A notion of inversion number associated to certain quiver flag varieties

Lucas Fresse*

Université de Lorraine, CNRS
Institut Elie Cartan de Lorraine, UMR 7502
Vandoeuvre-lès-Nancy, F-54506, France

lucas.fresse@univ-lorraine.fr

Submitted: May 1, 2018; Accepted: Aug 22, 2018; Published: Sep 7, 2018

Abstract

We define an algebraic variety $X(d,A)$ consisting of matrices whose rows and columns are partial flags. This is a smooth, projective variety, and we describe it as an iterated bundle of Grassmannian varieties. Moreover, we show that $X(d,A)$ has a cell decomposition, in which the cells are parametrized by certain matrices of sets and their dimensions are given by a notion of inversion number. On the other hand, we consider the Spaltenstein variety of partial flags which are stabilized by a given nilpotent endomorphism. We partition this variety into locally closed subvarieties which are affine bundles over certain varieties called Y_T, parametrized by semistandard tableaux T. We show that the varieties Y_T are in fact isomorphic to varieties of the form $X(d,A)$. We deduce that each variety Y_T has a cell decomposition, in which the cells are parametrized by certain row-increasing tableaux obtained by permuting the entries in the columns of T and their dimensions are given by the inversion number recently defined by P. Drube for such row-increasing tableaux.

Mathematics Subject Classifications: 05A05, 05A19, 14M15

1 Introduction

Given the following data:

- a $p \times q$ matrix of nonnegative integers $d = (d_{i,j})$ which is nondecreasing along the rows and the columns, i.e., $d_{i,j} \leqslant d_{i',j'}$ whenever $i \leqslant i'$, $j \leqslant j'$,
- a chain of $\mathbb{C}$-vector spaces $A = (A_1 \subset \ldots \subset A_q)$ such that $\dim A_j = d_{p,j}$,

*Supported in part by the ANR project GeoLie (ANR-15-CE40-0012).

we define $X(d, A)$ as the set of $p \times q$ matrices of vectors spaces $V = (V_{i,j})$ such that

$$V_{i,j} \text{ is a } d_{i,j}\text{-dimensional subspace of } A_j \text{ for all } i, j, \tag{1}$$
$$V_{i,j} \subset V_{i',j'} \text{ whenever } i \leqslant i',\ j \leqslant j'. \tag{2}$$

Thus $X(d, A)$ consists of representations of the $p \times q$ rectangular quiver

$$\begin{array}{ccccccccc}
\bullet_{1,1} & \to & \bullet_{1,2} & \to & \bullet_{1,3} & \cdots & \to & \bullet_{1,q} \\
\downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\bullet_{2,1} & \to & \bullet_{2,2} & \to & \bullet_{2,3} & \cdots & \to & \bullet_{2,q} \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots \\
\downarrow & & \downarrow & & \downarrow & \cdots & & \downarrow \\
\bullet_{p,1} & \to & \bullet_{p,2} & \to & \bullet_{p,3} & \cdots & \to & \bullet_{p,q}
\end{array}$$

in the subcategory of vector spaces where we retain only inclusion morphisms. Clearly $X(d, A)$ is a closed subset of the projective variety

$$\prod_{i,j} \mathrm{Grass}_{d_{i,j}}(A_j)$$

where $\mathrm{Grass}_k(H)$ stands for the Grassmannian variety of k-dimensional subspaces of a vector space H.

We outline some general facts on the variety $X(d, A)$, which are explained in more detail in the rest of the paper:

1) $X(d, A)$ is an iterated bundle of base type the following collection of Grassmannian varieties

$$\{\mathrm{Grass}_{d_{i-1,j}-d_{i-1,j-1}}(\mathbb{C}^{d_{i,j}-d_{i-1,j-1}}) : 2 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q\}$$

(where $d_{i,0} = 0$), in particular $X(d, A)$ is smooth, irreducible, and its Poincaré polynomial is explicitly determined; see Theorem 1.

2) For certain dimension matrices d, $X(d, A)$ is a resolution of a Schubert variety in a natural way; see Remark 5 (a). In fact, in the case where the chain A is maximal, $X(d, A)$ is a Bott-Samelson variety of special type; see Remark 5 (b). The definition of $X(d, A)$ is related to the combinatorial construction of Bott-Samelson varieties given in [10]; see Remark 5 (c).

3) We define $\mathcal{W} = \mathcal{W}(d)$ as the set of $p \times q$ matrices of sets $\omega = (\omega_{i,j})$ such that

$$\omega_{i,j} \text{ is a subset of } \{1, \ldots, d_{p,j}\} \text{ of cardinality } d_{i,j} \text{ for all } i, j, \tag{3}$$
$$\omega_{i,j} \subset \omega_{i',j'} \text{ whenever } i \leqslant i',\ j \leqslant j'. \tag{4}$$

In Section 3, we introduce a notion of inversion number $n_{\mathrm{inv}}(\omega)$ for the elements of $\mathcal{W}$, and we show that the elements of $\mathcal{W}$ parametrize a cell decomposition $X(d, A) = \bigsqcup_{\omega \in \mathcal{W}} C(\omega)$ such that $\dim C(\omega) = n_{\mathrm{inv}}(\omega)$.

4) Our main original motivation in considering the variety $X(d,A)$ is the study of Spaltenstein varieties. A Spaltenstein variety $\mathrm{Fl}_{k,u}$ is a variety of partial flags (for dimension vector k) which are preserved by a given nilpotent endomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ (see Section 4.1). This variety is in general not irreducible. As it is recalled in Section 4.2, there is a natural partition of $\mathrm{Fl}_{k,u}$ into locally closed subsets

$$\mathrm{Fl}_{k,u} = \bigsqcup_{T\in\mathrm{STab}_k(\lambda(u))} \mathrm{Fl}_{k,u,T}$$

parametrized by semistandard tableaux T whose shape is the Jordan form of u (seen as a Young diagram). Moreover, the closures $\overline{\mathrm{Fl}_{k,u,T}}$ are the irreducible components of $\mathrm{Fl}_{k,u}$. In Proposition 17, we show that for each subvariety $\mathrm{Fl}_{k,u,T}$, there is an affine bundle

$$\varphi_T : \mathrm{Fl}_{k,u,T} \to Y_T$$

where Y_T is a certain projective variety (realized as the subvariety of $\mathrm{Fl}_{k,u,T}$ formed by flags which are homogeneous with respect to a grading adapted to the filtration $\mathbb{C}^n = \bigcup_{j=1}^q \ker u^j$). The main results of Section 4 concern the structure of the variety Y_T. In Theorem 20 we show that

$$Y_T \text{ is isomorphic to a variety of the form } X(d,A).$$

In Theorem 27, relying on Section 3, we then show that Y_T has a cell decomposition

$$Y_T = \bigsqcup_{\tau\in\mathrm{RTab}(T)} Y(\tau)$$

parametrized by the set $\mathrm{RTab}(T)$ of all row-increasing tableaux obtained by permuting entries in the columns of T, moreover the cell decomposition is such that the dimensions of the cells $Y(\tau)$ coincide with the inversion numbers $n_{\mathrm{inv}}(\tau)$ defined by P. Drube [4] for such row-increasing tableaux.

