An orthosymplectic Pieri rule

Anna Stokke*

Department of Mathematics and Statistics
University of Winnipeg, Canada
a.stokke@uwinnipeg.ca

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Abstract

The classical Pieri formula gives a combinatorial rule for decomposing the product of a Schur function and a complete homogeneous symmetric polynomial as a linear combination of Schur functions with integer coefficients. We give a Pieri rule for describing the product of an orthosymplectic character and an orthosymplectic character arising from a one-row partition. We establish that the orthosymplectic Pieri rule coincides with Sundaram's Pieri rule for symplectic characters and that orthosymplectic characters and symplectic characters obey the same product rule.

Mathematics Subject Classifications: 05E05, 05E10

1 Introduction

The celebrated Littlewood-Richardson rule gives a combinatorial description of the coefficients that appear when the product $s_\lambda s_\mu$ of two Schur functions is written as a linear combination of Schur functions in the ring of symmetric polynomials [8, 12]. In the case where one of the partitions is a one-row partition $\mu = (k)$, the expansion of the product takes an especially nice form and the combinatorial description of the coefficients is known as Pieri's rule. Pieri's rule states that the product $s_\lambda s_{(k)}$ is the sum of Schur functions s_ν, where the Young diagram of shape ν can be obtained from that of λ by adding k boxes, where no two of the added boxes are in the same column.

Berele and Regev [3] introduced hook Schur functions, which are characters of general linear Lie superalgebras. Remmel [9] proved a Jacobi-Trudi identity for hook Schur functions, via lattice paths, that coincides with that for ordinary Schur functions and also proved that hook Schur functions and ordinary Schur functions satisfy the same Pieri rule, thus establishing that hook Schur functions obey the Littlewood-Richardson rule.

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Symplectic Schur functions, which are characters of irreducible representations of symplectic groups, can be described in terms of the semistandard symplectic tableaux of King [7]. Berele [2] gave an insertion algorithm for inserting a letter into a symplectic tableau using a combination of Knuth-Schensted row insertion and the jeu de taquin algorithm. Sundaram [15] gave a Pieri rule for decomposing the product $sp_\lambda sp_{(k)}$ of two symplectic Schur functions, one of which is associated to a one-row partition, as a linear combination of symplectic Schur functions using Berele's row-insertion algorithm.

Benkart, Shader and Ram [1] gave combinatorial descriptions of characters of representations of orthosymplectic Lie superalgebras $spo(2m, n)$, or orthosymplectic Schur functions, in terms of *spo*-tableaux, which have both a symplectic part and a row-strict part. An *spo*-insertion algorithm for inserting a letter into an *spo*-tableau to produce a new *spo*-tableau is also described in [1].

The aim of this paper is to present a Pieri rule for orthosymplectic Schur functions using *spo*-tableaux. Further, we prove that the orthosymplectic Pieri rule coincides with that for symplectic Schur functions. In [1], a Jacobi-Trudi identity for orthosymplectic Schur functions is given that mirrors the symplectic Jacobi-Trudi identity; this result is proved using lattice path arguments in [14]. It follows that the coefficients arising in the expansion of the product of two orthosymplectic Schur functions are the same as those that appear in the expansion of the product of the corresponding symplectic Schur functions.

We begin with a preliminary section before discussing the *spo*-insertion algorithm in Section 3. We then present a definition for the combinatorial product of an *spo*-tableau and a one-row *spo*-tableau and set about proving several technical lemmas in Section 4. Our main result, Theorem 4.1, describes the integer coefficients of the orthosymplectic characters in the expansion of the product $spo_\lambda spo_{(k)}$ of an orthosymplectic character and an orthosymplectic character associated to a one-row tableau.

2 Preliminaries

A *partition* of a positive integer N is a k-tuple of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ and $|\lambda| = \sum_{i=1}^{k} \lambda_i = N$. The *Young diagram* of shape λ contains N boxes in k left-justified rows with λ_i boxes in the ith row. For convenience, we will also denote the Young diagram of shape λ by λ. The *length* of the Young diagram of shape λ, denoted $\ell(\lambda)$, is equal to the number of rows in λ. The *conjugate* of λ is the partition $\lambda^t = (\lambda_1^t, \lambda_2^t, \dots, \lambda_s^t)$ where λ_i^t denotes the number of boxes in the ith column of the Young diagram of shape λ. A λ-tableau is obtained by filling the Young diagram of shape λ with entries from a set $\{1, 2, \dots, n\}$ of positive integers. A λ-tableau is *semistandard* if the entries in the rows are weakly increasing from left to right and the entries in the columns are strictly increasing from top to bottom.

Given partitions λ and μ, we write $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for $i \geqslant 1$. The *skew diagram* of shape λ/μ is formed by removing the Young diagram of shape μ from the upper left-hand corner of the Young diagram of shape λ and $|\lambda/\mu| = |\lambda| - |\mu|$. A *skew tableau* of shape λ/μ is obtained by filling the skew diagram of shape λ/μ with entries from a set $\{1, 2, \dots, n\}$

and is semistandard if the entries in the rows increase weakly from left to right and the entries in the columns increase strictly from top to bottom. A *horizontal strip* is a skew diagram λ/μ with no two boxes in the same column.

For a λ-tableau T, let $a_i(T)$ denote the number of entries equal to i in T. The *weight* of T is the monomial in the variables $X = \{x_1, x_2, \ldots, x_n\}$ defined by $\mathrm{wt}(T) = \prod_{i=1}^{n} x_i^{a_i(T)}$. The *Schur function* corresponding to λ is

$$s_\lambda(X) = \sum_T \mathrm{wt}(T),$$

where the sum runs over all semistandard λ-tableaux T with entries in $\{1, 2, \ldots, n\}$. The *weight* of a skew tableau with entries from $\{1, \ldots, n\}$ and the *skew Schur polynomial*, $s_{\lambda/\mu}(x_1, \ldots, x_n)$, are defined similarly.

Suppose that λ is a partition in which the Young diagram of shape λ has no more than m rows. A *semistandard symplectic λ-tableau* (see [7]) has entries in the set $B_0 = \{1, \overline{1}, 2, \overline{2}, \ldots, m, \overline{m}\}$, is semistandard with respect to the ordering $1 < \overline{1} < 2 < \overline{2} < \cdots < m < \overline{m}$ and satisfies the property (*symplectic condition*) that the entries in the ith row of T are greater than or equal to i, for each $1 \leqslant i \leqslant m$.

Let $\overline{X} = \{x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}\}$. The weight of a symplectic λ-tableau T is $\mathrm{wt}(T) = \prod_{i=1}^{m} x_i^{a_i(T)-a_{\overline{i}}(T)}$ where $a_i(T)$ (respectively $a_{\overline{i}}(T)$) is equal to the number of entries equal to i (respectively $\overline{i}$) in T. The *symplectic Schur function* is

$$sp_{\lambda,2m}(\overline{X}) = \sum_T \mathrm{wt}(T),$$

where the sum runs over the semistandard symplectic λ-tableaux T with entries in $\overline{X}$.

