Strong chromatic index of graphs with maximum degree four

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Abstract

A strong edge-coloring of a graph G is a coloring of the edges such that every color class induces a matching in G. The strong chromatic index of a graph is the minimum number of colors needed in a strong edge-coloring of the graph. In 1985, Erdős and Nešetřil conjectured that every graph with maximum degree Δ has a strong edge-coloring using at most $\frac{5}{4}\Delta^2$ colors if Δ is even, and at most $\frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$ if Δ is odd. Despite recent progress for large Δ by using an iterative probabilistic argument, the only nontrivial case of the conjecture that has been verified is when $\Delta = 3$, leaving the need for new approaches to verify the conjecture for any $\Delta \geqslant 4$. In this paper, we apply some ideas used in previous results to an upper bound of 21 for graphs with maximum degree 4, which improves a previous bound due to Cranston in 2006 and moves closer to the conjectured upper bound of 20.

Mathematics Subject Classifications: 05C15

1 Introduction

All graphs considered in this paper are finite, loopless, undirected, and may have multiple edges. For a graph G, we use $V(G)$ and $E(G)$ to denote the set of vertices and edges of

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G, respectively, and we use $\Delta(G)$ to denote the maximum degree of G. First introduced by Fouquet and Jolivet [11], a *strong edge-coloring* of a graph G is an assignment of colors to the edges of G such that if edges e_1 and e_2 receive the same color, they cannot be incident with one another nor can they be incident with a common edge. Thus, every color class in a strong edge-coloring induces a matching in G. The *strong chromatic index* of a graph G, denoted by $\chi'_s(G)$, is the minimum number of colors necessary for a strong edge-coloring of G. Observe that the strong chromatic index of G is equivalent to the chromatic number of $L^2(G)$, which is the square of the line graph of G.

Via the greedy algorithm, we see that $\chi'_s(G) \leqslant 2\Delta^2 - 2\Delta + 1$ for every graph G with maximum degree Δ. In 1985, Erdős and Nešetřil [9] conjectured the following upper bounds:

Conjecture 1. (Erdős and Nešetřil [9]) For every graph G with maximum degree Δ,

$$\chi'_s(G) \leqslant \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

Erdős and Nešetřil showed further that this conjecture, if true, is best possible by constructing a particular blow-up of C_5. It is worth noting that if a graph G is $2K_2$-free, then $\chi'_s(G) = |E(G)|$. In 1990, Chung, Gyárfás, Trotter, and Tuza [7] showed that the maximum number of edges in a $2K_2$-free graph with maximum degree Δ is $\frac{5}{4}\Delta^2$ for even Δ, and $\frac{5}{4}\Delta^2 - \frac{1}{2} + \frac{1}{4}$ for odd Δ; furthermore, the aforementioned blow-up of C_5 is the unique graph that attains this maximum.

While Conjecture 1 has been the impetus for many other conjectures and results in the area of strong edge-colorings (see [3, 6, 10, 13, 14, 17, 18, 20, 21, 22] for only a few), not much progress has been made in regards to proving this conjecture directly. The first nontrivial case of Conjecture 1 (i.e., for graphs with maximum degree at most three) was verified by Andersen [1] and independently by Horák, Qing, and Trotter [16]. For graphs with maximum degree at most four, Horák [15] first proved an upper bound of 23 in 1990. This was later improved by Cranston [8] in 2006, who showed that 22 colors suffice, which is 2 away from the conjectured bound 20.

For graphs with large enough Δ, exciting progress has been made. In 1997, Molloy and Reed [19] showed that such a graph G has $\chi'_s(G) \leqslant 1.998\Delta^2$. In 2015, Bruhn and Joos [4] improved this bound to $1.93\Delta^2$. Very recently, Bonamy, Perrett, and Postle [5] improved it to $1.835\Delta^2$. All of these proofs considered the coloring of $L^2(G)$, in which each vertex has a sparse neighborhood (with at most $0.75\binom{2\Delta^2}{2}$ edges), and then used an iterative coloring procedure. However, as pointed out in [19], this method is not sufficient to prove the conjecture. Therefore, it is necessary to explore new approaches and ideas to attack the conjecture.

We turn to the first unsolved case, $\Delta = 4$. We develop some ideas hidden in [1] by Andersen and prove the following.

Theorem 2. *For every graph G with maximum degree four, $\chi'_s(G) \leqslant 21$.*

According to a result by van Batenburg and Kang [2], Theorem 2 implies that for claw-free graphs with clique number at most four, their squared chromatic numbers are at most 21.

The idea of the proof is as follows. For a minimum counterexample G, we construct a partition $V(G) = L \cup M \cup R$ such that:

(1) For any $u \in L$ and $v \in R$, the distance between u and v is at least two, and

(2) the vertices in M are all within distance two from a fixed vertex.

By (1), we can color the edges in $G[L]$ and $G[R]$ independently, but also 'collaboratively', and by (2), a coloring on $G[L]$ and $G[R]$ can be extended to the whole graph, because the edges incident with M have clear structures. We hope this idea can stimulate new ideas to attack Conjecture 1.

The paper is organized as follows. In Section 2 we introduce some notation and prove various strutural statements about a minimal counterexample G. In particular, we show that the girth of G is at least six, whose proof is in Section 5. In Section 3, we obtain the partition described above. In Section 4, we show how to color the edges in $G[L]$ and $G[R]$ 'collaboratively', and extend it to a coloring of the whole graph; this completes the proof of Theorem 2.

2 Notation and some properties of minimal counterexamples

We will use the following notation. For two disjoint subsets of $V(G)$, call them X and Y, we let $E(X, Y)$ denote the set of edges of G with one end in X and the other end in Y. For an edge $e = uv$, we let $N_1(e)$ be the set of edges incident with u or v in $G - e$, and we let $N_2(e)$ be the set of edges not in $N_1(e)$ that have an endpoint adjacent to either u or v in $G - e$. We denote the set of edges of $N_1(e) \cup N_2(e)$ by $N(e)$, so that $N(e)$ contains at most 24 edges in a graph with maximum degree at most four. Furthermore, if $e' \in N(e)$, we will say that e *sees* e' and vice-versa.

A *partial strong edge-coloring* (or we will sometimes say a *good partial coloring*) of G is a coloring of any subset of $E(G)$ such that if any two colored edges e_1 and e_2 see one another in G, then e_1 and e_2 receive different colors. In particular, if a partial strong edge-coloring spans all of $E(G)$, then it is a strong edge-coloring of G. Given a partial strong edge-coloring of G, call it ϕ, we define $A_\phi(e)$ to be the set of colors available for edge e.

In the rest of this paper, we assume that G is a minimal counterexample with $|V(G)| + |E(G)|$ minimized. Here are some structural lemmas regarding G.

Lemma 3. *G is 4-regular.*

Proof. Suppose on the contrary that v is a vertex of degree at most three with $N(v) \subseteq \{u_1, u_2, u_3\}$. By the minimality of G, $G - v$ has a good coloring. Observe that $|A(u_i v)| \geqslant 3$ for $i \in [3]$. Thus, we can color the remaining edges in any order to obtain a good coloring of G. This is a contradiction. □

Lemma 4. *G contains no edge cut with at most 3 edges.*

Proof. Suppose otherwise that G contains a smallest edge cut with at most $t \leqslant 3$ edges, say $e_1 = a_1b_1, \ldots, e_t = a_tb_t$. By the minimality of G, G is connected. So $G - \{e_1, \ldots, e_t\}$ contains two components, say G_1 and G_2, so that $a_1, \ldots, a_t \in G_1$ and $b_1, \ldots, b_t \in G_2$. Note that a_t's and b_t's may be not distinct. Let G_1' be the graph obtained from G_1 by adding vertex z_1 and edges $z_1a_1, \ldots, z_1a_t$. Similarly, let G_2' be the graph obtained from G_2 by adding vertex z_2 and edges $z_2b_1, \ldots, z_2b_t$. By the minimality of G, both G_1' and G_2' can be colored with 21 colors.

By renaming the colors, we may assume that z_1a_s and z_2b_s have the color s for each $1 \leqslant s \leqslant t \leqslant 3$. Again by renaming colors, we may assume that the colors appearing on edges incident with $a_1, a_2, \ldots, a_t, b_1, \ldots, b_t$ are all different, which is possible, since there are at most 18 such edges but there are $21 - t \geqslant 18$ colors other than $1, \ldots, t$. Now, we can obtain a coloring of G by combining the colorings of G_1' and G_2': keep the colors of the edges in G_1 and G_2, and color $e_1, \ldots, e_t$ with $1, \ldots, t$, respectively. This is a contradiction. □

The girth of a graph G is the length of its shortest cycle.

Lemma 5. *The graph G has girth at least six.*

Since the proof of this lemma is long, we devote Section 5 to it. The reader may skip the proof for now.

By Lemma 5, we may assume that G is a simple graph.

3 A partition of the vertices

Let x be any vertex of G. In this section, we consider a coloring strategy that leads to a partition of $V(G)$ into sets L, M, and R, such that there are no edges between L and R, the numbers of the edges in $E(L, M)$ and $E(M, R)$ are relatively small, and M only contains some vertices within distance 2 from x. By Lemma 3, G is 4-regular. So we let $N(x) = \{u, v, w, y\}$ and for $z \in N(x)$, $N(z) = \{z_1, z_2, z_3, x\}$. By Lemma 5, above all these vertices are distinct. Furthermore, we let $N(z_i) = \{z_{i1}, z_{i2}, z_{i3}, z\}$ for $z \in N(x)$ (see Figure 1). Note that for $i, j, k, \ell \in \{1, 2, 3\}$ and $a, b \in \{u, v, w, y\}$, $a_{ij}, b_{k\ell}$ may be identical when $a \neq b$.

We now give a partial strong edge-coloring of G, call it ψ, using three colors: assign the edges uu_1, vv_1, ww_1 with the color 1, assign the edges uu_2, vv_2 with the color 2, and assign the edges uu_3, vv_3 with the color 3.

Consider the sequence S_0 of edges: w_2w_{21}, w_3w_{31}, ww_2, ww_3, xu, xv, xy, xw. We extend S_0 to a sequence S of uncolored edges such that the following hold:

(i) S contains S_0, where S_0 is at the end of S;

(ii) for each edge e of $S - S_0$, at least 4 edges of $N(e)$ fall behind it in S;

(iii) among all sequences satisfying (i) and (ii), S is longest.

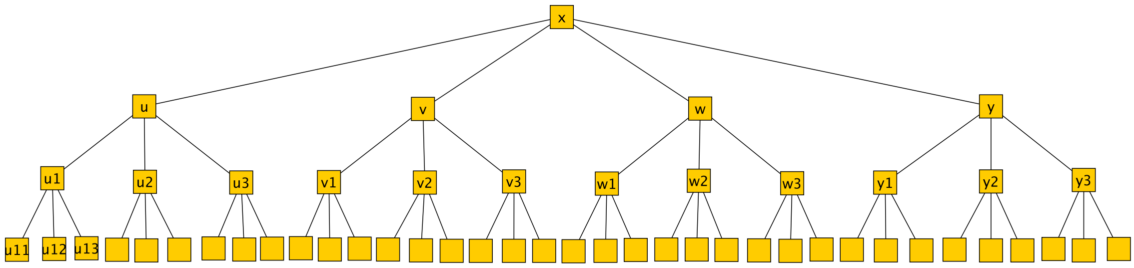

Figure 1: 4-regular graph

Observe that no edge outside of S can see four edges in S, otherwise it could be added to the start of S and contradict (iii).