Spaltenstein varieties are considered in [12, 13]. Computations of their Poincaré polynomials can also be deduced from [3, 11]. In the present paper, we are able to provide closed formulas for the Poincaré polynomials of the varieties Y_T and the Spaltenstein variety $\mathrm{Fl}_{k,u}$ (see Corollaries 21, 29, and Remark 30). This generalizes similar results obtained for Springer fibers in [6]. The results obtained in Section 4 also give a geometric interpretation of the recent results of [4, 5].

Notation and mathematical background

All the algebraic and geometric constructions are made over $\mathbb{C}$. By $|M|$ we denote the cardinality of a set M. For a positive integer k, we consider the polynomials $[k]_x := 1 + x + \ldots + x^{k-1}$ and $[k]_x! := [1]_x \cdots [k]_x$.

Given an algebraic variety Y, we denote by $H^i(Y,\mathbb{Q})$ its cohomology spaces (considering sheaf cohomology with rational coefficients) and by $H^i_c(Y,\mathbb{Q})$ its cohomology with compact support. Note that $H^i(Y,\mathbb{Q}) = H^i_c(Y,\mathbb{Q})$ whenever Y is projective. Let $P(Y)(t) := \sum_{i\geqslant 0} \dim H^i(Y,\mathbb{Q})\, t^i$ be the Poincaré polynomial.

In fact, all varieties considered in this paper satisfy the parity vanishing condition $H^i(Y,\mathbb{Q}) = 0$ whenever i is odd. Hence we may renormalize the Poincaré polynomial as $P(Y)(x) = \sum_{i\geqslant 0} \dim H^{2i}(Y,\mathbb{Q})\, x^i$, setting $x = t^2$.

A sufficient condition for a projective variety Y to have this parity vanishing condition is the existence of a cell decomposition. By *cell decomposition*, we mean here a partition into finitely many locally closed subsets that can be numbered as $Y = Y_1 \sqcup \ldots \sqcup Y_k$ so that $Y_1 \sqcup \ldots \sqcup Y_\ell$ is closed for all ℓ and each Y_ℓ is isomorphic to an affine space $\mathbb{A}^{\dim Y_\ell}$. Then

$$P(Y)(t) = P(Y)(x) = \sum_{\ell=1}^{k} x^{\dim Y_\ell}. \tag{5}$$

For example, the decomposition $\mathbb{P}(\mathbb{C}^n) = \bigsqcup_{\ell=1}^n (\mathbb{P}(\mathbb{C}^\ell) \setminus \mathbb{P}(\mathbb{C}^{\ell-1}))$ is a cell decomposition, hence $P(\mathbb{P}(\mathbb{C}^n))(x) = [n]_x$. More generally, letting $B \subset \mathrm{GL}_n(\mathbb{C})$ be the subgroup of upper triangular matrices, by the Bruhat decomposition, the partition into B-orbits of any variety of partial flags of $\mathbb{C}^n$ is a cell decomposition. In the case of the Grassmannian variety $\mathrm{Grass}_k(\mathbb{C}^n)$, the cells are parametrized by the subsets $I \subset \{1,\ldots,n\}$ with k elements: the cell $C(I)$ is the B-orbit of the subspace $\mathbb{C}^I \in \mathrm{Grass}_k(\mathbb{C}^n)$ spanned by the vectors ε_i $(i \in I)$ of the standard basis of $\mathbb{C}^n$. Moreover

$$\dim C(I) = |\{(i,j) : 1 \leqslant j < i \leqslant n,\ i \in I,\ j \notin I\}|. \tag{6}$$

If X, Y, F are projective varieties, Y, F are smooth, irreducible, satisfy the aforementioned parity vanishing condition, and $\varphi : X \to Y$ is a locally trivial fiber bundle with fiber isomorphic to F, then X is smooth, irreducible, satisfies the parity vanishing condition, and $P(X)(x) = P(Y)(x) \cdot P(F)(x)$.

The notion of *iterated bundle* is defined by induction. An iterated bundle of base type $\{Y_1\}$ is a variety isomorphic to Y_1; for $k \geqslant 2$, we say that X is an iterated bundle of base type $\{Y_1,\ldots,Y_k\}$ if (up to renumbering the Y_ℓ's), there is a locally trivial fiber bundle $X \to Y_k$ whose typical fiber is an iterated bundle of base type $\{Y_1,\ldots,Y_{k-1}\}$. Assume that $X, Y_1,\ldots,Y_k$ are projective. If $Y_1,\ldots,Y_k$ are smooth, irreducible, satisying the parity vanishing condition, then so is X, and $P(X)(x) = \prod_{\ell=1}^k P(Y_\ell)(x)$.

For instance the variety of complete flags $\mathrm{Fl}(\mathbb{C}^n)$ is an iterated bundle of base type $\{\mathbb{P}(\mathbb{C}^\ell) : 1 \leqslant \ell \leqslant n\}$ (indeed, the map $\mathrm{Fl}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$, $(V_0,\ldots,V_n) \mapsto V_1$ is locally trivial, of fiber isomorphic to $\mathrm{Fl}(\mathbb{C}^{n-1})$), hence

$$P(\mathrm{Fl}(\mathbb{C}^n))(x) = [n]_x!.$$

The map $\mathrm{Fl}(\mathbb{C}^n) \to \mathrm{Grass}_k(\mathbb{C}^n)$, $(V_0,\ldots,V_n) \mapsto V_k$ is also locally trivial, of fiber $\mathrm{Fl}(\mathbb{C}^k) \times \mathrm{Fl}(\mathbb{C}^{n-k})$, whence

$$P(\mathrm{Grass}_k(\mathbb{C}^n))(x) = \frac{[n]_x!}{[k]_x![n-k]_x!}.$$

2 Structure of the variety $X(d, A)$

The variety $X(d, A)$ is endowed with a natural action of the group

$$P_A := \{g \in \mathrm{GL}(A_q) : g(A_j) = A_j \text{ for all } j = 1, \ldots, q\},$$

which is a parabolic subgroup of $\mathrm{GL}(A_q)$.

