we can partition a large part of S such that each pair of parts forms a homogeneous pair and such that the 'quotient graph' of this partition belongs to a more restricted hereditary graph class than the one we started with. This allows us to push through an induction step which gives a linear sized homogeneous pair.

Structure. The rest of the paper is organised as follows. In Section 2, we provide all necessary definitions and tools that we use throughout the paper. In Section 3, we prove weighted versions of Theorem 6 and Theorem 7. In Section 4, we prove Theorem 5, using Theorem 10 and a structural result (cf. Theorem 20) which we prove in Section 6. As a warm-up for this technical result and to illustrate our method, we prove a simpler result in Section 5 (cf. Theorem 21). In Section 7, we prove Theorem 8. We close the paper with some concluding remarks in Section 8.

2 Preliminaries

In this section, we fix notation and terminology, and we prove a lemma which we will use several times throughout the paper.

A graph $G = (V, E)$ consists of a set $V(G) := V$ of vertices and a set $E(G) := E$ of edges, where an edge is an unordered pair of vertices. A multigraph is defined in the same way except that we allow $E(G)$ to be a multiset. A directed multigraph $D = (V, A)$ consists of a vertex set V and an arc multiset A, where an arc is an ordered pair of vertices. For graphs and multigraphs we set $v(G) := |V(G)|$ and $e(G) := |E(G)|$. We denote the *complement* of a graph G by G^c where $V(G^c) := V(G)$ and $e \in E(G^c)$ if and only $e \notin E(G)$. For an edge $e \in E(G)$, we write $G \setminus e$ for the graph on the same vertex set as G and with edge set $E(G) \setminus \{e\}$.

Let $X \subseteq V(G)$ be a subset of the vertices of a graph G. We denote by $G[X]$ the *induced subgraph of* G *on* X i.e. the graph with vertex set X and edge set $E(G[X]) := \{uv \in E(G) : u, v \in X\}$. We write $G - X$ for the induced subgraph of G on $V(G) \setminus X$. Let $N_G(X) := \{u \in V(G) \setminus X : uv \in E(G) \text{ for some } v \in X\}$ denote the *(open) neighbourhood of* X and let $N_G[X] := X \cup N_G(X)$ denote the *closed neighbourhood of* X. We omit the subscript if the graph G is clear from context. We write $N(v) := N(\{v\})$ and $N[v] := N[\{v\}]$. Furthermore, for a set $A \subseteq V(G)$ we define $N_A(X) := N(X) \cap A$ and $N_A[X] := N[X] \cap A$. For brevity, if $X = \{x, y\}$, we write $N(x, y)$ instead of $N(\{x, y\})$. For a graph G and two disjoint sets $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of edges of G with one endpoint in X and one endpoint in Y.

A graph H is called *a subgraph of* G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G[V(H)])$. It is called an *induced subgraph of* G if $E(H) = E(G[V(H)])$. A *k-vertex path*, denoted by P_k, is the graph on k vertices $\{v_1, \ldots, v_k\}$ with edge set $E(P_k) := \{v_i v_{i+1} : 1 \leqslant i < k\}$. We call a graph *connected* if for every pair of vertices x, y there exists a k-vertex path P for some $k \geqslant 1$ that is a subgraph of G and that contains both x and y. A *component* in G is a maximally connected subgraph of G.

A *complete graph* is one where all possible edges are present. A *clique* in a graph G is a subset $X \subseteq V(G)$ such that $G[X]$ is a complete graph and an *independent set* in G is a subset $X \subseteq V(G)$ such that $G[X]$ is the empty graph. In each case, we may also refer to the subgraph $G[X]$ as a clique or independent set.

We already defined the line graph of a graph, and more generally if G is a multigraph, the line graph $L(G)$ of G is the graph with vertex set $E(G)$ where ef is an edge of $L(G)$ if and only if e and f share a vertex in G. A graph G' is called a line graph if it is the line graph of some (multi)graph G.

Given four distinct vertices x, a, b, c of a graph G, we say that $(x; a, b, c)$ *is a claw in* G, if $G[\{x, a, b, c\}]$ is isomorphic to a claw with x being the degree-three vertex.

Hooks. For $k \geqslant 0$, recall that a k-hook, denoted by H_k, is a $(k+3)$-vertex path, say on vertex set $\{v_1, \ldots, v_{k+3}\}$ and edges $v_i v_{i+1}$ for $1 \leqslant i \leqslant k+2$, together with a pendant edge $v_{k+1} v_{k+4}$. The vertex v_1 is called the *active vertex* of the k-hook. Note that a 0-hook, denoted H_0, is the four-vertex path P_4, with one of its interior vertices designated as an active vertex.

When constructing an induced k-hook in a graph G we often start with a 0-hook i.e. a copy of an induced P_4, and then "grow" a path by adding edges subsequently to the active vertex. The following notion is helpful. An *active k-hook* in a graph G is a pair (X, R), where $X, R \subseteq V(G)$, $X \cap R = \emptyset$, $G[X]$ is isomorphic to a k-hook, $G[R]$ is connected, and $N(R) \cap X$ consists of exactly one vertex, namely the active vertex of the k-hook $G[X]$.

Modules. Frequently, we will encounter sets in our graph G that "behave like a single vertex" in the following way. A set $X \subseteq V(G)$ is a *module* in G if for every $x, y \in X$ we have $N(x) \setminus X = N(y) \setminus X$. For a partition $V(G) = X_1 \uplus X_2 \uplus \ldots \uplus X_r$ into nonempty modules $X_1, \ldots, X_r$, observe that, for every $i \neq j$, the pair (X_i, X_j) is a homogeneous pair. For such a partition, the quotient graph G_q is defined to be the graph with vertex set $\{X_1, X_2, \ldots, X_r\}$ where two sets X_i and X_j are connected by an edge in G_q if and only if they form an adjacent pair in G. Note that a quotient graph is necessarily isomorphic to some induced subgraph of G, namely one formed by taking exactly one vertex from every set X_i.

The following simple lemma is used frequently throughout the paper.

Lemma 9. *Let G be a graph, μ be a probability measure on $V(G)$, $\delta > 0$, and $X \subseteq V(G)$ such that $\mu(X) > 3\delta$. Then there exists either a set $P \subseteq X$ such that $(P, X \setminus P)$ is an anti-adjacent pair and $\mu(P), \mu(X \setminus P) > \delta$, or the largest component of $G[X]$ has measure at least $\mu(X) - \delta$.*

Proof. Let C be the vertex set of a component of $G[X]$ such that $\mu(C)$ is maximal. If $\mu(C) \geqslant \mu(X) - \delta$, then we are done. Also, if $\delta < \mu(C) < \mu(X) - \delta$, then we are done by taking $P := C$. In the remaining case, when all components of $G[X]$ have measure at most δ, we proceed as follows. We initiate $P := \emptyset$, and iterate over components of $G[X]$ one-by-one, putting them into the set P until $\mu(P)$ exceeds δ. Since every component of $G[X]$ has measure at most δ, we have $\delta < \mu(P) \leqslant 2\delta$ at the end of the process. Since $\mu(X) > 3\delta$, we have $\mu(X \setminus P) > \delta$. Furthermore, by construction, P and $X \setminus P$ form an anti-adjacent pair. This concludes the proof of the lemma. □

3 Line graphs and claw-free Berge graphs

In this section, we state and prove weighted versions of Theorem 6 and Theorem 7 from which those theorems immediately follow.

A graph class $\mathcal{G}$ has *the weighted strong Erdős-Hajnal property* if there exists a constant $\delta > 0$ such that every $G \in \mathcal{G}$ satisfies the following property. For every probability measure μ on $V(G)$ satisfying $\mu(v) \leqslant 1 - 2\delta$ for all $v \in V(G)$, there exists a homogeneous pair (P, Q) in G with $\mu(P), \mu(Q) \geqslant \delta$. The condition that $\mu(v) \leqslant 1 - 2\delta$ for all $v \in V(G)$ is necessary since for degenerate measures, where most of the mass is concentrated on one vertex only, we cannot hope to find a homogeneous pair of sufficient mass, for any graph G.

We shall prove the following two theorems.

Theorem 10. *The class of claw-free Berge graphs has the weighted strong Erdős-Hajnal property.*

Theorem 11. *The class of line graphs has the weighted strong Erdős-Hajnal property.*

Theorem 11 is an immediate corollary of the next lemma, where we prove that in any line graph with vertex weights, we find either an anti-adjacent pair or a *clique* of positive mass.

Lemma 12. *Let $\delta_1 = \frac{1}{14}$. Then for every graph G that is a line graph of some multigraph H, and every probability measure μ on $V(G)$, there exists either a clique K in G with $\mu(K) \geqslant 3\delta_1$ or an anti-adjacent pair (P, Q) in G with $\mu(P), \mu(Q) > \delta_1$.*

Recall that for a graph G and two disjoint sets X, Y of $V(G)$, we denote by $E_G(X, Y)$ the set of edges of G with one endpoint in X and the other in Y.

Proof. Fix a multigraph H, let G be its line graph, and fix a probability measure μ on $V(G)$. By definition, μ is a probability measure on the edges of H. We can find a partition of $V(H) = L \uplus R$ such that $w := \mu(E_H(L,R)) \geqslant \frac{1}{2}$. Such a partition exists since for a uniformly random partition $V(H) = V_1 \uplus V_2$ the expected value of $\mu(E_H(V_1,V_2))$ is $\frac{1}{2}$. Let H' be the bipartite subgraph of H with $V(H') = V(H)$ and $E(H') = E_H(L,R)$. Notice that the line graph of H' is an induced subgraph of G. We define a function $f : V(H) \to [0,1]$ by $f(v) = \sum_{u:uv \in E(H')} \mu(uv)$. We naturally extend f to subsets of $V(H)$ by summation over the elements of the subset. Notice that $w := f(L) = f(R) \geqslant \frac{1}{2}$.

Assume that G has no clique of measure at least $3\delta_1$. Since the set of edges adjacent to a single vertex in H' forms a clique in G, we may deduce that $f(v) < 3\delta_1$ for every vertex in H'. We find a partition $L = L_1 \uplus L_2$ of L such that $\frac{w}{2} - \frac{3}{2}\delta_1 < f(L_1), f(L_2) < \frac{w}{2} + \frac{3}{2}\delta_1$ in the following way. Pick $u \in L$, set $L_1 := \{u\}$, and add vertices from L to L_1, one at a time, until $f(L_1) > \frac{w}{2} - \frac{3}{2}\delta_1$. Since we add less than $3\delta_1$ to $f(L_1)$ each time, the upper bound on $f(L_1)$ follows. Since $f(L_2) = w - f(L_1)$, the same inequalities hold for $f(L_2)$. In the same way, we find a partition $R = R_1 \uplus R_2$ of R such that $\frac{w}{2} - \frac{3}{2}\delta_1 < f(R_1), f(R_2) < \frac{w}{2} + \frac{3}{2}\delta_1$.

For $i,j \in \{1,2\}$, let $\mu_{ij} := \mu(E_{H'}(L_i,R_j))$. We have $\mu_{i1} + \mu_{i2} = f(L_i) \geqslant \frac{w}{2} - \frac{3}{2}\delta_1$. Likewise, $\mu_{1i} + \mu_{2i} = f(R_i) \geqslant \frac{w}{2} - \frac{3}{2}\delta_1$.

If $\mu_{12} < \delta_1$ or $\mu_{21} < \delta_1$, then both $\mu_{11}, \mu_{22} > \frac{w}{2} - \frac{5}{2}\delta_1 \geqslant \delta_1$. In that case, we may take $P = E_{H'}(L_1,R_1), Q = E_{H'}(L_2,R_2)$, since then (P,Q) is an anti-adjacent pair in G. In the other case, if $\mu_{12}, \mu_{21} \geqslant \delta_1$, we may take $P = E_{H'}(L_1,R_2)$ and $Q = E_{H'}(L_2,R_1)$, and again observe that (P,Q) is an anti-adjacent in G. □

We now turn to the proof of Theorem 10. We resort to some known structural results on claw-free graphs. Let us first recall some standard terminology that we need. For a graph G, a pair (T,β) is called a *tree decomposition of* G if T is a tree, $\beta : V(T) \to 2^{V(G)}$ is a function, and the following conditions hold:

1. $V(G) = \bigcup_{t \in V(T)} \beta(t)$;
2. for every $v \in V(G)$ the set $T_v = \{t : v \in \beta(t)\}$ induces a connected subgraph of T;
3. for every edge $uv \in E(G)$ there exists $t \in V(T)$ such that $u,v \in \beta(t)$.