Lemma 6. *With 21 colors, we may extend ψ to a partial strong edge-coloring of G that inlcudes all edges of S.*

Proof. Using 21 colors, greedily color the edges of S in order, and let e be the first edge of S that cannot be colored. Let ϕ denote this partial strong edge-coloring of G. Observe that e must be in $\{w_2w_{21}, w_3w_{31}, xu, xv, xy, xw\}$, as otherwise $|A_\phi(e)| \geqslant 21 - (|N(e)| - 4) = 1$, so that e can be colored. Further, by the repetition of colors on the pre-colored edges, $e \notin \{xu, xv, xy, xw\}$. Thus, it suffices to consider $e \in \{w_2w_{21}, w_3w_{31}\}$.

Without loss of generality, assume that $e = w_2w_{21}$. Since e cannot be colored, it follows that the 21 colored edges in $N(w_2w_{21})$ must be assigned 21 different colors. Thus, we can remove the color 1 from ww_1 and assign it to w_2w_{21}. Observe that in this new partial strong edge-coloring, w_3w_{31} sees at least 4 uncolored edges, and ww_1 sees at least 6 uncolored edges. Hence, we can color w_3w_{31} and recolor ww_1. Since xw sees w_2w_{21} colored with 1, there is a color available for the remainder of S by the repetition of colors on the pre-colored edges. □

By Lemma 6, if S contains all uncolored edges of G under ψ, then we are done. So we assume that S does not contain all uncolored edges of G. Let H be the set of uncolored edges not in S, and let L be the set of endpoints of the edges in H. Then $L \neq \emptyset$. By the maximality of S, w_2w_{22} appears in S since w_2w_{21}, ww_2, ww_3 and xw are in S_0. Similarly, $w_2w_{23}, w_3w_{32}, w_3w_{33}, yy_1, yy_2, yy_3$ appear in some order in S. So, all edges incident with x, u, v, w, y, w_2, w_3 are either pre-colored or in S. By the definition of L, $x, u, v, w, y, w_2, w_3 \notin L$.

Lemma 7. $E(G[L]) = H$.

Proof. Suppose otherwise that there exists an edge $e \in E(G[L])$ with endpoints a and b such that $a, b \in L$ but $e \notin H$. Let $N(a) = \{a_1, a_2, a_3, b\}$ and $N(b) = \{b_1, b_2, b_3, a\}$ where $aa_1, bb_1 \in H$. Since $x, u, v, w, y, w_2, w_3 \notin L$, every pre-colored edge and every edge of S_0 cannot join two vertices of L. So, $e \in S - S_0$. By the definition of S, at least 4 edges, say e_1, e_2, e_3 and e_4, of $N(e)$ are in S. If say e_1 belongs to $N_1(e)$, then either aa_1 or bb_1 sees

e, e_1, and two edges from $\{e_2, e_3, e_4\}$. That is, either aa_1 or bb_1 can be added to S, which contradicts the maximality of S.

Therefore, $e_1, e_2, e_3, e_4 \in N_2(e)$. Furthermore, we claim that exactly two of these edges are incident with vertices in $\{a_1, a_2, a_3\}$, otherwise either aa_1 or bb_1 sees three of these edges along with e, and so is in S. Without loss of generality, assume that e_1, e_2 are incident with vertices in $\{a_1, a_2, a_3\}$. Let's further assume that e_1 is behind e_2 in the sequence S, and let $e_1 = a_i a_{i1}$ for some $i \in [3]$. Observe that $aa_2, aa_3 \notin S$, as otherwise aa_1 would see four edges in S, and so be in S.

We now show that e_1 is not in S_0, as otherwise one of the endpoints of e_1 is incident with four edges in S. Thus, aa_i would see each of these four edges and so be in S, which is a contradiction.

By the definition of S, at least three edges of $N(e_1)$ different from e are behind e_1 in S. We next assume that there is at least one edge of these edges incident with a_{i1}, a_{i2}, a_{i3}. Since e_1, e_2 and e are in S, $aa_i \in S$, a contradiction. So, all these three edges are incident with $N(a_{i1})\backslash\{a_i\}$. However, all four edges incident with a_{i1} would see these three edges together with e_1, so that four edges incident with a_{i1} are in S. Thus, $aa_i \in S$, again a contradiction. □

Let $F = E(L, G - L)$ and $A = \{u_1, u_2, u_3, v_1, v_2, v_3, w_1\}$. We present the relationship between edges of F and vertices of A as follows.

Lemma 8. *Each edge of F is incident with exactly one vertex of A, and each vertex in A is incident with at most two edges of F. Moreover, no vertex in L is incident with two edges of F.*

Proof. First note that if e is an edge in F with endpoints $z \in L$ and $z' \in V(G - L)$, then z' must be incident with a pre-colored edge by ψ. If not, then every edge incident with z' is in S, and consequently, every edge incident with z is in S, by the maximality of S. Yet this contradicts $z \in L$.

Now suppose e is an edge of F. Then e is incident with at most one vertex of A. Otherwise, the girth of G is at most 5, contrary to Lemma 5. Now we show that e is incident with at least one vertex of A. As shown above, one of the endpoints of e must be incident with a pre-colored edge. We are done unless $e \in \{xu, xv, xw\}$. Yet $x, u, v, w \in V(G - L)$, which contradict that e is an edge of F. Therefore, each edge of F is incident with exactly one vertex of A.

Next we show that each vertex in A is incident with at most two edges of F. Suppose otherwise that a vertex $a \in A$ is incident with three edges of F. Assume that $a \in L$. Since $u, v, w \in V(G-L)$, one edge of these three edges is pre-colored and other two edges are uncolored. Let aa' be such an uncolored edge where $a' \in V(G - L)$. By Lemma 5, a' is not incident with a pre-colored edge. Thus, every edge incident with a' is in S. Yet this would imply that every uncolored edge incident with a is also in S, contrary to the assumption that $a \in L$.

So we assume that $a \in V(G - L)$. Since $u, v, w \in V(G - L)$, three edges, say $e_1 = aa_1, e_2 = aa_2$ and $e_3 = aa_3$ where $a_1, a_2, a_3 \in L$, are the three uncolored edges of F

incident with a. Since $a \in V(G-L)$, $e_1, e_2, e_3 \in S$, and further, $e_1, e_2, e_3 \notin S_0$. We assume, without loss of generality, that both e_1 and e_2 preceed e_3 in the sequence S. By the definition of S, at least 4 edges of $N(e_3)$ come after e_3 in S. However, one of these four edges together with e_1, e_2, e_3, are seen by all four edges incident with either a_1, a_2, or a_3. Thus, at least one of a_1, a_2, a_3 is incident with four edges in S, which contradicts $a_1, a_2, a_3 \in L$.

We finally show that no vertex in L is incident with two edges of F. Suppose otherwise that the vertex $z \in L$ is incident with two edges of F, and let z' and z'' be the other endpoints of these edges. As shown at the start of this proof, z' and z'' are incident with pre-colored edges. As a consequence, $z \notin A$, and further $z \neq x$. Thus, $z', z'' \in A \cap V(G-L)$. So z' and z'' are surrounded by 3 edges in S, respectively. Yet every edge incident with z sees these edges in S, and so $z \notin L$, a contradiction. □

Let V_F, V_F' be the endpoints of F in $G-L$ and in L, respectively. So $F = E(G-L, L) = E(V_F, V_F')$. Let $M = \{x, u, v, w\} \cup V_F$ and $R = V(G) - L - M$. (See Figure 2 for an example.) Observe that $E(L, R) = \emptyset, E(L, M) = F$ and $y, w_2, w_3 \in R$. Furthermore, if $z \in V_F$, then z is incident with a pre-colored edge under ψ; for otherwise, z is incident with four edges in S and its neighbor in L would then be incident with four edges in S, a contradiction. Thus, $V_F \subseteq A \cup \{x, u, v, w\}$.

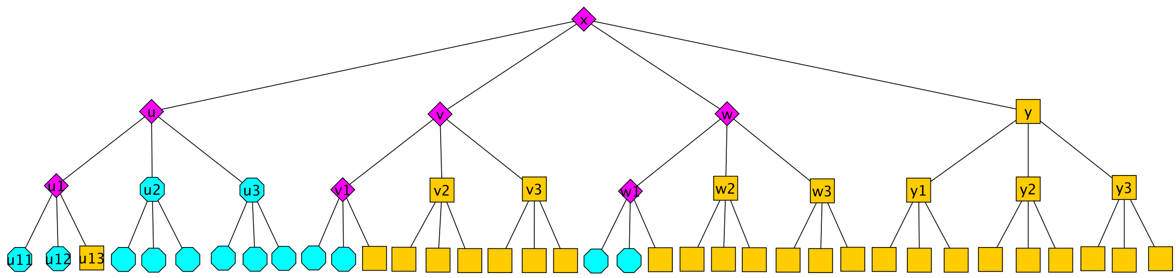

Figure 2: A possible partition of the vertices with $M = \{x, u, v, w, u_1, v_1, w_1\}$ (diamond vertice), R (square vertices) and L (octagon vertices), where $F = \{uu_2, uu_3, u_1u_{11}, u_1u_{12}, v_1v_{11}, v_1v_{12}, w_1w_{11}, w_1w_{12}\}$ and $V_F = \{u, u_1, v_1, w_1\}$.

An important observation is that no edges from $G[L]$ and $G[R]$ see each other, so they can be colored independently and be combined together without the need of changing their colors. Now we state some straightforward results as follows.

Lemma 9. *For $z \in \{u, v, w\}$ and $i, j, k \in [3]$, each of the following holds.*

(1) If $z_i \in M$, then for some $k \neq j$, $z_iz_{ij} \in F$ and $z_iz_{ik} \in E(M, R)$.

(2) If $z_iz_{ij} \in F$, then $z_i \in M, z_{ij} \in L$ and three edges incident with z_{ij} are in $G[L]$.

(3) If $z_i \in L$, then $z_iz_{ij} \in E(G[L])$.

(4) If $z_i \in R$, then $z_iz_{ij} \in E(G[R])$. Further, $yy_j \in E(G[R])$.

(5) If $z_iz_{ij} \in E(M,R)$, then $z_i \in M, z_{ij} \in R$ and at least one edge incident with z_{ij} is in $G[R]$.

(6) If $z_i \neq w_1$ and $z_iz_{ij}, z_iz_{ik} \in E(M,R)$, then at least three of the eight edges incident with z_{ij} and z_{ik} are in $G[R]$.

Proof. Observe that if $z_i \in M$, then $z_i \in V_F$ and consequently, $z_i \in A$ by Lemmas 5 and 8.

Lemma 8 implies (1) as every vertex in A is incident with at most two edges of F.