Given nonnegative integers $a \leqslant b \leqslant c$, we denote

$$\mathrm{Grass}\left(\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right) := \mathrm{Grass}_{b-a}(\mathbb{C}^{c-a}) \quad \text{and} \quad \left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right]_x := \frac{[c-a]_x!}{[c-b]_x!\,[b-a]_x!}.$$

Whenever $V_a \subset V_c$ are vector spaces of dimensions a and c, respectively, the variety of b-dimensional spaces H such that $V_a \subset H \subset V_c$ is isomorphic to $\mathrm{Grass}_{b-a}(V_c/V_a) \cong \mathrm{Grass}\left(\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right)$, and its Poincaré polynomial is $\left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right]_x$. Its dimension is

$$\dim \mathrm{Grass}\left(\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right) = \deg \left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}\right]_x = (c-b)(b-a).$$

Theorem 1. *The variety $X(d, A)$ is an iterated bundle of base type the sequence of Grassmannian varieties*

$$\left\{\mathrm{Grass}\begin{pmatrix} d_{i-1,j-1} & d_{i-1,j} \\ & d_{i,j} \end{pmatrix} : (i,j) \in \{2, \ldots, p\} \times \{1, \ldots, q\}\right\}$$

(where $d_{i,0} := 0$). In particular $X(d, A)$ is smooth, irreducible, of dimension

$$\dim X(d, A) = \sum_{\substack{2 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} (d_{i,j} - d_{i-1,j})(d_{i-1,j} - d_{i-1,j-1}),$$

its cohomology spaces $H^m(X(d, A), \mathbb{Q})$ vanish in odd degrees, and its Poincaré polynomial is given by

$$P(X(d, A))(x) := \sum_{m \geqslant 0} \dim H^{2m}(X(d, A), \mathbb{Q})\, x^m = \prod_{\substack{2 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} \begin{bmatrix} d_{i-1,j-1} & d_{i-1,j} \\ & d_{i,j} \end{bmatrix}_x.$$

Example 2. For $d = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$ we get

$$\begin{aligned} P(X(d, A))(x) &= \left[\begin{smallmatrix} 0 & 0 \\ & 0 \end{smallmatrix}\right]_x \left[\begin{smallmatrix} 0 & 0 \\ & 2 \end{smallmatrix}\right]_x \left[\begin{smallmatrix} 0 & 1 \\ & 3 \end{smallmatrix}\right]_x \left[\begin{smallmatrix} 0 & 0 \\ & 1 \end{smallmatrix}\right]_x \left[\begin{smallmatrix} 0 & 2 \\ & 2 \end{smallmatrix}\right]_x \left[\begin{smallmatrix} 2 & 3 \\ & 4 \end{smallmatrix}\right]_x \\ &= 1 \cdot 1 \cdot [3]_x \cdot 1 \cdot 1 \cdot [2]_x = (1+x)(1+x+x^2). \end{aligned}$$

Lemma 3. *Assume that $p \geqslant 2$.*

(a) *Assume that $d_{p,j} = d_{p-1,j}$ for all $j \in \{1, \ldots, q\}$. Then, $X(d, A) \cong X(\hat{d}, A)$, where $\hat{d} = (\hat{d}_{i,j})$ is the $(p-1) \times q$ matrix given by $\hat{d}_{i,j} = d_{i,j}$ for all $(i,j) \in \{1, \ldots, p-1\} \times \{1, \ldots, q\}$.*

(b) *Assume that* $d_{p,j_0} > d_{p-1,j_0}$ *for some* $j_0 \in \{1,\ldots,q\}$ *and choose* j_0 *minimal for this property. The map*

$$\varphi : X(d,A) \to \mathrm{Grass}_{d_{p-1,j_0}-d_{p-1,j_0-1}}(A_{j_0}/A_{j_0-1}),\ (V_{i,j}) \mapsto V_{p-1,j_0}$$

is a locally trivial fiber bundle, of fiber $\varphi^{-1}(H) \cong X(\widetilde{d},\widetilde{A})$, *where*

$$\widetilde{d}_{i,j} = \begin{cases} d_{i,j} & \text{if } (i,j) \neq (p,j_0), \\ d_{p-1,j_0} & \text{if } (i,j) = (p,j_0), \end{cases} \qquad \widetilde{A}_j = \begin{cases} A_j & \text{if } j \neq j_0, \\ H & \text{if } j = j_0. \end{cases}$$

In fact φ *is trivial over each* B*-orbit of* $\mathrm{Grass}_{d_{p-1,j_0}-d_{p-1,j_0-1}}(A_{j_0}/A_{j_0-1})$ *whenever* $B \subset \mathrm{GL}(A_{j_0}/A_{j_0-1})$ *is a Borel subgroup.*

Proof of Lemma 3. In the situation of Lemma 3 (a), for every $V = (V_{i,j}) \in X(d,A)$ we have $V_{p-1,j} = A_j$. We obtain a canonical isomorphism

$$X(d,A) \to X(\hat{d},A),\ V \mapsto \hat{V}$$

where $\hat{V} := (\hat{V}_{i,j})$ denotes the $(p-1)\times q$ matrix of spaces given by $\hat{V}_{i,j} = V_{i,j}$ for all $(i,j) \in \{1,\ldots,p-1\}\times\{1,\ldots,q\}$.