Example 2.1. Let $\lambda = (3, 2, 1)$ and $m = 4$. The first λ-tableau below is semistandard symplectic, while the second is not, and $\mathrm{wt}(T_1) = x_2 x_3^{-2} x_4$.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & \overline{1} & 4 \\ \hline 2 & \overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array}, \quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & \overline{3} \\ \cline{1-2} \overline{2} \\ \cline{1-1}\end{array}$$

Schur functions and symplectic Schur functions are characters of irreducible polynomial representations of the general linear groups $GL(n, \mathbb{C})$ and symplectic groups $Sp(2m, \mathbb{C})$, respectively. We now turn our attention to the combinatorial properties of characters of orthosymplectic Lie superalgebras. For background on the representation theory of Lie superalgebras, see [5]. Characters of orthosymplectic Lie superalgebras $spo(2m, n)$ are hybrids of symplectic Schur functions and ordinary Schur functions and can be defined in terms of $spo(2m, n)$-tableaux [1], which we now define.

Let m and n be positive integers, $B_0 = \{1, \overline{1}, 2, \overline{2}, \ldots, m, \overline{m}\}$, $B_1 = \{1^\circ, 2^\circ, \ldots, n^\circ\}$, $B = B_0 \cup B_1$ and order B as follows:

$$1 < \overline{1} < 2 < \overline{2} < \cdots < m < \overline{m} < 1^\circ < 2^\circ < \cdots < n^\circ.$$

An *$spo(2m, n)$-tableau* T of shape λ (or an *spo*-tableau) is a filling of the Young diagram of shape λ with entries from B that satisfies the following conditions:

1. the portion S of T that contains entries only from B_0 is semistandard symplectic;
2. the skew tableau formed by removing S from T is strictly increasing across the rows from left to right and weakly increasing down columns from top to bottom.

Let $\overline{X} = \{x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \overline{X} \cup Y$ and let λ be a partition. The *orthosymplectic character* corresponding to λ is given by

$$spo_\lambda(Z) = \sum_{\substack{\mu \subseteq \lambda \\ \ell(\mu) \leqslant m}} sp_\mu(\overline{X}) s_{\lambda^t/\mu^t}(Y),$$

where $sp_\mu(\overline{X})$ is a symplectic Schur polynomial in the variables $\overline{X}$ and $s_{\lambda^t/\mu^t}(Y)$ is a skew Schur polynomial in the variables Y [1, Theorem 4.24].

An orthosymplectic character can also be described in terms of *spo*-tableau. To define the weight of an *spo*-tableau T, replace each $i \in \{1, 2, \ldots, m\}$ that belongs to T with x_i, each $\bar{i} \in \{\overline{1}, \ldots, \overline{m}\}$ in T with x_i^{-1} and each $i^\circ \in \{1^\circ, \ldots, n^\circ\}$ in T with y_i and let $\mathrm{wt}(T)$ be the product of these variables. Then $spo_\lambda(Z) = \sum_T \mathrm{wt}(T)$, where the sum is over the *spo*-tableaux of shape λ with entries from Z [1, Theorem 5.1].

Example 2.2. The following is an *spo*$(8, 3)$-tableau of shape $\lambda = (3, 3, 2, 2)$ and $\mathrm{wt}(T) = x_1^{-1} x_4 y_1^2 y_2 y_3^3$.

$$T = \begin{array}{|c|c|c|} \hline \overline{1} & 2 & 3^\circ \\ \hline \overline{2} & 1^\circ & 3^\circ \\ \hline 4 & 1^\circ \\ \cline{1-2} 2^\circ & 3^\circ \\ \cline{1-2} \end{array}$$

Example 2.3. Let $\lambda = (1, 1)$, $m = 2$ and $n = 1$. Then $\overline{X} = \{x_1, x_1^{-1}, x_2, x_2^{-1}\}$, $Y = \{y_1\}$ and

$$spo_\lambda(Z) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} + 1 + x_1y_1 + x_1^{-1}y_1 + x_2y_1 + x_2^{-1}y_1 + y_1^2.$$

3 Orthosymplectic insertion and the product of tableaux

We first recall ordinary (Schensted-Knuth) *row insertion* for semistandard λ-tableaux (see [6, 10, 11, 13]). Given a semistandard λ-tableau T and a positive integer x, construct a new tableau, denoted $T \leftarrow x$, which has one more box than T as follows: if x is greater than or equal to every entry in the first row, add x in a new box at the end of the first row, giving a new tableau $T \leftarrow x$. If this is not the case, replace the left-most entry in the first row that is strictly larger than x with x and bump the displaced entry into the second row by repeating the process with the displaced positive integer in row two. Continue the process for successive rows until an entry lands in a new box at the end of a row or until an entry bumps into a new box at the bottom of the tableau, forming a new row. The resulting tableau is $T \leftarrow x$ and the new box in $T \leftarrow x$ belongs to an *outer corner* of the Young diagram, which means that there is not a box directly below it or directly to its right.

Insertion for *spo*-tableaux has a similar flavour to the insertion algorithm for (k,ℓ)-semistandard tableaux [3,9] ((k,ℓ)-semistandard tableaux arise in the description of characters of general linear Lie superalgebras). In that setting, insertion consists of both ordinary row insertion and ordinary column insertion; insertion for *spo*-tableaux incorporates Berele's row insertion [2] for the symplectic part of a tableau (entries from B_0) and ordinary column insertion for the remaining entries from B_1. To define Berele's insertion algorithm, we first define a version of Schutzenberger's jeu de taquin algorithm for *spo*-tableau, which will also apply to symplectic tableaux. A *punctured spo-tableau* is obtained from an *spo*-tableau by removing the entry from one box. An orthosymplectic version of jeu de taquin gives a way to move an empty box in the first column of a punctured *spo*-tableau through the tableau until the empty box lands in an outer corner. A *forward slide* is achieved using the following moves:

$$\begin{array}{|c|c|}\hline & a \\ \hline b \\ \cline{1-1}\end{array} \;\rightarrow\; \begin{cases} \begin{array}{|c|c|}\hline a & \\ \hline b \\ \cline{1-1}\end{array} & \text{if } a < b \text{ or } a = b \text{ and } a, b \in B_1 \\[2ex] \begin{array}{|c|c|}\hline b & a \\ \hline \\ \cline{1-1}\end{array} & \text{if } b < a \text{ or } a = b \text{ and } a, b \in B_0. \end{cases}$$

When a forward slide is performed on an empty box in a punctured *spo*-tableau, the result is a punctured *spo*-tableau. Furthermore, given a punctured *spo*-tableau, where the empty box occurs in the first column, forward slides can be continued until the empty box lands in an outer corner [1]. The procedure is also reversible and we will refer to the slides that reverse the procedure as *reverse slides*.

To describe Berele row insertion, consider a semistandard symplectic tableau T and $x \in B_0$. If ordinary row insertion of x into T does not introduce a violation of the symplectic condition, then the new tableau produced by ordinary row insertion is the result of Berele row insertion. If a symplectic violation is introduced, it must be the case that, for some i, an $\bar{i}$ was bumped out of row i and into row $i+1$ by an i. Find the least i for which this occurs and, at this stage, instead of replacing the $\bar{i}$ with an i, remove the $\bar{i}$ from the tableau, leaving a hole in the box that contained it. Apply jeu de taquin forward slides to the empty box until it lands in an outer corner, giving a semistandard symplectic tableau that has one less box than T. This will be referred to as a cancellation and this new tableau is the result of the Berele insertion of x into T.

