Cubic edge-transitive bi-p-metacirculants

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Abstract

A graph is said to be a *bi-Cayley graph* over a group H if it admits H as a group of automorphisms acting semiregularly on its vertices with two orbits. For a prime p, we call a bi-Cayley graph over a metacyclic p-group a *bi-p-metacirculant*. In this paper, the automorphism group of a connected cubic edge-transitive bi-p-metacirculant is characterized for an odd prime p, and the result reveals that a connected cubic edge-transitive bi-p-metacirculant exists only when $p = 3$. Using this, a classification is given of connected cubic edge-transitive bi-Cayley graphs over an inner-abelian metacyclic 3-group. As a result, we construct the first known infinite family of cubic semisymmetric graphs of order twice a 3-power.

Mathematics Subject Classifications: 05C25, 20B25

1 Introduction

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For a graph Γ, we denote by $V(\Gamma)$ the set of all vertices of Γ, by $E(\Gamma)$ the set of all edges of Γ, by $A(\Gamma)$ the set of all arcs (ordered paries of adjacent vertices) of Γ, and by $\mathrm{Aut}\,(\Gamma)$ the full automorphism group of Γ. For $u, v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in Γ. For the group-theoretic and the graph-theoretic terminology not defined here we refer the reader to [2, 24].

Let Γ be a graph. If $\mathrm{Aut}\,(\Gamma)$ is transitive on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, then Γ is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive*, respectively. An arc-transitive graph is also called a symmetric graph. A graph Γ is said to be *semisymmetric* if Γ has regular valency and is edge- but not vertex-transitive.

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Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is the subgroup of G fixing the point α. We say that G is *semiregualr* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. A graph is said to be a *bi-Cayley graph* over a group H if it admits H as a semiregular automorphism group with two orbits (Bi-Cayley graph is sometimes called *semi-Cayley graph*). Note that every bi-Cayley graph admits the following concrete realization. Given a group H, let R, L and S be subsets of H such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* over H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H, R, L, S)$, is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0,\ g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1,\ g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0,\ g_1\} \mid gh^{-1} \in S\}$. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$. For $g \in H$, define a permutation $R(g)$ on the vertices of Γ by the rule

$$h_i^{R(g)} = (hg)_i, \forall i \in \mathbb{Z}_2, h \in H.$$

Then $R(H) = \{R(g) \mid g \in H\}$ is a semiregular subgroup of $\mathrm{Aut}\,(\Gamma)$ which is isomorphic to H and has H_0 and H_1 as its two orbits. When $R(H)$ is normal in $\mathrm{Aut}\,(\Gamma)$, the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$ will be called a *normal bi-Cayley graph* over H (see [27]).

A natural problem in the study of bi-Cayley graphs is: for a given finite group H, to classify bi-Cayley graphs with specific symmetry properties over H. Some partial answers for this problem have been obtained. For example, in [1] Boben et al. studied some properties of cubic 2-type bi-Cayley graphs over cyclic groups and the configurations arising from these graphs, in [20] Pisanski classified cubic bi-Cayley graphs over cyclic groups, in [14] Kovács et al. gave a classification of arc-transitive one-matching abelian bi-Cayley graphs, and more recently, Zhou et al. [26] gave a classification of cubic vertex-transitive abelian bi-Cayley graphs. In this paper, we shall investigate cubic edge-transitive bi-Cayley graphs over metacyclic p-groups where p is an odd prime. Following up [8], we call a bi-Cayley graph over a metacyclic p-group a *bi-p-metacirculant*.

Another motivation for us to consider bi-Cayley graphs over metacyclic p-groups is the observation that the Gray graph [4], the smallest trivalent semmisymmetric graph, is a bi-Cayley graph over a non-abelian metacyclic group of order 27. In [8], the cubic edge-transitive bi-Cayley graphs over abelian groups have been classified. So, we shall restrict our attention to bi-Cayley graphs over non-abelian metacyclic p-groups.

Our first result characterizes the automorphism groups of cubic edge-transitive bi-p-metacirculants.

Theorem 1. *Let Γ be a connected cubic edge-transitive bi-Cayley graph over a non-abelian metacyclic p-group H with p an odd prime. Then $p = 3$, and either Γ is isomorphic to the Gray graph or Γ is a normal bi-Cayley graph over H.*

Applying the above theorem, our second result gives a classification of connected cubic edge-transitive bi-Cayley graphs over a inner-abelian metacyclic p-group. A non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian.

Theorem 2. *Let Γ be a connected cubic edge-transitive bi-Cayley graph over an inner-abelian metacyclic 3-group H. Then Γ is isomorphic to either Γ_t or Σ_t (see Section 5.1 for the construction of these two families of graphs).*

Theorem 1 also enables us to give a short proof of the main result in [17].

Corollary 3. [17, Theorem 1.1] *Let p be a prime. Then, with the exception of the Gray graph on 54 vertices, every cubic edge-transitive graph of order $2p^3$ is vertex-transitive.*

2 Preliminaries

In this section, we first introduce the notation used in this paper. For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n and by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For a finite group G, the full automorphism group, the center, the derived subgroup and the Frattini subgroup of G will be denoted by $\mathrm{Aut}\,(G)$, $Z(G)$, G' and $\Phi(G)$, respectively. For $x, y \in G$, denote by $[x, y]$ the commutator $x^{-1}y^{-1}xy$. For a subgroup H of G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M.

Below, we restate some group-theoretic results, of which the first is usually called the N/C-theorem.

Proposition 4. [13, Chapter 1, Theorem 4.5] *Let H be a subgroup of a group G. Then $C_G(H)$ is normal in $N_G(H)$, and the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of* $\mathrm{Aut}\,(H)$.

Now we give two results regarding metacyclic p-groups.

Proposition 5. [22, Lemma 2.4] *Let P be a split metacyclic p-group:*

$$P = \langle x, y \mid x^{p^m} = y^{p^n} = 1, yxy^{-1} = x^{1+p^l}\rangle, \text{ where } 0 < l < m,\ m - l \leqslant n.$$

Then the automorphism group $\mathrm{Aut}\,(P)$ *of P is a semidirect product of a normal p-subgroup and the cyclic subgroup $\langle\sigma\rangle$ of order $p-1$, where $\sigma(x) = x^r$ and $\sigma(y) = y$, r is a primitive $(p-1)$th root of unity modulo p^m.*

Proposition 6. [22, Proposition 2.3] *Let G be a finite group with a non-abelian metacyclic Sylow p-subgroup P. If P is nonsplit, then G has a normal p-complement.*

Next, we give some results about graphs. Let Γ be a connected graph with an edge-transitive group G of automorphisms and let N be a normal subgroup of G. The *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there exists an edge in Γ between the vertices lying in those two orbits. Below we introduce two propositions, of which the first is a special case of [15, Theorem 9].

Proposition 7. *Let Γ be a cubic graph and let $G \leqslant \mathrm{Aut}\,(\Gamma)$ be arc-transitive on Γ. Then G is an s-arc-regular subgroup of $\mathrm{Aut}\,(\Gamma)$ for some integer s. If $N \trianglelefteq G$ has more than two orbits in $V(\Gamma)$, then N is semiregular on $V(\Gamma)$, Γ_N is a cubic symmetric graph with G/N as an s-arc-regular subgroup of automorphisms.*

The next proposition is a special case of [16, Lemma 3.2].

Proposition 8. *Let Γ be a cubic graph and let $G \leqslant \mathrm{Aut}\,(\Gamma)$ be transitive on $E(\Gamma)$ but intransitive on $V(\Gamma)$. Then Γ is a bipartite graph with two partition sets, say V_0 and V_1. If $N \trianglelefteq G$ is intransitive on each of V_0 and V_1, then N is semiregular on $V(\Gamma)$, Γ_N is a cubic graph with G/N as an edge- but not vertex-transitive group of automorphisms.*

The next proposition is basic for bi-Cayley graphs.

Proposition 9. [26, Lemma 3.1] *Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over a group H. Then the following hold:*

(1) *H is generated by $R \cup L \cup S$.*

(2) *Up to graph isomorphism, S can be chosen to contain the identity of H.*

(3) *For any automorphism α of H, $\mathrm{BiCay}(H, R, L, S) \cong \mathrm{BiCay}(H, R^\alpha, L^\alpha, S^\alpha)$.*

(4) $\mathrm{BiCay}(H, R, L, S) \cong \mathrm{BiCay}(H, L, R, S^{-1})$.