For a tree decomposition (T,β) and a node $t \in V(T)$, the set $\beta(t)$ is called *a bag*. It is a standard fact about tree decompositions that if K is a clique of G then $V(K)$ is contained in some bag. Indeed for each $v \in V(K)$, T_v is connected and hence is a subtree of T. Also for each $u,v \in V(K)$ we have that $T_u \cap T_v \neq \emptyset$. The Helly property of subtrees of a tree implies that $\cap_{v \in V(K)} T_v \neq \emptyset$ and so taking t in this intersection, we see $V(K) \subseteq \beta(t)$.

A subset $S \subseteq V(G)$ is called *a separator (of G)* if there exist two vertices $x,y \in V(G) \setminus S$ such that x and y lie in different components of $G - S$. A *clique separator in* G is a set $S \subseteq V(G)$ that is a separator of G and such that $G[S]$ forms a clique. The following result on the existence of a clique separator decomposition is considered to be folklore, see e.g. [4]. Since in most references it is phrased as a recursive graph decomposition instead of a tree decomposition, we provide the short proof for completeness.

Lemma 13. *For every graph G there exists a tree decomposition of G where every bag induces a graph without clique separators.*

Proof. We prove the claim by induction on $|V(G)|$. In the base case, G does not contain any clique separator, so we can create a tree decomposition (T, β) where T consists of a single node t and $\beta(t) := V(G)$.

Otherwise, let S be a clique separator in G such that $|S|$ is minimal. By minimality, there exists a component A of $G - S$ such that $N_G(A) = S$. Let $G_1 := G[A \cup S]$ and $G_2 := G - A$. By induction, for $i = 1, 2$, there exists a tree decomposition (T_i, β_i) of the graph G_i. Since $G[S]$ is a clique, and S appears both in G_1 and G_2, for every $i = 1, 2$, there exists a bag $t_i \in V(T_i)$ such that $S \subseteq \beta_i(t_i)$. To conclude, let T be the tree formed by taking the disjoint union of T_1 and T_2 and adding the edge t_1t_2. Set $\beta(t) := \beta_i(t)$ if $t \in T_i$, and observe that (T, β) is a suitable tree decomposition of G. $\square$

The following lemma provides the main reason for considering tree decompositions (with additional suitable properties) when studying the strong Erdős-Hajnal property. It is considered folklore in the unweighted case, and we refer to it as the *central bag argument*.

Lemma 14. *Let $0 < \delta \leqslant \frac{1}{4}$ be a constant, let G be a graph, let μ be a probability measure on $V(G)$, and let (T, β) be a tree decomposition of G. Then there exists an anti-adjacent pair (P, Q) in G with $\mu(P), \mu(Q) > \delta$, or a bag $\beta(t)$ with $\mu(\beta(t)) \geqslant \frac{1}{2} - \delta$.*

Proof. Extend β to subsets of nodes of T by setting $\beta(S) := \bigcup_{t \in S} \beta(t)$, for every $S \subseteq V(T)$. We define an orientation of the edges of T as follows. For an edge $t_1t_2 \in E(T)$, let T_i be the component of $T \setminus t_1t_2$ that contains t_i, for $i = 1, 2$. Now, orient the edge t_1t_2 from t_1 to t_2 if $\mu(\beta(V(T_1))) \leqslant \mu(\beta(V(T_2)))$, and orient the edge t_1t_2 from t_2 to t_1 otherwise. Since the tree T has fewer edges than nodes, there exists a node $t \in V(T)$ of out-degree zero. For every component C of $G - \beta(t)$ there exists a component T_C of $T - \{t\}$ such that $C \subseteq \beta(V(T_C))$, by the properties of the tree decomposition. Therefore, $\mu(C) \leqslant \mu(\beta(V(T_C))) \leqslant \frac{1}{2}$, since the edge between T_C and t is oriented towards t.

If $\mu(\beta(t)) \geqslant \frac{1}{2} - \delta$, then we are done, so assume otherwise. Note that then $\mu(V(G) \setminus \beta(t)) > \frac{1}{2} + \delta \geqslant 3\delta$ since, by assumption, $\delta \leqslant \frac{1}{4}$. Therefore, by Lemma 9, there is an anti-adjacent pair (P, Q) in $V(G) \setminus \beta(t)$ such that $\mu(P), \mu(Q) > \delta$, or there is a component C in $G - \beta(t)$ with $\mu(C) \geqslant \mu(V(G) \setminus \beta(t)) - \delta > \frac{1}{2}$. Since the second outcome is a contradiction, we indeed find an anti-adjacent pair of desired size. $\square$

The previous two lemmas allow us to pass to a linear subset $Y \subseteq V(G)$ of a graph G with the additional property that $G[Y]$ has no clique separators, provided that G has no anti-adjacent pair of linear size. In light of Theorem 10, we search for a good characterisation of claw-free Berge graphs. Chvátal and Sbihi [17] show that a claw-free graph without clique separators is Berge if and only if it is either "elementary" or "peculiar". A graph is called *elementary* if its edges can be coloured by two colours in such a way that edges xy and yz have distinct colours whenever x and z are nonadjacent. We decide not to give the exact definition of a peculiar graph here, but rather we point out that the vertex set of a peculiar graph can be partitioned into nine parts that each form a clique. The following is then an immediate implication of Theorem 2 in [17].

Theorem 15. *Let G be a graph that is claw-free, Berge and that has no clique separator. Then G is either elementary or the vertex set $V(G)$ can be partitioned into nine sets $V(G) = \bigcup_{i=1}^{9} V_i$ such that $G[V_i]$ is a clique for each $1 \leqslant i \leqslant 9$.*

We now resort to a characterisation of elementary graphs due to Maffray and Reed [32] that suits our purposes better than the original definition. We use the following terminology from [32]. Let G be a graph. We call an edge a *flat edge in G* if it does not appear in any triangle of G. Let xy be a flat edge in G, let X, Y be two disjoint sets such that $X \cap V(G) = Y \cap V(G) = \emptyset$, and let $B = (X, Y; E_{XY})$ be a cobipartite graph, that is a graph on vertex set $X \uplus Y$, where $B[X]$ and $B[Y]$ form cliques, such that there is at least one edge between X and Y in B. We can build a new graph G obtained from $G - \{x, y\}$ and B by adding all possible edges between X and $N(x) \setminus \{y\}$ and between Y and $N(y) \setminus \{x\}$. We say that G is *augmented along xy*, that x and y are *augmented*, and that *x is replaced by X* and *y is replaced by Y*. Intuitively, we replace the vertices x and y by cliques, and the edge xy by a (non-empty) bipartite graph. It is easy to see that, if x_1y_1 and x_2y_2 are independent edges in G, then the graph obtained by first augmenting G along x_1y_1 and then the resulting graph along x_2y_2 is the same as if we had first augmented x_2y_2 and then x_1y_1. This leads to the following definition.

An *augmentation of a graph G* is a graph G' that is obtained by augmenting G along the edges of some matching of flat edges in G.

Theorem 16 ([32])**.** *A graph G is elementary if and only if it is an augmentation of a line graph of a bipartite multigraph.*

We now prove the equivalent of Lemma 12 for elementary graphs.

Lemma 17. *Let $\delta_2 = \frac{1}{28}$. Then, for every graph G that is an elementary graph, and every probability measure μ on $V(G)$, there exists either a clique K in G with $\mu(K) \geqslant 3\delta_2$ or an anti-adjacent pair (P, Q) in G with $\mu(P), \mu(Q) \geqslant \delta_2$.*

Proof. Let G' be a line graph of a bipartite multigraph B such that G is an augmentation of G', which exists by Theorem 16. We define a probability measure μ' on $V(G')$ in the natural way by setting $\mu'(x) := \mu(X)$ if $x \in V(G')$ was augmented and replaced by X, and $\mu'(x) := \mu(x)$ otherwise.

We apply Lemma 12 to G' to find either an anti-adjacent pair (P, Q) in G' with $\mu'(P), \mu'(Q) \geqslant \delta_1 = 2\delta_2$, or a clique, say on vertex set K, such that $\mu'(K) \geqslant 3\delta_1$. In the first outcome, note that some vertices of P and Q may have been replaced by cliques in the augmentation G, say $x_1, \ldots, x_p \in P$ are replaced by $X_1, \ldots, X_p$, and $y_1, \ldots, y_q$ are replaced by $Y_1, \ldots, Y_q$. Set $P_G := (P \setminus \{x_1, \ldots, x_p\}) \cup X_1 \cup \ldots \cup X_p$ and $Q_G := (Q \setminus \{y_1, \ldots, y_q\}) \cup Y_1 \cup \ldots \cup Y_q$, and note that (P_G, Q_G) is an anti-adjacent pair in G with $\mu(P_G), \mu(Q_G) \geqslant 2\delta_2$. In the second outcome, we consider two cases, depending on $|K|$. If $|K| \leqslant 2$, then the heaviest vertex of K corresponds to a clique in G of measure at least $3\delta_1/2 = 3\delta_2$. Otherwise, if K consists of at least three vertices, then none of its edges is flat, and therefore it remains a clique of measure at least $3\delta_1$ in G. □

We now deduce Theorem 10 as a corollary.

Proof of Theorem 10. We prove the weighted strong Erdős-Hajnal property with constant $\delta_3 = \frac{1}{58}$. Let G be a graph that is claw-free and Berge, and let μ be a probability measure on $V(G)$ such that for every vertex v of G we have that $\mu(v) \leqslant 1 - 2\delta_3$. In fact, we may assume that $\mu(v) < \delta_3$ for all $v \in V(G)$. Indeed if we let $v \in V(G)$ be a vertex such that $\delta_3 \leqslant \mu(v) \leqslant 1 - 2\delta_3$ then $\mu(N(v)) \geqslant \delta_3$ or $\mu(V(G) \setminus N[v]) \geqslant \delta_3$. That is, either $(\{v\}, N(v))$ is an adjacent pair in G of sufficient mass, or $(\{v\}, V(G) \setminus N[v])$ is an anti-adjacent pair in G of sufficient mass; in either case we are done and so we may assume $\mu(v) < \delta_3$.

Let (T, β) be a tree-decomposition of G such that every bag induces a subgraph of G without a clique separator, which exists by Theorem 13. By Lemma 14, either there is an anti-adjacent pair (P, Q) in G with $\mu(P), \mu(Q) > \delta_3$ and we are done, or there is a bag Y with $\mu(Y) \geqslant \frac{1}{2} - \delta_3$. In the second case, we apply Theorem 15 to $G[Y]$ to infer that $G[Y]$ is either elementary or its vertex set can be partitioned into nine cliques. In the latter case, $G[Y]$ contains a clique of measure at least $\frac{\mu(Y)}{9} \geqslant 3\delta_3$ and we are done. If $G[Y]$ is elementary, then by Lemma 17, $G[Y]$ contains an anti-adjacent pair (P, Q) with $\mu(P), \mu(Q) > \frac{1}{28}\mu(Y)$ or a clique K with $\mu(K) \geqslant \frac{3}{28}\mu(Y)$. In both cases we are done by the choice of δ_3, as $\frac{1}{28}\mu(Y) \geqslant \frac{0.5-\delta_3}{28} = \delta_3$. $\square$

4 Double hooks have the strong Erdős-Hajnal property

In this section, we state two technical lemmas and show how Theorem 5 can be derived from them. The lemmas will be proved in Sections 5 and 6.

Fix $k \geqslant 1$ and let $\mathcal{G} := \{G : G \text{ is } \mathcal{H}^2_{\geqslant k}\text{-free}\}$ i.e. the class of all graphs G that are $\{H^2_\ell, (H^2_\ell)^{\mathrm{C}}\}$-free, for all $\ell \geqslant k$. To prove Theorem 5, we show that $\mathcal{G}$ has the *strong* Erdős-Hajnal property. That is, we need to find a $\delta > 0$ such that every $G \in \mathcal{G}$ contains a homogeneous pair (P, Q) with $|P|, |Q| \geqslant \delta \cdot v(G)$. Similarly as in [8], our starting point is to pass down to an induced subgraph that is very sparse or very dense. The *edge density* of a graph G is the fraction $e(G)/\binom{v(G)}{2}$. The following is due to Fox and Sudakov [25], improving an earlier result of Rödl [34].

Theorem 18. *For every $0 < \varepsilon < 1/2$ and every graph H on at least two vertices there exists a constant $\delta = \delta(\varepsilon, H)$ such that every H-free graph on n vertices contains an induced subgraph on at least δn vertices with edge density either at most ε or at least $1 - \varepsilon$.*

In case the induced subgraph is particularly sparse we find a special structure within it. Let G be a graph, and let $\mathcal{S} = (A, B, S)$ be a triple of non-empty subsets of $V(G)$. We call the pair $(G, \mathcal{S})$ an *ε-structured pair* if

1. $A \uplus B \uplus S = V(G)$, i.e., the sets A, B, S form a partition of $V(G)$;
2. $G[A]$ and $G[B]$ are connected;
3. $N(A) = N(B) = S$, in particular, there is no edge between A and B; and
4. for every $v \in V(G)$ it holds that $|N_S[v]| \leqslant \varepsilon|S|$.

