

Note on the multicolour size-Ramsey number for paths

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Abstract

The size-Ramsey number $\hat{R}(F, r)$ of a graph F is the smallest integer m such that there exists a graph G on m edges with the property that any colouring of the edges of G with r colours yields a monochromatic copy of F . In this short note, we give an alternative proof of the recent result of Krivelevich that $\hat{R}(P_n, r) = O((\log r)r^2n)$. This upper bound is nearly optimal, since it is also known that $\hat{R}(P_n, r) = \Omega(r^2n)$.

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1 Introduction

Following standard notation, we write $G \rightarrow (F)_r$ if any r -edge-colouring of G (that is, any colouring of the edges of G with r colours) yields a monochromatic copy of F . We define the *size-Ramsey number* of F as $\hat{R}(F, r) = \min\{|E(G)| : G \rightarrow (F)_r\}$; that is, $\hat{R}(F, r)$ is the smallest integer m such that there exists a graph G on m edges such that $G \rightarrow (F)_r$. For two colours (that is, for $r = 2$) the size-Ramsey number was first studied by Erdős, Faudree, Rousseau and Schelp [9].

In this note, we are concerned with the size-Ramsey number of the path P_n on n vertices. It is obvious that $\hat{R}(P_n, 2) = \Omega(n)$ and it is easy to see that $\hat{R}(P_n, 2) = O(n^2)$; for example, $K_{2n} \rightarrow (P_n)_2$. The exact behaviour of $\hat{R}(P_n, 2)$ was not known for a long time. In fact, Erdős [8] offered \$100 for a proof or disproof that $\hat{R}(P_n, 2)/n \rightarrow \infty$ and $\hat{R}(P_n, 2)/n^2 \rightarrow 0$. This problem was solved by Beck [1] in 1983 who, quite surprisingly, showed that $\hat{R}(P_n, 2) < 900n$. (Each time we refer to inequality such as this one, we

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mean that the inequality holds for sufficiently large n .) A variant of his proof, provided by Bollobás [5], gives $\hat{R}(P_n, 2) < 720n$. Recently, the authors of this paper [6] used a different and more elementary argument that shows that $\hat{R}(P_n, 2) < 137n$. The argument was subsequently tuned by Letzter [12] who showed that $\hat{R}(P_n, 2) < 91n$, and then further refined by the authors of this paper [7] who showed that $\hat{R}(P_n, 2) \leq 74n$. On the other hand, the first nontrivial lower bound was provided by Beck [2] and his result was subsequently improved by Bollobás [4] who showed that $\hat{R}(P_n, 2) \geq (1 + \sqrt{2})n - O(1)$. The strongest known lower bound, $\hat{R}(P_n, 2) \geq 5n/2 - O(1)$, was proved in [7].

Let us now move to the multicolour version of this graph parameter. It was proved in [7] that $\frac{(r+3)r}{4}n - O(r^2) \leq \hat{R}(P_n, r) \leq 33r^4n$. It follows that $\hat{R}(P_n, r)$ is linear for any fixed value of r but the two bounds are quite apart from each other in terms of their dependence on r . Subsequently, Krivelevich [11] showed that in fact the dependence on r is (nearly) quadratic; that is, $\hat{R}(P_n, r) = r^{2+o_r(1)}n$. Furthermore, for $r \geq 6$ he also improved the lower bound and showed that $\hat{R}(P_n, r) \geq (r-2)^2n + O(\sqrt{n})$. Here is the precise statement of his upper bound:

Theorem 1 ([11]). *For any $C > 5$, $r \geq 2$, and all sufficiently large n we have*

$$\hat{R}(P_n, r) < 400^5 C r^{2+\frac{1}{C-4}} n.$$

It is straightforward to see that $C = C(r)$ that minimizes the upper bound in this theorem is of order $\log r$. As a result we get that $\hat{R}(P_n, r) = O((\log r)r^2n)$. In this note, we give an alternative proof of this fact.

Theorem 2. *For any integer $r \geq 2$ and all sufficiently large n we have*

$$\hat{R}(P_n, r) < 600(\log r)r^2n.$$

It will follow from the proof that the constant 600 is not optimal. Since we believe that the factor $\log r$ is not necessary, we do not attempt to optimize it.

2 Proof

Before we move to the proof of Theorem 2, we need one, straightforward, auxiliary result.

Proposition 3. *For any integer $r \geq 2$ there exists an integer $N = N(r)$ such that the following holds. For any integer $n \geq N$, there exists a graph $G = (V, E)$ such that*

- (i) $|V| = 7rn$,
- (ii) $500(\log r)r^2n < |E| < 600(\log r)r^2n$, and
- (iii) *for every two disjoint sets $S, T \subseteq V$, $|S| = |T| = n$, the number of edges induced by $S \cup T$ with at least one endpoint in S is at most $70(\log r)n$.*

Proof. The proof is an easy application of random graphs. Recall that the *binomial random graph* $\mathcal{G}(n, p)$ is a distribution over the class of graphs with vertex set $[n]$ in which every pair $\{i, j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p , which may (and usually does) tend to zero as n tends to infinity. Furthermore, we say that events A_n in a probability space hold *asymptotically almost surely* (or *a.a.s.*), if the probability that A_n holds tends to 1 as n goes to infinity.

Fix any integer $r \geq 2$. It suffices to show that the random graph $G \in \mathcal{G}(7rn, p)$ with $p = 22(\log r)/n$ a.a.s. satisfies properties (ii) and (iii). (Property (i) trivially holds.) Indeed, if this is the case, then there exists an integer $N = N(r)$ such that the desired properties hold with probability at least $1/2$ for $G \in \mathcal{G}(7rn, p)$ for all $n \geq N$. This implies that for each $n \geq N$, there exists at least one graph with these properties.