Next, we collect several results about the automorphisms of the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$. Recall that for each $g \in H$, $R(g)$ is a permutation on $V(\Gamma)$ defined by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h,\ g \in H, \tag{1}$$

and $R(H) = \{R(g) \mid g \in H\} \leqslant \mathrm{Aut}\,(\Gamma)$. For an automorphism α of H and $x, y, g \in H$, define two permutations on $V(\Gamma) = H_0 \cup H_1$ as following:

$$\begin{array}{ll} \delta_{\alpha,x,y}: & h_0 \mapsto (xh^\alpha)_1,\ h_1 \mapsto (yh^\alpha)_0,\ \forall h \in H, \\ \sigma_{\alpha,g}: & h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (gh^\alpha)_1,\ \forall h \in H. \end{array} \tag{2}$$

Set

$$\begin{array}{lll} \mathrm{I} & = & \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}\,(H)\ s.t.\ R^\alpha = x^{-1}Lx,\ L^\alpha = y^{-1}Ry,\ S^\alpha = y^{-1}S^{-1}x\}, \\ \mathrm{F} & = & \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}\,(H)\ s.t.\ R^\alpha = R,\ L^\alpha = g^{-1}Lg,\ S^\alpha = g^{-1}S\}. \end{array} \tag{3}$$

Proposition 10. [27, Theorem 3.4] *Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over the group H. Then $N_{\mathrm{Aut}\,(\Gamma)}(R(H)) = R(H) \rtimes F$ if $I = \emptyset$ and $N_{\mathrm{Aut}\,(\Gamma)}(R(H)) = R(H)\langle F, \delta_{\alpha,x,y}\rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, for any $\delta_{\alpha,x,y} \in I$, we have the following:*

(1) *$\langle R(H), \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$;*

(2) *if* α *has order* 2 *and* $x = y = 1$, *then* Γ *is isomorphic to the Cayley graph* $\mathrm{Cay}(\bar{H},\ R \cup \alpha S)$, *where* $\bar{H} = H \rtimes \langle \alpha \rangle$.

Proposition 11. [8, Proposition 5.2] *Let* n, m *be two positive integers such that* $nm^2 \geqslant 3$. *Let* $\lambda = 0$ *if* $n = 1$, *and let* $\lambda \in \mathbb{Z}_n^*$ *be such that* $\lambda^2 - \lambda + 1 \equiv 0 \pmod{n}$ *if* $n > 1$. *Let*

$$\mathcal{H}_{m,n} = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_{nm} \times \mathbb{Z}_m,$$
$$\Gamma_{m,n,\lambda} = \mathrm{BiCay}(\mathcal{H}_{m,n}, \emptyset, \emptyset, \{1, x, x^\lambda y\}).$$

Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ *be a connected cubic normal edge-transitive bi-Cayley graph over an abelian group* H. *Then* $\Gamma \cong \Gamma_{m,n,\lambda}$ *for some integers* m, n, λ.

Finally, we give some results about cubic edge-transitive graphs.

Proposition 12. [9, Theorem 3.2] *Let* Γ *be a connected cubic symmetric graph of order* $2p^n$ *with* p *an odd prime and* n *a positive integer. If* $p \neq 5, 7$, *then every Sylow* p*-subgroup of* $\mathrm{Aut}\,(\Gamma)$ *is normal.*

Proposition 13. [17, Proposition 2.4] *Let* Γ *be a connected cubic edge-transitive graph and let* $G \leqslant \mathrm{Aut}\,(\Gamma)$ *be transitive on the edges of* Γ. *For any* $v \in V(\Gamma)$, *the stabilizer* G_v *has order* $2^r \cdot 3$ *with* $r \geqslant 0$.

3 A few technical lemmas

In this section, we shall give two easily proved lemmas about metacyclic p-groups that are useful in this paper.

Lemma 14. *Let* p *be an odd prime, and let* H *be a metacyclic* p*-group generated by* a, b *with the following defining relations:*

$$a^{p^m} = b^{p^n} = 1, b^{-1}ab = a^{1+p^r},$$

where m, n, r *are positive integers such that* $r < m \leqslant n + r$. *Then the following hold:*

(1) *For any* $i \in \mathbb{Z}_{p^m}, j \in \mathbb{Z}_{p^n}$, *we have*

$$a^i b^j = b^j a^{i(1+p^r)^j}.$$

(2) *For any positive integer* k *and for any* $i \in \mathbb{Z}_{p^m}, j \in \mathbb{Z}_{p^n}$, *we have*

$$(b^j a^i)^k = b^{kj} a^{i \sum_{s=0}^{k-1} (1+p^r)^{sj}}.$$

(3) *For any* $i_1, i_2 \in \mathbb{Z}_{p^m}$, $j_1, j_2 \in \mathbb{Z}_{p^n}$, *we have*

$$(b^{j_1} a^{i_1})(b^{j_2} a^{i_2}) = b^{j_1+j_2} a^{i_1(1+p^r)^{j_2}+i_2}.$$

Proof. For any $i \in \mathbb{Z}_{p^m}, j \in \mathbb{Z}_{p^n}$, since $b^{-1}ab = a^{1+p^r}$, we have $b^{-j}ab^j = a^{(1+p^r)^j}$, and then $b^{-j}a^ib^j = a^{i(1+p^r)^j}$. It follows that $a^ib^j = b^ja^{i(1+p^r)^j}$, and so (1) holds.

For any positive integer k and for any $i \in \mathbb{Z}_{p^m}, j \in \mathbb{Z}_{p^n}$, if $k = 1$, then (2) is clearly true. Now we assume that $k > 1$ and (2) holds for any positive integer less than k. Then $(b^ja^i)^{k-1} = b^{(k-1)j}a^{i\sum_{s=0}^{k-2}(1+p^r)^{sj}}$, and then

$$\begin{aligned}(b^ja^i)^k &= b^ja^i(b^ja^i)^{k-1}\\ &= b^ja^i[b^{(k-1)j}a^{i\sum_{s=0}^{k-2}(1+p^r)^{sj}}]\\ &= b^j[a^ib^{(k-1)j}]a^{i\sum_{s=0}^{k-2}(1+p^r)^{sj}}\\ &= b^j[b^{(k-1)j}a^{i(1+p^r)^{(k-1)j}}]a^{i\sum_{s=0}^{k-2}(1+p^r)^{sj}}\\ &= b^{kj}a^{i\sum_{s=0}^{k-1}(1+p^r)^{sj}}.\end{aligned}$$

By induction, we have (2) holds.

For any $i_1, i_2 \in \mathbb{Z}_{p^m}$ and $j_1, j_2 \in \mathbb{Z}_{p^n}$, from (1) it follows that

$$(b^{j_1}a^{i_1})(b^{j_2}a^{i_2}) = b^{j_1}(a^{i_1}b^{j_2})a^{i_2} = b^{j_1}(b^{j_2}a^{i_1(1+p^r)^{j_2}})a^{i_2} = b^{j_1+j_2}a^{i_1(1+p^r)^{j_2}+i_2},$$

and so (3) holds. $\square$

Lemma 15. *Let p be an odd prime, and let H be an inner-abelian metacyclic p-group generated by a, b with the following defining relations:*

$$a^{p^m} = b^{p^n} = 1, b^{-1}ab = a^{1+p^r},$$

where m, n, r are positive integers such that $m \geqslant 2, n \geqslant 1$ and $r = m - 1$. Then the following hold:

(1) *For any positive integer k, we have*

$$a^{(1+p^r)^k} = a^{1+kp^r}.$$

(2) *For any $i \in \mathbb{Z}_{p^m}, j \in \mathbb{Z}_{p^n}$, we have*

$$(b^ja^i)^p = b^{jp}a^{ip}.$$

(3) *$H' \cong \mathbb{Z}_p$.*

Proof. For (1), the result is clearly true if $k = 1$. In what follows, assume $k \geqslant 2$. Since $r = m - 1$ and $m \geqslant 2$, we have $2r \geqslant m$. This implies that $a^{p^{2r}} = 1$, and hence $a^{p^{\ell r}} = 1$ for any $\ell \geqslant 2$. It then follows that

$$\begin{aligned}a^{(1+p^r)^k} &= a^{[C_k^0\cdot 1^k\cdot(p^r)^0 + C_k^1\cdot 1^{k-1}\cdot(p^r)^1 + C_k^2\cdot 1^{k-2}\cdot(p^r)^2 + \cdots + C_k^k\cdot 1^0\cdot(p^r)^k]}\\ &= a^{C_k^0\cdot(p^r)^0}\cdot a^{C_k^1\cdot(p^r)^1}\cdot a^{C_k^2\cdot(p^r)^2}\cdot\ldots\cdot a^{C_k^k\cdot(p^r)^k}\\ &= a\cdot(a^{p^r})^{C_k^1}\cdot(a^{p^{2r}})^{C_k^2}\cdot\ldots\cdot(a^{p^{kr}})^{C_k^k}\\ &= a\cdot a^{kp^r}\\ &= a^{1+kp^r},\end{aligned}$$

and so (1) holds. (Here for any integers $N \geqslant \ell \geqslant 0$, we denote by C_N^ℓ the binomial coefficient, that is, $C_N^\ell = \frac{N!}{\ell!(N-\ell)!}$.)