If $z_iz_{ij} \in F$, then $z_i \notin L$, else z_i would be incident with two edges of F, namely z_iz_{ij} and z_iz, contradicting Lemma 8. Therefore, $z_i \in M$ and $z_{ij} \in L$. Further, every edge incident with z_{ij} other than z_iz_{ij} must be in $E(G[L])$. This proves (2).

If $z_i \in L$, then $z_i \in A$ since $w_2, w_3 \in V(G-L)$. Thus, $zz_i \in F$, and every other edge incident with z_i must be in $E(G[L])$ by Lemma 8. This proves (3)

If $z_i \in R$ and $z_{ij} \notin R$, then $z_{ij} \in M$. In particular, $z_{ij} \in V_F$ so that z_{ij} is incident with a pre-colored edge under ψ. Yet this contradicts Lemma 5. Thus, $z_iz_{ij} \in E(G[R])$. Further, notice that $y \in R$. If $y_j \notin R$, then $y_j \in M$. So $y_j \in V_F$. By Lemma 8, y_j is incident with a vertex of A. This contradicts Lemma 5. Thus, $yy_j \in E(G[R])$. This proves (4).

If $z_iz_{ij} \in E(M,R)$ and $z_{ij} \in M$, then $z_{ij} \in V_F$ and is incident with a pre-colored edge under ψ. This contradicts Lemma 5 as previously. So $z_{ij} \in R$ and $z_i \in M$, and furthermore, $z_i \in A$. Observe that z_{ij} has no neighbors in $\{x,u,v,w\}$, as this would contradict Lemma 5. Thus, if the three neighbors of z_{ij} other than z_i are in M, then are all in V_F and are incident with pre-colored edges under ψ. However, this implies that z_{ij} has two neighbors in $\{a, a_1, a_2, a_3\}$ for some $a \in \{u,v,w\}$, which contradicts Lemma 5. This proves (5).

If $z_i \neq w_1$ and $z_iz_{ij}, z_iz_{ik} \in E(M,R)$, then $z_i \in M$ and $z_{ij}, z_{ik} \in R$ by (5). Suppose that z_{ij} and z_{ik} each have two neighbors other than z_i in M. By Lemma 5, z_{ij} and z_{ik} have four distinct neighbors other than z_i in M, and furthermore, none of these four vertices are in $\{x,u,v,w\}$. Hence they must be in A. Since $z_i \neq w_1$, we may assume without loss of generality that $z_i = u_i$. By Lemma 5, neither z_{ij} nor z_{ik} can have a neighbor in $\{u_1,u_2,u_3\}$ other than u_i. Thus, the four neighbors previously described must be v_1, v_2, v_3, w_1, which contradicts Lemma 5. This proves (6). □

4 How to color the vertices in L and R 'collaboratively'

In this section, we prove Theorem 2. Before doing so, we first prove some lemmas that show $M \cap A \neq \emptyset$ and potential properties of the vertices in $M \cap A$. In each of the following lemmas, we aim to color $E(G[L])$ and $E(G[R])$ and order the edges incident with M so that each edge e has at most 20 different colors in $N(e)$, which leads to a strong edge-coloring of G. We also remove the colors placed on the edges of G by ψ so that G is completely uncolored.

Lemma 10. *There is no vertex $z \in \{u,v,w\}$ such that $z_i \in L$, $z_j \in R$ and $z_k \in L \cup R$ for $i,j,k \in [3]$. In particular, $w_1 \notin L$.*

Proof. Suppose otherwise that for some $z \in \{u, v, w\}$, $z_1 \in L$ and $z_3 \in R$. So $zz_1 \in F$. By Lemma 9(3)-(4), for each $j \in [3]$, $z_1z_{1j} \in E(G[L])$, $z_3z_{3j} \in E(G[R])$ and $yy_j \in E(G[R])$. By Lemma 4, $|F| \geqslant 4$. So, there are at least three edges different from zz_1 in F. Assume that aa' is such an edge where $a \in V_F$ and $a' \in V'_F$. Consider two graphs G_L and G_R as follows:

$$V(G_L) = L \text{ and } E(G_L) = E(G[L]) \cup \{z_1a'\};$$

$$V(G_R) = R \text{ and } E(G_R) = E(G[R]) \cup \{z_3y\}.$$

Note that if z_1a' already exists, then we add a parallel edge with endpoints z_1 and a'. Recall that $x, z \in M$ so that z_1 has at most three neighbors in L, y and z_3 have at most three neighbors in R. Thus, G_L and G_R both have maximum degree at most 4. By the minimality of G, both G_L and G_R have strong edge-colorings with 21 colors. In G_L, let the colors of the three edges incident with z_1 in $G[L]$ (other than the new z_1a') be $1, 2, 3$, and the color of potentially new z_1a' be d, respectively. In G_R, by renaming colors, let the colors of the three edges incident with z_3 in $G[R]$ be $1, 2, 3$ and the color of yy_1 be d, respectively.

We now color the edges in G by giving the edges in $G[L]$ and $G[R]$ the same colors as in G_L and G_R. As observed before Lemma 9, this yields a partial strong edge-coloring, which we will call ϕ. Thus, the edges uncolored by ϕ are exactly those in $F \cup E(G[M]) \cup E(M, R)$. In particular, these are the edges incident with vertices in M, and recall that $M \subseteq A \cup \{x, u, v, w\}$. Observe that the edges incident with u, v, w, and x are all uncolored.

We now extend ϕ to some of the uncolored edges. For $z' \in \{u, v, w\} - z$ and $i, j \in \{1, 2, 3\}$, assign $z'_iz'_{ij}$ with an available color if it is not colored yet, and assign $z'z'_i$ an available color. This can be done as each of the aforementioned edges sees at least four uncolored edges. This yields a new, partial strong edge-coloring, which we will call ρ. Observe that the edges incident with z_2 other than zz_2 are the colored edges under ρ. Recall also that the edges incident with z_1 other than zz_1, and the edges incident with z_3 other than zz_3, are colored with 1,2, and 3. Also, yy_1 is colored with d.

We finally color the remaining edges based on whether or not d occurs on an edge incident with z_2. Let $\{u, v, w\} - z = \{z', z''\}$.

- If d occurs at an edge incident with z_2, then color the remaining edges in the following order: xz', xz'', xy, zz_1, zz_3, zz_2, xz.
- If d does not occur at the edges incident with z_2 in $G[R]$, then color zz_1 with d, and color the remaining edges in the following order: xz', xz'', xy, zz_3, zz_2, xz.

Note that in each case we always have a color available on the edges in the above sequence. In particular, xw will see four pairs of edges colored with 1,2,3, and d. Thus, G has a strong edge-coloring with 21 colors, a contradiction. □

Lemma 11. *$M \cap A \neq \emptyset$.*

Proof. Suppose otherwise $M \cap A = \emptyset$. Then the vertices of A must be partitioned amongst L and R, and furthermore $V_F \subseteq \{u, v, w\}$. By Lemma 10, $w_1 \in R$, and for each $z \in \{u, v\}$, $z_1, z_2, z_3 \in L$ or $z_1, z_2, z_3 \in R$. Thus, F contains all or none of edges in $\{zz_1, zz_2, zz_3\}$. Note that F is an edge-cut and $ww_1, ww_2, ww_3 \in E(M, R)$. This implies that $V_F \subseteq \{u, v\}$, and additionally, $F \subseteq \{zz_i : z \in \{u, v\}, i \in \{1, 2, 3\}\}$. However, this implies that $\{xu, xv\}$ is also an edge-cut, contrary to Lemma 4. □

Remark 1: For $z \in \{u, v, w\}$, if $z_i \in M$ (and so is in V_F), then by Lemma 9(1), z_i is incident with an edge in F and an edge in $E(M, R)$; we may assume, as a convention, that $z_i z_{i1} \in F$ and $z_i z_{i3} \in E(M, R)$.

Lemma 12. *There exists some vertex $z_i \in M \cap A$ such that at least three of the eight edges incident with z_{i2} and z_{i3} are in $E(G[R])$.*

Proof. Suppose otherwise that for each vertex $z_i \in M \cap A$, at most two of the eight edges incident with z_{i2} and z_{i3} are in $G[R]$.

Case 1. $w_1 \in M$ and there is only one edge incident with w_{13} in $G[R]$.

In this case, since $w_{13} \in R$, the other three edges incident with w_{13} must be in $E(M, R)$. In particular, w_{13} has at least two neighbors in $M \cap A$ other than w_1. By Lemma 5, w_{13} can be adjacent to at most one vertex in each of $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. Without loss of generality, we may assume that w_{13} is adjacent to u_1 and v_1. Then $u_1, v_1 \in M$. So, $u_1 u_{11},\ v_1 v_{11},\ w_1 w_{11} \in F$, $u_1 u_{13}, v_1 v_{13}, w_1 w_{13} \in E(M, R)$, where $u_{13} = v_{13} = w_{13}$.

Since $w_1 w_{11} \in F$, by Lemma 9(2), the three edges incident with w_{11} (other than $w_1 w_{11}$) are in $G[L]$. Similarly, there are three edges incident with u_{11} (other than $u_1 u_{11}$) that are in $G[L]$. In particular, u_{11} is not adjacent to either w_{11} or w_{12}, as this would contradict Lemma 5. In addition, there exists $u_{11} u' \in E(G[L])$ where $u' \notin \{w_{11}, w_{12}\}$.

Let G_L and G_R be the following graphs:

$$V(G_L) = L \text{ and } E(G_L) = E(G[L]) \cup \{w_{11} u_{11}\};$$

$$V(G_R) = R \text{ and } E(G_R) = E(G[R]) \cup \{w_{13} w_2, w_{13} w_3, w_{13} y\}.$$

Observe that $\Delta(G_L)$ and $\Delta(G_R)$ are both at most four. By the minimality of G, each of G_L and G_R has a strong edge-coloring with 21 colors. In G_L, let the colors of the three edges incident with w_{11} in $G[L]$ be $1, 2, 3$, respectively, and the color of $u_{11} u'$ be d. In G_R, let the color of the edge incident with w_{13} in $G[R]$ be 1, the color of $w_{13} w_2$ be 2, the color of $w_{13} w_3$ be 3, and the color of $w_{13} y$ be d. Clearly, $d \notin \{1, 2, 3\}$.

We now color the edges of G by assigning the edges in $G[L]$ and $G[R]$ the same colors as in G_L and G_R, respectively. Observe that this yields a partial strong edge-coloring of G in which the only uncolored edges are incident with vertices in $M \subseteq A \cup \{x, u, v, w\}$. Recall that $u_1, v_1, w_1 \in M$.

Since $y w_{13}, w_2 w_{13}$, and $w_3 w_{13}$ are colored with $d, 2$, and 3, respectively in G_R, we color xy, ww_2, ww_3 with $d, 2, 3$, respectively.