Property (iii): Fix any two disjoint subsets $S, T \subseteq V$, both of cardinality n . Let $X_{S,T}$ be the random variable counting the number of edges induced by $S \cup T$ with at least one endpoint in S . Clearly, $X_{S,T}$ has the binomial distribution $\text{Bin}(|S| \cdot |T| + \binom{|S|}{2}, p)$ with $\mathbb{E}(X_{S,T}) = (3/2 + o(1))n^2p = (33 + o(1))(\log r)n$. It follows from Chernoff's bound (see, for example, Corollary 21.7 in [10]) that

$$\Pr(X_{S,T} \geq 70(\log r)n) \leq \Pr(X_{S,T} \geq 2\mathbb{E}(X_{S,T})) \leq \exp(-\mathbb{E}(X_{S,T})/3) \leq \exp(-10.9(\log r)n).$$

Thus, the probability that there exist S and T such that $X_{S,T} \geq 70(\log r)n$ is, by the union bound, at most

$$\begin{aligned} \binom{7rn}{n}^2 \exp(-10.9(\log r)n) &\leq (7er)^{2n} \exp(-10.9(\log r)n) \\ &\leq \exp\left(n\left(2\log(7er) - 10.9\log r\right)\right) = o(1), \end{aligned}$$

since $(7er)^2 < r^{10.9}$ for any $r \geq 2$. Property (iii) holds a.a.s.

Property (ii): This property is straightforward to prove. Note that $|E|$ is distributed as $\text{Bin}\left(\binom{7rn}{2}, p\right)$ with $\mathbb{E}(|E|) = (539 + o(1))(\log r)r^2n$. It follows immediately from Chernoff's bound that property (ii) holds a.a.s. The proof of the proposition is finished. \square

Proof of Theorem 2. The proof is based on the depth first search algorithm (DFS), applied several times, and it is a variant of the previous approach taken in [7] where it was proved that $\hat{R}(P_n, r) \leq 33r^4n$. Ben-Eliezer, Krivelevich and Sudakov [3] were the first to successfully apply the DFS algorithm to a Ramsey-type problem.

Fix $r \geq 2$ and suppose that n is sufficiently large so that Proposition 3 can be applied. Let $G = (V, E)$ be a graph satisfying properties (i)–(iii) from Proposition 3. We will show that $G \rightarrow (P_n)_r$ which implies the desired upper bound as $|E| < 600(\log r)r^2n$ by property (ii). Consider any r -colouring of the edges of G . By an averaging argument, there is a colour (say blue) such that the number of blue edges is at least $|E(G)|/r$. For a contradiction, suppose that there is no monochromatic copy of P_n ; in particular, there is no blue copy of P_n .

From now on, we restrict ourselves to the subgraph of G induced by blue edges, denoted $G_1 = (V_1 = V, E_1 \subseteq E)$. We perform the following algorithm on G_1 to construct a path

P . Let v_1 be an arbitrary vertex of G_1 , let $P = (v_1)$, $U = V \setminus \{v_1\}$, and $W = \emptyset$. If there exists an edge from v_1 to some vertex in U (say from v_1 to v_2), we extend the path as $P = (v_1, v_2)$ and remove v_2 from U . We continue extending the path P this way for as long as possible. Since there is no P_n in the blue graph, we must reach a point of the process in which P cannot be extended, that is, there is a path from v_1 to v_k ($k < n$) and there is no edge from v_k to U (including the case when U is empty). This time, v_k is moved to W and we try to continue extending the path from v_{k-1} , reaching another critical point in which another vertex will be moved to W , etc. If P is reduced to a single vertex v_1 and no edge to U is found, we move v_1 to W and simply re-start the process from another vertex from U , again arbitrarily chosen.

Observe that during this algorithm there is never an edge between U and W . Moreover, in each step of the process, the size of U decreases by 1 or the size of W increases by 1. The algorithm ends when U becomes empty and all vertices from P are moved to W . However, we will finish it prematurely, distinguishing $7r$ phases; phase i starts with graph $G_i = (V_i, E_i)$ and ends when for the first time $|W| = n$. Before we move to the next phase, we set $S_i = W$, $T_i = V(P)$, and F_i to be all edges incident to W . Then, we set $V_{i+1} = V_i \setminus W$ and $G_{i+1} = G_i[V_{i+1}]$, the graph induced by V_{i+1} (in other words, G_{i+1} is formed from G_i by removing vertices from W together with F_i , all edges incident with them). Phase i ends now and we move to phase $i+1$ where we run the algorithm on G_{i+1} .

We make a few important observations. Note that, by property (i), $|V| = |V_1| = 7rn$ so the last phase, phase $7r$, finishes with $U = \emptyset$ and $T_{7r} = \emptyset$. As a result, family $(F_i : 1 \leq i \leq 7r)$ is a partition of E_1 . By construction, $|S_i| = n$ for all i and, since there is no path on n vertices in G_1 (and so also in any G_i), $|T_i| < n$ for all i . Hence, $|F_i| < 70(\log r)n$ by property (iii). Putting these things together and using property (ii) in the very last inequality, we get the desired contradiction:

$$|E|/r \leq |E_1| = |F_1| + |F_2| + \cdots + |F_{7r}| \leq 7r \cdot 70(\log r)n < 500(\log r)rn < |E|/r.$$

The proof is finished. □

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