Note that if $(G, \mathcal{S})$ is ε-strucutred, then it is ε'-structured for every $\varepsilon' \geqslant \varepsilon$.

Lemma 19. *Fix $0 < \varepsilon < \frac{1}{10}$ and let G be a graph on n vertices such that every vertex has at most εn neighbours. Then*

(*a*) *there exists a homogeneous pair (P, Q) in G with $|P|, |Q| \geqslant n/10$; or*

(*b*) *there exist subsets $A, B, S \subseteq V(G)$ such that $|S| \geqslant n/10$ and the pair $(G[A \cup B \cup S], (A, B, S))$ is a 10ε-structured pair.*

Proof. Assume that there is no homogeneous pair (P, Q) with $|P|, |Q| \geqslant n/10$ in G, and let G_1 be the largest component of G. By Lemma 9, G_1 has at least $9n/10$ vertices. Pick an arbitrary vertex x_A in G_1, and set $A := \{x_A\}$. Now add vertices one by one to A, keeping A connected, until $|N[A]|$ exceeds $n/2$. Then, $|N[A]| \leqslant n/2 + \varepsilon n < 3n/5$, since we add at most εn vertices to $N[A]$ in each step. Thus, $G_1 - N[A]$ has at least $3n/10$ vertices. Let B be the largest component in $G_1 - N[A]$. By Lemma 9, we may assume that $|B| \geqslant n/10$ and therefore we must also have that $|A| < n/10$ (otherwise (A, B) is an anti-adjacent pair of sufficient size). Thus, $|N(A)| \geqslant 2n/5$. Furthermore, $|N(A) \setminus N[B]| < n/10$, since otherwise, $(B, N(A) \setminus N[B])$ is an anti-adjacent pair of sufficient size. Setting $S := N(A) \cap N(B)$, we see that $|S| \geqslant 3n/10$ by the above discussion, and for every $v \in V(G)$ we have that $|N_S(v)| \leqslant |N(v)| \leqslant \varepsilon n \leqslant 10\varepsilon|S|$. Furthermore it is easy to see that in $G' = G[A \cup B \cup S]$ we have $N(A) = N(B) = S$ and that $G'[A]$ and $G'[B]$ are connected. Thus the pair $(G[A \cup B \cup S], (A, B, S))$ is a 10ε-structured pair. □

The following theorem is the crucial step in our proof. It states that within an ε-structured pair $(G, (A, B, S))$, we either find the desired homogeneous pair of linear size, or a very structured subset $\hat{S} \subseteq S$ of linear size, or an ℓ-hook for some $\ell \geqslant k$ which we can potentially extend to a double ℓ-hook. Recall that an *active ℓ-hook* in a graph G is a pair (X, R), where $X, R \subseteq V(G)$, $X \cap R = \emptyset$, $G[X]$ is isomorphic to an ℓ-hook, $G[R]$ is connected, and $N(R) \cap X$ consists of exactly one vertex, being the active vertex of the ℓ-hook $G[X]$.

Theorem 20. *For every $k \geqslant 0$, there exists a constant ε_0 such that for every $0 < \varepsilon \leqslant \varepsilon_0$ and in every ε-structured graph $(G, (A, B, S))$ there exists either*

1. *an anti-adjacent pair (P, Q) in G with $P, Q \subseteq S$, $|P|, |Q| \geqslant \varepsilon|S|$; or*

2. *an active ℓ-hook (X, R) in G with $\ell \geqslant k$, $R \subseteq S$, and $|R| \geqslant 2\varepsilon|S|$; or*

3. *a subset $\hat{S} \subseteq S$ with $|\hat{S}| \geqslant |S|/5$ and a partition $\hat{S} = S_1 \uplus S_2 \uplus \ldots \uplus S_m$, for some $m \geqslant 2$, such that*

   (a) *$|S_i| \leqslant \varepsilon|S|$ for every $1 \leqslant i \leqslant m$;*

   (b) *every set S_i is a module of $G[\hat{S}]$; and*

   (c) *the quotient graph of this partition of the vertex set of $G[\hat{S}]$ is a claw-free Berge graph.*

Remark - We note that any active ℓ-hook (X, R) of a graph G for $\ell \geqslant k$ can easily be turned into an active k-hook. Indeed let $v_1, \ldots, v_{\ell+4}$ be the vertices of X with $v_1 v_2 \cdots v_{\ell+3}$ forming the long path in the hook, and let v_1 be the active vertex. Taking $X' = \{v_{\ell-k+1}, \ldots, v_{\ell+4}\}$ (which induces a k-hook) and $R' = R \cup \{v_1, \ldots, v_{\ell-k}\}$ (which remains connected) we see (X', R') is an active k-hook with $v_{\ell-k+1}$ being the active vertex.

We delay the proof of this theorem until Section 6. Informally, the idea is as follows. Let $(G, (A, B, S))$ be an ε-structured pair and assume that G does not contain a homogeneous pair of linear size or an active ℓ-hook. After some filtering, we partition the vertices in S into equivalence classes according to their neighbourhoods in $A \cup B$. Assuming certain subgraphs like the hook are forbidden in G, it turns out that edges and non-edges between pairs of vertices in S correspond to a certain behaviour of the neighbourhoods of those vertices in A and B. This allows us to deduce that the equivalence classes of the partition on S are in fact modules. Furthermore, the quotient graph turns out to have an even more restricted structure in terms of the induced subgraphs that are forbidden.

We believe that these methods can be of further use to approach similar problems. Since the proof of Theorem 20 is rather technical, we present the following as a warm-up in Section 5 to illustrate our methods, although we will need many of the lemmas from Section 5 later.

Theorem 21. *For every $1/10$-structured graph $(G, (A, B, S))$ such that G is both claw-free and C_5-free, there exists a subset $\hat{S} \subseteq S$ with $|\hat{S}| \geqslant |S|/5$ and a partition $\hat{S} = S_1 \uplus S_2 \uplus \ldots \uplus S_\ell$ such that:*

1. *every set S_i is contained in a neighbourhood of some vertex in A;*
2. *every set S_i is a module of $G[\hat{S}]$;*
3. *the quotient graph of this partition of the vertex set of $G[\hat{S}]$ is a line graph of a triangle-free graph.*

We now prove our main result.

Proof of Theorem 5. Note that if the theorem holds for $k = r$ then it holds for all $1 \leqslant k \leqslant r$. Hence it is sufficient to prove the theorem for all $k \geqslant 2$. Thus, fix $k \geqslant 2$ and let $\mathcal{G} := \{G : G \text{ is } \mathcal{H}^2_{\geqslant k}\text{-free}\}$ and set $\varepsilon_0 = \varepsilon_0(k)$ to be the constant from Theorem 20. We shall prove the following claim.

Claim 22. *Suppose $G \in \mathcal{G}$ has n vertices and maximum degree $\varepsilon_0 n/100$. Then either G has a homogenous pair (P, Q) where $|P|, |Q| \geqslant \varepsilon_0 n/3000$ or we can find an active k-hook (X, R), where $|R| \geqslant \varepsilon_0 n/50$.*

Proof of Claim. By Lemma 19, either there is a homogeneous pair (P, Q) in G with $|P|, |Q| \geqslant n/10$ (in which case we are done) or there is an $\frac{\varepsilon_0}{10}$-structured pair $(G[A \cup B \cup S], (A, B, S))$ with $|S| \geqslant n/10$.

Set $G_1 := G[A \cup B \cup S]$. By Theorem 20, there is either 1. an anti-adjacent pair (P, Q) in S with $|P|, |Q| \geqslant \frac{\varepsilon_0}{10}|S| \geqslant \frac{\varepsilon_0}{100} n$ (in which case we are done); or 2. an active ℓ-hook

(X, R) with $\ell \geqslant k$, $R \subseteq S$ and $|R| \geqslant 2\frac{\varepsilon_0}{10}|S|$; or 3. a subset $\hat{S} \subseteq S$ with $|\hat{S}| \geqslant |S|/5$ and a partition $\hat{S} = S_1 \uplus S_2 \uplus \ldots \uplus S_m$, for some $m \geqslant 2$, such that

(*a*) $|S_i| \leqslant \frac{\varepsilon_0}{10}|S| \leqslant \frac{\varepsilon_0}{2}|\hat{S}|$ for every $1 \leqslant i \leqslant m$;

(*b*) every set S_i is a module of $G[\hat{S}]$; and

(*c*) the quotient graph of this partition of the vertex set of $G[\hat{S}]$ is a claw-free Berge graph.

In the third outcome, we consider the quotient graph G_q that has vertex set $V_q := \{S_i : 1 \leqslant i \leqslant m\}$, and where S_iS_j forms an edge in G_q if and only if (S_i, S_j) is an adjacent pair. We define a probability measure μ on V_q in the natural way by setting $\mu(S_i) := |S_i|/|\hat{S}|$. Note that, by Property (a), $\mu(S_i) \leqslant \varepsilon_0/2$ for every vertex S_i in G_q. By Property (c), the graph G_q is claw-free and Berge. We now invoke Theorem 10 to see that there is either a homogeneous pair (P_q, Q_q) in G_q with $\mu(P_q), \mu(Q_q) \geqslant \delta_{cB}$, or there is a vertex $S_i \in V_q$ with $\mu(S_i) \geqslant 1 - 2\delta_{cB}$, where δ_{cB} is a constant that can be taken to be $\frac{1}{58}$ (see Section 3). Since for every $1 \leqslant i \leqslant m$ we have that $\mu(S_i) \leqslant \varepsilon_0/2 \leqslant 1 - 2\delta_{cB}$ the first outcome must hold for G_q. Consider the sets $P := \bigcup_{S_i \in P_q} S_i$ and $Q := \bigcup_{S_i \in Q_q} S_i$. Note that since the S_i's are modules and (P_q, Q_q) is a homogeneous pair in G_q, then (P, Q) is a homogeneous pair in $G[\hat{S}]$ and hence in G. Furthermore $|P|, |Q| \geqslant \delta_{cB}|\hat{S}| \geqslant \frac{\delta_{cB}}{50} \cdot n \geqslant \frac{n}{3000}$, giving us the homogeneous pair of the desired size.

Thus we may assume the second outcome holds, where we find an active ℓ-hook (X, R), for some $\ell \geqslant k$, such that $R \subseteq S$ and $|R| \geqslant 2\frac{\varepsilon_0}{10}|S| \geqslant \frac{\varepsilon_0}{50}n$. By the remark after Theorem 20, we may assume $\ell = k$ as required. $\square$

To prove the theorem, we must show that there exists a constant $\delta > 0$ such that every $G \in \mathcal{G}$ contains a homogeneous pair (P, Q) with $|P|, |Q| \geqslant \delta \cdot v(G)$. Fix $\varepsilon = \varepsilon_0^2/(60000(k + 4))$, set $\delta_0 = \delta(\varepsilon, H_k^2)$ where $\delta(\cdot, \cdot)$ is the constant from Theorem 18 and set $\delta = \delta_0\varepsilon_0^2/(10^7(k + 4))$. Thus since both G and G^c are H_k^2-free, Theorem 18 implies that either G or G^c contains an induced subgraph, say G_0, on at least $\delta_0 \cdot v(G)$ vertices with edge density at most ε. Assume without loss of generality that G_0 is an induced subgraph of G. By a simple averaging argument, G_0 contains an induced subgraph G_1, with $v(G_1) \geqslant v(G_0)/2 \geqslant \delta_0 \cdot v(G)/2$ and where G_1 has maximum degree at most $4\varepsilon \cdot v(G_1)$.

By Claim 22, either G_1 (and hence G) has a homogeneous pair of size at least $\varepsilon_0 \cdot v(G_1)/3000 \geqslant \delta \cdot v(G)$ (and we are done) or G_1 has an active k-hook (X, R) with $|R| \geqslant \varepsilon_0 \cdot v(G_1)/50 \geqslant \delta_0\varepsilon_0 \cdot v(G)/100$. That is, $G[X]$ is isomorphic to a k-hook, say with active vertex x, and $N(R) \cap X = \{x\}$.