- If some edge incident with w_{12} has been colored with d, then we first color the edges $u_2u_{2j}, u_3u_{3j}, v_2v_{2j}, v_3v_{3j}$ where $j \in [3]$ (if they are not colored) with available colors, and color the remaining edges in the following order:

$$u_1u_{11},\ u_1u_{12},\ uu_1,\ uu_2,\ uu_3,\ xu,\ v_1v_{11},\ v_1v_{12},\ vv_2,\ vv_3,$$
$$xv,\ vv_1,\ v_1v_{13},\ u_1u_{13},\ xw,\ w_1w_{11},\ w_1w_{12},\ w_1w_{13},\ ww_1.$$

- If the edges incident with w_{12} are not colored with d (including the case that they are not colored), then color w_1w_{13} with d, color the edges $u_2u_{2j}, u_3u_{3j}, v_2v_{2j}, v_3v_{3j}$ where $j \in [3]$ (if they are not colored) with available colors, and color the remaining edges in the order (recall that $u_{11}u'$ is colored with d):

$$u_1u_{11},\ u_1u_{12},\ uu_1,\ uu_2,\ uu_3,\ xu,\ v_1v_{11},\ v_1v_{12},\ vv_2,$$
$$vv_3,\ xv,\ vv_1,\ v_1v_{13},\ u_1u_{13},\ xw,\ w_1w_{11},\ w_1w_{12},\ ww_1.$$

So, G has a strong edge-coloring with 21 colors. It is a contradiction.

Case 2. $w_1 \in M$ and there are exactly two edges incident with w_{13} in $G[R]$.

Recall from Remark 1, that $w_1w_{11} \in F$ so that $w_{11} \in L$, and $w_1w_{13} \in E(M, R)$. If $w_{12} \in G - L$, then it must be in R, and by Lemma 9(5), w_{12} would have an edge incident with it in $G[R]$. Yet, we are assuming that at most two of the eight edges incident with w_{12} and w_{13} are in $G[R]$, a contradiction. So $w_{12} \in L$ and $w_1w_{12} \in F$.

Since w_{13} has exactly two neighbors in R, we may assume without loss of generality, that w_{13} is adjacent to u_1. Then $u_1 \in M$, and consequently, $u_1u_{11} \in F$ and $u_1u_{13} \in E(M, R)$, where $u_{13} = w_{13}$. Consider two graphs G_L and G_R as follows:

$$V(G_L) = L \text{ and } E(G_L) = E(G[L]) \cup \{w_{11}w_{12}\};$$
$$V(G_R) = R \text{ and } E(G_R) = E(G[R]) \cup \{w_{13}w_2, w_{13}w_3\}.$$

Notice that $w_{11}, w_{12} \in L$, $w_1 \in M$, $w_2, w_3 \in R$ and w_{13} has exactly two neighbors in R. So, both graphs G_L and G_R have maximum degree at most four. By the minimality of G, both G_L and G_R have strong edge-colorings with 21 colors. In G_L, let the colors of the three edges incident with w_{11}, other than $w_{11}w_{12}$, be $1, 2, 3$, and let the color of one edge incident with w_{12}, other than $w_{11}w_{12}$, be d (these edges exist by Lemma 9(2)). Clearly, $d \neq 1, 2, 3$. In G_R, by renaming colors, let the color of the edges incident with w_{13}, other than $w_{13}w_2$ and $w_{13}w_3$, be 1 and 2, let the color of $w_{13}w_2$ be 3, and let the color of w_3w_{31} be d.

We now color the edges of G by assigning the edges in $G[L]$ and $G[R]$ the same colors as in G_L and G_R, respectively. Observe that this yields a partial strong edge-coloring of G in which the only uncolored edges are incident with vertices in $M \subseteq A \cup \{x, u, v, w\}$. Recall that $u_1, w_1 \in M$.

Next we color ww_2 with 3. We assign u_ju_{jk}, v_jv_{jk} for $j,k \in [3]$, if not colored yet, with available colors except for u_1u_{13}, and assign uu_i, vv_i for $i \in [3]$ with available colors. Finally, we color the remaining edges in the order:

$$xv,\ xy,\ xu,\ u_1u_{13},\ xw,\ ww_3,\ w_1w_{11}, w_1w_{12}, w_1w_{13},\ ww_1.$$

So, G has a strong edge-coloring with 21 colors. It is a contradiction.

Case 3. $w_1 \notin M$. By the symmetry of u and v, we may assume that $u_1 \in M \cap A$.

Since $u_1u_{11} \in F$ and $u_1u_{13} \in E(M,R)$, by Lemma 9(6), u_1u_{12} must be in F. By Lemma 9(2), the three edges incident with u_{11} other than u_1u_{11} and the three edges incident with u_{12} other than u_1u_{12} are in $G[L]$. Since $u_1u_{13} \in E(M,R)$, by Lemma 9(5), at least one edge incident with u_{13} is in $G[R]$.

Since $w_1 \notin M$, Lemma 10 implies that $w_1 \in R$. By Lemma 9(4), $w_1w_{1j} \in G[R]$ for $j \in [3]$. Since we are assuming that at most two of the eight edges incident with u_{12} and u_{13} are in $G[R]$, u_{13} must have a neighbor other than u_1 in M. Since $w_1 \notin M$, we may assume it is v_1 so that $v_1 \in M$. Note that by Lemma 5, u_{13} cannot have any other neighbors in M, as they would be in $\{u_2, u_3, v_2, v_3\}$. Thus, there are exactly two edges incident with u_{13} in $G[R]$.

By a similar argument to the above, $u_1u_{11}, u_1u_{12}, v_1v_{11}, v_1v_{12} \in F$, $u_1u_{13}, v_1v_{13} \in E(M,R)$, and $u_{13} = v_{13}$. By Lemma 5, $u_{11}, u_{12}, v_{11}, v_{12}$ are all distinct.

Consider two graphs G_L and G_R as follows:

$$V(G_L) = L \text{ and } E(G_L) = E(G[L]) \cup \{u_{11}u_{12}\};$$

$$V(G_R) = R \cup \{b\} \text{ and } E(G_R) = E(G[R]) \cup \{w_1b, w_2b, w_3b, u_{13}b, u_{13}y\}.$$

Notice that $x, w, u_1 \in M$, $w_1, w_2, w_3, y \in R$ and u_{13} has exactly two neighbors in R. So, both graphs G_L and G_R have maximum degree at most four. By the minimality of G, both G_L and G_R have strong edge-colorings with 21 colors. In G_L, let the colors of the three edges incident with u_{11} (other than $u_{11}u_{12}$) be $1,2,3$, and the color of $u_{11}u_{12}$ be d. Clearly, $d \neq 1,2,3$. In G_R, by renaming colors, let the colors of the two edges incident with u_{13} (other than $u_{13}b, u_{13}y$) be $1,2$, the color of $u_{13}y$ be 3, the color of $u_{13}b$ be d, and the colors of w_1b, w_2b, w_3b be d_1, d_2, d_3, respectively. Clearly, $\{d_1, d_2, d_3\} \cap \{1,2,3,d\} = \emptyset$.

Claim: the colors $1,2,3$ appear on edges incident with v_{11} or v_{12} in G_L.

Proof. Suppose otherwise that at least one of colors $1,2,3$ does not appear. If 3 appears on an edge incident with v_{11} or v_{12} but 1 does not, then switch the colors 3 and 1 in G_L so that 3 is missing. We do a similar switch if 3 appears, but 2 does not. Thus, we may assume that 3 does not appear on edges incident with v_{11} or v_{12}.

We now color the edges of G by assigning the edges in $G[L]$ and $G[R]$ the colors used in G_L and G_R, respectively. Note that $v_{13}y \notin E(G)$, by Lemma 5. So we can color v_1v_{13} and xy with 3. We next color ww_1, ww_2, ww_3 with d_1, d_2, d_3, respectively. We assign $u_2u_{2j}, u_3u_{3j}, v_2v_{2j}, v_3v_{3j}$ for $j \in [3]$ with available colors if they are not colored yet. Finally, we color the remaining edges in the order:

$$v_1v_{11},\ v_1v_{12},\ vv_1,\ vv_2,\ vv_3,\ xv,\ xw,\ xu,\ uu_2,\ uu_3,\ u_1u_{11},\ u_1u_{12},\ u_1u_{13},\ uu_1.$$

So, G has a strong edge-coloring with 21 colors. It is a contradiction. □

We now color the edges of G by assigning the edges in $G[L]$ and $G[R]$ the same colors as in G_L and G_R, respectively. Observe that this yields a partial strong edge-coloring of G in which the only uncolored edges are incident with vertices in $M \subseteq A \cup \{x, u, v, w\}$. Recall that $u_1, v_1 \in M$. We next color xy with 3, color xw and u_1u_{13} with d, and color ww_1, ww_2, ww_3 with d_1, d_2, d_3, respectively. We assign $u_2u_{2j}, u_3u_{3j}, v_2v_{2j}, v_3v_{3j}$ for $j \in [3]$ with available colors if they are not colored yet. Finally, we color the remaining edges in the order:

$$uu_2,\ uu_3,\ u_1u_{11},\ u_1u_{12},\ xu,\ xv,\ vv_2,\ vv_3,\ v_1v_{11},\ v_1v_{12},\ v_1v_{13},\ vv_1,\ uu_1.$$

With the claim, it is easy to check that each edge has an available color. So, G has a strong edge-coloring with 21 colors. It is a contradiction. □

Lemma 13. *If there exists $z_i \in M \cap A$ such that at least three of the eight edges incident with z_{i2} and z_{i3} are in $G[R]$, then $z_j \notin R$ for all $j \in [3]$. In particular, w_1 is not such a vertex in $M \cap A$.*

Proof. Suppose otherwise that for some $z_i \in M \cap A$, at least three of the eight edges incident with z_{i2} and z_{i3} are in $G[R]$ and $z_j \in R$. Without loss of generality, we may assume that $i = 1$ and that $z_3 \in R$. Recall that by our convention in Remark 1, $z_1z_{11} \in F$, $z_1z_{13} \in E(M, R)$, and by Lemma 9(2), three edges incident with z_{11} are in $G[L]$.

By Lemma 4, $|F| \geqslant 4$. It follows that at least three edges other than z_1z_{11} are in F. Assume that aa' is such an edge with $a \in V_F$ and $a' \in V_F'$. Consider the graph G_L:

$$V(G_L) = L \text{ and } E(G_L) = E(G[L]) \cup \{z_{11}a'\},$$

where if $z_{11}a'$ already exists, then we add a parallel edge with endpoints z_{11} and a'. Observe that $\Delta(G_L) \leqslant 4$. By the minimality of G, G_L has a strong edge-coloring with 21 colors. In G_L, let the colors of the three edges incident with z_{11} (other than the new copy of $z_{11}a'$) be $1, 2, 3$, and the color of new copy of $z_{11}a'$ be d, respectively.

We may assume that either z_{13} is incident with three edges in $G[R]$, or both z_{12} and z_{13} are incident with at most two edges in $G[R]$. Consider G_R with $V(G_R) = R$ and

$$E(G_R) = \begin{cases} E(G[R]) \cup \{z_{13}z_3\}, & \text{if } z_{13} \text{ is incident with three edges in } G[R]; \\ E(G[R]) \cup \{z_{13}z_{12}, z_{13}z_3\}, & \text{otherwise.} \end{cases}$$

Observe that $\Delta(G_R) \leqslant 4$. By the minimality of G, G_R have a strong edge-coloring with 21 colors. In G_R, let the colors of any three edges in $G[R]$ incident with z_{12}, z_{13} be $1, 2, 3$, respectively. By Lemma 9(4), z_3 is incident with three edges in $G[R]$. So we may assume that one of them, say z_3z_{31}, is colored with d (up to renaming it), which is possible even if z_{12} is incident with less than three edges in $G[R]$.