For (2), for any positive integer k and for any $i \in \mathbb{Z}_{p^m}, j \in \mathbb{Z}_{p^n}$, by Lemma 14 (1) − (2), we have

$$\begin{aligned}(b^j a^i)^p &= b^{jp} a^{i[1+(1+p^r)^j+(1+p^r)^{2j}+\cdots+(1+p^r)^{(p-1)j}]} \\ &= b^{jp} a^{i[1+(1+j\cdot p^r)+(1+2j\cdot p^r)+\cdots+(1+(p-1)\cdot jp^r)]} \\ &= b^{jp} a^{i(p+\frac{1}{2}p(p-1)\cdot jp^r)} \\ &= b^{jp} a^{ip}.\end{aligned}$$

Hence (2) holds.

From [25] we can obtain (3). □

4 Proof of Theorem 1

We shall prove Theorem 1 by a series of lemmas. We first prove three lemmas regarding cubic edge-transitive graphs of order twice a prime power.

Lemma 16. *Let Γ be a connected cubic edge-transitive graph of order $2p^n$ with p an odd prime and $n \geqslant 2$. Let $G \leqslant \mathrm{Aut}(\Gamma)$ be transitive on the edges of Γ. Then any minimal normal subgroup of G is an elementary abelian p-group.*

Proof. Let N be a minimal normal subgroup of G. If G is transitive on the arcs of Γ, then by [9, Lemma 3.1], N is an elementary abelian p-group, as required.

In what follows, assume that G is not transitive on the arcs of Γ. Then since Γ has valency 3, Γ is semisymmetric and so it is bipartite. Let B_0 and B_1 be the two partition sets of $V(\Gamma)$. Then B_0, B_1 are just the two orbits of G on $V(\Gamma)$ and have size p^n. Recalling that $N \trianglelefteq G$, each orbit of N has size dividing p^n. So, if N is solvable, then N must be an elementary abelian p-group, as required.

Suppose that N is non-solvable. By Proposition 13, we have $|G| = 2^r \cdot 3 \cdot p^n$, where $r \geqslant 0$. If $p = 3$, then by Burnside $p^a q^b$-theorem, G would be solvable, which is impossible because N is non-solvable. Thus, $p > 3$. Since N is a minimal normal subgroup of G, N is a product of some isomorphic non-abelian simple groups. Observing that $3^2 \nmid |G|$, by [12, pp.12-14], we obtain that $N \cong A_5$ or $\mathrm{PSL}(2,7)$. Then $p = 5$ or 7, and $p^2 \nmid |N|$. Since $n \geqslant 2$, it follows that N is intransitive on each bipartition sets of Γ. By Proposition 8, N is semiregular on $V(\Gamma)$, and so $|N| \mid p^n$, which is impossible. This completes the proof of our lemma. □

Lemma 17. *Let $p \geqslant 5$ be a prime and let Γ be a connected cubic edge-transitive graph of order $2p^n$ with $n \geqslant 1$. Let $A = \mathrm{Aut}(\Gamma)$ and let H be a Sylow p-subgroup of A. Then Γ is a bi-Cayley graph over H, and moreover, if $p \geqslant 11$, then Γ is a normal bi-Cayley graph over H.*

Proof. By Proposition 13, the stabilizer of any $v \in V(\Gamma)$ in A has order dividing $2^r \cdot 3$ with $r \geqslant 0$. Recalling H is a Sylow p-subgroup of A, H must be semiregular on $V(\Gamma)$ since $p \geqslant 5$. Since Γ is edge-transitive, Γ is either arc-transitive or semisymmetric, and so $p^n \mid |A|$. It follows that $p^n \mid |H|$, and so $|H| = p^n$. Thus, H has two orbits on $V(\Gamma)$, and hence Γ is a bi-Cayley graph over H.

Now suppose that $p \geqslant 11$. We shall prove the second assertion. It suffices to prove that $H \trianglelefteq A$. Use induction on n. If $n = 1$, then Γ is symmetric by [11, Theorem 2], and then by [18, Theorem 1] (see also [5, Table 1] or [9, Proposition 2.8]), we have $H \trianglelefteq A$, as required. Assume $n \geqslant 2$. Take N to be a minimal normal subgroup of A. By Lemma 16, N is an elementary abelian p-group and $|N| \mid p^n$. Consider the quotient graph Γ_N of Γ corresponding to the orbits of N. If $|N| = p^n$, then $H = N \trianglelefteq A$, as required. Suppose that $|N| < p^n$. Then each orbit of N has size at most p^{n-1}, and by Propositions 8 and 7, N is semiregular, and Γ_N is of valency 3 with A/N as an edge-transitive group of automorphisms of Γ_N. Clearly, Γ_N has order $2p^m$ with $m < n$. By induction, we have any Sylow p-subgroup of $\mathrm{Aut}(\Gamma_N)$ is normal. It follows that $H/N \trianglelefteq A/N$ because H/N is a Sylow p-subgroup of A/N. Therefore, $H \trianglelefteq A$, as required. □

Lemma 18. *Let Γ be a connected cubic edge-transitive graph of order $2p^n$ with $p = 5$ or 7 and $n \geqslant 2$. Let $Q = O_p(A)$ be the maximal normal p-subgroup of $A = \mathrm{Aut}(\Gamma)$. Then $|Q| = p^n$ or p^{n-1}.*

Proof. Let $|Q| = p^m$ with $m \leqslant n$. Suppose that $n - m \geqslant 2$. Then by Propositions 7 and 8, the quotient graph Γ_Q is a connected cubic graph of order $2p^{n-m}$ with A/Q as an edge-transitive group of automorphisms. Let N/Q be a minimal normal subgroup of A/Q. By Lemma 16, N/Q is an elementary abelian p-group. It follows that $N \trianglelefteq A$ and $Q < N$, contrary to the maximality of Q. Thus $n - m \leqslant 1$, and so $|Q| = p^n$ or p^{n-1}. □

Now we are ready to consider cubic edge-transitive bi-Cayley graphs over a metacyclic p-group. We first prove that $p = 3$.

Lemma 19. *Let Γ be a connected cubic edge-transitive bi-Cayley graph over a non-abelian metacyclic p-group H with p an odd prime. Then $p = 3$.*

Proof. Suppose to the contrary that $p > 3$. Let $A = \mathrm{Aut}(\Gamma)$. Then $R(H)$ is a Sylow p-subgroup of A. We shall first prove the following claim.

Claim.. $R(H) \trianglelefteq A$.

Suppose to the contrary that $R(H)$ is not normal in A. By Lemma 17, we have $p = 5$ or 7. Let N be the maximal normal p-subgroup of A. Then $N \leqslant R(H)$, and by Lemma 18, we have $|R(H) : N| = p$. Then the quotient graph Γ_N is a cubic graph of order $2p$ with A/N as an edge-transitive automorphism group. By [6, 7], if $p = 5$, then Γ_N is the Petersen graph, and if $p = 7$, then Γ_N is the Heawood graph. Since A/N is transitive on the edges of Γ_N and $R(H)/N$ is non-normal in A/N, it follows that

$$\begin{array}{ll} A_5 \lesssim A/N \lesssim S_5, & \text{if } p = 5; \\ \mathrm{PSL}(2,7) \lesssim A/N \lesssim \mathrm{PGL}(2,7), & \text{if } p = 7. \end{array}$$

Let B/N be the socle of A/N. Then B/N is also edge-transitive on Γ_N, and so B is also edge-transitive on Γ. Let $C = C_B(N)$. By Proposition 4, $B/C \lesssim \mathrm{Aut}(N)$. And

$C/(C\cap N)\cong CN/N\trianglelefteq B/N$. Since B/N is non-abelian simple, one has $CN/N=1$ or B/N.

Suppose first that $CN/N=1$. Then $C\leqslant N$, and so $C=C\cap N=C_N(N)=Z(N)$. Then $B/Z(N)=B/C\lesssim \mathrm{Aut}\,(N)$. Since $R(H)$ is a metacyclic p-group, N is also a metacyclic p-group. If N is non-abelian, then by Proposition 5 and [22, Lemma 2.6], $\mathrm{Aut}\,(N)$ is solvable. It follows that $B/Z(N)$ is solvable, and so B is solvable. This is contrary to the fact that B/N is non-abelian simple.