Next we construct $A \subseteq R \cup \{x\}$ iteratively as follows: start with $A = \{x\}$ and add vertices of R to A ensuring each time that $G[A]$ is connected (this is possible since $G[R]$ is connected) until $|N[A] \cap R| \geqslant |R|/2$. We have $\Delta(G_1) \leqslant 4\varepsilon v(G_1) \leqslant 200\varepsilon\varepsilon_0^{-1}|R| \leqslant |R|/6$ by our choice of small enough ε, and so $|N[A] \cap R| \leqslant (|R|/2) + \Delta(G_1) \leqslant \frac{2}{3}|R|$. Thus $|R - N[A]| \geqslant |R|/3 \geqslant \delta_0\varepsilon_0 v(G)/300 \geqslant \delta v(G)$.

Writing $R' = R - N[A]$ and $G_2 = G_1[R']$, we have $|R'| \geqslant |R|/3 \geqslant \delta_0\varepsilon_0 \cdot v(G)/300$. Note that $\Delta(G_2) \leqslant \Delta(G_1) \leqslant 4\varepsilon v(G_1) \leqslant 600\varepsilon\varepsilon_0^{-1} v(G_2) \leqslant \varepsilon_0 v(G_2)/100$ by our choice of ε

small. By Claim 22, either we obtain a homogeneous pair in G_2 and hence G of size at least $\varepsilon_0 v(G_2)/3000 \geqslant \delta_0\varepsilon_0^2/900000 \geqslant \delta v(G)$ and we are done, or else we obtain a k-hook (X^*, R^*) with $|R^*| \geqslant \varepsilon_0 \cdot v(G_2)/50 \geqslant (\delta_0\varepsilon_0^2/15000) \cdot v(G) \geqslant \delta v(G)$ by our choice of δ small. Thus, $G_2[X^*] = G[X^*]$ is isomorphic to a k-hook, say with active vertex x^*, and $N(R^*) \cap X^* = \{x^*\}$.

Note that $|N(X^*) \cap R| \leqslant (k+4)\Delta(G_1) \leqslant (4\varepsilon)(k+4)v(G_1) \leqslant |R|/4$ by our choice of ε small. Hence $|N[A] - N(X^*)| \geqslant |N(A)| - N(X^*) \geqslant |R|/4 \geqslant \delta v(G)$ by our choice of δ small enough. We may assume there is an edge between R^* and $N[A] - N(X^*)$; otherwise we have an anti-adjacent pair $(R^*, N[A] - N(X^*))$ of size $\delta v(G)$ and we are done.

Now let $T = R^* \cup (N[A] - N(X^*))$, so $x, x^* \in T$. Note $G[T]$ is connected because $G[N[A] - N(X^*)]$ is connected (since $G[A]$ is connected and $N(X^*) \subseteq N(A)$) and $G[R^*]$ is connected and there is an edge between R^* and $N[A] - N[X^*]$. Writing P for the shortest path from x to x^* in $G[T]$, we have that in G, $X \cup P \cup X^*$ induces a copy of a double ℓ-hook for some $\ell \geqslant k$. $\square$

5 Warm up: Proof of Theorem 21

The aim of this section is to provide a proof of Theorem 21 that will serve as a warm-up before proving the main technical step of this paper, namely Theorem 20. The proofs of Theorems 21 and 20 follow the same general outline, while the technical details in this section are much simpler. To exhibit the similarities between the proofs, we use nearly the same subsection structure in this section and the next one, even though here some subsections will consist only of a single simple observation.

Let us fix a 1/10-structured pair $(G, (A, B, S))$ such that G is claw-free and C_5-free. Define a binary relation $R^=$ on S as $R^=(x, y)$ if and only if $N_{A\cup B}(x) = N_{A\cup B}(y)$ and we note that this is an equivalence relation. Our approach consists of the following steps:

1. We start with filtering out vertices $x \in S$ that have large neighbourhood in A or in B. Since every vertex $p \in A \cup B$ satisfies $|N_S(p)| \leqslant |S|/10$, a standard averaging argument shows that the number of such vertices is small.

2. Second, for every $x \in S$, we study nonedges inside neighbourhoods $N_A(x)$; such nonedges turn out to be good starting points to construct either a claw (in the case of Theorem 21) or a hook (in the case of Theorem 20).

3. Then, for every two vertices $x, y \in S$, we investigate how the neighbourhoods $N_A(x)$ and $N_A(y)$ differ, depending on whether xy is an edge or a nonedge. Intuitively, we want to prove that if $xy \in E(G)$, then the neighbourhoods in A and B cannot change much, while if $xy \notin E(G)$, then they should change much or not at all.

4. We then collect the main properties we need from the aforementioned steps in the definition of a *nice* ε-structured pair. We prove that the relevant ε-structured pair is nice, both in the proof of Theorem 21 and Theorem 20. In this section we show that it is sufficient for the relevant ε-structured pair to be nice in order to find a large set

$\hat{S} \subseteq S$, such that if we restrict the relation $R^{=}$ to $\hat{S}$, the equivalence classes of this relation form a decomposition of $G[\hat{S}]$ into modules.

5. Finally, we show that in the case of Theorem 21, the quotient graph of the aforementioned decomposition is diamond-free; this, together with being claw-free, implies that the quotient graph is in fact a line graph of a triangle-free graph, concluding the proof of Theorem 21.

In the proofs of Theorems 21 and 20, if a statement is accompanied with a sign $(A \leftrightarrow B)$, then we also claim that the same statement holds with the roles of A and B swapped.

5.1 Filtering step

Let $S_A = \{x \in S : A \subseteq N(x)\}$ and similarly define S_B. A standard averaging argument shows the following:

Claim 23. $|S_A|, |S_B| \leqslant |S|/10$.

Proof. Consider the following random experiment: independently and uniformly at random pick a vertex $p \in A$ and $x \in S$. Since every vertex in A is adjacent to at most $|S|/10$ vertices of S, the probability that $px \in E(G)$ is at most $1/10$. On the other hand, once $x \in S_A$, we have $px \in E(G)$ regardless of the choice of p. Consequently, the probability that $x \in S_A$ is at most $1/10$. □

Define now $S' = S \setminus (S_A \cup S_B)$ and $G' = G \setminus (S_A \cup S_B)$. Since $|S_A \cup S_B| \leqslant |S|/5$, we have that $(G', (A, B, S'))$ is an $1/8$-structured pair.

By restricting ourselves to the structured pair $(G', (A, B, S'))$, it suffices to prove the conclusion of Theorem 21 with stronger condition $|\hat{S}| \geqslant |S|/4$, but with the additional assumption

$$\forall_{x \in S} (N_A(x) \subsetneq A) \wedge (N_B(x) \subsetneq B). \tag{5.1}$$

To simplify the notation, in the rest of this section we assume that the input structured graph is only $1/8$-structured, but satisfies already (5.1).

5.2 Neighbourhoods in $A \cup B$

5.2.1 Nonedges inside a neighbourhood in A

In the case of claw-free graphs, there are simply no edges inside neighbourhoods in A.

Claim 24 ($A \leftrightarrow B$). *For every $x \in S$ the set $N_A(x)$ is a clique.*

Proof. Assume the contrary, let $p, q \in N_A(x)$, $p \neq q$, and $pq \notin E(G)$. Let $z \in N_B(x)$ be any vertex (it exists since $N(B) = S$). Then $(x; p, q, z)$ is a claw in G, a contradiction. □

5.2.2 Neighbourhoods along a nonedge in S

Claim 25 ($\boldsymbol{A \leftrightarrow B}$)**.** *For every $x, y \in S$ with $x \neq y$, $xy \notin E(G)$, there is no edge between $N_A(x) \cap N_A(y)$ and $A \setminus N_A(x, y)$.*

Proof. Assume the contrary, let $p \in N_A(x) \cap N_A(y)$ and $q \in A \setminus N_A(x, y)$ with $pq \in E(G)$. Then $(p; x, y, q)$ is a claw in G, a contradiction. ☐

5.2.3 Neighbourhoods along an edge in S

Claim 26. *For every $xy \in E(G[S])$, either $N_A(x) \setminus N_A(y)$ or $N_B(x) \setminus N_B(y)$ is empty.*

Proof. Assume the contrary, let $p_\Gamma \in N_\Gamma(x) \setminus N_\Gamma(y)$ for $\Gamma \in \{A, B\}$. Then $(x; y, p_A, p_B)$ is a claw in G, a contradiction. ☐

5.3 Niceness of an ε-structure and its corollaries

In the following definition, we extract some properties of the ε-structured pair $(G, (A, B, S))$ that were proven in Claims 24, 25, and 26, and then show what can be deduced from these properties only. Exactly the same properties will be proven in the next section, in the more general setting of Theorem 20, and hence we will be able to reuse the statements obtained here.

Definition 27. An ε-structured pair $(G, (A, B, S))$ is called *nice* if the following holds:

(NE1) for every $x \in S$ we have $A \not\subseteq N(x)$ and $B \not\subseteq N(x)$;

(NE2) $(A \leftrightarrow B)$ for every $x, y \in S$ with $x \neq y$ and $xy \notin E(G)$, if $N_B(x) \neq N_B(y)$, then there is no edge between $N_A(x) \cap N_A(y)$ and $A \setminus N_A(x, y)$;

(NE3) for every $x, y \in S$ with $x \neq y$, $xy \notin E(G)$, and $N_A(x) \subsetneq N_A(y)$, the sets $N_A(x)$ and $N_A(y) \setminus N_A(x)$ are fully adjacent;

(E1) for every $x, y \in S$ such that $xy \in E(G[S])$, either $N_A(x) \setminus N_A(y) = \emptyset$ or $N_B(x) \setminus N_B(y) = \emptyset$.

Note that we have used here the notation $(A \leftrightarrow B)$, denoting that the particular condition is required to hold also with the roles of A and B swapped. Whenever ε is unimportant for the analysis we shall drop it from the notation and speak only of a *(nice) structured pair.*

Let us now formally verify that the considered structured pair $(G, (A, B, S))$ is nice.

Claim 28. *The structured pair $(G, (A, B, S))$ is nice.*

Proof. Property (NE1) is equivalent to (5.1), Property (NE2) is strictly weaker than the statement of Claim 25, Property (NE3) is a special case of the statement of Claim 24, while Property (E1) is exactly the statement of Claim 26. ☐

We start our analysis of nice structured graphs with the following observation.

Lemma 29. *If a structured pair $(G,(A,B,S))$ satisfies Properties (NE1) and (NE2), then for every two distinct vertices $x,y \in S$ with $xy \notin E(G)$ we have $N_A(x) = N_A(y)$ if and only if $N_B(x) = N_B(y)$.*

Proof. Assume by contradiction that for some $x,y \in S$ with $x \neq y$ and $xy \notin E(G)$ we have $N_A(x) = N_A(y)$ but $N_B(x) \neq N_B(y)$. By Property (NE2), there is no edge between $N_A(x) \cap N_A(y) = N_A(x)$ and $A \setminus N_A(x,y) = A \setminus N_A(x)$. However, by Property (NE1) and the assumption $N(A) = S$, both $N_A(x)$ and $A \setminus N_A(x)$ are nonempty. This contradicts the connectivity of $G[A]$. □

We now move to a deeper study of the situation treated in Property (NE3).

Lemma 30 ($\boldsymbol{A \leftrightarrow B}$). *If $(G,(A,B,S))$ is a nice structured pair, then there do not exist three distinct vertices $x,y,z \in S$ with $xy, yz \notin E(G)$ and $N_A(x) \subsetneq N_A(y) \subsetneq N_A(z)$.*

Proof. Assume the contrary, and let x,y,z be as in the statement. Let p be any vertex of $N_A(x)$ and q be any vertex of $N_A(z) \setminus N_A(y)$. By Lemma 29 applied to the pair (x,y), we have $N_B(x) \neq N_B(y)$ since $N_A(x) \neq N_A(y)$. By Property (NE2) applied to the pair (x,y), we have $pq \notin E(G)$, since $p \in N_A(x) = N_A(x) \cap N_A(y)$ and $q \in N_A(z) \setminus N_A(y) \subseteq A \setminus N_A(x,y)$. However, Property (NE3) applied to the pair (y,z) implies that $pq \in E(G)$, a contradiction. □

Recall that we have defined the relation $R^=$ on the set S as $R^=(x,y)$ if and only if $N_{A\cup B}(x) = N_{A\cup B}(y)$. We now introduce a number of other binary relations on the set S that describe the relation between neighbourhoods in $A \cup B$. For two vertices $x,y \in S$ we have

$\boldsymbol{R^{\neq}(x,y)}$ if and only if $N_A(x)$ and $N_A(y)$ are incomparable with respect to inclusion, and $N_B(x)$ and $N_B(y)$ are incomparable with respect to inclusion;

$\boldsymbol{R^=_A(x,y)}$ if and only if $N_A(x) = N_A(y)$ and $N_B(x) \neq N_B(y)$;

$\boldsymbol{R^=_B(x,y)}$ if and only if $N_B(x) = N_B(y)$ and $N_A(x) \neq N_A(y)$;

$\boldsymbol{R^{\subsetneq}(x,y)}$ if and only if $N_A(x) \subsetneq N_A(y)$ and $N_B(x) \supsetneq N_B(y)$;

$\boldsymbol{R^{\supsetneq}(x,y)}$ if and only if $N_A(x) \supsetneq N_A(y)$ and $N_B(x) \subsetneq N_B(y)$.