In the situation of Lemma 3 (b), we have a canonical isomorphism

$$\varphi^{-1}(H) \to X(\widetilde{d},\widetilde{A}),\ (V_{i,j}) \mapsto (\widetilde{V}_{i,j}) \text{ given by } \widetilde{V}_{i,j} = \begin{cases} V_{i,j} & \text{if } (i,j) \neq (p,j_0), \\ H & \text{if } (i,j) = (p,j_0). \end{cases}$$

It remains to show that φ is trivial over the B-orbit of H whenever B is a Borel subgroup of $\mathrm{GL}(A_{j_0}/A_{j_0-1})$ (this fact guarantees that φ is locally trivial, since there is a Borel subgroup B for which the orbit $B\cdot H$ is open). By the properties of Schubert cells (see, e.g., [2]), there is a unipotent subgroup $U \subset B$ such that the map

$$\psi : U \to B\cdot H,\ u \mapsto u(H)$$

is an isomorphism of algebraic varieties. Moreover there is a natural embedding of $\mathrm{GL}(A_{j_0}/A_{j_0-1})$ into P_A, which yields an action of $\mathrm{GL}(A_{j_0}/A_{j_0-1})$ on $X(d,A)$, such that the map φ is $\mathrm{GL}(A_{j_0}/A_{j_0-1})$-equivariant. Whence a commutative diagram

$$\begin{tikzcd} \varphi^{-1}(B\cdot H) \arrow[d, "\varphi"] & U \times \varphi^{-1}(H) \arrow[l, "\sim"', "\xi"] \arrow[d, "\mathrm{pr}_1"] \\ B\cdot H & U \arrow[l, "\sim"', "\psi"] \end{tikzcd}$$

where pr_1 is the projection on the first factor while the isomorphism ξ is given by $\xi(u,V) = u\cdot V$ for all $(u,V) \in U\times\varphi^{-1}(H)$ and $\xi^{-1}(V) = (\psi^{-1}\circ\varphi(V), (\psi^{-1}\circ\varphi(V))^{-1}\cdot V)$ for all $V \in \varphi^{-1}(B\cdot H)$. Therefore the restriction of φ to $\varphi^{-1}(B\cdot H)$ is trivial. $\square$

Proof of Theorem 1. The proof is done by induction on the tuple $(p, d_{p,1},\ldots,d_{p,q})$ (considering lexicographic order), with immediate initialization for $p=1$ (in which case $X(d,A)$ is reduced to a point). The induction step is yielded by Lemma 3. $\square$

Remark 4. If G is an algebraic group and G' is a closed subgroup acting on an algebraic variety Y, then we let G' act on $G\times Y$ by $g'\cdot(g,y)=(gg'^{-1},g'y)$ and denote by $G\times_{G'}Y:=(G\times Y)/G'$ the quotient variety. The latter variety is equipped with a G-action in a natural way.

Lemma 3 (b) yields the following inductive formula, in terms of a P_A-equivariant isomorphism of varieties

$$X(d,A)\cong P_A\times_{P_{A,H}}X(\widetilde{d},\widetilde{A}),$$

where H is any d_{p-1,j_0}-dimensional space such that $A_{j_0-1}\subset H\subset A_{j_0}$. As before $P_A\subset \mathrm{GL}(A_q)$ is the parabolic subgroup of elements which fix the partial flag $A=(A_1\subset\ldots\subset A_q)$, while by $P_{A,H}$ we denote the (parabolic) subgroup $P_{A,H}=\{g\in P_A: g(H)=H\}$.

Remark 5. (a) We consider the space $\mathbb{C}^n$ ($n\geqslant 1$) and its standard basis $(\varepsilon_1,\ldots,\varepsilon_n)$, and let $A=(A_1,\ldots,A_q)$ be a standard partial flag, i.e., $A_j=\langle\varepsilon_s:1\leqslant s\leqslant \ell_j\rangle_{\mathbb{C}}$, for some $1\leqslant\ell_1<\ell_2<\ldots<\ell_q=n$. Thus P_A is a standard parabolic subgroup.

Given a sequence of positive integers $k=(k_1<\ldots<k_p=n)$, let $\mathrm{Fl}_k(\mathbb{C}^n)$ be the variety of partial flags $F=(F_1\subset\ldots\subset F_p=\mathbb{C}^n)$ with $\dim F_i=k_i$ for all i. A permutation $w\in\mathfrak{S}_n$ gives rise to the element

$$F_w:=(\langle\varepsilon_{w_r}:1\leqslant r\leqslant k_i\rangle_{\mathbb{C}})_{i=1}^p.$$

Let $d^w=(d^w_{i,j})$ be the $p\times q$ matrix given by

$$d^w_{i,j}:=|\{w_1,w_2,\ldots,w_{k_i}\}\cap\{1,2,\ldots,\ell_j\}|.$$

The P_A-orbit of F_w is given by

$$P_A\cdot F_w=\{F=(F_1,\ldots,F_p)\in\mathrm{Fl}_k(\mathbb{C}^n):\dim F_i\cap A_j=d^w_{i,j}\ \forall i,j\}$$

and its closure is the Schubert variety

$$\overline{P_A\cdot F_w}=\{F=(F_1,\ldots,F_p)\in\mathrm{Fl}_k(\mathbb{C}^n):\dim F_i\cap A_j\geqslant d^w_{i,j}\ \forall i,j\}.$$

Then, the map

$$X(d^w,A)\to\overline{P_A\cdot F_w},\ V=(V_{i,j})\mapsto(V_{1,q},\ldots,V_{p,q})$$

is a resolution of singularities of the Schubert variety $\overline{P_A\cdot F_w}$ (this map is proper since $X(d^w,A)$ is projective; it is birational since its restriction over $P_A\cdot F_w$ is an isomorphism; finally it follows from Theorem 1 that the variety $X(d^w,A)$ is smooth).

(b) Now assume that $\ell_j=k_j=j$ for all j, hence $\mathrm{Fl}(\mathbb{C}^n):=\mathrm{Fl}_k(\mathbb{C}^n)$ is the variety of complete flags, $A=(A_1,\ldots,A_n)=(\langle\varepsilon_1,\ldots,\varepsilon_j\rangle_{\mathbb{C}})_{j=1}^n=F_{\mathrm{id}}$ is the standard complete flag, and $B:=P_A$ is the Borel subgroup of upper triangular matrices. For a permutation $w\in\mathfrak{S}_n$, the matrix $d^w=(d^w_{i,j})_{1\leqslant i,j\leqslant n}$ is here given by

$$d^w_{i,j}=|\{w_1,\ldots,w_i\}\cap\{1,\ldots,j\}|\ \text{ for all } i,j\in\{1,\ldots,n\}.$$

For every $j \in \{1, \ldots, n-1\}$, let $P_j := \{g \in \mathrm{GL}(\mathbb{C}^n) : g(A_{j'}) = A_{j'}\ \forall j' \neq j\}$ be the corresponding minimal parabolic subgroup. Write $s_j = (j, j+1) \in \mathfrak{S}_n$. A reduced decomposition

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$$

gives rise to the Bott-Samelson variety

$$Z_{(i_1,\ldots,i_\ell)} := P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell}/B$$

and to the resolution

$$Z_{(i_1,\ldots,i_\ell)} = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell}/B \to (P_{i_1} P_{i_2} \cdots P_{i_\ell}) \cdot F_{\mathrm{id}} = \overline{B \cdot F_w}$$

(see [2]).