If N is abelian, then $C=Z(N)=N$. Let

$$\mathrm{Aut}^{\Phi}(N)=\langle \alpha\in\mathrm{Aut}\,(N)\mid g^{\alpha}\Phi(N)=g\Phi(N),\forall g\in N\rangle,$$

where $\Phi(N)$ is the Frattini subgroup of N. Recall that $\mathrm{Aut}^{\Phi}(N)$ is a normal p-subgroup of $\mathrm{Aut}\,(N)$ and $\mathrm{Aut}\,(N)/\mathrm{Aut}^{\Phi}(N)\leqslant \mathrm{Aut}\,(N/\Phi(N))$ (see [19]). Let $K/C=(B/C)\cap \mathrm{Aut}^{\Phi}(N)$. Then $K/C\trianglelefteq B/C$, and so $K\trianglelefteq B$. It follows that

$$B/K\cong (B/C)/(K/C)\cong((B/C)\cdot\mathrm{Aut}^{\Phi}(N))/\mathrm{Aut}^{\Phi}(N)\leqslant\mathrm{Aut}\,(N/\Phi(N)).$$

Clearly, K/C is a p-group. Since $C=N$, K is also a p-group. As N is the maximal normal p-subgroup of A, N is also the maximal normal p-subgroup of B. This implies that $K=N$. If N is cyclic, then $N/\Phi(N)\cong\mathbb{Z}_p$, and so $B/N=B/K\lesssim\mathrm{Aut}\,(N/\Phi(N))\cong\mathbb{Z}_{p-1}$, again contrary to the fact that B/N is a non-abelian simple group. If N is not cyclic, then $N/\Phi(N)\cong\mathbb{Z}_p\times\mathbb{Z}_p$. It follows that $B/N=B/K\lesssim\mathrm{Aut}\,(N/\Phi(N))\cong\mathrm{GL}(2,p)$. This forces that either $A_5\leqslant\mathrm{GL}(2,5)$ with $p=5$, or $\mathrm{PSL}(2,7)\leqslant\mathrm{GL}(2,7)$ with $p=7$. However, each of these can not happen by Magma [3], a contradiction.

Suppose now that $CN/N=B/N$. Since $C\cap N=Z(N)$, we have $1<C\cap N\leqslant Z(C)$. Clearly, $Z(C)/(C\cap N)\trianglelefteq C/(C\cap N)\cong CN/N$. Since $CN/N=B/N$ is non-abelian simple, $Z(C)/C\cap N$ must be trivial. Thus $C\cap N=Z(C)$, and hence $B/N=CN/N\cong C/C\cap N=C/Z(C)$. If $C=C'$, then $Z(C)$ is a subgroup of the Schur multiplier of B/N. However, the Schur multiplier of A_5 or $\mathrm{PSL}(2,7)$ is $\mathbb{Z}_2$, a contradiction. Thus, $C\neq C'$. Since $C/Z(C)$ is non-abelian simple, one has $C/Z(C)=(C/Z(C))'=C'Z(C)/Z(C)\cong C'/(C'\cap Z(C))$, and then we have $C=C'Z(C)$. It follows that $C''=C'$. Clearly, $C'\cap Z(C)\leqslant Z(C')$, and $Z(C')/(C'\cap Z(C))\trianglelefteq C'/(C'\cap Z(C))$. Since $C'/(C'\cap Z(C))\cong C/Z(C)$ and since $C/Z(C)$ is non-abelian simple, it follows that $Z(C')/(C'\cap Z(C))$ is trivial, and so $Z(C')=C'\cap Z(C)$. As $C/(C\cap N)\cong CN/N$ is non-abelian, we have $C/(C\cap N)=(C/(C\cap N))'=(C/Z(C))'\cong C'/(C'\cap Z(C))=C'/Z(C')$. Since $C'=C''$, $Z(C')$ is a subgroup of the Schur multiplier of CN/N. However, the Schur multiplier of A_5 or $\mathrm{PSL}(2,7)$ is $\mathbb{Z}_2$, forcing that $Z(C')\cong\mathbb{Z}_2$. This is impossible because $Z(C')=C'\cap Z(C)\leqslant C\cap N$ is a p-subgroup. Claim is proved.

If H is non-split, then by Proposition 6, A has a normal p-complement Q. By Propositions 7 and 8, the quotient graph Γ_Q would be cubic graph of odd order, a contradiction.

Thus, H is split. Then we may assume that

$$H=\langle a,b\mid a^{p^m}=b^{p^n}=1,a^b=a^{1+p^r}\rangle,$$

where m,n,r are positive integers such that $r<m\leqslant n+r$.

By Claim, $R(H) \trianglelefteq A$. Since Γ is edge-transitive, we assume that $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$. By Proposition 9, we may assume that $S = \{1, g, h\}$ with $g, h \in H$. By Proposition 10, there exists $\sigma_{\alpha,x} \in \mathrm{Aut}\,(\Gamma)_{1_0}$, where $\alpha \in \mathrm{Aut}\,(H)$ and $x \in H$, such that $\sigma_{\alpha,x}$ cyclically permutates the three elements in $\Gamma(1_0) = \{1_1, g_1, h_1\}$. Without loss of generality, assume that $(\sigma_{\alpha,x})_{|\Gamma(1_0)} = (1_1\ g_1\ h_1)$. Then $g_1 = (1_1)^{\sigma_{\alpha,x}} = x_1$, implying that $x = g$. Furthermore, $h_1 = (g_1)^{\sigma_{\alpha,x}} = (gg^{\alpha})_1$ and $1_1 = (h_1)^{\sigma_{\alpha,x}} = (gh^{\alpha})_1$. It follows that $g^{\alpha} = g^{-1}h$, $h^{\alpha} = g^{-1}$. This implies that α is an automorphism of H order dividing 3. If α is trivial, then $h = g^{-1}$ and $g = g^{-1}h = g^{-2}$, and then $g^3 = 1$. Since $p > 3$, we must have $h = g = 1$, a contradiction. Thus, α has order 3. By Proposition 5, we must have $3 \mid p-1$. Furthermore, α is conjugate to the following automorphism of H induced by the following map:

$$\beta : a \mapsto a^s, b \mapsto b,$$

where s is an element of order 3 of $\mathbb{Z}_{p^m}^*$.

Assume that $\beta = \pi^{-1}\alpha\pi$ for $\pi \in \mathrm{Aut}\,(H)$. Consider the graph $\Gamma^{\pi} = \mathrm{BiCay}(H, \emptyset, \emptyset, S^{\pi})$. By Proposition 10 (3), we have $\Gamma^{\pi} \cong \Gamma$, and $\sigma_{\beta,g^{\pi}}$ cyclically permutates the three elements in $\Gamma^{\pi}(1_0) = \{1_1^{\pi}, g_1^{\pi}, h_1^{\pi}\}$. For convenience of the statement, we may assume that π is trivial and $\alpha = \beta$.

Let $g = b^j a^i$, where $i \in \mathbb{Z}_{p^m}$, $j \in \mathbb{Z}_{p^n}$. Then $h = gg^{\alpha} = b^j a^i b^j a^{is}$. Since Γ is connected, we have $H = \langle S \rangle = \langle g, h \rangle = \langle b^j a^i, b^j a^i b^j a^{is} \rangle = \langle b^j, a^i, a^{is} \rangle = \langle a^i, b^j \rangle$, implying that i, j are coprime to p. Then there exists an integer u such that $ui \equiv 1 \pmod{p^m}$. It is easy to check that the map $\gamma : a \mapsto a^u, b \mapsto b$ can induce an automorphism of H, and then $(a^i)^{\gamma} = a^{ui} = a$. Again, by Proposition10 (3), we have $\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^{\gamma})$, where $S^{\gamma} = \{1, b^j a, b^j a b^j a^s\}$. Let $\Gamma' = \mathrm{BiCay}(H, \emptyset, \emptyset, S^{\gamma})$. Then $\sigma_{\gamma^{-1}\alpha\gamma, g^{\gamma}} \in \mathrm{Aut}\,(\Gamma')$ cyclically permutates the elements in $\Gamma'(1_0) = \{1_1, (b^j a)_1, (b^j a b^j a^s)_1\}$.

It is easy to check that $a^{\gamma^{-1}\alpha\gamma} = (a^i)^{\alpha\gamma} = (a^{is})^{\gamma} = a^s$ and $b^{\alpha^{\gamma}} = b$. It then follows that $1_1^{\sigma_{\alpha\gamma, b^j a}} = (b^j a)_1$, $(b^j a)_1^{\sigma_{\alpha\gamma, b^j a}} = (b^j a b^j a^s)_1$, and $(b^j a b^j a^s)_1^{\sigma_{\alpha\gamma, b^j a}} = (b^j a (b^j a b^j a^s)^{\alpha^{\gamma}})_1 = (b^j a b^j a^s b^j a^{s^2})_1 = (b^{3j} a^{(1+p^r)^{2j} + s(1+p^r)^j + s^2})_1 \neq 1_1$. This is a contradiction. Thus $p = 3$. $\square$

In what follows, we consider cubic edge-transitive bi-Cayley graph over the group H, where H is a non-abelian metacyclic 3-group.