Example 3.1. In the following example, the entry $\bar{1}$ is inserted into the symplectic tableau T.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & \mathbf{2} & 3 \\ \hline \bar{2} & 3 \\ \cline{1-2} \bar{3} & 4 \\ \cline{1-2}\end{array} \leftarrow \bar{1} \qquad \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & 3 \\ \hline & 3 \\ \cline{1-2} \bar{3} & 4 \\ \cline{1-2}\end{array} \;;\; \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & 3 \\ \hline 3 & 4 \\ \cline{1-2} \bar{3} \\ \cline{1-1}\end{array}$$

We now define insertion of a letter $x \in B_0 \cup B_1$ into an *spo*-tableau T, as in [1, §5.3]. We will refer to the procedure as *spo*-insertion and if $x \in B_0$, insertion of x into T will be denoted $T \leftarrow x$, while if $x \in B_1$, insertion of x into T will be denoted $x \rightarrow T$. This is consistent with notation used in [11], since entries in B_0 will be inserted into rows, while entries in B_1 will be inserted into columns.

We first describe how to insert an entry x into a given row or column. If $x \in B_1$ is greater than or equal to every entry in a given column, place x in a new box at the end of that column. If not, replace the least entry y in the column that satisfies $y > x$ with x, which displaces y. For entries x belonging to B_0, insert x into a given row by placing it in a new box at the end of the row if x is larger than every entry in the row. Otherwise, replace the least entry y in the row that satisfies $y > x$ with x, displacing y, provided that doing so does not cause x to replace an entry $\overline{x}$ in row x. If this is the case, remove $\overline{x}$ from row x, leaving an empty box.

In general, to insert x into an *spo*-tableau T, begin by inserting x into the first row of T if $x \in B_0$ and insert x into the first column of T if $x \in B_1$, as described above. If doing so displaces an entry $y \in B_0$, insert y into the subsequent row using the same procedure, while if an entry $y \in B_1$ is displaced, insert y into the subsequent column. On the other hand, if an empty box arises, move the empty box to an outer corner of the tableau using jeu de taquin slides. Repeat the procedure until an entry lands in a new box at the end of a row or until an empty box moves to an outer corner. The process results in an *spo*-tableau $T \leftarrow x$, if $x \in B_0$, and $x \rightarrow T$, if $x \in B_1$. When a cancellation occurs, the new tableau $T \leftarrow x$ has one less box than T. If a cancellation does not occur, $T \leftarrow x$ (respectively $x \rightarrow T$) has one more box than T and the new box belongs to an outer corner.

Example 3.2. The following example illustrates the steps in the *spo*-insertion process, where $\overline{1}$ is inserted into the *spo*-tableau T.

$$T = \begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \overline{2} \\ \hline 2 & 3 & 4 \\ \hline \overline{3} & 1^\circ \\ \cline{1-2} 1^\circ & 2^\circ \\ \cline{1-2}\end{array} \leftarrow \overline{1} \quad \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{2} \\ \hline 2 & \mathbf{3} & 4 \\ \hline \overline{3} & 1^\circ \\ \cline{1-2} 1^\circ & 2^\circ \\ \cline{1-2}\end{array} \leftarrow 2 \quad \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{2} \\ \hline 2 & 2 & 4 \\ \hline \mathbf{\overline{3}} & 1^\circ \\ \cline{1-2} 1^\circ & 2^\circ \\ \cline{1-2}\end{array} \leftarrow 3 \quad \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{2} \\ \hline 2 & 2 & 4 \\ \hline & 1^\circ \\ \cline{1-2} 1^\circ & 2^\circ \\ \cline{1-2}\end{array}; \quad \begin{array}{|c|c|c|}\hline 1 & \overline{1} & \overline{2} \\ \hline 2 & 2 & 4 \\ \hline 1^\circ & 2^\circ \\ \cline{1-2} 1^\circ \\ \cline{1-1}\end{array}$$

Let $x \in B_0$, suppose that x does not cause a cancellation when inserted into an *spo*-tableau T and suppose that $T \leftarrow x$ has shape ν. Define the *sp-bumping route* of x to be the collection of boxes from the Young diagram of shape ν that consist of those boxes from which an element from B_0 was bumped from a row, together with the last box in which an element from B_0 lands. This last box will be referred to as the final box in the bumping route for x. As in [6], we say that an *sp*-bumping route R_1 sits strictly right (or weakly right) of an *sp*-bumping route R_2 if for every row in the Young diagram that contains a box of R_1, R_2 contains a box that is right of the box in R_1. We define R_1 to be above (or weakly above) R_2 similarly.

Example 3.3. The *sp*-bumping route for the row-insertion of 3 into the given tableau T is marked in bold.

$$\begin{array}{|c|c|c|}\hline \overline{1} & \overline{3} & 5^\circ \\ \hline 4 & 4^\circ \\ \cline{1-2} 3^\circ & 4^\circ \\ \cline{1-2}\end{array} \leftarrow 3 = \begin{array}{|c|c|c|c|}\hline \overline{1} & \mathbf{3} & 4^\circ & 5^\circ \\ \hline \mathbf{\overline{3}} & 3^\circ \\ \cline{1-2} \mathbf{4} & 4^\circ \\ \cline{1-2}\end{array}$$

In what follows, we will associate each term in an orthosymplectic Schur function spo_λ with an *spo*-tableau T of shape λ, and each term in an orthosymplectic Schur function $spo_{(k)}$, where k is a positive integer, with a one-row *spo*-tableau U with k boxes. The result of multiplying a term from spo_λ with a term from $spo_{(k)}$ will correspond to a certain ν-tableau, which has shape corresponding to the *product* $T \cdot U$, which we now define.

Given an *spo*-tableau T and a one-row *spo*-tableau $U = \boxed{a_1|a_2|\cdots|a_r|b_1|b_2|\cdots|b_t}$, where $a_i \in B_0$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$ and $b_i \in B_1$ with $b_1 < b_2 < \cdots < b_t$, define

$$T \cdot U = (b_1 \to \cdots \to b_t \to T) \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_r.$$

The tableau resulting from each stage of the *spo*-insertion process is an *spo*-tableau [1, Lemma 5.4] so $T \cdot U$ is an *spo*-tableau.

Example 3.4. The tableaux involved in each step of the construction of $T \cdot U$, where $U = \boxed{1|3|5|2^\circ|5^\circ}$ are given below.

$$T = \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ \\ \cline{1-2} 2^\circ \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ \\ \cline{1-2} 2^\circ \\ \cline{1-1} 5^\circ \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & \overline{4} & 5 \\ \hline 3 & 1^\circ \\ \cline{1-2} 4 & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \cline{1-3} 4 & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ \\ \cline{1-1} \end{array} = T \cdot U$$

4 An orthosymplectic Pieri rule

Our main result is a Pieri rule for orthosymplectic Schur functions.