Now we color the edges in G. First of all, the edges in $G[L]$ and $G[R]$ keep their colors in G_L and G_R. For $z' \in \{u,v,w\}-z$, we assign $z'_iz'_{ij}$ for $i,j \in [3]$ with an available color if it is not colored yet, and then assign $z'z'_i$ for $i \in [3]$ with an available color. We then color the edges z_2z_{2j} for $j \in [3]$ with an available color (note that the edges z_3z_{3j} are colored). Finally, we color the remaining edges according to whether the color d appears on the edges incident with z_{12} (let $\{u,v,w\}-z=\{z',z''\}$):

- If the color d does not appear at the edges incident with z_{12}, then color z_1z_{11} with d, and color the remaining edges in the following order

$$xz',\ xz'',\ xy,\ xz,\ zz_2,\ zz_3,\ z_1z_{12},\ z_1z_{13},\ zz_1.$$

- If the color d appears at an edge incident with z_{12}, then color the remaining edges in the following order:

$$xz',\ xz'',\ xy,\ xz,\ zz_2,\ zz_3,\ z_1z_{11},\ z_1z_{12},\ z_1z_{13},\ zz_1.$$

So, G has a strong edge-coloring with 21 colors. It is a contradiction. □

We are now ready to finish the proof of Theorem 2.

Proof of Theorem 2. By Lemmas 12 and 13, there exists some vertex $z_i \in (M \cap A) \setminus \{w_1\}$ such that at least three of the eight edges incident with z_{i2} and z_{i3} are in $G[R]$. Without loss of generality, we will assume $z_i = u_1$. Recall that from Remark 1, we will assume $u_1u_{11} \in F$ and $u_1u_{13} \in E(M,R)$. Thus, by Lemma 9(2) and (5) three edges incident with u_{11} are in $G[L]$ and there is at least one edge incident with u_{13} in $G[R]$. Note that by Lemma 13, $u_2, u_3 \notin R$. So, we consider the following cases.

Case 1. $u_2 \in L$ or $u_3 \in L$. Without loss of generality, let $u_2 \in L$.

By Lemma 9(3), u_2u_{2j} is in $G[L]$ for each $j \in [3]$. We consider graph G_L:

$$V(G_L) = L \text{ and } E(G_L) = E(G[L]) \cup \{u_{11}u_2\}.$$

Observe that $\Delta(G_L) \leqslant 4$. By the minimality of G, G_L has a strong edge-coloring with 21 colors. In G_L, let the colors of the three edges in $G[L]$ incident with u_{11} be $1, 2, 3$, and the color of u_2u_{21} be d, respectively.

We may assume that either u_{13} is incident with three edges in $G[R]$, or both u_{12} and u_{13} are incident with at most two edges in $G[R]$. Consider G_R with $V(G_R) = R$ and

$$E(G_R) = \begin{cases} E(G[R]) \cup \{u_{13}y\}, & \text{if } u_{13} \text{ is incident with three edges in } G[R]; \\ E(G[R]) \cup \{u_{13}u_{12}, u_{13}y\}, & \text{otherwise.} \end{cases}$$

Observe that $\Delta(G_R) \leqslant 4$. By the minimality of G, G_R has a strong edge-coloring with 21 colors. In G_R, by renaming colors, let the colors of (any) three edges incident with u_{13}, u_{12} in $G[R]$ be $1, 2, 3$, and the color of $u_{13}y$ be d, respectively.

Now we color the edges in G, where the edges in $G[L]$ and $G[R]$ keep their colors in G_L and G_R. We then assign $v_iv_{ij}, w_iw_{ij}, u_3u_{3j}$ for $i, j \in [3]$ with an available color if it is not colored yet, and assign vv_i, ww_i for $i \in [3]$ with available colors. Finally, we color the remaining edges according to whether the color d appears on the edges incident with u_{12}:

- If the color d does not appear at the edges incident with u_{12}, then color u_1u_{13} with d, and color the remaining edges in the following order:

$$xv,\ xw,\ xy,\ xu,\ uu_2,\ uu_3,\ u_1u_{11},\ u_1u_{12},\ uu_1.$$

- If the color d appears at an edge incident with u_{12}, then color the remaining edges in the following order:

$$xv,\ xw,\ xy,\ xu,\ uu_2,\ uu_3,\ u_1u_{11},\ u_1u_{12},\ u_1u_{13},\ uu_1.$$

In either case, G has a strong edge-coloring with 21 colors. It is a contradiction.

Case 2. $u_2, u_3 \in M$.

By Lemma 9, we have $u_1u_{11}, u_2u_{21}, u_3u_{31} \in F$ and $u_1u_{13}, u_2u_{23}, u_3u_{33} \in E(M, R)$.

Subcase 2.1. For some $i, j \in [3]$ with $i \neq j$, $u_iu_{i2} \in F$ and $u_ju_{j2} \in E(M, R)$.

Assume that $u_1u_{12} \in F$ and $u_2u_{22} \in E(M, R)$. Consider two graphs G_L and G_R as follows:

$$V(G_L) = L \cup \{a\} \text{ and } E(G_L) = E(G[L]) \cup \{u_{11}a, u_{12}a, u_{21}a, u_{31}a\};$$

$$V(G_R) = R \cup \{b\} \text{ and } E(G_R) = E(G[R]) \cup \{u_{13}b, u_{22}b, u_{23}b, u_{33}b\}.$$

Observe that $\Delta(G_L) \leqslant 4$ and $\Delta(G_R) \leqslant 4$. By the minimality of G, both G_L and G_R can be colored with 21 colors. Let the colors of $au_{11}, au_{12}, au_{21}$ be $1, 2, 3$, respectively. We rename colors of edges in G_R so that the colors of $bu_{23}, bu_{22}, bu_{13}$ are $1, 2, 3$, respectively. We further assume that an edge incident with u_{31} (other than $u_{31}a$) and an edge incident with u_{33} (other than $u_{33}b$) have the same color, say d. Clearly, $d \neq 1, 2, 3$.

Now we color the edges in G. First of all, the edges in $G[L]$ and $G[R]$ keep their colors in G_L and G_R. Then we color u_1u_{11} and u_2u_{23} with 1, color u_1u_{12} and u_2u_{22} with 2, and color u_1u_{13} and u_2u_{21} with 3. We assign v_iv_{ij}, w_iw_{ij} for $i, j \in [3]$ with available colors if they are not colored yet, and assign vv_i, ww_i for $i \in [3]$ with available colors. Finally, we color the remaining edges in the order:

$$xv,\ xw,\ xy,\ xu,\ u_3u_{31},\ u_3u_{32},\ u_3u_{33},\ uu_1,\ uu_2,\ uu_3.$$

So, G has a strong edge-coloring with 21 colors. It is a contradiction.

Subcase 2.2. $u_1u_{12}, u_2u_{22} \in F$, or $u_1u_{12}, u_2u_{22} \in E(M, R)$.

If $u_1u_{12}, u_2u_{22} \in F$, then clearly, the three edges incident with u_{13} other than u_1u_{13} are in $G[R]$. Consider two graphs G_L and G_R as follows:

$$V(G_L) = L \cup \{a\} \text{ and } E(G_L) = E(G[L]) \cup \{u_{11}a, u_{12}a, u_{21}a, u_{22}a\};$$

$$V(G_R) = R \text{ and } E(G_R) = E(G[R]) \cup \{u_{13}u_{23}\}.$$

Observe that $\Delta(G_L) \leqslant 4$ and $\Delta(G_R) \leqslant 4$. By the minimality of G, both G_L and G_R have strong edge-colorings with 21 colors. In G_L, let the colors of the three edges incident with u_{11} in $G[L]$ be $1, 2, 3$, and the color of $u_{11}a$ be d, respectively. In G_R, by renaming the colors of edges in G_R, let the colors of the three edges incident with u_{13} in $G[R]$ be $1, 2, 3$, and the color of $u_{13}u_{23}$ be d, respectively.

If $u_1u_{12}, u_2u_{22} \in E(M, R)$, consider G_L with $V(G_L) = L$ and G_R with $V(G_R) = R \cup \{b\}$ and

$$E(G_L) = E(G[L]) \cup \{u_{11}u_{21}\};$$

$$E(G_R) = \begin{cases} E(G[R]) \cup \{u_{12}b, u_{13}b, u_{22}b, u_{23}b\}, & \text{if } u_{13} \text{ is incident with three edges in } G[R]; \\ E(G[R]) \cup \{u_{12}b, u_{13}b, u_{22}b, u_{23}b, u_{12}u_{13}\}, & \text{otherwise.} \end{cases}$$

Observe that $\Delta(G_L) \leqslant 4$ and $\Delta(G_R) \leqslant 4$. By the minimality of G, both G_L and G_R have strong edge-colorings with 21 colors. In G_L, let the colors of the three edges incident with u_{11} in $G[L]$ be $1, 2, 3$, and the color of $u_{11}u_{21}$ be d, respectively. In G_R, let the colors on any three edges in $G[R]$ incident with u_{12}, u_{13} be $1, 2, 3$, and the color of $u_{23}b$ be d, respectively.

In either case, we color the edges in G in the following procedure. First of all, the edges in $G[L]$ and $G[R]$ keep their colors in G_L and G_R. Next, we color u_1u_{11} and u_2u_{23} with d, and assign v_iv_{ij}, w_iw_{ij} for $i, j \in [3]$ with available colors if they are not colored yet, and assign vv_i, ww_i for $i \in [3]$ with available colors. Finally, we color the remaining edges in the following order:

$$xv, xw, xy,\ u_2u_{21},\ u_2u_{22},\ u_3u_{31},\ u_3u_{32},\ u_3u_{33},\ xu,\ uu_2,\ uu_3,\ u_1u_{12},\ u_1u_{13}, uu_1.$$

So, G has a strong edge-coloring with 21 colors. It is a contradiction. □

5 Proof of Lemma 5

In this section, we proof Lemma 5 in a series of lemmas. In these proofs, we will often remove vertices and edges from G to obtain a strong edge-coloring, say ϕ, of the remaining graph that use at most 21 colors. Often, we will consider $|A_\phi(e)|$ for each uncolored edge e of G with the purpose of applying the well-known result of Hall [12] in terms of systems of distinct representatives. This yields a coloring of the remaining uncolored edges such that they will receive distinct colors, which ultimately produces a strong edge-coloring of G. Thus, when in a situation in which we can apply this result of Hall, we will say that we obtain a strong edge-coloring of G by SDR.

Let $\mathcal{U}_\phi(v)$ to be the set of colors used on edges incident with a vertex v. For adjacent vertices u and v, let $\Upsilon_\phi(u,v)$ be $\mathcal{U}_\phi(u) \setminus \{\phi(uv)\}$. That is, $\Upsilon_\phi(u,v)$ is the set of colors used on the edges incident with u other than uv. Observe that $\Upsilon_\phi(u,v)$ and $\Upsilon_\phi(v,u)$ are disjoint. Often, we will refer to only one partial strong edge-coloring that will not be named. In such cases we will suppress the subscripts used in the above notations.