A standard way of producing a particular reduced decomposition $[w] := (i_1, \ldots, i_\ell)$ of w is as follows:

- $[\mathrm{id}] := \varnothing$;
- if $w_n = n$, then set $[w] := [\hat{w}]$ where $\hat{w} := w|_{\{1,\ldots,n-1\}} \in \mathfrak{S}_{n-1}$;
- if $w_n < n$, then write $w = s_{w_n} w'$ with $w'_n = w_n + 1$ and set $[w] := (w_n, i'_2, \ldots, i'_\ell)$ where $[w'] = (i'_2, \ldots, i'_\ell)$.

Then, the variety $X(d^w, A)$ is (B-equivariently) isomorphic to the Bott-Samelson variety $Z_{[w]}$ associated to the reduced decomposition $[w]$ just defined. This fact can be shown directly by induction, by relying on the inductive formula given in Remark 4. It also follows from part (c) of the present Remark.

(c) Following the terminology of [10], a *subset family* is a collection D of subsets of $\{1, \ldots, n\}$. A *flagged representation of* D is a sequence of subspaces $(V_C)_{C \in D}$ of $\mathbb{C}^n$ such that $\dim V_C = |C|$ and $V_C \subset V_{C'}$ whenever $C \subset C'$. In fact, we focus on subset families such that $\{1, \ldots, j\} \in D$ for all $j \in \{1, \ldots, n\}$, and on flagged representations (V_C) such that $V_{\{1,\ldots,j\}} = A_j$ for all j. Given a subset family D, let $\mathcal{I}_D^B$ be the set of all such flagged representations; it is a projective variety endowed with a natural action of B.

Given a permutation $w \in \mathfrak{S}_n$, let D^w be the subset family

$$D^w = \big\{\, \{w_1, \ldots, w_i\} \cap \{1, \ldots, j\} : 1 \leqslant i, j \leqslant n \big\}.$$

Then, there is a natural isomorphism $\mathcal{I}_{D^w}^B \xrightarrow{\sim} X(d^w, A)$.

If $\mathbf{i} = (i_1, \ldots, i_\ell)$ is a reduced decomposition of w, it is shown in [10, Theorem 7] that there is a B-equivariant isomorphism $Z_{\mathbf{i}} \cong \mathcal{I}_{D_{\mathbf{i}}^+}^B$, where $D_{\mathbf{i}}^+$ is the subset family

$$D_{\mathbf{i}}^+ := \big\{\, s_{i_1} \cdots s_{i_t}(\{1, \ldots, i_t\}) : 1 \leqslant t \leqslant \ell \big\} \cup \big\{\, \{1, \ldots, j\} : 1 \leqslant j \leqslant n \big\}.$$

If $\mathbf{i} = [w]$, then it is easy to show (by induction) that both subset families D^w and $D_{[w]}^+$ coincide, whence B-equivariant isomorphisms $X(d^w, A) \cong \mathcal{I}_{D^w}^B \cong Z_{[w]}$.

3 Discrete data and inversion number

In this section we set $n := d_{p,q}(= \dim A_q)$ and fix a basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of the space A_q such that $A_j = \langle \varepsilon_a : a \in \{1, \ldots, d_{p,j}\}\rangle_{\mathbb{C}}$ for all $j \in \{1, \ldots, q\}$.

Definition 6. Let $\omega \in \mathcal{W}(d)$, i.e., $\omega = (\omega_{i,j})$ is a $p \times q$ matrix of subsets of $\{1, \ldots, n\}$ which fulfills (3) and (4). Thus, for every $j \in \{1, \ldots, q\}$, the j-th column of ω yields a filtration

$$\omega_{1,j} \subset \omega_{2,j} \subset \ldots \subset \omega_{p,j} = \{1, \ldots, d_{p,j}\}. \tag{7}$$

(a) Let $a, b \in \{1, \ldots, n\}$ such that $a \neq b$. For $j \in \{1, \ldots, q\}$, we write $a <_j b$ if one of the following two conditions is fulfilled:

- a appears before b in the filtration (7), i.e., there is $i \in \{1, \ldots, p\}$ such that $a \in \omega_{i,j}$ and $b \notin \omega_{i,j}$; or
- b does not appear in the filtration (7) and is greater than a, i.e., $b \notin \omega_{p,j}$ (that is, $b > d_{p,j}$) and $b > a$.

We write $a \sim_j b$ if the following condition is fulfilled:

- a, b appear simultaneously in the filtration (7), i.e., $a, b \in \{1, \ldots, d_{p,j}\}$ and $\min\{i = 1, \ldots, p : a \in \omega_{i,j}\} = \min\{i = 1, \ldots, p : b \in \omega_{i,j}\}$.

For $j = 0$, we write $a <_0 b$ whenever $a < b$; hence $1 <_0 2 <_0 \ldots <_0 n$.

(b) Let $j \in \{1, \ldots, q\}$. Let $\mathrm{Inv}_j(\omega)$ be the set of couples $(a, b) \in \{1, \ldots, n\}^2$ satisfying, for some $j' \in \{0, \ldots, j-1\}$,

$$a < b \quad \text{and} \quad \begin{cases} a <_j b, \ b <_{j'} a, \ \text{and } a \sim_{j''} b \text{ whenever } j' < j'' < j \\ \text{or} \\ b <_j a, \ a <_{j'} b, \ \text{and } a \sim_{j''} b \text{ whenever } j' < j'' < j. \end{cases}$$

Note that $\mathrm{Inv}_j(\omega) \subset \{1, \ldots, d_{p,j}\}^2$.