Lemma 20. *Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected cubic edge-transitive bi-Cayley graph over a non-abelian metacyclic 3-group H with $|H| = 3^s$, where $s \geqslant 4$. Then Γ is a normal bi-Cayley graph over H.*

Proof. Let $A = \mathrm{Aut}\,(\Gamma)$ and let P be a Sylow 3-subgroup of A such that $R(H) \leqslant P$. By Proposition 13, we have $|A| = 3^{s+1} \cdot 2^r$ with $r \geqslant 0$. This implies that $|P| = 3|R(H)|$, and so $|P_{1_0}| = |P_{1_1}| = 3$. Thus, P is transitive on the edges of Γ. Clearly, $R(H) \trianglelefteq P$. This implies that the two orbits H_0, H_1 of $R(H)$ do not contain the edges of Γ, and so $R = L = \emptyset$.

Claim. $P \trianglelefteq A$.

Let $M \trianglelefteq A$ be maximal subject to that M is intransitive on both H_0 and H_1. By Proposition 7 and Proposition 8, M is semiregular on $V(\Gamma)$ and the quotient graph Γ_M of

Γ relative to M is a cubic graph with A/M as an edge-transitive group of automorphisms. Assume that $|M| = 3^t$. Then $|V(\Gamma_M)| = 2 \cdot 3^{s-t}$. If $s - t \leqslant 2$, then by [6, 7], Γ_M is isomorphic to F006A or the Pappus graph F018A, and then Aut (Γ_M) has a normal Sylow 3-subgroup. It follows that $P/M \trianglelefteq A/M$, and so $P \trianglelefteq A$, as claimed.

Now assume that $s - t > 2$. Take a minimal normal subgroup N/M of A/M. By Lemma 16, N/M is an elementary abelian 3-group. By the maximality of M, N is transitive on at least one of H_0 and H_1, and so $3^s \mid |N|$. If $3^{s+1} \mid |N|$, then $P = N \trianglelefteq A$, as claimed. Assume that $|N| = 3^s$. If N is transitive on both H_0 and H_1, then N is semiregular on both H_0 and H_1, and then Γ_M would be a cubic bi-Cayley graph on N/M. Since Γ_M is connected, by Proposition 9, N/M is generated by two elements, and so $N/M \cong \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. This implies that $|V(\Gamma_M)| = 6$ or 18, contrary to the assumption that $|V(\Gamma_M)| = 2 \cdot 3^{s-t} > 18$. Thus, we may assume that N is transitive on H_0 but intransitive on H_1. Then $N/M \neq R(H)M/M$, and so $NR(H)M/M = P/M$. Since $|P/M : R(H)M/M| \mid 3$, one has $|N/M : (N/M \cap R(H)M/M)| \mid 3$, and since H is metacyclic, one has $N/M \cap R(H)M/M$ is also metacyclic and so is a two-generator group. This implies that $|N/M| \mid 3^3$, and so $|N/M| = 3^3$ because $|N/M| = 3^{s-t} > 9$. Then $|V(\Gamma_M)| = 2 \cdot |N/M| = 54$. Since $s \geqslant 4$, we have $|M| \geqslant 3$. If $M \nleqslant R(H)$, then $P = MR(H)$ and then $N/M \leqslant R(H)M/M$. As H is metacyclic, N/M is also metacyclic, and so $|N/M| = 3$ or 9, a contradiction. Thus, $M \leqslant R(H)$, and hence M is metacyclic. Then $M/\Phi(M) \cong \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\Phi(M)$ is characteristic in M, one has $\Phi(M) \trianglelefteq A$ because $M \trianglelefteq A$. Then the quotient graph $\Gamma_{\Phi(M)}$ is a cubic graph of order $2 \cdot 3^4$ or $2 \cdot 3^5$ with $A/\Phi(M)$ as an edge-transitive group of automorphisms. By [6, 7] and Magma [3], we obtain that every Sylow 3-subgroup of Aut $(\Gamma_{\Phi(M)})$ is normal. This implies that $P/\Phi(M) \trianglelefteq A/\Phi(M)$, and so $P \trianglelefteq A$, completing the proof of our claim.

Now we are ready to finish the proof of our lemma. By Claim, we have $P \trianglelefteq A$. Since $|P : R(H)| = 3$, one has $\Phi(P) \leqslant R(H)$. As H is non-abelian, one has $\Phi(P) < R(H)$ for otherwise, we would have P is cyclic and so H is cyclic which is impossible. Then $\Phi(P)$ is intransitive on both H_0 and H_1, the two orbits of $R(H)$ on $V(\Gamma)$. Since $\Phi(P)$ is characteristic in P, $P \trianglelefteq A$ gives that $\Phi(P) \trianglelefteq A$. By Propositions 7 and 8, the quotient graph $\Gamma_{\Phi(P)}$ of Γ relative to $\Phi(P)$ is a cubic graph with $A/\Phi(P)$ an edge-transitive group of automorphisms. Furthermore, $P/\Phi(P)$ is transitive on the edges of $\Gamma_{\Phi(P)}$. Since $P/\Phi(P)$ is abelian, it is easy to see that $\Gamma_{\Phi(P)} \cong K_{3,3}$, and so $P/\Phi(P) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $|P| = 3^{s+1} \geqslant 3^5$, one has $|\Phi(P)| = 3^{s-1} \geqslant 3^3$.

Let Φ_2 be the Frattini subgroup of $\Phi(P)$. Then $\Phi_2 \trianglelefteq A$ because Φ_2 is characteristic in $\Phi(P)$ and $\Phi(P) \trianglelefteq A$. Clearly, $\Phi_2 \leqslant \Phi(P) < R(H)$, so Φ_2 is intransitive on both H_0 and H_1. Consider the quotient graph Γ_{Φ_2} of Γ relative to Φ_2. By Propositions 7 and 8, Γ_{Φ_2} is a cubic graph with A/Φ_2 as an edge-transitive group of automorphisms. Furthermore, Γ_{Φ_2} is a bi-Cayley graph over the group $R(H)/\Phi_2$. Again, since H is a metacyclic group, we have $\Phi(P)/\Phi_2 \cong \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. If $\Phi(P)/\Phi_2 \cong \mathbb{Z}_3$, then $\Phi(P)$ is a cyclic 3-group, and so Γ is an edge-transitive cyclic cover of $\Gamma_{\Phi(P)} \cong K_{3,3}$. By Feng et al. [10, 23], we have Γ is isomorphic to either $K_{3,3}$ or the Pappus graph, a contradiction.

Thus, $\Phi(P)/\Phi_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $|\Phi(P)| = 3^{s-1} \geqslant 3^3$, one has $|\Phi_2| \geqslant 3$. Let Φ_3 be the Frattini subgroup of Φ_2. Then Φ_3 is characteristic in Φ_2, and so normal in A because

$\Phi_2 \trianglelefteq A$. As $\Phi_2 \leqslant R(H)$, one has $\Phi_2/\Phi_3 \cong \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, and so $|R(H)/\Phi_3| = 3^4$ or 3^5. Clearly, Φ_3 is intransitive on both H_0 and H_1. Again, by Propositions 7 and 8, the quotient graph Γ_{Φ_3} is a cubic graph of order 162 or 486 with A/Φ_3 as an edge-transitive group of automorphisms. Observe that $R(H)/\Phi_3$ is metacyclic semiregular on $V(\Gamma_{\Phi_3})$ with two orbits.

If $|\Gamma_{\Phi_3}| = 486$, then by [6, 7], Γ_{Φ_3} is semisymmetric or symmetric. For the former, by Magma [3], all semiregular subgroups of $\mathrm{Aut}\,(\Gamma_{\Phi_2})$ of order 243 are normal, and so $R(H)/\Phi_3 \trianglelefteq \mathrm{Aut}\,(\Gamma_{\Phi_3})$. It follows that $R(H)/\Phi_3 \trianglelefteq A/\Phi_3$, and so $R(H) \trianglelefteq A$, as required. If Γ_{Φ_3} is symmetric, then by [6], $\Gamma_{\Phi_3} \cong$ F486A, F486B, F486C or F486D. By Magma [3], if $\Gamma_{\Phi_3} \cong$ F486B, F486C or F486D, then $\mathrm{Aut}\,(\Gamma_{\Phi_3})$ does not have a metacyclic semiregular subgroup of order 243, a contradiction. If $\Gamma_{\Phi_3} \cong$ F486A, then by Magma [3], all semiregular subgroups of $\mathrm{Aut}\,(\Gamma_{\Phi_3})$ of order 243 are normal, and so $R(H)/\Phi_3 \trianglelefteq \mathrm{Aut}\,(\Gamma_{\Phi_3})$. It follows that $R(H)/\Phi_3 \trianglelefteq A/\Phi_3$, and so $R(H) \trianglelefteq A$, as required.