Theorem 4.1. *Let λ be a partition and k be a positive integer. Then*

$$spo_\lambda \cdot spo_{(k)} = \sum_\nu \alpha_\nu spo_\nu, \text{ where}$$

$$\alpha_\nu = |\{\mu \mid \mu \subseteq \lambda, \mu \subseteq \nu, \lambda/\mu, \nu/\mu \text{ are horizontal strips and } |\nu/\mu| + |\lambda/\mu| = k\}|.$$

Theorem 4.1 establishes that orthosymplectic Schur functions obey the same Pieri rule as Sundaram's Pieri rule for symplectic Schur functions [15, Theorem 4.1]. Since the Jacobi-Trudi identity for orthosymplectic Schur functions matches that for symplectic Schur functions (see [1] and [14, Corollary 4.3]), the structure coefficients that appear in the expansion of the product of two orthosymplectic Schur functions are the same as those that appear in the expansion of the product of the corresponding symplectic Schur functions. This situation is similar to that which occurs for characters of general linear Lie superalgebras [3, 9].

Corollary 4.2. *Suppose that $sp_\lambda \cdot sp_\mu = \sum \alpha_\nu sp_\nu$. Then $spo_\lambda \cdot spo_\mu = \sum \alpha_\nu spo_\nu$.*

We now set about proving several lemmas from which Theorem 4.1 follows. Lemmas 4.3 – 4.13 allow us to describe the shape ν of a tableau that is the product of an *spo*-tableau and a one-row *spo*-tableau in Theorem 4.14.

Lemmas 4.3 and 4.5 follow from known results about ordinary insertion into semistandard tableaux. Proofs can be found in, for instance, [6, §1.1].

Lemma 4.3. *Suppose that $x_1, x_2 \in B_1 = \{1^\circ, \dots, n^\circ\}$, with $x_1 > x_2$ and that the box added to the Young diagram of T in $x_1 \to T$ is S_1, and the box added to the shape of $x_1 \to T$ in $x_2 \to (x_1 \to T)$ is S_2. Then S_2 is weakly above and strictly right of S_1.*

Corollary 4.4. *Suppose that $x_1, \dots, x_r \in B_1$, with $x_1 > x_2 > \cdots > x_r$, that T is an spo-tableau of shape λ, and that $S = x_r \to \cdots \to x_1 \to T$ has shape μ. Then μ/λ is a horizontal strip.*

Lemma 4.5. *Suppose that $x_1, x_2 \in B_0 = \{1, \overline{1}, \dots, m, \overline{m}\}$ with $x_1 \leqslant x_2$ and that neither insertion of x_1 into an spo-tableau T nor insertion of x_2 into the resulting tableau $T \leftarrow x_1$ introduce cancellations. Let R_1 be the sp-bumping route for x_1 and R_2 the sp-bumping route for x_2. Then the final box in R_2 is weakly above and strictly right of the final box in R_1.*

Lemma 4.6. *Suppose that $x \in B_1$ is inserted into an spo-tableau T giving a new box S_1 in $x \to T$, and insertion of $y \in B_0$ into $x \to T$ does not cause a cancellation and gives a new box S_2 in $(x \to T) \leftarrow y$. Then S_2 is weakly above and strictly right of S_1.*

Proof. Let $U = x \to T$ and suppose that S_1 belongs to column j of U and contains $x_j \in B_1$. Then there are entries $x_1, \dots, x_{j-1}$, where x_i belongs to column $i < j$ of U and $x_i \in B_1$, such that $x_1 < x_2 < \cdots < x_j$. If the *sp*-bumping route of y ends in a column strictly right of column j, the result follows so suppose that the *sp*-bumping route of y ends in column k, where $k \leqslant j$. Then, since each of columns $1, \dots, k$ contain entries from B_1, the entry in the final box of the *sp*-bumping route of y upon insertion into U bumps an entry $a \in B_1$, with $a \leqslant x_k$, out of column k and into column $k+1$, which then bumps an entry from B_1 that is less than or equal to x_{k+1} into column $k+2$, et cetera. This continues until an entry from B_1 that is less than or equal to x_j is bumped out of column j and into column $j+1$, which necessarily bumps into a box in the same row or in a row above the row containing S_1. It follows that S_2 belongs to column strictly right of column j and is weakly above the box S_1. □

Lemma 4.7. *Suppose that $x_1, x_2 \in B_0$ with $x_1 \leqslant x_2$ and that neither insertion of x_1 into an spo-tableau T nor insertion of x_2 into the resulting tableau $T \leftarrow x_1$ introduce cancellations. Let S_1 denote the new box in $U = T \leftarrow x_1$ that results from the x_1 insertion and S_2 the new box in $U \leftarrow x_2$ that results from the x_2 insertion. Then S_2 is strictly right of and weakly above S_1.*

Proof. Suppose that neither insertion of x_1 into T nor insertion of x_2 into U displace entries from B_1. Then S_1 is the final box in the *sp*-bumping route of x_1 and S_2 is the final box in the *sp*-bumping route of x_2 so the result follows from Lemma 4.5.

Suppose that insertion of x_1 does not displace an entry from B_1, but insertion of x_2 into U does displace an entry from B_1. Then S_1 is the final box in the *sp*-bumping route of x_1 and the final box in the *sp*-bumping route of x_2 is strictly right of and weakly above S_1 by Lemma 4.5. But x_2 insertion bumps an entry from B_1 out of this final box in the *sp*-bumping route into a box in the next column that is strictly right of and weakly above it and this continues until the new box S_2 is added to U, so S_2 is strictly right of and weakly above S_1.

If insertion of x_1 into T displaces an entry from B_1, but insertion of x_2 into U does not, then S_2 must belong to a column strictly right of that which contains S_1. Indeed, since insertion of x_1 displaces an entry from B_1, each outer corner in columns $1, \ldots, j-1$ of U must contain an entry from B_1. Since S_2 is an outer corner and insertion of x_2 does not displace an entry from B_1, S_2 must belong to a column to the right of column j. Since S_2 contains an entry from B_0, while S_1 does not, S_2 must belong to a row above S_1.

