Linear Bound for Majority Colourings of Digraphs

Fiachra Knox*
Department of Mathematics
Simon Fraser University
Burnaby, B.C., Canada
fknox@sfu.ca

Robert Šámal†
Computer Science Institute
Charles University
Prague, Czech Republic
samal@iuuk.mff.cuni.cz

Submitted: Jan 21, 2017; Accepted: Jul 2, 2018; Published: Aug 24, 2018

Abstract

Given $\eta \in [0,1]$, a colouring C of $V(G)$ is an *η-majority colouring* if at most $\eta d^+(v)$ out-neighbours of v have colour $C(v)$, for any $v \in V(G)$. We show that every digraph G equipped with an assignment of lists L, each of size at least k, has a $2/k$-majority L-colouring. For even k this is best possible, while for odd k the constant $2/k$ cannot be replaced by any number less than $2/(k+1)$. This generalizes a result of Anholcer, Bosek and Grytczuk, who proved the cases $k = 3$ and $k = 4$ and claim a weaker result for general k.

Mathematics Subject Classifications: 05C20,05C15

1 Introduction

Given a digraph G, we write $V(G)$ and $E(G)$ for the vertex and edge set of a digraph G, respectively. For $v \in V(G)$, we denote by $d^+(v)$ the out-degree of v. Given $\eta \in [0,1]$, a (not necessarily proper) colouring C of $V(G)$ is an *η-majority colouring* if at most $\eta d^+(v)$ out-neighbours of v have colour $C(v)$, for any $v \in V(G)$. A 1/2-majority colouring is referred to simply as a *majority colouring*. This concept was introduced in connection to neural networks by van der Zypen [5], who asked whether every digraph has a majority colouring with a bounded number of colours. This question was answered by Kreutzer, Oum, Seymour, van der Zypen and Wood [4], who showed that 4 colours always suffice and ask, whether 3 colours do.

We consider the list-colouring version of this problem. For a set S, we denote by $\mathcal{P}(S)$ the power set of S. Given a digraph G and an assignment $L : V(G) \to \mathcal{P}(\mathbb{N})$ of lists to vertices of G, an *L-colouring* $C : V(G) \to \mathbb{N}$ of G is a colouring of $V(G)$

*Research of the first author was supported by a PIMS postdoctoral fellowship.
†The second author was supported by grant GA ČR 16-19910S.

such that $C(v) \in L(v)$ for every $v \in V(G)$. If G has an η-majority L-colouring for any such assignment L whose lists are all of size at least k, we say that G is *η-majority k-choosable*. Anholcer, Bosek and Grytczuk [1] showed that every digraph G is $1/2$-majority 4-choosable. As noted by David Wood (personal communication), their method can be extended to show that every digraph is $1/k$-majority k^2-choosable for every $k \geqslant 2$. Our Theorem 1 improves on this result.

Theorem 1. *For any integer $k \geqslant 2$, every digraph G is $2/k$-majority k-choosable.*

Theorem 1 was proved independently by Girão, Kittipassorn and Popielarz [2]. The case $k = 2$ is trivial. Previously, Anholcer, Bosek and Grytczuk [1] showed that Theorem 1 holds in the cases $k = 3$ and $k = 4$ and conjectured that $2/k$ can be replaced by $1/2$ when $k = 3$. Theorem 1 is best possible when k is even, as shown by the example of a $k/2$-regular tournament on $k+1$ vertices (that is, all vertices have both in-degree and out-degree equal to $k/2$). If we make all lists equal, then some vertex must have an out-neighbour of the same colour, and this out-neighbour represents $2/k$ of its out-neighbourhood. When k is odd, a similar example shows that we cannot replace $2/k$ by any number less than $2/(k+1)$.

2 Proof of Theorem 1

We denote by vw an edge from a vertex v of a digraph to another vertex w. The proof of Theorem 1 relies on the following lemma.

Lemma 2. *Let $k \geqslant 2$ be an integer and let G be a digraph on a vertex set $V(G) = S \cup T$, such that $G[S]$ is strongly connected, $G[T]$ is edgeless and there are no edges from T to S. Let C_T be any colouring of T and let $L : S \to \mathcal{P}(\mathbb{N})$ be an assignment of lists, each of size at least k, to vertices in S. Then there exists an extension C of C_T to $V(G)$ with $C(v) \in L(v)$ for each $v \in S$, such that no vertex $v \in S$ has more than $2d^+(v)/k$ out-neighbours with the same colour as v.*

Proof. For any colouring C of $V(G)$, we define the function $f_C : S \to \mathbb{R}$ by

$$f_C(v) = \frac{|\{w \in N^+(v) \mid C(w) = C(v)\}|}{d^+(v)}$$

for each vertex $v \in S$; i.e., $f_C(v)$ is the proportion of out-neighbours of v which have the same colour as v under C. Given $v \in S$, we write $d_S^+(v) = |N^+(v) \cap S|$.