(c) Finally set $n_{\mathrm{inv}}(\omega) := |\mathrm{Inv}_1(\omega)| + |\mathrm{Inv}_2(\omega)| + \ldots + |\mathrm{Inv}_q(\omega)|$.

Example 7. For $\omega = \begin{pmatrix} \varnothing & \{1\} & \{1,3\} \\ \{2\} & \{1,2,3\} & \{1,2,3\} \\ \{1,2\} & \{1,2,3\} & \{1,2,3\} \end{pmatrix}$ we get

$$1 <_0 2 <_0 3, \quad 2 <_1 1 <_1 3, \quad 1 <_2 2 \sim_2 3, \quad 1 \sim_3 3 <_3 2,$$

hence $\mathrm{Inv}_1(\omega) = \{(1,2)\}$, $\mathrm{Inv}_2(\omega) = \{(1,2)\}$, $\mathrm{Inv}_3(\omega) = \{(2,3)\}$, and so $n_{\mathrm{inv}}(\omega) = 3$.

Definition 8. For $\omega = (\omega_{i,j}) \in \mathcal{W}(d)$, we define

$$V_\omega := (\langle \varepsilon_a : a \in \omega_{i,j}\rangle_{\mathbb{C}})_{i,j},$$

which is an element of the variety $X(d, A)$. Clearly, $V_\omega = V_{\omega'}$ iff $\omega = \omega'$.

Recall that $P_A \subset \mathrm{GL}(A_q)$ is the parabolic subgroup of elements which preserve the partial flag $A = (A_1, \ldots, A_q)$. Hence

$$S_A := \{g \in \mathrm{GL}(A_q) : g(\varepsilon_a) \in \mathbb{C}^* \varepsilon_a \text{ for all } a = 1, \ldots, n\}$$

is a maximal torus of P_A.

Theorem 9.

(a) *The elements V_ω, for $\omega \in \mathcal{W}(d)$, are exactly the S_A-fixed points of the variety $X(d, A)$.*

(b) *There is a cell decomposition $X(d, A) = \bigsqcup_{\omega \in \mathcal{W}(d)} C(\omega)$ such that*

   (i) *$V_\omega \in C(\omega)$ for all $\omega \in \mathcal{W}(d)$;*

   (ii) *$C(\omega)$ is isomorphic to the affine space $\mathbb{A}^{n_{\mathrm{inv}}(\omega)}$ of dimension $n_{\mathrm{inv}}(\omega)$.*

The proof relies on a discrete version of Lemma 3 (b).

Lemma 10. *Assume that $p \geqslant 2$. Assume that there is $j_0 \in \{1, \ldots, q\}$ (chosen minimal) such that $d_{p,j_0} > d_{p-1,j_0}$.*

- *Let $\widetilde{d} = (\widetilde{d}_{i,j})$ be the $p \times q$ matrix as in Lemma 3* (b)*.*
- *Let $\mathcal{W}_0$ denote the set of subsets ω_0 such that $\{1, \ldots, d_{p,j_0-1}\} \subset \omega_0 \subset \{1, \ldots, d_{p,j_0}\}$ and $|\omega_0| = d_{p-1,j_0}$, and let us consider the map*

$$\phi : \mathcal{W}(d) \to \mathcal{W}_0,\ \omega = (\omega_{i,j}) \mapsto \omega_{p-1,j_0}.$$

- *Fix $\omega_0 \in \mathcal{W}_0$, and let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the bijection such that*

$$\begin{cases} \sigma(a) = a \quad \textit{unless } d_{p,j_0-1} < a \leqslant d_{p,j_0}, \\ \sigma \textit{ is increasing on } \omega_0 \textit{ and on } \omega_0^c := \{1, \ldots, d_{p,j_0}\} \setminus \omega_0, \\ \sigma(\omega_0) = \{1, \ldots, d_{p-1,j_0}\} \textit{ and } \sigma(\omega_0^c) = \{d_{p-1,j_0} + 1, \ldots, d_{p,j_0}\}. \end{cases}$$

Then, the map

$$\phi^{-1}(\omega_0) \to \mathcal{W}(\widetilde{d}),\ \omega \mapsto \widetilde{\omega} := (\widetilde{\omega}_{i,j}) \textit{ with } \begin{cases} \widetilde{\omega}_{i,j} = \sigma(\omega_{i,j}) \textit{ if } (i,j) \neq (p, j_0), \\ \widetilde{\omega}_{p,j_0} = \sigma(\omega_0) (= \{1, \ldots, \widetilde{d}_{p,j_0}\}) \end{cases}$$

is a bijection such that, for every $\omega \in \phi^{-1}(\omega_0)$,

$$n_{\mathrm{inv}}(\omega) = n_{\mathrm{inv}}(\widetilde{\omega}) + |J_0|, \tag{8}$$

where $J_0 := \{(a, b) : d_{p,j_0-1} < a < b \leqslant d_{p,j_0},\ b \in \omega_0,\ a \notin \omega_0\}$.

Proof of Lemma 10. The definitions of ϕ, $\mathcal{W}(\widetilde{d})$, and σ easily ensure that the map $\omega \mapsto \widetilde{\omega}$ is well defined and bijective. It remains to show (8).

Let $\widetilde{<}_j$, resp. $\widetilde{\sim}_j$, be the analogues of $<_j$, resp. $\sim_j$, relative to $\widetilde{\omega}$. Given $a, b \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, q\}$, we say that the relative position of (a, b) in the j-th column of ω is $<$, $\sim$, or $>$ depending on whether $a <_j b$, $a \sim_j b$, or $b <_j a$. For $j \in \{1, \ldots, q\}$, we say that the relative position of (a, b) before the j-th column of ω is $<$ or $>$ depending on whether $a <_{j'} b$ or $b <_{j'} a$ where $j' < j$ is maximal such that $a \not\sim_{j'} b$.