Suppose that both insertions displace entries from B_1 and suppose that the final box in the *sp*-bumping route for x_2 belongs to column k. By Lemma 4.5, the final box for the *sp*-bumping route of x_1 is strictly left of column k. If the box S_1 that is added to T by x_1 insertion belongs to column $j \leqslant k$, then, since x_2 insertion bumps an entry from column k into column $k+1$ and S_1 and S_2 are both at the end of columns, S_1 belongs to a row below S_2. Otherwise, suppose that insertion of x_1 into T adds the box S_1 to a column $j > k$. Then x_1 insertion bumps an entry $s_1 \in B_1$ from column k and into column $k+1$, replacing s_1 with an entry $s_2 < s_1$, where $s_2 \in B_1$. Since the final box for the *sp*-bumping route of x_2 is in column k, x_2 insertion displaces the smallest entry $s_3 \in B_1$ in column k and bumps it into column $k+1$. But $s_3 \leqslant s_2 < s_1$ and the final box produced by x_1 insertion is determined by insertion of s_1 into column $k+1$ and the remaining columns of T, while the final box produced by s_2 insertion is determined by insertion s_3 into column $k+1$ and the remaining columns of $T \leftarrow x_1$. The result follows from Lemma 4.3. □

Lemma 4.8. [15, Lemma 3.2] *Suppose that T is an spo-tableau and that $x_1, x_2 \in B_0$ with $x_1 \leqslant x_2$. If insertion of x_2 into $T \leftarrow x_1$ causes a cancellation, then insertion of x_1 into T also causes a cancellation.*

Corollary 4.9. *Let T be an spo-tableau and U a one-row spo-tableau. Entries in U that cause cancellations when $T \cdot U$ is constructed occur in an initial strip at the beginning of U.*

Lemma 4.10. *Suppose that $x_1 \in B_1$ and that insertion of x_1 into T adds a new box to row one of the Young diagram of T. Then if $x_2 \in B_0$ and insertion of x_2 into $U = x_1 \rightarrow T$ causes a cancellation, the tableau $U \leftarrow x_2$ and U have the same number of boxes in the first row. In other words, the cancellation does not result in the removal of a box from the first row.*

Proof. The added box in row one of U contains an entry from B_1. Suppose that column k is the leftmost column in which an entry from B_1 bumps into the first row during the x_1 insertion process and let $a \in B_1$ denote the entry that belongs to column k in the first row of U. Then column $k-1$ of U contains an entry $b \in B_1$, where $b < a$. Suppose that insertion of x_2 into U causes a cancellation and that moving the resulting empty box through the tableau via jeu de taquin results in the removal of a box from row one of U. Then the cancellation must occur in the first row of T and the resulting empty box moves to the end of row one via jeu de taquin. This then moves the entry a in row one above the entry b, which is a contradiction, since $b < a$. □

Lemma 4.11. [15, Lemma 3.3] *Suppose that $x_1, x_2 \in B_0$ with $x_1 \leqslant x_2$ and that T is a symplectic tableau. Suppose that x_1 causes a cancellation when inserted into T via the Berele insertion algorithm and that x_2 causes a cancellation when inserted into the tableau $T \leftarrow x_1$. Then the box removed from $T \leftarrow x_1$ by insertion of x_2 sits weakly below and strictly left of the box removed from T by insertion of x_1 into T.*

Lemma 4.12. *Suppose that $x_1, x_2 \in B_0$ with $x_1 \leqslant x_2$ and suppose that x_1 causes a cancellation when inserted into an spo-tableau T and that x_2 causes a cancellation when inserted into $U = T \leftarrow x_1$. Then the box removed from U by insertion of x_2 belongs to a different column than the box removed from T by insertion of x_1.*

Proof. The empty box caused by cancellation when x_1 is inserted into T slides from the first column through the *sp*-portion of the tableau and into the remaining portion of the tableau via jeu de taquin. Let S_1 denote the last box in the jeu de taquin path of the empty box such that an entry from B_0 slides out of S_1. Then S_1 either belongs to an outer corner of T, in which case the insertion procedure is complete or an entry from B_1 slides into S_1. If S_2 is the last box in the jeu de taquin path of the empty box created by x_2 such that an entry from B_0 slides out of S_2, then S_2 is weakly below and strictly left of the box S_1 by Lemma 4.11 so the two boxes belong to different columns of T. If one or both boxes are the boxes removed through orthosymplectic insertion via x_1 and x_2 insertion respectively, the result follows. If the jeu de taquin process continues through the remaining portion of the tableau by sliding entries from B_1 into the boxes S_1 and S_2 and continuing, the two vacant boxes migrate to outer corners in different columns. This follows from [16], where the author proves jeu de taquin results through tableau switching. ☐

Lemma 4.13. *Suppose that $y \in B_1$ is inserted into an spo-tableau T and that $x_1, \ldots, x_r \in B_0$ with $x_1 \leqslant \ldots \leqslant x_r$ are subsequently inserted into the remaining tableaux producing $(y \to T) \leftarrow x_1 \leftarrow \cdots \leftarrow x_{r-1} \leftarrow x_r$ and suppose that x_{r-1} causes a cancellation while x_r does not. Then insertion of x_r into $(y \to T) \leftarrow x_1 \leftarrow \cdots \leftarrow x_{r-1}$ produces a new box in a column that is strictly right of the box added by insertion of y into T.*

Proof. Suppose that insertion of y augments the tableau T by a box in column j. Then each of columns 1 through j in $y \to T$ contain an entry from B_1. Furthermore, if a_1 is the smallest entry from B_1 that belongs to the first column of $y \to T$, then entries $a_2, \ldots, a_j \in B_1$ can be chosen, where a_i in column i, with $a_1 < a_2 < \cdots < a_j$. Take a_2 in the second column of $y \to T$ to be minimal with the property that $a_1 < a_2$ and, generally, take a_{i+1} to be minimal in column $i+1$ with the property that $a_i < a_{i+1}$, for $2 \leqslant i \leqslant j$. Then a_{i+1} belongs to a row weakly above a_i in $y \to T$. When a cancellation occurs and an empty box in column one slides through the tableau, none of the entries a_i move out of their respective columns since, otherwise, an entry a_i would slide into column $i-1$ above a_{i-1}, which is impossible. Thus, if $S = (y \to T) \leftarrow x_1 \cdots \leftarrow x_{r-1}$ then a_i also belongs to column i of S for $1 \leqslant i \leqslant j$.

If the final box in the *sp*-bumping route for x_r insertion into S belongs to a column strictly right of column j, the result follows so suppose that the final box in the *sp*-bumping route for x_r is in column i, which is weakly left of column j. Then an entry from B_0 displaces an entry from B_1 that is in this final box in column i and moves it into column $i+1$. Since the displaced entry is the smallest entry in column i that belongs to B_1, it is less than or equal to a_i and cannot land at the bottom of column $i+1$ since there is at least one entry in column $i+1$ – namely a_{i+1} – that is larger than it. Since there are entries $a_{i+1} < a_{i+2} < \cdots < a_j$ in each of the subsequent columns, which belong to B_1, an entry in B_1 cannot get appended to the bottom of any column prior to, and including, column j. Thus the new box belongs to a column strictly right of column j. □

The above lemmas show that the shape ν of $T \cdot U$ takes a particular form.