If $|\Gamma_{\Phi_3}| = 162$, then by [6, 7], Γ_{Φ_3} is symmetric, and is isomorphic to F162A, F162B or F162C. By Magma [3], if $\Gamma_{\Phi_3} \cong$ F162C, then $\mathrm{Aut}\,(\Gamma_{\Phi_3})$ does not have a metacyclic semiregular subgroup of order 81, a contradiction. If $\Gamma_{\Phi_3} \cong$ F162A or F162B, then by Magma [3], all semiregular subgroups of $\mathrm{Aut}\,(\Gamma_{\Phi_3})$ of order 81 are normal, and so $R(H)/\Phi_3 \trianglelefteq \mathrm{Aut}\,(\Gamma_{\Phi_3})$. It follows that $R(H)/\Phi_3 \trianglelefteq A/\Phi_3$, and so $R(H) \trianglelefteq A$, as required. □

Proof of Theorem 1. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected cubic edge-transitive bi-Cayley graph over a non-abelian metacyclic p-group H with p an odd prime. By Lemma 19, we have $p = 3$, and since H is a non-abelian metacyclic 3-group, we have $|H| = 3^s$ with $s \geqslant 3$. If $s = 3$, then Γ has order 54, and by [6, 7], Γ is isomorphic to F054 or the Gray graph. However, by Magma [3], Aut (F054) does not have a non-abelian metacyclic 3-subgroup which acts semiregularly on the vertex set of F054 with two orbits. It follows that Γ is isomorphic to Gray graph. If $s > 3$, then by Lemma 20, $R(H) \trianglelefteq \mathrm{Aut}\,(\Gamma)$, as required. □

5 A class of cubic edge-transitive bi-3-metacirculants

In this section, we shall use Theorem 1 to give a characterization of connected cubic edge-transitive bi-Cayley graphs over inner-abelian metacyclic 3-groups.

5.1 Construction

We shall first construct two classes of connected cubic edge-transitive bi-Cayley graphs over inner-abelian metacyclic 3-groups.

Construction 1. Let t be a positive integer, and let

$$\mathcal{G}_t = \langle a, b \mid a^{3^{t+1}} = b^{3^t} = 1, b^{-1}ab = a^{1+3^t} \rangle.$$

Let $S = \{1, a, a^{-1}b\}$, and let $\Gamma_t = \mathrm{BiCay}(\mathcal{G}_t, \emptyset, \emptyset, S)$.

Lemma 21. *For any integer t, the graph Γ_t is semisymmetric.*

Proof. We first prove the following three claims.

Claim 1. $\mathcal{G}_t$ has an automorphism α mapping a, b to $a^{-2}b, a^{3^t-3}b$, respectively.

Let $x = a^{-2}b$ and $y = a^{3^t-3}b$. Then,

$$\begin{array}{l}(yx^{-1})^{3^t+1} = [(a^{3^t-3}b)(a^{-2}b)^{-1}]^{3^t+1} = (a^{3^t-1})^{3^t+1} = a^{-1},\\ ((yx^{-1})^{3^t+1})^{-2} \cdot x = a^2 \cdot a^{-2}b = b,\end{array}$$

and hence $\langle a, b\rangle = \langle x, y\rangle$.

By Lemma 15 (2), we have $x^{3^{t+1}} = (a^{-2}b)^{3^{t+1}} = 1$ and $y^{3^t} = (a^{3^t-3}b)^{3^t} = 1$. Furthermore, we have

$$x^{1+3^t} = (a^{-2}b)^{1+3^t} = (a^{-2}b)(a^{-2}b)^{3^t} = a^{-2}ba^{-2\cdot 3^t} = a^{-2-2\cdot 3^t}b = a^{3^t-2}b,$$

and

$$\begin{array}{rcl}y^{-1}xy & = & (a^{3^t-3}b)^{-1}(a^{-2}b)(a^{3^t-3}b)\\ & = & (b^{-1}a^{3-3^t}a^{-2}b)a^{3^t-3}b\\ & = & (b^{-1}a^{1-3^t}b)a^{3^t-3}b\\ & = & a^{(1+3^t)(1-3^t)}a^{3^t-3}b\\ & = & a^{3^t-2}b\\ & = & x^{1+3^t}.\end{array}$$

It follows that x and y have the same relations as do a and b. Thus, the map $\alpha : a \mapsto a^{-2}b, b \mapsto a^{3^t-3}b$ induces an automorphism of $\mathcal{G}_t$, as claimed.

Claim 2. $\mathcal{G}_t$ has no automorphism mapping a, b to $a^{-1}, a^{3t}b^{-1}$, respectively.

Suppose to the contrary that $\mathcal{G}_t$ has an automorphism, say β, such that $a^\beta = a^{-1}, b^\beta = a^{3t}b^{-1}$. Then $(b^{-1}ab)^\beta = (a^{3^t+1})^\beta$, and so

$$\begin{array}{rcl}a^{-3^t-1} & = & (a^{3^t+1})^\beta = (b^{-1}ab)^\beta\\ & = & (a^{3^t}b^{-1})^{-1} \cdot a^{-1} \cdot (a^{3^t}b^{-1})\\ & = & ba^{-1}b^{-1} = a^{-(1+3^t)^{3^t-1}} = a^{-1+3^t}.\end{array}$$

It follows that $a^{2\cdot 3^t} = 1$, and so $3^{t+1} \mid 2\cdot 3^t$, a contradiction.

Claim 3. $\mathcal{G}_t$ has no automorphism mapping a, b to $b^{-1}a, b^{-1}$, respectively.

Suppose to the contrary that there exists $\gamma \in \mathrm{Aut}\,(\mathcal{G}_t)$ such that $a^\gamma = b^{-1}a, b^\gamma = b^{-1}$. Then $(b^{-1}ab)^\gamma = (a^{1+3^t})^\gamma$, and then

$$b^{-1}a^{3^t+1} = (b^{-1}a)^{1+3^t} = (a^{1+3^t})^\gamma = (b^{-1}ab)^\gamma = b(b^{-1}a)b^{-1} = ab^{-1}.$$

It follows that $b^{-1}a^{3^t+1}b = a$, and so $a^{3^{2t}+2\cdot 3^t+1} = a^{2\cdot 3^t+1} = a$, forcing that $3^{t+1} \mid 2\cdot 3^t$, a contradiction.

Now we are ready to finish the proof. By Claim 1, there exists $\alpha \in \mathrm{Aut}\,(\mathcal{G}_t)$ such that $a^\alpha = a^{-2}b$ and $b^\alpha = a^{3^t-3}b$. Then $(a^{-1}b)^\alpha = (a^{-2}b)^{-1}(a^{3^t-3}b) = b^{-1}a^{3^t-1}b = a^{-1}$. It then follows that

$$S^\alpha = \{1^\alpha, a^\alpha, (a^{-1}b)^\alpha\} = \{1, a^{-2}b, a^{-1}\} = a^{-1}S.$$

By Proposition 10, $\sigma_{\alpha,a}$ is an automorphism of Γ_t fixing 1_0 and cyclically permutating the three neighbors of 1_0. Set $B = R(\mathcal{G}_t) \rtimes \langle\sigma_{\alpha,a}\rangle$. Then B acts regularly on the edges of Γ_t.

If $t = 1$, then by Magma [3], Γ_1 is isomorphic to the Gray graph, which is semisymmetric. In what follows, assume that $t > 1$. By Theorem 1, Γ_t is a normal bi-Cayley graph over $R(\mathcal{G}_t)$. Suppose that Γ_t is vertex-transitive. Then Γ_t is also arc-transitive. So, there exist $f \in \mathrm{Aut}\,(\mathcal{G}_t), g, h \in \mathcal{G}_t$ so that $\delta_{f,g,h}$ is an automorphism of Γ_t taking the arc $(1_0, 1_1)$ to $(1_1, 1_0)$. By the definition of $\delta_{f,g,h}$, one may see that $g = h = 1$ and $S^f = S^{-1}$, namely,

$$\{1, a, a^{-1}b\}^f = \{1, a^{-1}, b^{-1}a\}.$$

So, f takes $(a, a^{-1}b)$ either to $(a^{-1}, b^{-1}a)$ or to $(b^{-1}a, a^{-1})$. However, this is impossible by Claims 2-3. Therefore, Γ_t is semisymmetric. □

Construction 2. Let t be a positive integer, and let

$$\mathcal{H}_t = \langle a, b \mid a^{3^{t+1}} = b^{3^{t+1}} = 1, b^{-1}ab = a^{1+3^t}\rangle.$$

Let $T = \{1, b, b^{-1}a\}$, and let $\Sigma_t = \mathrm{BiCay}(\mathcal{H}_t, \emptyset, \emptyset, T)$.

Lemma 22. *For any positive integer t, the graph Σ_t is symmetric.*

Proof. We first prove the following two claims.

Claim 1. $\mathcal{H}_t$ has an automorphism α mapping a, b to $a^{2\cdot3^t+1}b^{-3}, a^{2\cdot3^t+1}b^{-2}$, respectively.

Let $x = a^{2\cdot3^t+1}b^{-3}$ and $y = a^{2\cdot3^t+1}b^{-2}$. Note that $((y^{-1}x)^{-1}) = b$ and $xb^3 = a^{2\cdot3^t+1}$. This implies that $\langle x, y\rangle = \langle a, b\rangle = \mathcal{H}_t$.