Observe that the relations $R^=$, $R^{\neq}$, $R^=_A$, and $R^=_B$ are symmetric, while $R^{\subsetneq}$ and $R^{\supsetneq}$ are strongly antisymmetric, and $R^{\subsetneq}(x,y)$ if and only if $R^{\supsetneq}(y,x)$. Furthermore, all six defined relations are pairwise disjoint. Note that this is not an exhaustive list of all possible neighbourhood relations that can occur in general graphs, but we will see that these cover all cases in our particular situation.

Lemma 30 implies that, along nonedges in S, the neighbourhoods in A cannot create chains with respect to inclusions. As a corollary, we can obtain the following:

Lemma 31. *If $(G,(A,B,S))$ is a nice structured pair, then there exists a set $\hat{S} \subseteq S$ of size at least $|S|/4$ such that for every $x, y \in \hat{S}$ with $x \neq y$ and $xy \notin E(G)$, either $R^{=}(x,y)$ or $R^{\neq}(x,y)$.*

Proof. Consider an auxiliary directed multigraph G_A defined as follows: we take $V(G_A) = S$ and for every $x, y \in S$ with $x \neq y$ and $xy \notin E(G)$ we add an arc (x,y) if $N_A(x) \subsetneq N_A(y)$. Let S_A^+ be the set of vertices of S that have positive out-degree in G_A, and let S_A^- be the set of vertices of S that have positive in-degree. Symmetrically, define G_B and sets S_B^+ and S_B^-. For $\alpha, \beta \in \{+,-\}$, define $S^{\alpha\beta} = S \setminus (S_A^\alpha \cup S_B^\beta)$.

Lemma 30 implies that $S_A^+ \cap S_A^- = \emptyset$ and $S_B^+ \cap S_B^- = \emptyset$, which in turn implies that $S^{++} \cup S^{+-} \cup S^{-+} \cup S^{--} = S$. Consequently, by setting $\hat{S}$ to be the largest of the sets $S^{\alpha\beta}$, we have $|\hat{S}| \geqslant |S|/4$. The definition of the sets $S^{\alpha\beta}$ ensures that $G_A[\hat{S}]$ and $G_B[\hat{S}]$ are arcless, that is, for every $x, y \in \hat{S}$ with $xy \notin E(G)$ it cannot happen that $N_A(x) \subsetneq N_A(y)$ or $N_B(x) \subsetneq N_B(y)$. However, Lemma 29 ensures that once $N_{A\cup B}(x) \neq N_{A\cup B}(y)$ for some $x, y \in S$ with $xy \notin E(G)$, then both $N_A(x) \neq N_A(y)$ and $N_B(x) \neq N_B(y)$, and, consequently, $R^{\neq}(x,y)$ if $x, y \in \hat{S}$. This finishes the proof of the lemma. $\square$

Summarizing, we obtain the following statement, which says that for every distinct $x, y \in \hat{S}$, the existence or non-existence of an edge xy can be determined by examining the neighbourhoods of x and y in $A \cup B$.

Theorem 32. *For every nice structured pair $(G,(A,B,S))$ there exists a set $\hat{S} \subseteq S$ with $|\hat{S}| \geqslant |S|/4$ such that for every $x, y \in \hat{S}$ with $N_{A\cup B}(x) \neq N_{A\cup B}(y)$ the following holds.*

1. *$xy \in E(G)$ if and only if exactly one of the following holds: $R_A^{=}(x,y)$, $R_B^{=}(x,y)$, $R^{\subsetneq}(x,y)$, or $R^{\supsetneq}(x,y)$.*

2. *$xy \notin E(G)$ if and only if $R^{\neq}(x,y)$.*

Proof. We obtain the set $\hat{S}$ from Lemma 31. The "if" part of the assertion for edges xy and the "only if" part of the assertion for nonedges xy is straightforward from Lemma 31, while the remaining two implications follow from Property (E1). $\square$

As mentioned at the beginning of this section, we now partition $\hat{S}$ according to the relation $R^{=}$. Observe that due to Theorem 32, the presence or absence of an edge between two vertices $x, y \in S$ is determined by $N_{A\cup B}(x)$ and $N_{A\cup B}(y)$ unless $R^{=}(x,y)$. An immediate corollary is the following.

Corollary 33. *Let $(G,(A,B,S))$ be a nice structured pair, let $\hat{S} \subseteq S$ be the set obtained from Theorem 32, and let $S_1, S_2, \ldots, S_r$ be the equivalence classes of the relation $R^{=}$ restricted to $\hat{S}$. Then every set S_i is a module of $G[\hat{S}]$.*

As a last step in our analysis of nice structured graphs, we investigate P_3's in the quotient graph of the aforementioned partition of $G[\hat{S}]$ into modules.

Lemma 34. *Let $(G,(A,B,S))$ be a nice structured pair, let $\hat{S} \subseteq S$ be the set obtained from Theorem 32, and let $x,y,z \in \hat{S}$ be three distinct vertices belonging to different equivalence classes of the relation $R^=$, such that $xy \in E(G)$, $yz \in E(G)$, and $xz \notin E(G)$. Then one of the following holds:*

- *$R^=_A(x,y)$ and $R^=_B(y,z)$;*
- *$R^=_B(x,y)$ and $R^=_A(y,z)$;*
- *$R^{\subsetneq}(x,y)$ and $R^{\subsetneq}(z,y)$;*
- *$R^{\supsetneq}(x,y)$ and $R^{\supsetneq}(z,y)$;*

Proof. Since $xz \notin E(G)$, we have $R^{\neq}(x,z)$; in particular the sets $N_A(x)$ and $N_A(z)$ are incomparable with respect to inclusion. If $R^=_A(x,y)$, then the only option from Theorem 32 for the edge yz that allows this property to happen is $R^=_B(y,z)$; symmetrical claims follow if we swap the roles of A and B and/or the roles of x and z. In the remaining case, if neither (x,y) nor (y,z) belongs to $R^=_A \cup R^=_B$, then the only way to ensure incomparability of $N_A(x)$ and $N_A(z)$ is to have $R^{\subsetneq}(x,y)$ and $R^{\subsetneq}(z,y)$ or $R^{\supsetneq}(x,y)$ and $R^{\supsetneq}(z,y)$. □

In the next lemma we remark that Lemma 34 already implies that the quotient graph of the partition of $G[\hat{S}]$ into equivalence classes of the relation $R^=$ is Berge.

Lemma 35. *Let $(G,(A,B,S))$ be a nice structured pair and let $\hat{S} \subseteq S$ be the set obtained from Theorem 32. Then the quotient graph of the partition of $G[\hat{S}]$ into equivalence classes of the relation $R^=$ is Berge.*

Proof. Assume that the set $\hat{S}$ contains a sequence $x_1, x_2, \ldots, x_h$ of vertices for some odd integer $h \geqslant 5$, such that $x_i x_{i+1} \in E(G)$ and $x_i x_{i+2} \notin E(G)$ for every $1 \leqslant i \leqslant h$ and for indices behaving cyclically modulo h. Furthermore, assume that no two vertices x_i are in relation $R^=$.

Consider the edge x_1x_2, and let us consider four cases, depending on which option of Theorem 32 holds for this edge. By symmetry between the sides A and B, we need only consider the cases $R^=_A(x_1,x_2)$ and $R^{\subsetneq}(x_1,x_2)$. If $R^=_A(x_1,x_2)$, then Lemma 34 applied to the P_3 x_1, x_2, x_3 implies that $R^=_B(x_2,x_3)$. Inductively, we infer that $R^=_A(x_i,x_{i+1})$ if i is odd and $R^=_B(x_i,x_{i+1})$ if i is even. However, this leads to a contradiction as h is odd. A similar situation happens if $R^{\subsetneq}(x_1,x_2)$: we have $R^{\subsetneq}(x_i,x_{i+1})$ for odd i and $R^{\supsetneq}(x_i,x_{i+1})$ for even i, again yielding a contradiction

We infer that no such sequence $x_1, x_2, \ldots, x_h$ exists. However, note that such a sequence is present in any odd hole in the quotient graph in the question (take the subsequent vertices on the hole) and is present in any odd anti-hole as well (if the anti-hole consists of h vertices $y_1, y_2, \ldots, y_h$ in this order, take $x_i = y_{(i\lfloor h/2 \rfloor) \bmod h}$). We infer that the quotient graph in the question does not contain any odd hole nor anti-hole, and is thus Berge. □

Let us now wrap up what our analysis of nice structured graphs implies for the proof of Theorem 21. Recall that we are dealing with a structured pair $(G,(A,B,S))$ where

G is claw-free and C_5-free. Claim 28 implies that (after the filtering step) we are in fact dealing with a nice structured pair. Theorem 32 provides us with a candidate set $\hat{S}$, that we fix for the remainer of this proof. Corollary 33 implies that the relation $R^{=}$ partitions $G[\hat{S}]$ into modules. Moreover, by construction, every such module S_i is contained in a neighbourhood of some vertex from A. It remains to analyse the quotient graph of this partition.

5.4 The quotient graph: Excluding a diamond

Clearly, the quotient graph of the partition of $G[\hat{S}]$ into equivalence classes of the relation $R^{=}$ is claw-free, since G is claw-free. In the rest of this section we show that it is also diamond-free. This, together with a characterization from [28, 33] showing that the class of (claw,diamond)-free graphs is exactly the class of line graphs of triangle-free graphs, concludes the proof of Theorem 21.

We start by showing that the last two cases of Lemma 34 cannot appear if G is claw-free and C_5-free.

Claim 36. *Let x, y, z be as in the statement of Lemma 34. Then either $R^{=}_{A}(x, y)$ and $R^{=}_{B}(y, z)$ or $R^{=}_{B}(x, y)$ and $R^{=}_{A}(y, z)$. That is, the last two cases cannot happen.*

Proof. Assume the contrary; by swapping the sides A and B if needed, we can assume that $R^{\subsetneq}(x, y)$ and $R^{\subsetneq}(z, y)$. Since $xz \notin E(G)$, the sets $N_A(x)$ and $N_A(z)$ are incomparable with respect to inclusion; let $p \in N_A(x) \setminus N_A(z)$ and $q \in N_A(z) \setminus N_A(x)$. By Claim 24, we have $pq \in E(G)$, since $p, q \in N_A(y)$. Let $s \in N_B(y)$ be any vertex. Observe that $\{p, x, s, z, q\}$ induce a C_5 in G, a contradiction. ☐

We conclude with an observation that without the two cases of Lemma 34 excluded in Claim 36, we cannot have a diamond in the quotient graph.

Claim 37. *The quotient graph of the partition of $G[\hat{S}]$ into equivalence classes of the relation $R^{=}$ is diamond-free.*

Proof. Assume the contrary. Let $x, y, s, t \in \hat{S}$ be four distinct vertices that belong to four different equivalence classes of the relation $R^{=}$. Furthermore, assume that $G[\{x, y, s, t\}]$ is isomorphic to a diamond with $xy \notin E(G)$. By swapping the sides A and B if needed, by Claim 36 applied to the triple x, s, y, we can assume that $R^{=}_{A}(x, s)$ and $R^{=}_{B}(y, s)$.

Let us now consider two cases of Claim 36 applied to the triple x, t, y. If $R^{=}_{A}(x, t)$ and $R^{=}_{B}(y, t)$, then we have $N_A(s) = N_A(x) = N_A(t)$ and $N_B(s) = N_B(y) = N_B(t)$, giving $R^{=}(s, t)$, a contradiction. If $R^{=}_{B}(x, t)$ and $R^{=}_{A}(y, t)$, then we have

$$N_A(s) = N_A(x), \quad N_A(t) = N_A(y), \quad N_B(s) = N_B(y), \quad N_B(t) = N_B(x). \tag{5.2}$$

Since $xy \notin E(G)$, we have by Theorem 32 that $R^{\neq}(x, y)$. By (5.2) this implies that $R^{\neq}(s, t)$, a contradiction to the assumption $st \in E(G)$ and Theorem 32. ☐

6 Proof of Theorem 20

In this section we prove Theorem 20 using the same proof outline as for Theorem 21 from the previous section. In particular, after a filtering step we will prove that the ε-structured pair at hand is actually nice (c.f. Definition 27), which allows us to apply the tools developed in Section 5.3.