Lemma 14. *G has no multiple edges. That is, G is simple.*

Proof. Suppose on the contrary that there exists a parallel edge e with endpoints u, v. By the minimality of G, $G - e$ has a good coloring. Since e has at least five colors available, we can obtain a good coloring of G. □

Lemma 15. *G contains no triangles.*

Proof. Suppose on the contrary that G contains a triangle u_1, u_2, u_3. Since G is 4-regular, there exist $x_i, y_i \in N(u_i) \setminus \{u_1, u_2, u_3\}$. By the minimality of G, $G - \{u_1, u_2, u_3\}$ has a good coloring.

Observe that $|A(x_iu_i)|, |A(y_iu_i)| \geqslant 6$ for $i \in [3]$ and $|A(u_ju_{j+1})| \geqslant 9$ for $j \in [3]$ modulo 3. Thus, we obtain a good coloring of G by SDR. □

Lemma 16. *G contains no $K_{3,3}$.*

Proof. Suppose on the contrary that G contains $K_{3,3}$ as a subgraph with partite sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. For $i \in [3]$, let x_i denote the fourth neighbor of u_i not in $\{v_1, v_2, v_3\}$, and let y_i denote the fourth neighbor of v_i not in $\{u_1, u_2, u_3\}$.

Let G' be obtained from G by removing $u_1, u_2, u_3, v_1, v_2, v_3$. By the minimality of G, G' has a good coloring that we can impose onto G. Observe that $|A(u_ix_i)|, |A(v_iy_i)| \geqslant 9$ for $i \in [3]$, and $|A(u_jv_\ell)| \geqslant 15$ for $1 \leqslant j \leqslant \ell \leqslant 3$. We then obtain a good coloring of G by SDR. □

Lemma 17. *G contains no $K_{2,4}$.*

Proof. Suppose on the contrary that G contains $K_{2,4}$ as a subgraph with partite sets $\{u_1, u_2, u_3, u_4\}$ and $\{v_1, v_2\}$. For $i \in [4]$, let x_i, y_i denote the third and fourth neighbors of u_i not in $\{v_1, v_2\}$. Of course, $x_i \neq y_i$, and by Lemma 15, $x_i, y_i \notin \{u_1, u_2, u_3, u_4\}$ for $i \in [4]$. So $|\{x_1, y_1, \ldots, x_4, y_4\}| \geqslant 4$.

Let G' be obtained from G by removing v_1 and v_2. By the minimality of G, G' has a good coloring that we can impose onto G. Call it ϕ. Note that if $e, e' \in E(G')$, and $\phi(e) = \phi(e')$, then they are still sufficiently far apart in G. Thus, ϕ is a good partial coloring of G. Observe that $|A_\phi(u_iv_j)| \geqslant 7$ for $i \in [4], j \in [2]$.

If $|\bigcup_{i\in[4],j\in[2]} A_\phi(u_iv_j)| \geqslant 8$, then we can greedily color the remaining edges to obtain a good coloring of G. Therefore, since $|A_\phi(u_iv_j)| \geqslant 7$ for each $i \in [4]$ and $j \in [2]$, we may assume each $A_\phi(u_iv_j) = [7]$. Observe that this implies each u_ix_i and u_iy_i receives distinct colors. So without loss of generality, suppose they are colored with the colors from $[15]\setminus[7]$. Furthermore, we may assume $\Upsilon_\phi(x_i, u_i) \cup \Upsilon_\phi(y_i, u_i) = [21]\setminus[15]$, for each $i \in [4]$, as otherwise $A_\phi(u_iv_j) \neq [7]$ for some i and j. This also implies that $|\{x_1, y_1, \ldots, x_4, y_4\}| = 8$; that is, they are all distinct.

Thus, our goal is to recolor two edges among $\{u_ix_i, u_iy_i : i \in [4]\}$ to be the same and obtain a good partial coloring of G. If so, then we can color the remaining edges greedily to obtain a good coloring of G. As a result, if we uncolor an edge u_ix_i, then in the resulting good partial coloring, the only colors available on this edge must be contained in $[7] \cup \phi(u_ix_i)$.

Note that by Lemma 3, x_1 cannot be adjacent to every vertex in $\{x_2, y_2, x_3, y_3, x_4, y_4\}$. So we may assume $x_1x_2 \notin E(G)$. Uncolor the edges u_1x_1 and u_2x_2, and let σ be this good partial coloring of G. Since the colors on these edges in ϕ were distinct and in $[15] \setminus [7]$, we may assume they were 8 and 9. Observe that $|A_\sigma(u_ix_i)| \geqslant 5$ for $i \in [2]$, and $A_\sigma(u_iv_j) = [9]$ for $i \in [4], j \in [2]$.

As noted, $A_\sigma(u_1x_1) \cup A_\sigma(u_2x_2) \subseteq [9]$. However, since each edge now has at least 5 colors available, there must be some $\alpha \in A_\sigma(u_1x_1) \cup A_\sigma(u_2x_2)$. Thus, we can color these two edges with α to obtain a coloring ψ. Since $x_1x_2 \notin E(G)$, and since the x_i's and y_i's are all distinct, ψ is a good partial coloring of G in which $|A_\psi(u_iv_j)| \geqslant 8$ for $i \in [4], j \in [2]$. Thus, we can greedily color the remaining edges to obtain a good coloring of G. □

Lemma 18. *G contains no $K_{2,3}$.*

Proof. Suppose on the contrary that G contains a $K_{2,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2\}$. By Lemma 15, this subgraph is induced, and as G is 4-regular, for $i \in [3]$, there exist vertices x_i, y_i adjacent to u_i other than v_1, v_2, and vertices z_1, z_2 adjacent to u_1, u_2, respectively, other than u_1, u_2, u_3. By Lemma 17, $z_1 \neq z_2$, and by Lemma 15, z_1, z_2 are distinct from the x_i, y_i.

We define the following sets, for $j \in [2]$, let $\mathcal{Y}_j := \{u_iv_j : i \in [3]\}$, let $\mathcal{Y} := \mathcal{Y}_1 \cup \mathcal{Y}_2$, let $\mathcal{Z} := \{v_1z_1, v_2z_2\}$, and let $\mathcal{X} := \{u_ix_i, u_iy_i : i \in [3]\}$.

We proceed based on the existence of $z_1z_2 \in E(G)$.

Case 1. $z_1z_2 \in E(G)$.

Let G' be the graph obtained from G by deleting v_1 and v_2. By the minimality of G, G' has a good coloring ϕ.

Observe that $|A_\phi(e)| \geqslant 6$ for $e \in \mathcal{Y}$ and $|A_\phi(e')| \geqslant 4$ for $e' \in \mathcal{Z}$. Since $z_1z_2 \in E(G)$, we may assume $\mathcal{U}_\phi(z_1) = \{1, 2, 5\}$ and $\mathcal{U}_\phi(z_2) = \{3, 4, 5\}$ so that $\phi(z_1z_2) = 5$.

We can extend ϕ by coloring the edges of $\mathcal{Z}$, and denote this good partial coloring by σ. Note that neither edge in $\mathcal{Z}$ is colored with 1 or 2. Now, every edge in $\mathcal{Y}$ has at least four colors available on it. We proceed by considering which edges incident to some x_i or y_i are already colored with either 1 or 2.

We first claim that neither 1 nor 2 appear on any edge of $\mathcal{X}$ under σ. If 1 and 2 both appear, then each vertex in $\mathcal{Y}_1$ has at least six colors available, and we can extend σ by SDR. If only 1 appears on an edge of $\mathcal{X}$, then each edge in $\mathcal{Y}_1$ has at least five colors available. So if one edge in $\mathcal{Y}$ has at least six colors available, we can extend σ by SDR. This implies that for every $i \in [3]$, $2 \notin \mathcal{U}_\sigma(x_i) \cup \mathcal{U}_\sigma(u_i)$, else $|A_\sigma(u_iv_1)| \geqslant 6$. Thus, we can color any edge in $\mathcal{Y}_2$ with 2 to obtain a good partial coloring ψ. Observe that $|A_\psi(e)| \geqslant 4$

for $e \in \mathcal{Y}_2$, and $|A_\psi(e')| \geqslant 5$ for $e' \in \mathcal{Y}_1$. So we can extend ψ by SDR, which proves our claim.

We now return to σ. Suppose $1 \notin \mathcal{U}_\sigma(x_1) \cup \mathcal{U}_\sigma(y_1)$. If in addition, $2 \notin \mathcal{U}_\sigma(x_2) \cup \mathcal{U}_\sigma(y_2)$. Then we can color u_1v_2 and u_2v_2 with 1 and 2, respectively to obtain a good partial coloring of G. From here we can color the remaining four edges by SDR as every edge in $\mathcal{Y}_1$ still has at least four colors available. Then by a similar argument, we may assume $2 \in \mathcal{U}_\sigma(x_i) \cup \mathcal{U}_\sigma(y_i)$ for $i \in \{2, 3\}$. We again color u_1v_2 with 1 to obtain a good partial coloring of G. From here we can color the remaining five edges by SDR as v_1u_2 and v_1u_3 each have at least five colors available.

Thus, we can assume $1, 2 \in \mathcal{U}_\sigma(x_i) \cup \mathcal{U}_\sigma(y_i)$ for each $i \in [3]$. We then color the edges of $\mathcal{Y}$ be SDR as every edge in $\mathcal{Y}_1$ has at least six colors available. This completes the case.

Case 2. $z_1z_2 \notin E(G)$.

Let G' be the graph obtained from G by deleting v_1 and v_2 and adding the edge z_1z_2. By the minimality of G, G' has a good coloring, which ignoring z_1z_2, can be applied to G. We immediately extend this by coloring v_1z_1 and v_2z_2 with the color used on z_1z_2. Call this good partial coloring ϕ.

We may assume that $\mathcal{U}_\phi(z_1) = \{15, 16, 17, 21\}$ and $\mathcal{U}_\phi(z_2) = \{18, 19, 20, 21\}$ so that $\phi(v_1z_1) = \phi(v_2z_2) = 21$. Observe that $|A_\phi(e)| \geqslant 5$ for $e \in \mathcal{Y}$. If $|A_\phi(e)| \geqslant 6$ for any $e \in \mathcal{Y}$, we can color the remaining edges of G by SDR.

If 15, 16, or 17 appears in some $A_\phi(x_i) \cup A_\phi(y_i)$ for some $i \in [3]$, then $|A_\phi(v_1u_i)| \geqslant 6$, and we are done. So we may color the edges in $\mathcal{Y}_2$ with 15, 16, and 17, to obtain a good partial coloring of G, call it σ. Observe that $|A_\sigma(e)| \geqslant 5$ for each $e \in \mathcal{Y}_1$, so that we can color them greedily to obtain a good coloring of G.