By Lemma 15 (2), we have $x^{3^{t+1}} = (a^{-2}b)^{3^{t+1}} = 1$ and $y^{3^{t+1}} = (a^{2\cdot3^t+1}b^{-2})^{3^{t+1}} = 1$. Furthermore, we have

$$\begin{aligned} y^{-1}xy &= (a^{2\cdot3^t+1}b^{-2})^{-1}(a^{2\cdot3^t+1}b^{-3})(a^{2\cdot3^t+1}b^{-2}) \\ &= b^{-1}a^{2\cdot3^t+1}b^{-2} = b^{-1}a^{2\cdot3^t+1}bb^{-3} \\ &= a^{(2\cdot3^t+1)(3^t+1)}b^{-3} = ab^{-3} = x^{3^t}x \\ &= x^{3^t+1}. \end{aligned}$$

It follows that x and y have the same relations as do a and b. Therefore, $\mathcal{H}_t$ has an automorphism taking (a, b) to (x, y), as claimed.

Claim 2. $\mathcal{H}_t$ has an automorphism β mapping a, b to $a^{-1}, a^{-1}b$.

Let $x = a^{-1}$ and $y = a^{-1}b$. Clearly, $\langle a, b\rangle = \langle x, y\rangle$. By Lemma 15 (2), we have that $x^{3^{t+1}} = (a^{-1})^{3^{t+1}} = 1$ and $y^{3^{t+1}} = (a^{-1}b)^{3^{t+1}} = 1$. Furthermore, we have

$$y^{-1}xy = (a^{-1}b)^{-1}(a^{-1})(a^{-1}b) = b^{-1}a^{-1}b = a^{-3^t-1} = x^{3^t+1}.$$

It follows that x and y have the same relations as do a and b. Therefore, $\mathcal{H}_t$ has an automorphism β which takes (a, b) to $(a^{-1}, a^{-1}b)$, as claimed.

Now we are ready to finish the proof. By Claim 1, there exists $\alpha \in \mathrm{Aut}\,(\mathcal{H}_t)$ such that $a^\alpha = a^{2\cdot 3^t+1}b^{-3}$ and $b^\alpha = a^{2\cdot 3^t+1}b^{-2}$. Then

$$S^\alpha = \{1, b, b^{-1}a\}^\alpha = \{1, a^{2\cdot 3^t+1}b^{-2}, b^{-1}\}.$$

By an easy computation, we have $a^{2\cdot 3^t+1}b^{-2} = a^{2\cdot 3^t+1}b^{-3}b = b^{-3}a^{2\cdot 3^t+1}b = b^{-2}b^{-1}a^{2\cdot 3^t+1}b = b^{-2}a^{(2\cdot 3^t+1)(3^t+1)} = b^{-2}a$. It follows that

$$b^{-1}S = b^{-1}\{1, b, b^{-1}a\} = \{b^{-1}, 1, b^{-2}a\} = S^\alpha.$$

By Proposition 10, $\sigma_{\alpha,b}$ is an automorphism of Σ_t fixing 1_0 and cyclically permutating the three neighbors of 1_0. Set $B = R(\mathcal{H}_t) \rtimes \langle \sigma_{\alpha,b} \rangle$. Then B acts transitively on the edges of Σ_t.

By Claim 2, there exists $\beta \in \mathrm{Aut}\,(\mathcal{H}_t)$ such that $a^\beta = a^{-1}$ and $b^\beta = a^{-1}b$. Then $S^\beta = \{1, b, b^{-1}a\}^\beta = \{1, a^{-1}b, b^{-1}\} = S^{-1}$. By Proposition 10, $\delta_{\beta,1,1}$ is an automorphism of Σ_t swapping 1_0 and 1_1. Thus, Σ_t is vertex-transitive, and so Σ_t is symmetric. □

5.2 Classification

In this section, we shall give a classification of cubic edge-transitive bi-Cayley graph over an inner-abelian metacyclic 3-group.

Lemma 23. *Let H be an inner-abelian metacyclic 3-group, and let Γ be a connected cubic edge-transitive bi-Cayley graph over H. Then $\Gamma \cong \Gamma_t$ or Σ_t.*

Proof. Since H is an inner-abelian metacyclic 3-group, it has order at least 3^3. If $|H| = 3^3$, then $|\Gamma| = 54$ and by [6, 7], we know that Γ is isomorphic to Γ_1. In what follows, assume that $|H| > 3^3$. By Theorem 1, Γ is a normal bi-Cayley graph over H. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$. Since Γ is edge-transitive, the two orbits H_0, H_1 of $R(H)$ on $V(\Gamma)$ do not contain any edge of Γ, and so $R = L = \emptyset$. By Proposition 9, we may assume that $S = \{1, x, y\}$ for $x, y \in H$. Since Γ is connected, by Proposition 9, we have $H = \langle S \rangle = \langle x, y \rangle$.

Let $A = \mathrm{Aut}\,(\Gamma)$, since Γ is normal and since Γ is edge-transitive, by Proposition 10, there exists $\sigma_{\alpha,h} \in A_{1_0}$, where $\alpha \in \mathrm{Aut}\,(H)$ and $h \in H$, such that $\sigma_{\alpha,h}$ cyclically permutates the three elements in $\Gamma(1_0) = \{1_1, x_1, y_1\}$. Without loss of generality, assume that $(\sigma_{\alpha,h})_{|\Gamma(1_0)} = (1_1\ x_1\ y_1)$. Then $x_1 = (1_1)^{\sigma_{\alpha,h}} = h_1$, implying that $x = h$. Furthermore, $y_1 = (x_1)^{\sigma_{\alpha,h}} = (xx^\alpha)_1$ and $1_1 = (y_1)^{\sigma_{\alpha,h}} = (xy^\alpha)_1$. It follows that $x^\alpha = x^{-1}y$ and $y^\alpha = x^{-1}$. This implies that α is an automorphism of H order dividing 3. If α is trivial, then $x = y^{-1}$ and $x = x^{-1}y = y^2$, and then $y^3 = 1$ and $x^3 = 1$. This implies that $H \cong \mathbb{Z}_3$, contrary to the assumption that $|H| > 3^3$. Thus, α has order 3.

Since H is an inner-abelian 3-group, by elementary group theory (see also [21]), we may assume that

$$H = \langle a, b \mid a^{3^{t+1}} = b^{3^s} = 1, b^{-1}ab = a^{3^t+1} \rangle,$$

where $t \geqslant 2, s \geqslant 1$. We first prove the following claim.

Claim 1. $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^t}$, $\mathbb{Z}_{3^t} \times \mathbb{Z}_{3^{t-1}}$ or $\mathbb{Z}_{3^t} \times \mathbb{Z}_{3^{t+1}}$.

By Lemma 15 (3), we have the derived subgroup $R(H)'$ of $R(H)$ is isomorphic to $\mathbb{Z}_3$. Since $R(H)'$ is characteristic in $R(H)$, $R(H) \trianglelefteq A$ gives that $R(H)' \trianglelefteq A$. Consider the quotient graph $\Gamma_{R(H)'}$ of Γ relative to $R(H)'$. Clearly, $R(H)'$ is intransitive on both H_0 and H_1, the two orbits of $R(H)$ on $V(\Gamma)$. By Propositions 7 and 8, $\Gamma_{R(H)'}$ is a cubic graph with $A/R(H)'$ as an edge-transitive group of automorphisms. Clearly, $\Gamma_{R(H)'}$ is a bi-Cayley graph over the abelian group $R(H)/R(H)'$. Since $R(H)/R(H)' \trianglelefteq A/R(H)'$, by Proposition 11, we have $R(H)/R(H)' \cong \mathbb{Z}_{3^{m+m_1}} \times \mathbb{Z}_{3^m}$ for some integers m, m_1 satisfying the equality $\lambda^2 - \lambda + 1 \equiv 0 \pmod{3^{m_1}}$ with $\lambda \in \mathbb{Z}^*_{3^{m_1}}$. This implies that $m_1 = 0$ or 1, and so $R(H)/R(H)' \cong \mathbb{Z}_{3^m} \times \mathbb{Z}_{3^m}$ or $\mathbb{Z}_{3^{m+1}} \times \mathbb{Z}_{3^m}$.

Since $a^{3^t} = [a, b]$, one has $\langle aH' \rangle \cong \mathbb{Z}_{3^t}$, and since $H' \cap \langle b \rangle = 1$, one has $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^s}$. So, if $R(H)/R(H)' \cong \mathbb{Z}_{3^m} \times \mathbb{Z}_{3^m}$, then we have $m = s = t$, and if $R(H)/R(H)' \cong \mathbb{Z}_{3^{m+1}} \times \mathbb{Z}_{3^m}$, then $(t, s) = (m, m+1)$ or $(m+1, m)$. Claim 1 is proved.