It will be convenient for the proof to split the constant ε into three constants ε, δ, and γ in the following way. We show that, for every $k \geqslant 0$, if $\varepsilon, \delta, \gamma$ are small enough positive constants that satisfy $2\varepsilon < \gamma$ then in every ε-structured pair $(G, (A, B, S))$ there exists either

1. an anti-adjacent pair (P, Q) in G with $P, Q \subseteq S$, $|P|, |Q| \geqslant \delta|S|$; or
2. an active ℓ-hook (X, R) in G with $\ell \geqslant k$, $R \subseteq S$, and $|R| \geqslant \gamma|S|$; or
3. a subset $\hat{S} \subseteq S$ with $|\hat{S}| \geqslant |S|/5$ and a partition $\hat{S} = S_1 \uplus S_2 \uplus \ldots \uplus S_m$, for some $m \geqslant 2$, such that
   (a) $|S_i| \leqslant \varepsilon|S|$ for every $1 \leqslant i \leqslant m$;
   (b) every set S_i is a module of $G[\hat{S}]$; and
   (c) the quotient graph of this partition of the vertex set of $G[\hat{S}]$ is a claw-free Berge graph.

Instead of giving an explicit formula for ε, δ, and γ, we will state a number of inequalities that these constants should satisfy in the course of the proof. Every such inequality will be true for sufficiently small positive constants; in particular, taking $\varepsilon = \delta = \frac{1}{200(k+10)}$ and $\gamma = \frac{1}{100(k+10)}$ will suffice.

For two disjoint vertex sets Q and D in a graph G, $\texttt{reach}(Q \to D)$ denotes the set of vertices v of D such that in the graph $G[Q \cup D]$ there is a path from some vertex in Q to v. Equivalently $\texttt{reach}(Q \to D) = C \cap D$, where C is the union of all components of $G[Q \cup D]$ that contain at least one vertex of Q.

Let $(G, (A, B, S))$ be an ε-structured graph for some (small) constant $\varepsilon > 0$.

6.1 Filtering

In the proof of Theorem 20 we need a stronger filtering step than the one used for Theorem 21: we need not only to discard vertices of S that are adjacent to the entire set A or B, but all vertices that are adjacent to a large fraction of A or B. Furthermore, we need to use a non-uniform measure on A and B, as defined below.

For every $x \in S$, we fix one neighbour $\pi_A(x) \in N_A(x)$ and one neighbour $\pi_B(x) \in N_B(x)$. We define a probability measure μ_A on A by $\mu_A(X) = |\pi_A^{-1}(X)|/|S|$. That is, the measure μ_A corresponds to a random experiment where we choose a vertex $x \in S$ uniformly at random, and output $\pi_A(x)$. Similarly we define a probability measure μ_B on B using the function π_B.

Let $S_A = \{x \in S : \mu_A(N_A(x)) \geqslant 10\varepsilon\}$ and similarly let $S_B = \{x \in S : \mu_B(N_B(x)) \geqslant 10\varepsilon\}$. A standard averaging argument shows the following.

Claim 38. $|S_A|, |S_B| \leqslant |S|/10$.

Proof. Consider the following random experiment: independently choose $x \in S$ uniformly at random and $p \in A$ according to the measure μ_A. Since every vertex in A is adjacent to at most $\varepsilon|S|$ vertices of S, the probability that $px \in E(G)$ is at most ε. On the other hand, conditioning on $x \in S_A$, we have $px \in E(G)$ with probability at least 10ε by the definition of S_A. Consequently, the probability that $x \in S_A$ is at most $1/10$. The proof for S_B is symmetric. □

By Claim 38, we have $|S_A \cup S_B| \leqslant |S|/5$. Consequently, by considering the pair $(G \setminus (S_A \cup S_B), (A, B, S \setminus (S_A \cup S_B)))$ instead of $(G, (A, B, S))$, and by suitably adapting the constant ε, in the rest of the proof we can assume that our ε-structured pair $(G, (A, B, S))$ has the additional property that

$$\text{for all } x \in S: \ \mu_A(N_A(x)) < \varepsilon \text{ and } \mu_B(N_B(x)) < \varepsilon. \tag{6.1}$$

However, we now need to exhibit a set $\hat{S}$ of size at least $|S|/4$ (instead of $|S|/5$ in the statement of Theorem 20).

In the remainder of the proof, let us assume that, for some sufficiently small constants ε, δ, and γ, our input structured graph $(G, (A, B, S))$ does not admit the desired anti-adjacent pair nor the desired active hook; our goal is to prove that $(G, (A, B, S))$ is nice and use the results of Section 5.3 to obtain the set $\hat{S}$. Observe that (6.1) already implies Property (NE1) for $(G, (A, B, S))$.

6.2 A generic claim to find an active hook

We will encounter several situations that allow us to find an active hook in an ε-structured pair. We bundle the commonalities in the following claim.

Claim 39 ($\boldsymbol{A \leftrightarrow B}$)**.** *Assume there exist pairwise disjoint sets $Z, Q, D \subseteq V(G)$ such that:*

(i) $Q, D \subseteq A$;

(ii) (Z, D) *is an anti-adjacent pair;*

(iii) for every $q \in Q$, there exists an integer $i \geqslant 0$ and an i-hook in $G[\{q\} \cup Z]$ with q being the active vertex;

(iv) $(|Z| + k)\varepsilon + (k + 3)\delta + \gamma < 1$;

Then $\mu_A(\mathtt{reach}(Q \to D)) \leqslant |Z|\varepsilon + \delta$.

Proof. For a contradiction, assume that $\mu_A(\texttt{reach}(Q \to D)) > |Z|\varepsilon + \delta$. Our goal is to construct an active ℓ-hook (X, R) with $\ell \geqslant k$, $R \subseteq S$, and $|R| \geqslant \gamma|S|$.

Let S_0 be the vertex set of the largest component of $G[S \setminus N_S[Z]]$, and let $M = S \setminus (S_0 \cup N_S[Z])$. Note that $|N_S[Z]| \leqslant \varepsilon|Z||S|$ by Property 4. of an ε-structured pair, and so we have $|S_0 \cup M| > 3\delta|S|$ by assumption (iv). Thus, Lemma 9 implies that $|M| \leqslant \delta|S|$. Hence, we have

$$|S \setminus S_0| \leqslant (|Z|\varepsilon + \delta)|S| \tag{6.2}$$

and, by assumption (iv),

$$|S_0| > (k\varepsilon + (k+2)\delta + \gamma)|S|. \tag{6.3}$$

Since $\mu_A(\texttt{reach}(Q \to D)) > |Z|\varepsilon + \delta$, and by the definition of μ_A, there exists $x \in S_0$ with $\pi_A(x) \in \texttt{reach}(Q \to D)$. In particular, there exists a path from Q to S_0 with all internal vertices in D. Let L be a shortest such path; note that it is possible that L consists of a single edge, but L contains at least two vertices since $Q \subseteq A$ and $S_0 \subseteq S$.

Let q be the endpoint of L in Q, y be the second endpoint of L, and x be the neighbour of y on L (it is possible that $x = q$). Using assumption (iii), we find an integer $i_0 \geqslant 0$ and an i_0-hook with vertex set $X \subseteq \{q\} \cup Z$ and active vertex q. We lengthen this hook with the path L: define $i := i_0 + |V(L)| - 2$, $X_i := X \cup (V(L) \cap D)$, and $R_i := S_0$. Observe that, since Z and $D \cup S_0$ are fully anti-adjacent, and L is a shortest path from Q to S_0 via D, we have that $G[X_i]$ is an i-hook with x being the active vertex, and $N(R_i) \cap X_i = \{x\}$ Consequently, (X_i, R_i) is an active i-hook.

If $i \geqslant k$, then (6.3) ensures that (X_i, R_i) is a desired active hook, a contradiction. Otherwise, we use the path-growing argument of [8] to turn it into an active k-hook, using the slack in (6.3) in the process. More formally, we build a sequence of active j-hooks (X_j, R_j) for $j = i, i+1, \ldots, k$, with $X_i \subset X_{i+1} \subset \cdots \subset X_k$, $S_0 \supseteq R_i \supset R_{i+1} \supset \cdots \supset R_k$, and additionally maintain that

$$|R_j| > \left((k-j)(\varepsilon + \delta) + 2\delta + \gamma\right)|S|. \tag{6.4}$$

Clearly, (6.4) holds for $j = i$ (using (6.3) and the fact that $R_i = S_0$), while for $j = k$, (6.4) gives the desired lower bound on $|R_k|$ for the active k-hook (X_k, R_k).

Assume that an active j-hook (X_j, R_j) has been constructed for some $j < k$. Let $v_j \in X_j$ be the active vertex of this hook. Let R_{j+1} be the vertex set of the largest component of $G[R_j \setminus N(v_j)]$; by (6.4), we have that $|R_j \setminus N(v_j)| > 3\delta|S|$ as $R_j \subseteq S_0$, and Lemma 9 asserts that $|R_{j+1}| \geqslant |R_j| - \varepsilon|S| - \delta|S|$, proving (6.4) for R_{j+1}. We take v_{j+1} to be any vertex of $R_j \cap N(v_j) \cap N(R_{j+1})$; such a vertex exists by the connectivity of $G[R_j]$ and the assumption $v_j \in N(R_j)$. Let $X_{j+1} = X_j \cup \{v_{j+1}\}$. A direct check shows that the choice of v_{j+1}, X_{j+1}, and R_{j+1} ensures that $G[X_{j+1}]$ is a $(j+1)$-hook with active vertex v_{j+1}, and $N(R_{j+1}) \cap X_{j+1} = \{v_{j+1}\}$, finishing the description of the construction of (X_{j+1}, R_{j+1}). Hence, (X_k, R_k) is an active k-hook with $R_k \subseteq S$ and $|R_k| > \gamma|S|$, a contradiction. This concludes the proof of the claim. ☐

In the remainer of the proof we assume that the constants ε, δ, and γ are sufficiently small such that

$$2\varepsilon + 3(6\varepsilon + \delta) < 1. \tag{6.5}$$

In particular this means that assumption (iv) of Claim 39 is satisfied as long as $|Z| \leqslant 6$. It also means that the bound in the conclusion of Claim 39 is small for $|Z| \leqslant 6$; specifically, we can assume that any two neighbourhoods $N_A(x), N_A(y)$ of vertices $x, y \in S$, together with any three sets $\mathtt{reach}(Q \to D) \subseteq A$ obtained from Claim 39 (applied with $|Z| \leqslant 6$), cannot cover the entire set A.

6.3 Neighbourhoods in $\boldsymbol{A} \cup \boldsymbol{B}$

6.3.1 Non-edges inside an $\boldsymbol{A}$-neighbourhood

We start with proving an analogue of Claim 24.

Claim 40 **($\boldsymbol{A} \leftrightarrow \boldsymbol{B}$).** *For every $x \in S$ and $p, q \in N_A(x)$, if $pq \notin E(G)$ then $N_A(p) \setminus N_A(x) = N_A(q) \setminus N_A(x)$.*

Proof. By contradiction, and using the symmetry between vertices p and q, let us assume there exists $r \in A \setminus N_A(x)$ with $pr \in E(G)$ and $qr \notin E(G)$. Let $Z = \{p, q, r, x\}$ and observe that $G[X]$ is isomorphic to P_4, with x being one of the internal vertices. Consequently, the assumptions of Claim 39 are satisfied (with the roles of A and B swapped) for $Q = N_B(x)$ and $D = B \setminus N_B(x)$, and we have $\mu_A(\mathtt{reach}(Q \to D)) \leqslant 4\varepsilon + \delta$. However, the connectivity of B implies that $\mathtt{reach}(Q \to D) = D$, a contradiction to (6.5). $\square$

6.4 Neighbourhoods along a nonedge in $\boldsymbol{S}$

We start by proving Property (NE2).

Claim 41 **($\boldsymbol{A} \leftrightarrow \boldsymbol{B}$).** *For every $x, y \in S$ with $x \neq y$ and $xy \notin E(G)$, if $N_B(x) \neq N_B(y)$, then there is no edge between $N_A(x) \cap N_A(y)$ and $A \setminus N_A(x, y)$.*

Proof. Let $z \in N_B(x) \triangle N_B(y)$ be any vertex. Observe that the assumptions of Claim 39 are satisfied for $Z = \{x, y, z\}$, $Q = N_A(x) \cap N_A(y)$ and $D = A \setminus N_A(x, y)$: for every $q \in Q$ the graph $G[\{z, x, q, y\}]$ is a P_4 with q being one of its internal vertices. Hence, $\mu_A(\mathtt{reach}(Q \to D)) \leqslant 3\varepsilon + \delta$. Let us denote $F = \mathtt{reach}(Q \to D)$; our goal is to prove that $F = \emptyset$.