Let A be the non-negative real $S \times S$ matrix with entries $A_{vw} = 1/d_S^+(v)$ if vw is an edge of G and $A_{vw} = 0$ otherwise. We have $A\mathbf{j} = \mathbf{j}$ (where $\mathbf{j}$ is the vector of all 1's). On the other hand if $A\mathbf{y} = c\mathbf{y}$ for any vector $\mathbf{y}$, then choose $v \in S$ such that $|y_v|$ is maximal; now $|cy_v| = |\sum_{w \in S} A_{vw} y_w| \leqslant \sum_{w \in S} A_{vw}|y_v| = |y_v|$ and so $|c| \leqslant 1$. Thus, the spectral radius of A is 1.

By applying the Perron–Frobenius Theorem (see, e.g., [3, Theorem 8.8.1]) to $A^\top$, noting that $G[S]$ is strongly connected, we obtain an eigenvector $\mathbf{x}$ of $A^\top$ with positive entries and eigenvalue 1. We remark that by normalizing $\mathbf{x}$ we could obtain a stationary distribution of the uniform random walk on $G[S]$.

Consider an extension C of C_T with $C(v) \in L(v)$ for each $v \in S$ such that $\sum_{v \in S} x_v f_C(v)$ is minimized. We claim that C satisfies the requirements of the lemma. For brevity we write f for f_C. It suffices to show that $f(v) \leqslant 2/k$ for every $v \in S$. Observe that

$$\sum_{v \in S} x_v f(v) = \sum_{\substack{vw \in E(G) \\ C(v)=C(w)}} \frac{x_v}{d^+(v)}. \tag{1}$$

Fix a vertex $v \in S$. We define $g : L(v) \to \mathbb{R}$ by

$$g(i) = \sum_{\substack{w \in N^+(v) \\ C(w)=i}} \frac{x_v}{d^+(v)} + \sum_{\substack{u \in N^-(v) \\ C(u)=i}} \frac{x_u}{d^+(u)}$$

for $i \in L(v)$. Observe that if v were recoloured with colour i, then (1) would change by $g(i) - g(C(V(G)))$. By the minimality of C and the definition of g we have that $g(i) \geqslant g(C(v)) \geqslant x_v f(v)$. Since $A^\top \mathbf{x} = \mathbf{x}$,

$$x_v = \sum_{u \in N^-(v)} \frac{x_u}{d_S^+(u)} \geqslant \sum_{u \in N^-(v)} \frac{x_u}{d^+(u)}$$

and hence

$$2x_v \geqslant \sum_{i \in L(v)} g(i) \geqslant k x_v f(v).$$

Since $x_v > 0$, we have $f(v) \leqslant 2/k$. It follows immediately that C satisfies the requirements of the lemma. □

Proof of Theorem 1. Let $L : V(G) \to \mathcal{P}(\mathbb{N})$ be an assignment of lists, each of size at least k, to vertices of G. We partition $V(G)$ into strongly connected components $S_1, S_2, \ldots, S_r$, where there are no edges from S_i to S_j for any $i < j$. We write A_i for $\bigcup_{j \in [i]} S_j$ (taking $A_0 = \emptyset$); let C_0 be the unique colouring of A_0. For each $i = 0, 1, 2, \ldots, r-1$ in turn, we apply Lemma 2 to the digraph obtained from $G[A_{i+1}]$ by deleting the arcs in $G[A_i]$, with $S = S_{i+1}$ and $T = A_i$. This gives us an extension of C_i to an L-colouring C_{i+1} of A_{i+1} such that no $v \in S_{i+1}$ (and hence no $v \in A_{i+1}$) has more than $2d^+(v)/k$ out-neighbours of the same colour. At the end of this process we obtain C_r, which is the desired $2/k$-majority L-colouring of $V(G)$. □

3 Future Work

The main related open problem is the question, whether every digraph has a majority 3-coloring. We refer the reader to [4] for further questions and related results. We note that Anholcer, Bosek and Grytczuk [1] prove their result for $k = 4$ in a more general setting (with weights on the colors). As mentioned earlier, this approach can be extended to prove the existence of $1/k$-majority k^2-choosability of every digraph. We don't know whether $O(k)$-choosability is true in their more general setting.

Acknowledgements

The authors would like to thank David Wood for presenting this problem at the workshop "New Trends in Graph Colouring" in Banff, October 2016, and for many helpful comments. We are also grateful to the organizers of this wonderful workshop and to our colleagues at the workshop for helpful discussions.

References

[1] Marcin Anholcer, Bartłomiej Bosek, and Jarosław Grytczuk. Majority choosability of digraphs. *Electron. J. Combin.*, 24(3): #P3.57, 2017.

[2] António Girão, Teeradej Kittipassorn, and Kamil Popielarz. Generalized majority colourings of digraphs. *Combin. Probab. Comput.*, 26(6):850–855, 2017.

[3] Chris Godsil and Gordon F. Royle. *Algebraic Graph Theory*. Graduate Texts in Mathematics. Springer New York, 2013.

[4] Stephan Kreutzer, Sang-il Oum, Paul Seymour, Dominic van der Zypen, and David R. Wood. Majority colourings of digraphs. *Electron. J. Combin.*, 24(2): #P2.25, 2017.

[5] Dominic van der Zypen. Majority colouring for directed graphs, 2016. https://mathoverflow.net/questions/233014.