For any $h \in H$, denote by $o(h)$ the order of h. Let $n = \mathrm{Max}\{t+1, s\}$. By Lemma 15 (2), it is easy to see that 3^n is the exponent of H.

Claim 2. $o(x) = o(y) = o(x^{-1}y) = 3^n$ and $x^{3^{n-1}} \neq y^{3^{n-1}}$.

Observing that $x^\alpha = x^{-1}y$ and $y^\alpha = x^{-1}$, we have $o(x) = o(y) = o(x^{-1}y)$. By Lemma 15 (2), we must have $o(x) = o(y) = o(x^{-1}y) = 3^n$. Then $(x^{-1}y)^{3^{n-1}} \neq 1$, and again by Lemma 15 (2), we have $x^{-3^{n-1}} y^{3^{n-1}} \neq 1$, namely, $x^{3^{n-1}} \neq y^{3^{n-1}}$, as claimed.

By Claim 1, we shall consider the following three cases:

Case 1. $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^t}$.

In this case, we have $s = t$. By Claim 2, we have $o(x) = o(y) = o(x^{-1}y) = 3^{t+1}$ and $x^{3^t} \neq y^{3^t}$. As $H' \cong \mathbb{Z}_3$, we have $H' = \langle x^{3^t} \rangle = \langle y^{3^t} \rangle$, implying that $y^{3^t} = x^{-3^t}$. Thus $(xy)^{3^t} = x^{3^t} y^{3^t} = x^{3^t} x^{-3^t} = 1$. Since $[x, y] \in H'$ and $H' = \langle x^{3^t} \rangle$, we have $[x, y] = x^{3^t}$ or x^{-3^t}. It follows that $(xy)^{-1} \cdot x \cdot (xy) = y^{-1}xy = x^{1+3^t}$ or x^{1-3^t}.

If $(xy)^{-1} \cdot x \cdot (xy) = y^{-1}xy = x^{1+3^t}$, then

$$H = \langle x, xy \mid x^{3^{t+1}} = (xy)^{3^t} = 1, (xy)^{-1} \cdot x \cdot (xy) = x^{1+3^t} \rangle,$$

and $S = \{1, x, y\} = \{1, x, x^{-1}(xy)\}$. So, Γ is isomorphic to Γ_t (see Construction 1).

If $(xy)^{-1} \cdot x \cdot (xy) = y^{-1}xy = x^{1-3^t}$, then

$$H = \langle x, (xy)^{-1} \mid x^{3^{t+1}} = [(xy)^{-1}]^{3^t} = 1, (xy) \cdot x \cdot (xy)^{-1} = x^{1+3^t} \rangle,$$

and $S = \{1, x, y\} = \{1, x, x^{-1}[(xy)^{-1}]^{-1}\}$. By Proposition 9 (4), we have

$$\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S) \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^{-1}).$$

Note that $S^{-1} = \{1, x^{-1}, y^{-1}\} = \{1, x^{-1}, (xy)^{-1}x\}$. It is easy to check that the map

$$f : x \mapsto x^{-1}, (xy)^{-1} \mapsto (xy)^{-1}x^{3^t}$$

induces an automorphism of H such that $\{1, x, x^{-1}(xy)^{-1}\}^f = S^{-1}$. By Proposition 9 (3), we have

$$\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^{-1}) \cong \mathrm{BiCay}(H, \emptyset, \emptyset, \{1, x, x^{-1}(xy)^{-1}\}) \cong \Gamma_t,$$

as required.

Case 2. $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^{t-1}}$.

In this case, we have $s = t - 1$. Let $T = \langle R(h) \mid h \in H, h^{3^{t-1}} = 1 \rangle$. Then $T = \langle R(a)^9 \rangle \times \langle R(b) \rangle$ and T is characteristic in $R(H)$, and so normal in A for $R(H) \trianglelefteq A$. Furthermore, $R(H)/T \cong \mathbb{Z}_9$. By Propositions 7 and 8, the quotient graph Γ_T of Γ relative to T is a cubic edge-transitive graph of order 18. Clearly, $R(H)/T$ is semiregular on $V(\Gamma_T)$ with two orbits, so Γ_T is a bi-Cayley graph over the cyclic group $R(H)/T$ of order 9. Since $R(H)/T \trianglelefteq A/T$, by Proposition 11, there exists $\lambda \in \mathbb{Z}_{3^2}^*$ such that $\lambda^2 - \lambda + 1 \equiv 0 \pmod{3^2}$, which is impossible.

Case 3. $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{3^t} \times \mathbb{Z}_{3^{t+1}}$.

In this case, we have $s = t + 1$. Let $N = \langle h \mid h \in H, h^3 = 1 \rangle$. Then $N = \langle a^{3^t}, b^{3^t} \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. By Claim 2, we have $o(x) = o(y) = 3^{t+1}$. Since $H = \langle x, y \rangle$, one has $N = \langle x^{3^t}, y^{3^t} \rangle$. As $H' \cong \mathbb{Z}_3$, one has $H' \leqslant N$. So, $H' = \langle x^{3^t} \rangle$, $\langle y^{3^t} \rangle$, $\langle (xy)^{3^t} \rangle$ or $\langle (xy^{-1})^{3^t} \rangle$.

Recall that H has an automorphism α taking (x, y) to $(x^{-1}y, x^{-1})$. Suppose that one of the three subgroups: $\langle x \rangle, \langle y \rangle, \langle x^{-1}y \rangle$ is normal in H. Then all of them are normal in H. So $H = \langle x, y \rangle = \langle x \rangle \times \langle y \rangle$ because $|H| = 3^{2(t+1)}$. This is impossible because H is non-abelian. Thus, all of the three subgroups: $\langle x \rangle, \langle y \rangle, \langle x^{-1}y \rangle$ are not normal in H.

It then follows that $H' = \langle (xy)^{3^t} \rangle$. Then either $x^{-1}(xy)x = (xy)^{1+3^t}$ or $x^{-1}(xy)x = (xy)^{1-3^t}$. For the former, we have

$$H = \langle xy, x \mid (xy)^{3^{t+1}} = x^{3^{t+1}} = 1, x^{-1}(xy)x = (xy)^{3^t+1} \rangle,$$

and $S = \{1, x, y\} = \{1, x, x^{-1}(xy)\}$. Hence, $\Gamma \cong \Sigma_t$ (see Construction 2).

For the latter, we have

$$H = \langle xy, x^{-1} \mid (xy)^{3^{t+1}} = x^{-3^{t+1}} = 1, x(xy)x^{-1} = (xy)^{3^t+1} \rangle,$$

and $S = \{1, x, y\} = \{1, (x^{-1})^{-1}, x^{-1}(xy)\}$. By Proposition 9 (4), we have

$$\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S) \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^{-1}).$$

Note that $S^{-1} = \{1, x^{-1}, y^{-1}\} = \{1, x^{-1}, (xy)^{-1}x\}$. It is easy to check that the map

$$f' : x^{-1} \mapsto x^{-1}, xy \mapsto (xy)^{3^t-1}$$

induces an automorphism of H such that $\{1, x^{-1}, x(xy)\}^{f'} = S^{-1}$. By Proposition 9 (3), we have

$$\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^{-1}) \cong \mathrm{BiCay}(H, \emptyset, \emptyset, \{1, x^{-1}, x(xy)\}) \cong \Sigma_t,$$

as required. $\square$

6 Proof of Corollary 3

In this section we complete the proof of Corollary 3.

Proof of Corollary 3. Let p be a prime, and let Γ be a connected cubic edge-transitive graph of order $2p^3$. By [11], the smallest semisymmetric graph has 20 vertices. So, if $p = 2$, then Γ is vertex-transitive. If $p = 3$, then by [6, 7], we know that Γ is not vertex-transitive if and only if it isomorphic to the Gray graph.

Now assume that $p > 3$. By Lemma 17, Γ is a bi-Cayley graph over a group H of order p^3. Suppose that Γ is not vertex-transitive. Then Γ is bipartite with the two orbits of H as its two parts. So we may let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$. By Proposition 9, we may assume that $S = \{1, a, b\}$ form $a, b \in H$. If H is abelian, then H has an automorphism α which maps every element of H to its inverse. By Proposition 10, $\delta_{\alpha,1,1}$ is an automorphism of Γ swapping the two parts of Γ, and so Γ is vertex-transitive, a contradiction. If H is non-abelian, then H is either metacyclic or isomorphic to the following group:

$$J = \langle a, b, c \mid a^p = b^p = c^p = 1, c = [a, b], [a, c] = [b, c] = 1\rangle.$$

By Theorem 1, H is non-metacyclic. If $H \cong J$, then it is easy to see that J has an automorphism β taking (a, b) to (a^{-1}, b^{-1}). Again, by Proposition 10, $\delta_{\beta,1,1}$ is an automorphism of Γ swapping the two parts of Γ, and so Γ is vertex-transitive, a contradiction. This completes the proof of Corollary 3. □

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