Assume the contrary, let $p \in F$ and $q \in Q$ with $pq \in E(G)$. Let $D' = D \setminus F$ and $Q' = N_A(D')$; note that, by the definition of F, we have $Q' \subseteq N_A(x) \triangle N_A(y)$. Furthermore, Claim 40 implies that $qq' \in E(G)$ for every $q' \in Q'$: q' has a neighbour in D', while q does not have such a neighbour, and both q and q' belong either to $N_A(x)$ or to $N_A(y)$.

Let z_x be any vertex in $N_B(x)$, z_y be any vertex in $N_B(y)$, and $Z' = \{p, q, x, y, z_x, z_y\}$. We claim that the assumptions of Claim 39 are satisfied for Z', Q', and D': clearly Z' and D' are fully anti-adjacent by construction, so it remains only to check assumption (iii).

To this end, consider $q' \in Q'$. By symmetry between x and y, assume $q' \in N_A(x) \setminus N_A(y)$. If $pq' \in E(G)$, then $G[\{p, q', x, z_x\}]$ is a 0-hook with q' being the active vertex. If $pq' \notin E(G)$, then $G[\{q, q', p, y, z_y\}]$ is a 1-hook with q' being the active vertex.

By Claim 39, we infer that $\mu_A(\mathtt{reach}(Q' \to D')) \leqslant 5\varepsilon + \gamma$. However, by connectivity of B we have $\mathtt{reach}(Q' \to D') = D'$. This, together with $\mu_A(N_A(x, y)) \leqslant 2\varepsilon$ by (6.1) and $\mu_A(F) \leqslant 3\varepsilon + \delta$ contradicts (6.5). $\square$

Since the structured pair $(G,(A,B,S))$ satisfies Properties (NE1) and (NE2), we can use Lemma 29 in the following, where we prove Property (NE3).

Claim 42 ($\boldsymbol{A \leftrightarrow B}$)**.** *For every $x, y \in S$ with $x \neq y$ and $xy \notin E(G)$, if $N_A(x) \subsetneq N_A(y)$, then the sets $N_A(x)$ and $N_A(y) \setminus N_A(x)$ are fully adjacent.*

Proof. Since $N_A(x) \subsetneq N_A(y)$, Lemma 29 implies that $N_B(x) \neq N_B(y)$. Consequently, Claim 41 asserts that $D := A \setminus N_A(y)$ and $N_A(x)$ are fully anti-adjacent. That is, if we define $Q = N_A(D)$, then $Q \subseteq N_A(y) \setminus N_A(x)$.

By contradiction, assume there exists $z \in N_A(y)\setminus N_A(x)$ and $p \in N_A(x)$ with $pz \notin E(G)$. Claim 40 implies that $z \notin Q$, as $p \notin Q$ and $p, z \in N_A(y)$. Furthermore, Claim 40 also implies that z is fully adjacent to Q. We also know that p is fully adjacent to Q. We infer that the conditions of Claim 39 are satisfied for $Z = \{z, p, x\}$ and the sets Q and D: for every $q \in Q$, the graph $G[\{z, q, p, x\}]$ is a P_4 with q being one of its internal vertices. Consequently, $\mu_A(\texttt{reach}(Q \to D)) \leqslant 3\varepsilon + \delta$, which stands in contradiction with the connectivity of $G[A]$ and (6.5). □

6.5 Neighbourhoods along an edge in $\boldsymbol{S}$

In the next three claims we prove Property (E1).

Claim 43 ($\boldsymbol{A \leftrightarrow B}$)**.** *For every $x, y \in G[S]$, if there is no edge between $N_A(x)\triangle N_A(y)$ and $A \setminus N_A(x, y)$, then $N_A(x) = N_A(y)$.*

Proof. By contradiction, assume there exists $p \in N_A(x)\triangle N_A(y)$; by symmetry, assume $p \in N_A(x) \setminus N_A(y)$. Let $D = A \setminus N_A(x, y)$ and $Q = N_A(D) \subseteq N_A(x) \cap N_A(y)$. Let z be any vertex in $N_B(y)$, and let $Z = \{p, y, z\}$. Observe that Claim 40 implies that p is fully adjacent to Q, as they are both contained in $N_A(x)$ and p does not have any neighbour in D. Consequently, the assumptions of Claim 39 are satisfied for the sets Z, Q, and D: for every $q \in Q$, the graph $G[\{p, q, y, z\}]$ is a P_4 with q being one of the middle vertices. Hence, $\mu_A(\texttt{reach}(Q \to D)) \leqslant 3\varepsilon + \delta$. However, $\texttt{reach}(Q \to D) = D$ by the connectivity of A, and we have a contradiction with (6.5). □

Claim 44 ($\boldsymbol{A \leftrightarrow B}$)**.** *For every $xy \in E(G[S])$, if $N_A(x)\setminus N_A(y) \neq \emptyset$ but $\mu_A(\texttt{reach}(N_A(x)\setminus N_A(y) \to A \setminus N_A(x, y))) \leqslant 6\varepsilon + \delta$, then $N_B(x) = N_B(y)$.*

Proof. Let $F = \texttt{reach}(N_A(x) \setminus N_A(y) \to A \setminus N_A(x, y))$, $D = A \setminus (N_A(x, y) \cup F)$, and $Q = N_A(D) \subseteq N_A(y)$. Let p be any vertex in $N_A(x) \setminus N_A(y)$ and let z be any vertex in $N_B(y)$.

If $N_B(y) \nsubseteq N_B(x)$, then let z_1 be any vertex of $N_B(y) \setminus N_B(x)$ and define $Z = \{x, y, z, p, z_1\}$. Otherwise, unless $N_B(x) = N_B(y)$, Claim 43 implies that there exists an edge z_2z_3 with $z_2 \in N_B(x) \setminus N_B(y)$ and $B \setminus N_B(x, y)$, and we take $Z = \{x, y, z, p, z_2, z_3\}$.

We claim that in both cases the sets Z, Q, and D satisfy the assumptions of Claim 39. Clearly, D and Z are fully anti-adjacent, so it remains to check only assumption (iii). To this end, consider $q \in Q$. If $pq \in E(G)$, then $G[\{p, q, y, z\}]$ is a P_4 with q being one of the middle vertices. Otherwise, Claim 40 implies that $q \notin N_A(x)$, that is, $q \in N_A(y) \setminus N_A(x)$.

If the vertex z_1 exists, then $G[\{y, q, x, p, z_1\}]$ is a 1-hook with q being the active vertex. Finally, if the edge z_2z_3 exists, then $G[\{x, y, q, p, z_2, z_3]$ is a 2-hook with q being the active vertex.

We infer that $\mu_A(\texttt{reach}(Q \to D)) \leqslant 6\varepsilon + \delta$. However, the connectivity of $G[A]$ implies that $D = \texttt{reach}(Q \to D)$. This is in contradiction with (6.5) and the assumption $\mu_A(F) \leqslant 6\varepsilon + \delta$. □

Claim 45. *For every $xy \in E(G[S])$, either $N_A(x) \setminus N_A(y) = \emptyset$ or $N_B(x) \setminus N_B(y) = \emptyset$.*

Proof. Assume the contrary. Since $N_A(x) \neq N_A(y)$, Claim 44 applied to the side B instead of the side A asserts that $\mu_B(\texttt{reach}(N_B(x)\setminus N_B(y) \to B\setminus N_B(x, y))) > 6\varepsilon+\delta$; in particular, there exists an edge $z_1z_2 \in E(G)$ with $z_1 \in N_B(x) \setminus N_B(y)$ and $z_2 \in B \setminus N_B(x, y)$.

Define now $Z = \{x, y, z_1, z_2\}$, $Q = N_A(x) \setminus N_A(y)$, and $D = A \setminus N_A(x, y)$. Observe that the assumptions of Claim 39 are satisfied for these sets: for every $q \in Q$ the graph $G[\{x, q, y, z_1, z_2\}]$ is a 1-hook with q being its active vertex. Consequently, $\mu_A(\texttt{reach}(Q \to D)) \leqslant 4\varepsilon + \delta$, a contradiction to Claim 44 and the assumption $N_B(x) \neq N_B(y)$. □

6.6 Niceness and quotient graph

Summing up, we have so far proven the following.

Corollary 46. *The ε-structure $(G, (A, B, S))$ is nice.*

Let us define relations $R^=$, $R^{\neq}$, $R^{\subsetneq}$, $R^{\supsetneq}$, $R^=_A$, and $R^=_B$ on S as in Section 5.3. We apply Theorem 32, obtaining a set $\hat{S} \subseteq S$ of size at least $|S|/4$; by Corollary 33, the equivalence classes of $R^=$ restricted to $\hat{S}$ partition $\hat{S}$ into modules of $G[\hat{S}]$. Furthermore, Lemma 35 asserts that the quotient graph of this partition is Berge. Thus, to conclude the proof of Theorem 20, it suffices to show that the quotient graph of this partition is also claw-free.

6.7 Excluding a claw in the quotient graph

Claim 47. *The quotient graph of the partition of $G[\hat{S}]$ into equivalence classes of the relation $R^=$ is claw-free.*

Proof. By contradiction, assume there exists a claw $(t; x, y, z)$ in $G[\hat{S}]$ such that no pair of vertices from $\{t, x, y, z\}$ are in relation $R^=$.

We apply Lemma 34 to three P_3s contained in the claw $(t; x, y, z)$. Observe that if one of the first two outcomes happens for one of P_3s, say $R^=_A(x, t)$ and $R^=_B(y, t)$, then we have $R^=_A(z, t)$ by looking at the P_3 on vertices y, t, z. Thus we obtain $R^=_A(x, z)$, a contradiction to the properties of $\hat{S}$ obtained from Theorem 32. We infer that the only two possibilities are $R^{\subsetneq}(x, t)$, $R^{\subsetneq}(y, t)$, and $R^{\subsetneq}(z, t)$, or the symmetrical option $R^{\supsetneq}(x, t)$, $R^{\supsetneq}(y, t)$, and $R^{\supsetneq}(z, t)$. By swapping the sides A and B if needed, we may assume that the first option happens, that is, $N_A(x) \cup N_A(y) \cup N_A(z) \subseteq N_A(t)$ and $N_B(t) \subseteq N_B(x) \cap N_B(y) \cap N_B(z)$.

Let $D = A \setminus N_A(t)$, $Q_{xy} = N_A(t) \setminus (N_A(x) \triangle N_A(y))$, and similarly define Q_{yz} and Q_{xz}. Since $xy \notin E(G)$, by Theorem 32 we have $R^{\neq}(x, y)$ and there exists $p \in N_B(x) \setminus N_B(y)$. Furthermore, observe that also $p \notin N_B(t)$. We infer that the sets $Z = \{t, x, y, p\}$,

Q_{xy}, and D satisfy the assumptions of Claim 39: for every $q \in N_A(x) \cap N_A(y)$ the graph $G[\{p, x, q, y\}]$ is a P_4 with q being one of the middle vertices, while for every $q \in N_A(t) \setminus N_A(x, y)$ the graph $G[\{t, q, x, p, y\}]$ is a 1-hook with q being its active vertex. Consequently, $\mu_A(\texttt{reach}(Q_{xy} \to D)) \leqslant 4\varepsilon + \delta$. Symmetrically, the same conclusion holds for Q_{yz} and Q_{xz}.

Note now that $Q_{xy} \cup Q_{yz} \cup Q_{xz} = N_A(t)$, as $(X \triangle Y) \cap (Y \triangle Z) \cap (Z \triangle X) = \emptyset$ for any three sets X, Y, Z. Consequently, $\mu_A(\texttt{reach}(N_A(t) \to D)) \leqslant 3(4\varepsilon + \delta)$. However, $\texttt{reach}(N_A(t) \to D) = D$ by connectivity of $G[A]$, and we have a contradiction with (6.5). This concludes the proof of the claim, and of Theorem 20. $\square$

7 The strong Erdős-Hajnal property is much stronger

In this section, we prove Theorem 8. Both statements, (a) and (b), are implied by the following lemma.

Lemma 48. *Let $k > 2$ be fixed and let $\mathcal{H}$ be a family of graphs such that*

$(P1)$ *every $H \in \mathcal{H}$ contains a cycle of length at most k; or*

$(P2)$ *for every $H \in \mathcal{H}$, the complement of H contains a cycle of length at most k.*

Then the class of $\mathcal{H}$-free graphs does not have the strong Erdős-Hajnal property.