This completes the proof of the case, and hence, proves the lemma. □

Lemma 19. *G has no 4-cycles.*

Proof. Suppose on the contrary that $u_1u_2u_3u_4$ is a 4-cycle in G. By Lemmas 3 and 15, for each $i \in [4]$, there exists $x_i, y_i \in N(u_i) \setminus \{u_1, u_2, u_3, u_4\}$. By Lemmas 15 and 18, $x_1, y_1, \ldots, x_4, y_4$ are distinct. Define the sets $\mathcal{X} := \{x_iu_i, y_iu_i : i \in [4]\}$, $\mathcal{Y} := \{x_ix_{i+1} : i \in [4] \text{ modulo } 4\}$, $\mathcal{X}_i = \{x_iu_i, y_iu_i\}$ for $i \in [4]$.

By the minimality of G, $G - \{u_1, u_2, u_3, u_4\}$ has a good coloring ϕ that we can apply to G. Observe that $|A_\phi(e)| \geqslant 6$ for $e \in \mathcal{X}$ and $|A_\phi(e')| \geqslant 9$ for $e' \in \mathcal{Y}$. We proceed based on if we can extend ϕ by coloring the edges of $\mathcal{X}_1$ and $\mathcal{X}_3$ (or $\mathcal{X}_2$ and $\mathcal{X}_4$) with the same colors.

Case 1. We can extend ϕ by coloring the edges of $\mathcal{X}_1$ and $\mathcal{X}_3$ with 1 and 2.

Suppose we can extend ϕ by coloring x_1u_1, x_3u_3 with 1, and y_1u_1, y_3u_3 with 2. Call this good partial coloring σ. Observe that $|A_\sigma(e)| \geqslant 4$ for $e \in \mathcal{X}_2 \cup \mathcal{X}_4$ and $|A_\sigma(e')| \geqslant 7$ for $e' \in \mathcal{Y}$.

If there are at least eight colors available over all the edges of $\mathcal{Y}$ under σ, then we can obtain a good coloring of G by SDR. Thus, $A_\sigma(e_1) = A_\sigma(e_2)$ and $|A_\sigma(e_1)| = 7$ for all $e_1, e_2 \in \mathcal{Y}$. Without loss of generality, we may assume $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{3, 4, 5, 6, 7, 8\}$ for $i = 1, 3$, and $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{9, 10, 11, 12, 13, 14\}$ for $i = 2, 4$.

Now, x_2, y_2, x_4, y_4 cannot induce a $K_{2,2}$ by Lemma 18. So, say $x_2x_4 \notin E(G)$. If $|A_\sigma(x_2u_2) \cup A_\sigma(x_4u_4)| \geqslant 8$, then we obtain a good coloring of G by SDR. Thus, we can extend σ by coloring x_2u_2, x_4u_4 with the same color, and then further extend by SDR.

Case 2. We can extend ϕ by coloring an edge of $\mathcal{X}_1$ and an edge of $\mathcal{X}_3$ with 1.

We may assume that we can extend ϕ by coloring x_1u_1, x_3u_3 with 1. Suppose that we can further extend ϕ by coloring an edge of $\mathcal{X}_2$ and $\mathcal{X}_4$ with 2, say x_2u_2, x_4u_4. Call this good partial coloring σ. Observe that $|A_\sigma(e)| \geqslant 4$ for uncolored $e \in \mathcal{X}$ and $|A_\sigma(e')| \geqslant 7$ for $e' \in \mathcal{Y}$.

As in the previous case, we may assume that $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{3, 4, 5, 6, 7, 8\}$ for $i = 1, 3$, and $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{9, 10, 11, 12, 13, 14\}$ for $i = 2, 4$ so that $A_\sigma(e) = A_\sigma(e')$ and $|A_\sigma(e)| = 7$ for $e, e' \in \mathcal{Y}$.

Now, suppose $y_1y_3 \in E(G)$. Then $|A_\sigma(y_1u_1)|, |A_\sigma(y_3u_3)| \geqslant 7$, and we can obtain a good coloring of G by coloring $y_2u_2, y_4u_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, y_1u_1, y_3u_3$ in this order.

So $y_1y_3 \notin E(G)$, and by the previous case $A_\sigma(y_1u_1) \cap A_\sigma(y_3u_3) = \emptyset$. Thus, $|A_\sigma(y_1u_1) \cup A_\sigma(y_3u_3)| \geqslant 8$, and we can obtain a good coloring of G by SDR.

Thus, it remains to consider when we cannot extend ϕ by coloring an edge of $\mathcal{X}_2$ and $\mathcal{X}_4$ with a common color. By Lemma 18, we may assume $x_2x_4 \notin E(G)$. Let ψ denote the good partial coloring extending ϕ by coloring x_1u_1, x_3u_3 with 1.

Observe that $|A_\psi(e)| \geqslant 5$ for uncolored $e \in \mathcal{X}$ and $|A_\psi(e')| \geqslant 8$ for $e' \in \mathcal{Y}$. Since $x_2x_4 \notin E(G)$, we must have $|A_\psi(x_2u_2) \cup A_\psi(x_4u_4)| \geqslant 10$, otherwise we can color x_2u_2, x_4u_4 with a common color, a contradiction. Now, if there are at least nine colors available over all the edges of $\mathcal{Y}$ under ψ, then we obtain a good coloring of G by SDR. Thus, we have $A_\psi(e_1) = A_\psi(e_2)$ and $|A_\psi(e_1)| = 8$ for $e_1, e_2 \in \mathcal{Y}$.

As above, we may assume $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{2, 3, 4, 5, 6, 7\}$ for $i = 1, 3$, and $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{8, 9, 10, 11, 12, 13\}$ for $i = 2, 4$.

Suppose $y_2y_4 \notin E(G)$. By the previous case, $A_\sigma(y_2u_2) \cap A_\sigma(y_4u_4) = \emptyset$ so that $|A_\sigma(y_2u_2) \cup A_\sigma(y_4u_4)| \geqslant 10$. We then obtain a good coloring of G by SDR.

Thus, $y_2y_4 \in E(G)$ so that $|A_\sigma(y_2u_2)|, |A_\sigma(y_4u_4)| \geqslant 8$. Now if $y_2x_4 \in E(G)$, then $|A_\sigma(y_2u_2)| \geqslant 11$, and we obtain a good coloring of G by SDR. Thus, $y_2x_4 \notin E(G)$, and by symmetry, $x_2y_4 \notin E(G)$. By the previous case, we have $|A_\sigma(y_2u_2) \cup A_\sigma(x_4u_4)|, |A_\sigma(x_2u_2) \cup A_\sigma(y_4u_4)| \geqslant 13$, and we obtain a good coloring of G by SDR.

Case 3. We cannot extend ϕ by coloring an edge of $\mathcal{X}_1$ and an edge of $\mathcal{X}_3$ with the same color.

By symmetry, we may assume that the same holds for edges in $\mathcal{X}_2$ and $\mathcal{X}_4$. By Lemma 18, we may assume that $x_1x_3, x_2x_4 \notin E(G)$. Thus, by the previous case, $|A_\phi(x_1u_1) \cup A_\phi(x_3u_3)|, |A_\phi(x_2u_2) \cup A_\phi(x_4u_4)| \geqslant 12$.

Now, if we have at least ten colors available over all the edges in $\mathcal{Y}$, then we can obtain a good coloring of G by SDR. So, we have $A_\phi(e_1) = A_\phi(e_2)$ and $|A_\phi(e_1)| = 9$ for $e_1, e_2 \in \mathcal{Y}$. Thus, as above, we may assume that $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{1, 2, 3, 4, 5, 6\}$ for $i = 1, 3$, and $\Upsilon_\sigma(x_i, u_i) \cup \Upsilon_\sigma(y_i, u_i) = \{7, 8, 9, 10, 11, 12\}$ for $i = 2, 4$.

If $y_1y_3 \in E(G)$, then $|A_\phi(y_1u_1)|, |A_\phi(y_3u_3)| \geqslant 9$, and we obtain a good coloring of G by SDR. So, $y_1y_3 \notin E(G)$. By the previous case, we have $|A_\phi(y_1u_1) \cup A_\phi(y_3u_3)| \geqslant 12$, and we obtain a good coloring of G by SDR.

Thus, in any case, we can extend ϕ to a good coloring of G. $\square$

Proof of Lemma 5. By the previous lemmas, we know that the girth of G is at least five. So suppose on the contrary that $u_1u_2u_3u_4u_5$ is a 5-cycle. By Lemmas 3, 15, and 19, each u_i has neighbors x_i, y_i not on this 5-cycle. Furthermore, $x_1, y_1, \ldots, x_5, y_5$ are distinct and the only possibly adjacencies are between $\{x_i, y_i\}$ and $\{x_{i\pm2}, y_{i\pm2}\}$, $i \in [5]$ modulo 5. However, by Lemma 19, neither x_i nor y_i can be adjacent to both x_{i+2} and y_{i+2} (similarly, x_{i-2} and y_{i-2}). As a result, we may assume that $x_2y_4, x_4y_1, x_1y_3, x_3y_5 \notin E(G)$.

Let G' be the graph obtained from G by removing u_1, u_2, u_3, u_4, u_5. By the minimality of G, G' has a good coloring ϕ. Let $\mathcal{X}$ denote $\{x_iu_i, y_iu_i : i \in [5]\}$ and $\mathcal{Y}$ denote $\{u_iu_{i+1} : i \in [5]$ modulo $5\}$. Observe that $|A_\phi(e)| \geqslant 9$ for $e \in \mathcal{Y}$ and $|A_\phi(e')| \geqslant 6$ for $e' \in \mathcal{X}$.

Since $x_1y_3 \notin E(G)$, if $A_\phi(x_1u_1) \cap A_\phi(y_3u_3) \neq \emptyset$, then we can color edges x_1u_1, y_3u_3 with the same color. Similarly for the other three nonadjacencies. Let $\mathcal{S} := \{\{x_2u_2, y_4u_4\}, \{x_4u_4, y_1u_1\}, \{x_1u_1, y_3u_3\}, \{x_3u_3, y_5u_5\}\}$ so that each element of $\mathcal{S}$ is a pair of edges that can possibly receive the same color. Let $\mathcal{S}' \subseteq \mathcal{S}$ such that we can extend ϕ by coloring each pair of edges in $\mathcal{S}'$ with its own color, and suppose that $\mathcal{S}'$ is as large as possible. Color each pair of edges in $\mathcal{S}'$ with its own color, and call this good partial coloring σ.

Case 1. $\mathcal{S}' = \mathcal{S}$.

Observe that $|A_\sigma(e)| \geqslant 5$ for $e \in \mathcal{Y}$, and $|A_\sigma(y_2u_2)|, |A_\sigma(x_5u_5)| \geqslant 2$. Suppose there exists $\alpha \in A_\sigma(y_2u_2) \cap A_\sigma(u_4u_5)$. We then color y_2u_2, u_4u_5 with α, and then $x_5u_5, u_5u_1, u_1u_2, u_3u_4, u_2u_3$ in this order to obtain a good coloring of G. Thus, $|A_\sigma(y_2u_2) \cup A_\sigma(u_4u_5)| \geqslant 7$, and by symmetry, $|A_\sigma(x_5u_5) \cup A_\sigma(u_2u_3)| \geqslant 7$. We then obtain a good coloring of G by SDR.