Set $I = \{(a, b) \in \{1, \ldots, n\}^2 : a < b\}$ and let $\overline{\sigma} : I \to I$ be the bijection defined by

$$\overline{\sigma}(a, b) = \begin{cases} (\sigma(a), \sigma(b)) & \text{if } \sigma(a) < \sigma(b), \\ (\sigma(b), \sigma(a)) & \text{otherwise.} \end{cases}$$

By definition of $n_{\mathrm{inv}}(\omega)$ and $n_{\mathrm{inv}}(\widetilde{\omega})$, for proving (8), it suffices to check that for every couple $(a, b) \in I$ we have

$$\text{if } (a, b) \notin J_0 \text{ or } j \neq j_0: \quad (a, b) \in \mathrm{Inv}_j(\omega) \Leftrightarrow \overline{\sigma}(a, b) \in \mathrm{Inv}_j(\widetilde{\omega}); \tag{9}$$
$$\text{if } (a, b) \in J_0 \text{ and } j = j_0: \quad (a, b) \in \mathrm{Inv}_{j_0}(\omega) \text{ and } \overline{\sigma}(a, b) \notin \mathrm{Inv}_{j_0}(\widetilde{\omega}). \tag{10}$$

To this end, we need to compare the relative positions of (a, b) and $(\sigma(a), \sigma(b))$ in each column of ω and $\widetilde{\omega}$, respectively.

We make two observations. The first one follows from the construction of $\widetilde{\omega}$:

$$\begin{aligned} a \in \omega_{i,j} \Leftrightarrow \sigma(a) \in \widetilde{\omega}_{i,j} \text{ unless } j = j_0 \text{ and } a \in \omega_0^{\mathrm{c}}, \\ \text{in which case } a \in \omega_{p,j_0} \text{ but } \sigma(a) \notin \widetilde{\omega}_{p,j_0}. \end{aligned} \tag{11}$$

The second one follows from the construction of σ:

$$\begin{aligned} a < b \Rightarrow \sigma(a) < \sigma(b) \text{ unless } (a, b) \in J_0, \\ \text{in which case we have } a < b \text{ but } \sigma(a) > \sigma(b). \end{aligned} \tag{12}$$

Since the relative position of a couple (a, b) in the j-th column of ω depends only on the belonging of a, b in the subsets $\omega_{i,j}$ and on the fact that $a < b$ or $b < a$, for every couple $(a, b) \in I \setminus J_0$, we deduce from (12) and (11) that $\overline{\sigma}(a, b) = (\sigma(a), \sigma(b))$ and

$$\begin{aligned} &\text{the relative position of } (a, b) \text{ in the } j\text{-th column of } \omega \text{ coincides} \\ &\text{with the relative position of } (\sigma(a), \sigma(b)) \text{ in the } j\text{-th column of } \widetilde{\omega} \end{aligned} \tag{13}$$

for all $j \in \{0, \ldots, q\}$ when $a, b \notin \omega_0^{\mathrm{c}}$, resp., for all $j \in \{0, \ldots, q\} \setminus \{j_0\}$ when $a \in \omega_0^{\mathrm{c}}$ or $b \in \omega_0^{\mathrm{c}}$. Moreover:

- For $(a, b) \in I$ such that $b \in \omega_0^{\mathrm{c}} (= \{1, \ldots, d_{p,j_0}\} \setminus \omega_0)$ and $a \notin \omega_0^{\mathrm{c}}$, we have on one hand $a \in \omega_{p-1,j_0} (= \omega_0)$ and $b \notin \omega_{p-1,j_0}$, hence $a <_{j_0} b$, and we get on the other hand $\sigma(a) \in \widetilde{\omega}_{p-1,j_0} (= \sigma(\omega_{p-1,j_0}))$ and $\sigma(b) \notin \widetilde{\omega}_{p-1,j_0}$, hence $\sigma(a) \widetilde{<}_{j_0} \sigma(b)$.

- For $(a, b) \in I$ such that $b > d_{p,j_0}$, we have on one hand $b > \max\{d_{p,j_0}, a\}$, hence $a <_{j_0} b$, while we get on the other hand $\sigma(b) > \sigma(a)$ (by (12)) and $\sigma(b) = b > d_{p,j_0} > \widetilde{d}_{p,j_0}$ (by definition of σ), hence $\sigma(a) \widetilde{<}_{j_0} \sigma(b)$.

Thus in these two cases, (13) holds also for $j = j_0$. At this stage, because of (13), we obtain that every couple $(a, b) \in I$ such that $(a, b) \notin J_0$ and $(a, b) \notin J_0' := \{(a', b') \in I : a', b' \in \omega_0^c\}$ fulfills the equivalence in (9).

Next we consider a couple $(a, b) \in J_0'$. Thus $a, b \in \omega_0^c = \omega_{p,j_0} \setminus \omega_{p-1,j_0}$, and this yields $a \sim_{j_0} b$. For every $j < j_0$, we have $b > a > d_{p-1,j_0} \geqslant d_{p,j}$ hence $a <_j b$. On the other hand by (11)–(12) we have $\sigma(b) > \sigma(a)$ and $\sigma(b) \notin \widetilde{\omega}_{p,j_0}$, i.e., $\sigma(b) > \widetilde{d}_{p,j_0} \geqslant \widetilde{d}_{p,j}$ for all $j \leqslant j_0$, whence $\sigma(a) \widetilde{<}_j \sigma(b)$ for all $j \in \{0, \ldots, j_0\}$. Altogether we conclude that $(a, b) \notin \mathrm{Inv}_j(\omega)$ and $\overline{\sigma}(a, b) \notin \mathrm{Inv}_j(\widetilde{\omega})$ for all $j \in \{1, \ldots, j_0\}$, and that the relative positions of (a, b) and $\overline{\sigma}(a, b)$ before the $(j_0 + 1)$-th column of ω and $\widetilde{\omega}$, respectively, are both equal to $<$. These facts, combined with (13), guarantee that every couple $(a, b) \in J_0'$ fulfills the equivalence in (9).