Theorem 4.14. *Let T be an spo-tableau of shape λ and U a semistandard one-row (k)-tableau containing entries $x_s \leqslant \cdots \leqslant x_1 \in B_0$ and $a_1 < \cdots < a_{m_1} \in B_1$. Then $T \cdot U$ is formed in three stages, producing intermediate tableaux of shapes μ° and μ^-: an addition stage that yields a tableau $T_{\mu_0} = a_1 \rightarrow \cdots \rightarrow a_{m_1} \rightarrow T$, then a cancellation stage, which produces $T_{\mu^-} = T_{\mu_0} \leftarrow x_s \leftarrow \cdots \leftarrow x_{m_2+1}$, for some $0 \leqslant m_2 \leqslant s$, followed by another addition stage, which gives $T_\nu = T_{\mu^-} \leftarrow x_{m_2} \leftarrow \cdots \leftarrow x_1$. Furthermore, $T \cdot U$ has shape ν, where ν is the end shape in a sequence of Young diagrams $(\lambda, \mu^\circ, \mu^-, \nu)$ that satisfies the following properties:*

1. *μ°/λ, μ°/μ^- and ν/μ^- are horizontal strips;*
2. *$|\mu^\circ/\lambda| = m_1$, for some $0 \leqslant m_1 \leqslant k$;*
3. *$|\mu^\circ/\mu^-|=\ell$, for some $0 \leqslant \ell \leqslant k - m_1$, and if the first rows of λ and μ^0 do not contain the same number of boxes then the first rows of μ° and ν do contain the same number of boxes;*
4. *$|\nu/\mu^-| = k - m_1 - \ell$ and the leftmost box in ν/μ^- belongs to a column that is strictly right of the rightmost box in μ^0/λ.*

We will call a sequence $(\lambda, \mu^\circ, \mu^-, \nu)$, where the Young diagrams satisfy the properties in Theorem 4.14, an *spo-bumping sequence*. A triple (λ, μ, ν), where $\mu \subseteq \lambda$ and $\mu \subseteq \nu$ will be called an *sp-bumping sequence* if λ/μ and ν/μ are horizontal strips. The triple (λ, μ, ν), where $|\lambda/\mu| = \ell$ and $|\nu/\mu| = k - \ell$, for $k \geqslant 0$ and $0 \leqslant \ell \leqslant k$, is an *sp*-bumping sequence if and only if the symplectic Schur function sp_ν appears in the expansion of the product of $sp_\lambda sp_{(k)}$ [15]. The shape μ° in the *spo*-sequence $(\lambda, \mu^\circ, \mu^-, \nu)$ described in Theorem 4.14 is that obtained by column-insertion of the entries in U that belong to B_1. The triple (μ°, μ^-, ν) is an *sp*-bumping sequence so for symplectic tableaux, the result follows from [15].

If $S = T{\cdot}U$ and the *spo*-bumping sequence corresponding to the product is $(\lambda, \mu^\circ, \mu^-, \nu)$, then the *spo*-insertion algorithm can be reversed to decompose S as a product. Working from the sequence, $m_1 = |\mu^\circ/\lambda|$, $\ell = |\mu^\circ/\mu^-|$, $m_2 = |\nu/\mu^-|$ and $k = m_1 + m_2 + \ell$. If the entry x in the rightmost box of ν/μ^- belongs to B_0, then if x is in the first row of S, it bumps out of S. Otherwise, bump $x \in B_0$ into the row above it by replacing the largest entry in that row that is less than it. Then bump the displaced entry into the row above it and continue until an entry from B_0 bumps out of the first row of the tableau. If $x \in B_1$, bump x into the column immediately left of it by replacing the largest entry less than it in the column with x. If the displaced entry belongs to B_1, bump the displaced entry into the column immediately left of it in a similar way. On the other hand, if the displaced entry belongs to B_0, bump it into the row above it as described above. Continue bumping displaced entries in this way until an entry from B_0 bumps out of row one. The procedure can be continued with the entries in the remaining boxes of ν/μ^- until m_2 entries x_i have been bumped out of the tableau in succession with $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{m_2}$, since ν/μ^- is a horizontal strip. This yields a tableau T_{μ^-} of shape μ^-.

The next portion of the reverse algorithm, which is part of the argument used to prove the symplectic Pieri rule [15], produces the symplectic part of the one-row tableau U that causes cancellations during the insertion procedure. Add empty boxes to T_{μ^-} to produce a punctured tableau of shape μ° with ℓ empty boxes. Move the leftmost empty box through the tableau using reverse jeu de taquin slides until it lands in the highest box possible in column one without causing a symplectic violation. Then place an $\overline{i}$ in the empty box, where i is equal to the row number in which the empty box sits. Then bump an i into the row above the $\overline{i}$ by replacing the largest entry less than i in that row with $\overline{i}$ and continue bumping up through the rows until an entry x_{m_2+1} is bumped out of row one. Continue the procedure with the empty boxes until ℓ entries $x_{m_2+1}, \ldots, x_s$ from B_0 have been bumped out of the tableau in succession with $x_{m_2+1} \geqslant \cdots \geqslant x_s$. This yields a tableau T_{μ° of shape μ°.

Finally, consider the rightmost box in μ° that does not belong to λ; the entry in this box of T_{μ° necessarily belongs to B_1. If the entry belongs to the first column of T_{μ°, it bumps out of T_{μ°. Otherwise, bump the entry into the column immediately left of it by replacing the largest entry less than it and continue until an entry $a_1 \in B_1$ bumps out of the first column. Repeat the process for each of the entries in the boxes that belong to μ° but not to λ until entries $a_1 < a_2 < \cdots < a_{m_1}$ have been bumped out of the first column. The tableau remaining is T, which has shape λ, and $U = \boxed{x_s \mid \cdot\cdot \mid x_1 \mid a_1 \mid \cdot\cdot \mid a_{m_1}}$.

Example 4.15.

Suppose that $\lambda = (3,3,3,2)$, $\mu^\circ = (4,3,3,2,2)$, $\mu^- = (4,3,2,2,2)$ and $\nu = (6,3,2,2,2)$. Then $m_1 = 3, \ell = 1$ and $m_2 = 2$. The following details the procedure for decomposing the ν-tableau S as a product of a λ-tableau and a one-row (6)-tableau. The entries bumped out of the tableau are shown above the arrows at each stage.

$$S = \begin{array}{|c|c|c|c|c|c|} \hline 2 & 3 & 5 & 5 & 5^\circ & 6^\circ \\ \hline 3 & \overline{4} & 1^\circ \\ \cline{1-3} 4 & 4^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} \end{array} \xrightarrow{5} \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 5 & 5^\circ & 6^\circ \\ \hline 3 & \overline{4} & 1^\circ \\ \cline{1-3} 4 & 4^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} \end{array} \xrightarrow{3} \begin{array}{|c|c|c|c|} \hline 2 & \overline{4} & 5 & 6^\circ \\ \hline 3 & 1^\circ & 5^\circ \\ \cline{1-3} 4 & 4^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 2 & \overline{4} & 5 & 6^\circ \\ \hline 3 & 1^\circ & 5^\circ \\ \cline{1-3} 4 & 4^\circ & \\ \cline{1-3} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2} \end{array}$$

$$\longrightarrow \begin{array}{|c|c|c|c|}\hline & 2 & 5 & 6^\circ \\ \hline 3 & \overline{4} & 1^\circ \\ \cline{1-3} 4 & 4^\circ & 5^\circ \\ \cline{1-3} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2}\end{array} \xrightarrow{1} \begin{array}{|c|c|c|c|}\hline \overline{1} & 2 & 5 & 6^\circ \\ \hline 3 & \overline{4} & 1^\circ \\ \cline{1-3} 4 & 4^\circ & 5^\circ \\ \cline{1-3} 2^\circ & 5^\circ \\ \cline{1-2} 2^\circ & 5^\circ \\ \cline{1-2}\end{array} \xrightarrow{2^\circ} \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ & 6^\circ \\ \hline 2^\circ & 5^\circ \\ \cline{1-2} 4^\circ & 5^\circ \\ \cline{1-2}\end{array} \xrightarrow{4^\circ} \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ & 6^\circ \\ \hline 2^\circ & 5^\circ \\ \cline{1-2} 5^\circ \\ \cline{1-1}\end{array} \xrightarrow{5^\circ} \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ & 6^\circ \\ \hline 2^\circ & 5^\circ \\ \cline{1-2}\end{array}$$

$$S = \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 5 \\ \hline 3 & \overline{4} & 1^\circ \\ \hline 4 & 5^\circ & 6^\circ \\ \hline 2^\circ & 5^\circ \\ \cline{1-2}\end{array} \cdot \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 2^\circ & 4^\circ & 5^\circ \\ \hline\end{array}$$

The following Lemma is the key to proving the equivalence of the symplectic and orthosymplectic Pieri rules.