Proof. Assume first that $\mathcal{H}$ is a family of graphs with Property $(P1)$, i.e. every $H \in \mathcal{H}$ contains a cycle of length at most k. For every $\delta > 0$, we construct a graph G_δ, say on n vertices, that is $\mathcal{H}$-free and that does not contain a homogeneous pair (P, Q) with $|P|, |Q| \geqslant \delta n$.

Fix $\delta > 0$, let n be large enough, and let $G \sim G(n, p)$ be a random graph on n vertices where every edge is present independently at random with probability

$$p = \frac{50}{\delta^2 n}.$$

Let X^k be a random variable that counts the number of cycles of length at most k in G, and for $3 \leqslant \ell \leqslant k$, let X_ℓ be a random variable that counts the number of cycles of length ℓ in G. By linearity of expectation we have

$$\mathbf{E}(X^k) = \sum_{\ell=3}^{k} \mathbf{E}(X_\ell) \leqslant \sum_{\ell=3}^{k} (pn)^\ell \leqslant k \left(\frac{50}{\delta^2}\right)^k =: C.$$

Therefore, by Markov's Inequality,

$$\mathbf{Pr}(X^k \geqslant 3C) \leqslant \frac{1}{3}. \tag{7.1}$$

Let Z_δ be a random variable that counts the number of homogeneous pairs (P, Q) in G with $|P|, |Q| = \left\lfloor \frac{\delta}{2} n \right\rfloor$. Then

$$\mathbf{E}(Z_\delta) \leqslant 2^n \cdot 2^n \cdot (1-p)^{\delta^2 n^2/10} + 2^n \cdot 2^n \cdot p^{\delta^2 n^2/10},$$

where the first term is an upper bound on the expected number of anti-adjacent pairs (P, Q) and the second term is an upper bound on the expected number of adjacent pairs (P, Q). For n large enough we have $p < \frac{1}{2}$, so that we can deduce

$$\mathbf{E}(Z_\delta) \leqslant 2^{2n+1}(1-p)^{\delta^2 n^2/10} \leqslant 2^{2n+1} e^{-p\delta^2 n^2/10},$$

where we use $1 - x \leqslant e^{-x}$ in the last inequality. Therefore, by a standard first-moment argument and our choice of p,

$$\mathbf{Pr}(Z_\delta > 0) = \mathbf{Pr}(Z_\delta \geqslant 1) \leqslant \mathbf{E}(Z_\delta) \leqslant e^{(2n+1)\ln(2) - p\delta^2 n^2/10} \leqslant e^{-n}.$$

Therefore, with probability at most $\frac{1}{3} + o(1)$, G satisfies $X^k \geqslant 3C$ or $Z_\delta > 0$. That is, there exists a graph G' that has at most $3C$ cycles of length at most k, and that has no homogeneous pair (P, Q) with $|P|, |Q| = \lfloor \frac{\delta}{2} n \rfloor$. Remove a vertex from every cycle of length at most k to obtain a graph G_δ on $n' \geqslant n/2$ vertices with no homogeneous pair (P, Q) with $|P|, |Q| \geqslant \delta n'$. In particular, G_δ is $\mathcal{H}$-free, which proves the claim.

Assume now that the family $\mathcal{H}$ satisfies Property $(P2)$. Then the family $\mathcal{H}^c := \{H^c \colon H \in \mathcal{H}\}$ satisfies Property $(P1)$. So, by the first part, for every $\delta > 0$ we find a graph G_δ, say on n vertices, that is $\mathcal{H}^c$-free and has no homogeneous pair (P, Q) with $|P|, |Q| \geqslant \delta n$. But then, the collection of graphs G_δ^c shows that the family $\mathcal{H}$ cannot have the strong Erdős-Hajnal property either. □

We are ready to prove Theorem 8.

Proof of Theorem 8. First, observe that since $P_4^c = P_4$, the class of P_4-free graphs has the strong Erdős-Hajnal property by the result of [8]. To prove the implication in the other direction, notice that if H is not an induced subgraph of P_4 then either H or H^c contains a cycle. But then we can apply Lemma 48 to $\mathcal{H} = \{H\}$ and we are done. Thus, we proved statement (a). Statement (b) follows from Lemma 48 by taking $\mathcal{H} = \{H, H^c\}$. □

8 Conclusions

We proved in this paper that for every $k \geqslant 1$, the class of $\mathcal{H}^2_{\geqslant k}$-free graphs has the strong Erdős-Hajnal property. Specifically, there exists $\varepsilon(k) > 0$ such that every $\mathcal{H}^2_{\geqslant k}$-free n-vertex graph contains a clique or an independent set of size at lest $n^{\varepsilon(H)}$. This result extends e.g. the result on forbidding long paths and antipaths [8]. Furthermore, from combinations of known results, the only tree on six vertices for which the Conjecture 2 was not previously known to hold is the 2-hook, also known as the E-graph. Therefore, Conjecture 2 is now known to be true for every tree on at most six vertices. The question for general trees remains wide open.

The original inspiration for our work comes from a paper of Lokshtanov, Vatshelle, and Villanger [31], who used the framework of minimal separators and potential maximal cliques to give a polynomial-time algorithm for the (algorithmic) Independent Set problem in P_5-free graphs. For the interested reader, below we give some brief background to this framework without defining all the notions we make reference to.

A *minimal separator* in a connected graph is an inclusion-wise minimal set of vertices whose deletion leaves the graph disconnected. A *minimal triangulation* of a graph G is an inclusion-wise minimal set of edges F such that $G + F$, the graph obtained by adding the edges of F to G, is a chordal graph. A potential maximal clique of G is a set $K \subseteq V(G)$ which is a maximal clique in $G + F$ for some minimal triangulation F. These notions all turn out to be closely related through the notion of treewidth and tree decompositions.

Bouchitte and Todinca [6] studied the notion of potential maximal cliques from this perspective and showed that the natural dynamic programming algorithm for finding a maximum independent set in a graph of bounded treewidth can be modified to find such a set in time polynomial in the size of G and linear in the number of potential maximal cliques in G. In this way, they obtained a unified explanation for the existence of polynomial-time algorithms for the Independent Set problem in many hereditary graph classes. The work for P_5-free graphs [31] follows the same approach, but generalises it, by showing that in P_5-free graphs one needs to examine only a particular (polynomially-sized) set of potential maximal cliques in the aforementioned algorithm. Subsequent work [30] uses minimal separators and potential maximal cliques in a different way to develop a quasipolynomial-time algorithm for the Independent Set problem in P_6-free graphs.

Since the framework of minimal separators and potential maximal cliques has been successfully applied to the Independent Set problem for various hereditary graph classes, we wished to investigate to what extent these methods are useful for problems related to the Erdős-Hajnal conjecture. While our original proof of Theorem 3 followed this framework closely, we eventually found a simpler proof which circumvents most of the theory, although some artefacts remain.

The question of excluding pairs of graphs in the context of the Erdős-Hajnal conjecture was considered also in the directed setting (see [9]). The directed version of the conjecture is equivalent to the undirected one and was recently heavily investigated ([3, 11, 13, 10, 12]). In the directed setting the analogue of the complement of the graph is the graph obtained by reversing directions of all the edges. It would be interesting to see whether techniques presented in this paper can be applied in the directed setting to get generalisations of some of the known results.

References

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir. Crossing patterns of semi-algebraic sets. *Journal of Combinatorial Theory, Series A*, 111(2):310–326, 2005.

[2] N. Alon, J. Pach, and J. Solymosi. Ramsey-type theorems with forbidden subgraphs. *Combinatorica*, 21(2):155–170, 2001.

[3] E. Berger, K. Choromanski, and M. Chudnovsky. Forcing large transitive subtournaments. *Journal of Combinatorial Theory, Series B*, 112:1–17, 2015.

[4] A. Berry, R. Pogorelcnik, and G. Simonet. An introduction to clique minimal separator decomposition. *Algorithms*, 3(2):197–215, 2010.

[5] M. Bonamy, N. Bousquet, and S. Thomassé. The Erdős-Hajnal Conjecture for Long Holes and Anti-holes. *SIAM J. Discrete Math.*, 30:1159–1164, 2016.

[6] V. Bouchitté and I. Todinca. Treewidth and minimum fill-in: Grouping the minimal separators. *SIAM J. Comput.*, 31(1):212–232, 2001.

[7] N. Bousquet, A. Lagoutte, and S. Thomassé. Clique versus independent set. *European Journal of Combinatorics*, 40:73–92, 2014.

[8] N. Bousquet, A. Lagoutte, and S. Thomassé. The Erdős-Hajnal conjecture for paths and antipaths. *Journal of Combinatorial Theory, Series B*, 2015.

[9] K. Choromanski. Excluding pairs of tournaments. *Journal of Graph Theoty, to appear*, doi:10.1002/jgt.22250.

[10] K. Choromanski. Upper bounds for Erdős-Hajnal Coefficients of Tournaments. *Journal of Graph Theory*, 74(1):122–132, 2013.

[11] K. Choromanski. EH-suprema of tournaments with no nontrivial homogeneous sets. *Journal of Combinatorial Theory, Series B*, 114:97–123, 2015.

[12] K. Choromanski, M. Chudnovsky, and P. Seymour. Tournaments with near-linear transitive subsets. *Journal of Combinatorial Theory, Series B*, 109:228–249, 2014.

[13] K. Choromanski and T. Jebara. Coloring tournaments with forbidden substructures. *Submitted for publication.*

[14] M. Chudnovsky. The Erdős-Hajnal conjecture – A Survey. *Journal of Graph Theory*, 75(2):178–190, 2014.

[15] M. Chudnovsky and S. Safra. The Erdős-Hajnal conjecture for bull-free graphs. *Journal of Combinatorial Theory, Series B*, 98(6):1301–1310, 2008.

[16] M. Chudnovsky and P. Seymour. Excluding paths and antipaths. *Combinatorica*, pages 1–24, 2012.

[17] V. Chvátal and N. Sbihi. Recognizing claw-free perfect graphs. *Journal of Combinatorial Theory, Series B*, 44(2):154–176, 1988.

[18] P. Erdős. Some remarks on the theory of graphs. *Bulletin of the American Mathematical Society*, 53(4):292–294, 1947.

[19] P. Erdős, A. Hajnal, and J. Pach. A Ramsey-type theorem for bipartite graphs. *Geombinatorics*, 10(2):64–68, 2000.

[20] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[21] P. Erdős and A. Hajnal. On spanned subgraphs of graphs. *Contributions to Graph Theory and its Applications*, pages 80–96, 1977.

[22] P. Erdős and A. Hajnal. Ramsey-type theorems. *Discrete Applied Mathematics*, 25(1):37–52, 1989.

[23] J. Fox. A bipartite analogue of Dilworth's theorem. *Order*, 23(2-3):197–209, 2006.

[24] J. Fox and J. Pach. Erdős-Hajnal-type results on intersection patterns of geometric objects. In *Horizons of combinatorics*, pages 79–103. Springer, 2008.

[25] J. Fox and B. Sudakov. Induced Ramsey-type theorems. *Advances in Mathematics*, 219(6):1771–1800, 2008.

[26] J. Fox and B. Sudakov. Density theorems for bipartite graphs and related Ramsey-type results. *Combinatorica*, 29(2):153–196, 2009.

[27] M. Göös. Lower bounds for clique vs. independent set. In *Proceedings of the 2015 IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS '15)*, pages 1066–1076, 2015.

[28] T. Kloks, D. Kratsch, and H. Müller. Dominoes. In E. W. Mayr, G. Schmidt, and G. Tinhofer, editors, *WG 1994*, volume 903 of *LNCS*, pages 106–120. Springer, 1994.

[29] A. Lagoutte and T. Trunck. Clique-stable set separation in perfect graphs with no balanced skew-partitions. *Discrete Math.*, 339(6):1809–1825, 2016.

[30] D. Lokshtanov, M. Pilipczuk, and E. J. van Leeuwen. Independence and efficient domination on P_6-free graphs. *ACM Trans. Algorithms*, 14(1):3:1–3:30, 2018.

[31] D. Lokshtanov, M. Vatshelle, and Y. Villanger. Independent set in P_5-free graphs in polynomial time. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 570–581. SIAM, 2014.

[32] F. Maffray and B. A. Reed. A description of claw-free perfect graphs. *Journal of Combinatorial Theory, Series B*, 75(1):134–156, 1999.

[33] Y. Metelsky and R. Tyshkevich. Line graphs of Helly hypergraphs. *SIAM J. Discrete Math.*, 16(3):438–448, 2003.

[34] V. Rödl. On universality of graphs with uniformly distributed edges. *Discrete Mathematics*, 59(1):125–134, 1986.

[35] M. Yannakakis. Expressing combinatorial optimization problems by linear programs. *J. Comput. Syst. Sci.*, 43(3):441–466, 1991.