Case 2. $\mathcal{S}' \subset \mathcal{S}$.

Let $k' := |\mathcal{S}'|$ so that $0 \leqslant k' \leqslant 3$. Observe that $|A_\sigma(e)| \geqslant 9 - k'$ for $e \in \mathcal{Y}$, and $|A_\sigma(e')| \geqslant 6 - k'$ for uncolored $e' \in \mathcal{X}$. Since $\mathcal{S}'$ is the largest subset of $\mathcal{S}$ that we can color, we have $|A_\sigma(g) \cup A_\sigma(g')| \geqslant 12 - 2k'$ for all $\{g, g'\} \in \mathcal{S} \setminus \mathcal{S}'$.

Since $k' \leqslant 3$, there exists some uncolored $f \in \mathcal{X} \setminus \{y_2u_2, x_5u_5\}$ and $h \in \mathcal{Y} \setminus \{u_2u_3, u_4u_5\}$ such that f and h can receive the same color if $A_\sigma(f) \cap A_\sigma(h) \neq \emptyset$. Since $f \notin \{y_2u_2, x_5u_5\}$ and $k' \leqslant 3$, there exists an uncolored edge $f' \in \mathcal{X}$ such that $\{f, f'\} \in \mathcal{S}$.

Let $\mathcal{T} := \{\{y_2u_2, u_4u_5\}, \{x_5u_5, u_2u_3\}, \{f, h\}\}$ so that every element of $\mathcal{T}$ is a pair of edges that can possibly receive the same color. Let $\mathcal{T}' \subseteq \mathcal{T}$ such that we can extend σ by coloring each pair of edges in $\mathcal{T}'$ with its own color, and suppose that $\mathcal{T}'$ is as large as possible. Color each pair of edges in $\mathcal{T}'$ with its own color, and call this good partial coloring ψ. Let $t' := |\mathcal{T}'|$ so that $0 \leqslant t' \leqslant 3$.

Let $\mathcal{X}' \subset \mathcal{X}$ and $\mathcal{Y}' \subset \mathcal{Y}$ be the edges colored by ψ. So, $|\mathcal{X} \setminus \mathcal{X}'| = 10 - 2k' - t'$ and $|\mathcal{Y} \setminus \mathcal{Y}'| = 5 - t'$. Additionally, $|A_\psi(e_x)| \geqslant 6 - k' - t'$ for all $e_x \in \mathcal{X} \setminus \mathcal{X}'$ and $|A_\psi(e_y)| \geqslant 9 - k' - t'$ for all $e_y \in \mathcal{Y} \setminus \mathcal{Y}'$.

We now show that we can obtain a good coloring of G by SDR. Let $\mathcal{A}$ be a nonempty subset of $(\mathcal{X} \setminus \mathcal{X}') \cup (\mathcal{Y} \setminus \mathcal{Y}')$, and let $\bigcup_\mathcal{A} := \bigcup_{e \in \mathcal{A}} A_\psi(e)$. So $1 \leqslant |\mathcal{A}| \leqslant 15 - 2k' - 2t'$, and we aim to show that $|\bigcup_\mathcal{A}| \geqslant |\mathcal{A}|$.

Subcase 2.1. ψ does not color f or h.

Observe that $t' \leqslant 2$. If $1 \leqslant |\mathcal{A}| \leqslant 6 - k' - t'$, then $|\bigcup_{\mathcal{A}}| \geqslant 6 - k' - t'$.

If $7' - k' - t' \leqslant |\mathcal{A}| \leqslant 9 - k' - t'$ and $\mathcal{A} \cap (\mathcal{Y} \setminus \mathcal{Y}') \neq \emptyset$, then $|\bigcup_{\mathcal{A}}| \geqslant 9 - k' - t'$. So, we may assume $\mathcal{A} \subseteq (\mathcal{X} \setminus \mathcal{X}')$. Since $|\mathcal{S} \setminus \mathcal{S}'| = 4 - k' \geqslant 1$, if $\mathcal{A}$ contains at least $7 - k' - t'$ edges from $\mathcal{X} \setminus \mathcal{X}'$, it must include a pair of edges, say e_s, e'_s, that form an element of $\mathcal{S} \setminus \mathcal{S}'$. Thus, $|\bigcup_{\mathcal{A}}| \geqslant |A_\psi(e_s) \cup A_\psi(e'_s)| \geqslant 12 - 2k' - t'$, otherwise we could have colored e_s and e'_s with the same color and obtained a larger $\mathcal{S}' \subseteq \mathcal{S}$.

Since $|\mathcal{T} \setminus \mathcal{T}'| = 3 - t' \geqslant 1$, if $\mathcal{A}$ contains at least $13 - 2k' - t'$ edges, it must include a pair of edges, say e_t, e'_t, that form an element of $\mathcal{T} \setminus \mathcal{T}'$. Thus, $|\bigcup_{\mathcal{A}}| \geqslant |A_\psi(e_t) \cup A_\psi(e'_t)| \geqslant 15 - 2k' - 2t'$, otherwise we could have colored e_t and e'_t with the same color and obtained a larger $\mathcal{T} \subseteq \mathcal{T}'$.

So, it remains to consider when $10 - k' - t' \leqslant |\mathcal{A}| \leqslant 12 - 2k' - t'$. Thus, if $k' = 3$, we are done and obtain a good coloring of G by SDR. So $k' \leqslant 2$. Observe that $|(\mathcal{S} \setminus \mathcal{S}') \cup (\mathcal{T} \setminus \mathcal{T}')| = 7 - k' - t'$, and the only edge that is contained in an element of both $\mathcal{S} \setminus \mathcal{S}'$ and $\mathcal{T} \setminus \mathcal{T}'$ is f ($\{f, f'\} \in \mathcal{S} \setminus \mathcal{S}'$ and $\{f, h\} \in \mathcal{T} \setminus \mathcal{T}'$). Thus, we can find $6 - k' - t'$ elements in $(\mathcal{S} \setminus \mathcal{S}') \cup (\mathcal{T} \setminus \mathcal{T}')$ that are pairwise disjoint.

As a result, when $|\mathcal{A}| \geqslant 10 - k' - t'$, $\mathcal{A}$ must contain a pair of edges, say e, e', that forms an element of $(\mathcal{S} \setminus \mathcal{S}') \cup (\mathcal{T} \setminus \mathcal{T}')$. Thus, $|\bigcup_{\mathcal{A}}| \geqslant |A_\psi(e) \cup A_\psi(e')| \geqslant 12 - 2k' - t' \geqslant 10 - k' - t'$ for $k \leqslant 2$. So, in any case, we obtain a good coloring of G by SDR.

Subcase 2.2. ψ colors both f and h.

Observe that $t' \geqslant 1$ and $f' \in \mathcal{X} \setminus \mathcal{X}'$. If $1 \leqslant |\mathcal{A}| \leqslant 6 - k' - t'$, then $|\bigcup_{\mathcal{A}}| \geqslant 6 - k' - t'$.

If $|\mathcal{A}| = 7 - k' - t'$ and either $f' \in \mathcal{A}$ or $\mathcal{A} \cap (\mathcal{Y} \cap \mathcal{Y}') \neq \emptyset$, then $|\bigcup_{\mathcal{A}}| \geqslant 7 - k' - t'$. So, we may assume $\mathcal{A} \subseteq \mathcal{X} \setminus (\mathcal{X}' \cup \{f'\})$. Since there are exactly $3 - k'$ uncolored pairs in $\mathcal{S} \setminus \mathcal{S}'$, if $\mathcal{A}$ contains at least $7 - k' - t'$ edges from $\mathcal{X} \setminus (\mathcal{X}' \cup \{f'\})$, it must include a pair of edges, say e_s, e'_s, that form an element of $\mathcal{S} \setminus \mathcal{S}'$. Thus, $|\bigcup_{\mathcal{A}}| \geqslant |A_\psi(e_s) \cup A_\psi(e'_s)| \geqslant 12 - 2k' - t' \geqslant 7' - k' - t'$.

If $8 - k' - t' \leqslant |\mathcal{A}| \leqslant 9 - k' - t'$ and $\mathcal{A} \cap (\mathcal{Y} \setminus \mathcal{Y}') \neq \emptyset$, then $|\bigcup_{\mathcal{A}}| \geqslant 9 - k' - t'$. So we may assume $\mathcal{A} \subseteq (\mathcal{X} \setminus \mathcal{X}')$. However, in a similar manner to the above, $\mathcal{A}$ must contain a pair of edges that form an element of $\mathcal{S} \setminus \mathcal{S}'$. Thus, $|\bigcup_{\mathcal{A}}| \geqslant 12 - 2k' - t' \geqslant 9 - k' - t'$.

So, it remains to consider when $10 - k' - t' \leqslant |\mathcal{A}| \leqslant 15 - 2k' - 2t'$. Suppose that $t' \leqslant 2$ so that $|\mathcal{T} \setminus \mathcal{T}'| = 3 - t' \geqslant 1$. As in the previous subcase, if $\mathcal{A}$ contains at least $13 - 2k' - t'$ edges, it contains a pair of edges, say e_t, e'_t, that form an element of $\mathcal{T} \setminus \mathcal{T}'$. Thus, $|\bigcup_{\mathcal{A}}| \geqslant |A_\psi(e_t) \cup A_\psi(e'_t)| \geqslant 15 - 2k' - 2t'$. So, $10 - k' - t' \leqslant |\mathcal{A}| \leqslant 12 - 2k' - t'$. If $k' = 3$, we are done and obtain a good coloring of G by SDR. If $k' \leqslant 2$, then as in the previous subcase, we can find $6 - k' - t'$ elements in $(\mathcal{S} \setminus \mathcal{S}') \cup (\mathcal{T} \setminus \mathcal{T}')$ that are pairwise disjoint. Thus, when $|\mathcal{A}| \geqslant 10 - k' - t'$, $\mathcal{A}$ must contain a pair of edges that form an element of $(\mathcal{S} \setminus \mathcal{S}') \cup (\mathcal{T} \setminus \mathcal{T}')$, and $|\bigcup_{\mathcal{A}}| \geqslant 12 - 2k' - t' \geqslant 10 - k' - t'$ for $k \leqslant 2$. So, when $t' \leqslant 2$, we obtain a good coloring of G by SDR.

When $t' = 3$, we consider $7 - k' \leqslant |\mathcal{A}| \leqslant 9 - 2k'$. If $k' = 3$, we are done and obtain a good coloring of G by SDR. When $k' \leqslant 2$, $|\mathcal{S} \setminus \mathcal{S}'| = 3 - k' \geqslant 1$ so that if $\mathcal{A}$ contains at least $7 - k'$ edges, it contains a pair of edges, say e_s, e'_s that form an element of $\mathcal{S} \setminus \mathcal{S}'$, and $|\bigcup_{\mathcal{A}}| \geqslant 9 - 2k'$.

Thus, in any case we obtain a good coloring of G by SDR. □

6 Closing remarks

The essential part of the proof is to get a nice partition of the vertices described in the introduction. This partition is largely due to some kind of non-trivial edge-cuts. The study of existence of such edge-cuts may be of independent interest.

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