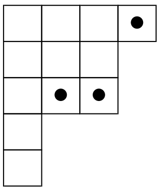
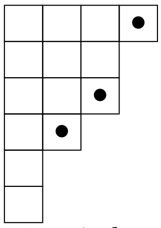
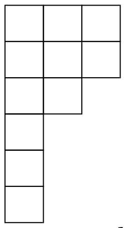
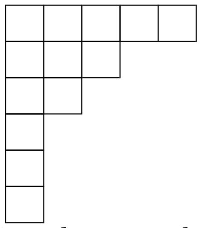
Lemma 4.16. *Let ℓ and k be nonnegative integers with $\ell \leqslant k$. There is a bijection between the set of pairs $((\lambda, \mu, \nu), m_1)$, consisting of sp-bumping sequences (λ, μ, ν) where $|\lambda/\mu| = \ell$ and $|\nu/\mu| = k-\ell$, and spo-bumping sequences $(\lambda, \mu^\circ, \mu^-, \nu)$, where $|\mu^\circ/\lambda| = m_1$, $|\mu^\circ/\mu^-| = \ell$, and $|\nu/\mu^-| = m_2 = k - \ell - m_1$.*

Proof. Given an *spo*-bumping sequence $(\lambda, \mu^\circ, \mu^-, \nu)$, if $\mu^\circ = \mu^-$, then $\ell = 0$ and the corresponding *sp*-bumping sequence is (λ, μ, ν), where $\mu = \mu^\circ$. Otherwise, $\mu^- \subset \mu^\circ$ and we can form an *sp*-bumping sequence (λ, μ, ν), where $|\lambda/\mu| = \ell$, by first placing dots in all boxes in μ° that belong to μ° but not to μ^-. If a box in the first row of μ° contains a dot, there must also be a box in that position in λ by Lemma 4.10 and all boxes containing dots are in different columns. Working from top to bottom and right to left, if a box containing a dot belongs to λ, place a dot in the box in λ. If the dotted box does not belong to λ, and belongs to row i in μ°, then place a dot in the rightmost box of row $i-1$ of λ that does not already contain a dot. In this way, a unique box in λ can be chosen for each of the ℓ dotted boxes in μ° and, at the end of the process, the dotted boxes are all in different columns of λ.

Once ℓ boxes contain dots in λ, remove the dotted boxes from λ to form $\mu \subseteq \nu$. Any box in ν/μ must correspond to a box of one of the following three types: a box that belongs to ν/μ^-, a box that belongs to μ°/λ but not μ°/μ^-, or a box that is dotted in λ, via the above process, but not dotted in μ°. Boxes in ν/μ of the first two types together form a horizontal strip in ν/μ. The rightmost box in ν/μ of the third type belongs to a column of ν that is strictly left of the leftmost column that contains a box from ν/μ^-. As well, any box in ν/μ of the third type corresponds to a dotted box in λ that belongs to a column of ν that does not contain a box corresponding to the second type, since boxes in μ°/λ form a horizontal strip. It follows that ν/μ is a horizontal strip and (λ, μ, ν) is an *sp*-bumping sequence.

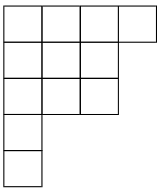
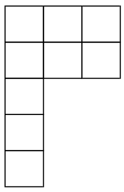
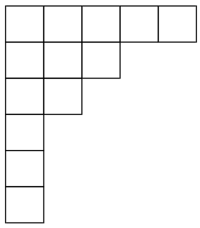
Given a pair $((\lambda, \mu, \nu), m_1)$ as in the statement of the lemma, we reverse the above procedure to produce $(\lambda, \mu^\circ, \mu^-, \nu)$. If $\mu = \nu$, then $k = \ell$, so $m_1 = m_2 = 0$. In this case, $\mu^\circ = \lambda$ and $\mu^- = \nu$. Otherwise, remove the rightmost m_2 boxes from ν that belong to ν but not to μ to give the shape μ^-, where $\mu \subseteq \mu^-$. If $m_2 = k - \ell$ so that $\mu = \mu^-$, then $m_1 = 0$ so let $\mu^\circ = \lambda$. Otherwise, place dots in all boxes in μ^- that belong to μ^- but not to μ; there are m_1 such boxes. Working from top to bottom and from left to right, if a dotted box does not belong to λ, add a box to λ in that position. If a dotted box does belong to λ, add a box to λ in the first row below this box. The new shape formed in this way is μ°, which contains m_1 more boxes than λ and the sequence $(\lambda, \mu^\circ, \mu^-, \nu)$ is an *spo*-bumping sequence. □

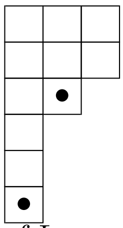
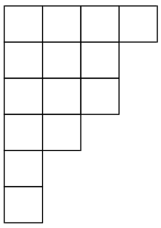
Example 4.17. Consider the *spo*-bumping sequence $(\lambda, \mu^\circ, \mu^-, \nu)$, where

$\lambda =$  $\mu^\circ =$  $\mu^- =$  $\nu =$ 

By considering the boxes in μ° that are not in μ^- and following the procedure in the proof of Lemma 4.16, with boxes dotted as above, we obtain $\mu = (3, 3, 1, 1, 1)$ and *sp*-bumping sequence (λ, μ, ν).

Example 4.18. Let $m_1 = m_2 = 2$ and $\ell = 3$ and consider the *sp*-bumping sequence (λ, μ, ν) below:

$\lambda =$  $\mu =$  $\nu =$ 

Then $\mu^- =$  and $\mu^\circ =$  , where the dotted boxes are determined as in the proof of Lemma 4.16.

Corollary 4.19. *Let T be an spo-tableau of shape λ and let U be a one-row (k)-tableau. Then $T \cdot U$ has shape ν, where ν is the end shape in an sp-bumping sequence (λ, μ, ν). In other words, λ/μ is a horizontal strip with $|\lambda/\mu| = \ell$, for some $0 \leqslant \ell \leqslant k$, and ν/μ is a horizontal strip with $|\nu/\mu| = k - \ell$.*

Theorem 4.20. *Consider an sp-bumping sequence (λ, μ, ν), where λ/μ is a horizontal strip with $|\lambda/\mu| = \ell$, for some $0 \leqslant \ell \leqslant k$, and ν/μ is a horizontal strip with $|\nu/\mu| = k - \ell$. If S is an spo-tableau of shape ν, then S can be decomposed as a product of an spo-tableau of shape λ and a (k)-tableau.*

Proof. We first associate a unique m_1 and m_2 to $(S, (\lambda, \mu, \nu))$, which will allow us to find the *spo*-bumping sequence corresponding to $((\lambda, \mu, \nu), m_1)$ in accordance with Lemma

4.16. We will then decompose S by reversing the *spo*-insertion algorithm. Since $\ell = |\lambda/\mu|$, $k - \ell = |\nu/\mu|$, and $m_1 + m_2 = k - \ell$, we need only determine a value for m_1 or m_2.

If S is a symplectic tableau (so does not contain entries from $B_1 = \{1^\circ, \ldots, n^\circ\}$) then $m_1 = 0$ and if $\mu = \nu$ then $m_1 = m_2 = 0$. Otherwise, consider the rightmost box in ν that does not belong to μ and begin reversing the *spo*-insertion process with the entry x that belongs to this box in S. If $x \in B_1$ and is in the first column of S or if $x \in B_0 = \{1, \overline{1}, \ldots, m, \overline{m}\}$ and is in the first row, x bumps out of the tableau and the box it occupies is removed from S, leaving a tableau S_1 of shape ν_1 with one less box. Otherwise, if $x \in B_1$, bump it into the column immediately to its left by displacing the largest entry in that column that is less than x and replacing it with x. If $x \in B_0$, bump it into the row above it by replacing the largest entry in that row that is less than it. Continue the procedure with the displaced entries until an entry bumps out of the first row or the first column, leaving a tableau S_1 of shape ν_1 with one less box. If this results in an entry from B_1 getting bumped out of the first column, then $m_2 = 0$. Otherwise, repeat the procedure for the entry in the rightmost box in S_1 that does not belong to μ until an entry bumps out of the first row or column. Repeat the procedure until $k - \ell$ entries have been bumped out of the tableau or until an entry in B_1 is bumped out of the first column. Let m_2 equal the number of entries in B_0 that are bumped out of the first row during this process. The tableau T_1 remaining after bumping these m_2 entries from the tableau will have shape μ^- in the *spo*-bumping sequence described below. If $m_2 = k - \ell$, then $\mu = \mu^-$ and $m_1 = 0$. Otherwise, if the rightmost box in T_1 that does not belong to μ is in column j, then the entry in this box belongs to B_1 and, since bumping this entry through the tableau causes an entry from B_1 to bump out of the first column, for each column i with $1 \leqslant i \leqslant j$ there is an entry $a_i \in B_1$ in column i and an entry $a_{i-1} \in B_1$ in column $i - 1$, with $a_{i-1} < a_i$, that belongs to a row below a_i.

We can now find the image of $((\lambda, \mu, \nu), m_1)$, using Lemma 4.16, which is an *spo*-bumping sequence $(\lambda, \mu^\circ, \mu^-, \nu)$. We now verify that the *spo*-insertion process can be reversed, as described in the paragraphs following Theorem 4.14, using this particular sequence and these values of m_1 and m_2. Reversing the *spo*-insertion process first bumps the m_2 entries from B_0 as described above, leaving T_1, which has shape μ^-.

Continuing the reversed *spo*-insertion algorithm then adds ℓ empty boxes to T_1, which are each moved through the tableau through the jeu de taquin process, resulting in a tableau T_2 of shape μ°. Provided the empty boxes each start in rows below the first, the tableau T_2 still contains an entry from B_1 in each of columns 1 through j since the jeu de taquin process cannot move an entry a_{i-1} below an entry a_i, where $a_{i-1} < a_i$. If an empty box starts in the first row of T_1, then μ^- has fewer boxes in the first row than μ° so, by Lemma 4.10, λ and μ° have the same number of boxes in the first row, which means that no entries are bumped out of the first row in the next stage of the reverse *spo*-algorithm.

Since μ°/λ is a horizontal strip with m_2 boxes corresponding to columns left of column $j+1$ in μ°, any entry in T_2 that belongs to a box of μ°/λ must be at the bottom of column j or at the bottom of a column left of column j, so contains an entry from B_1. Thus the final stage of the *spo*-insertion algorithm can be completed to bump m_2 entries from B_1 from T_2, producing a tableau T of shape λ and a (k)-tableau, whose product is S. □

Example 4.21. Consider the *spo*-tableau $\begin{array}{ccc} 2 & 1^\circ & 3^\circ \\ \overline{2} & & \\ 2^\circ & & \end{array}$ and the *sp*-bumping sequence $\left(\begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix} \right)$. Reversing the *spo*-insertion process for the entry in the $(1,3)$-box bumps a 2 out of the first row, and then repeating the process for the entry in the $(1,2)$-box in the resulting tableau bumps a 2° out of the first column. It follows that $m_2 = 1$ and, since $\ell = 2$, $m_1 = 1$. The corresponding *spo*-bumping sequence, via Lemma 4.16, is then $\left(\begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square&\square\\ \square&\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix} \right)$. We can use this *spo*-bumping sequence and the given tableau to reverse the *spo*-insertion process and decompose the tableau as a product:

$$\begin{array}{ccc} 2 & 1^\circ & 3^\circ \\ \overline{2} & & \\ 2^\circ & & \end{array} = \begin{array}{ccc} \overline{1} & \overline{1} & \overline{2} \\ 1^\circ & & \\ 3^\circ & & \end{array} \cdot \begin{array}{cccc} 1 & 1 & 2 & 2^\circ \end{array}.$$

Note that if we consider the same *spo*-tableau but *sp*-bumping sequence $\left(\begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix} \right)$, then the corresponding *spo*-bumping sequence is $\left(\begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square&\square\\ \square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square\\ \square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square&\square&\square\\ \square\\ \square\end{smallmatrix} \right)$, which yields

$$\begin{array}{ccc} 2 & 1^\circ & 3^\circ \\ \overline{2} & & \\ 2^\circ & & \end{array} = \begin{array}{ccc} \overline{1} & \overline{1} & 3^\circ \\ \overline{2} & & \\ 1^\circ & & \end{array} \cdot \begin{array}{cccc} 1 & 1 & 2 & 2^\circ \end{array}.$$

Theorem 4.1 now follows. Our final example illustrates the orthosymplectic Pieri rule.

Example 4.22. Let $\lambda = (3,1)$ and $k = 3$. Then $\ell = 0, 1, 2$ or 3. We have

$$spo_\lambda spo_{(k)} = spo_{(6,1)} + spo_{(5,2)} + spo_{(5,1,1)} + spo_{(4,3)} + spo_{(4,2,1)} + 2spo_{(4,1)} + 2spo_{(3,2)}$$
$$+spo_{(3,1,1)} + spo_{(2,2,1)} + spo_{(5)} + 2spo_{(2,1)} + spo_{(1,1,1)} + spo_{(3)} + spo_{(1)}.$$

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