Finally we consider a couple $(a, b) \in J_0$. Thus $\overline{\sigma}(a, b) = (\sigma(b), \sigma(a))$ in view of (12). This yields $b \in \omega_{p-1,j_0} (= \omega_0)$ and $a \notin \omega_{p-1,j_0}$, hence $b <_{j_0} a$, while for every $j < j_0$ we have $b > a > d_{p,j_0-1} \geqslant d_{p,j}$ hence $a <_j b$. On the other hand, by (11) we get $\sigma(b) \in \widetilde{\omega}_{p-1,j_0}$ and $\sigma(a) \notin \widetilde{\omega}_{p-1,j_0}$, hence $\sigma(b) \widetilde{<}_{j_0} \sigma(a)$, and for every $j < j_0$ we get $\sigma(a) > \sigma(b)$ and $\sigma(a) > d_{p,j_0-1} \geqslant \widetilde{d}_{p,j}$ hence $\sigma(b) \widetilde{<}_j \sigma(a)$. Altogether this implies that $(a, b) \notin \mathrm{Inv}_j(\omega)$ whenever $j < j_0$ and $(a, b) \in \mathrm{Inv}_{j_0}(\omega)$ on one hand, $\overline{\sigma}(a, b) \notin \mathrm{Inv}_j(\widetilde{\omega})$ whenever $j \leqslant j_0$ on the other hand; in addition the relative positions of the couples (a, b) and $(\sigma(a), \sigma(b))$ before the $(j_0 + 1)$-th column of ω and $\widetilde{\omega}$, respectively, are both equal to $>$. For $j > j_0$ we have $a, b \in \omega_{p,j_0} \subset \omega_{p,j}$. On the basis of (11) we deduce that the relative positions of (a, b) and $(\sigma(a), \sigma(b))$ in the j-th column of ω and $\widetilde{\omega}$, respectively, coincide. Hence, for $j > j_0$, we have $(a, b) \in \mathrm{Inv}_j(\omega)$ if and only if $(\sigma(b), \sigma(a)) \in \mathrm{Inv}_j(\widetilde{\omega})$. Therefore the couple (a, b) fulfills the equivalences in (9) and (10). The proof of the lemma is complete. $\square$

Proof of Theorem 9. (a) Clearly V_ω is fixed by S_A. Conversely, an element $V = (V_{i,j}) \in X(d, A)$ is fixed by S_A if and only if each subspace $V_{i,j}$ is S_A-stable, which means that $V_{i,j}$ is a sum of S_A-eigenspaces, i.e., is of the form $V_{i,j} = \langle \varepsilon_a : a \in \omega_{i,j} \rangle_{\mathbb{C}}$ for some subset $\omega_{i,j} \subset \{1, \ldots, n\}$. This subset must be of cardinality $\dim V_{i,j} = d_{i,j}$ and the inclusion $V_{i,j} \subset V_{i',j'}$ yields $\omega_{i,j} \subset \omega_{i',j'}$ whenever $i \leqslant i'$, $j \leqslant j'$; in addition, the equality $V_{p,j} = A_j$ yields $\omega_{p,j} = \{1, \ldots, d_{p,j}\}$ for all j. Hence $\omega := (\omega_{i,j})$ is an element of $\mathcal{W}(d)$ and we have $V = V_\omega$.

(b) The proof is done by induction on the tuple $(p, d_{p,1}, \ldots, d_{p,q})$ (considering lexicographic order) with immediate initialization if $p = 1$. So assume that $p \geqslant 2$ and let us distinguish two cases, as in the statement of Lemma 3.

Case 1: $d_{p,j} = d_{p-1,j}$ for all $j \in \{1, \ldots, q\}$.

We denote by $\hat{d}$ (resp., $\hat{V}$) (resp., $\hat{\omega}$) the matrix formed by the first $p - 1$ rows of d (resp., of $V \in X(d, A)$) (resp., of $\omega \in \mathcal{W}(d)$). The map $\theta : X(d, A) \to X(\hat{d}, A)$, $V \mapsto \hat{V}$ is an isomorphism of algebraic varieties (see Lemma 3), similarly the map $\mathcal{W}(d) \to \mathcal{W}(\hat{d})$, $\omega \mapsto \hat{\omega}$ is a bijection, and we have $\theta(V_\omega) = V_{\hat{\omega}}$ for all $\omega \in \mathcal{W}(d)$. Moreover, since $\omega_{p-1,j} = \omega_{p,j}$, we must have $\min\{i = 1, \ldots, p - 1 : a \in \hat{\omega}_{i,j}\} = \min\{i = 1, \ldots, p : a \in \omega_{i,j}\}$ for all $j \in \{1, \ldots, q\}$, all $a \in \hat{\omega}_{p-1,j} = \omega_{p,j}$, which ensures that $n_{\mathrm{inv}}(\omega) = n_{\mathrm{inv}}(\hat{\omega})$. The statement follows from these observations and the induction hypothesis.

Case 2: $d_{p,j_0} > d_{p,j_0-1}$ for some $j_0 \in \{1, \ldots, q\}$, chosen minimal.

Let Y be the variety of d_{p-1,j_0}-dimensional subspaces H such that $A_{j_0-1} \subset H \subset A_{j_0}$ and let us consider the map

$$\varphi : X(d, A) \to Y,\ V = (V_{i,j}) \mapsto V_{p-1,j_0}$$

as in Lemma 3. For $\omega_0 \in \mathcal{W}_0$, with $\mathcal{W}_0$ as in Lemma 10, let

$$H_{\omega_0} := \langle \varepsilon_a : a \in \omega_0 \rangle_{\mathbb{C}} \in Y.$$

Denoting by $B \subset \mathrm{GL}(A_q)$ the subgroup of automorphisms which are upper triangular in the basis $(\varepsilon_1, \ldots, \varepsilon_n)$, we have a cell decomposition

$$Y = \bigsqcup_{\omega_0 \in \mathcal{W}_0} B \cdot H_{\omega_0} \tag{14}$$

such that $\dim B \cdot H_{\omega_0} = |J_0|$ where J_0 is the set given in Lemma 10 (see (6)). Note that (14) yields a partition $X(d, A) = \bigsqcup_{\omega_0 \in \mathcal{W}_0} \varphi^{-1}(B \cdot H_{\omega_0})$. Note also that $V_\omega \in \varphi^{-1}(H_{\omega_0}) \subset \varphi^{-1}(B \cdot H_{\omega_0})$ whenever $\omega \in \phi^{-1}(\omega_0)$, with ϕ as in Lemma 10. Hence, for showing Theorem 9 (b), given any $\omega_0 \in \mathcal{W}_0$, it suffices to construct a cell decomposition

$$\varphi^{-1}(B \cdot H_{\omega_0}) = \bigsqcup_{\omega \in \phi^{-1}(\omega_0)} C(\omega)$$

which satisfies conditions (i) and (ii) of the statement.

Letting $\widetilde{d} = (\widetilde{d}_{i,j})$ and $\widetilde{A} = (\widetilde{A}_j)$ be as in Lemma 3 (b) (for $H = H_{\omega_0}$), we get by Lemma 3 (and its proof) a trivialization of φ over $B \cdot H_{\omega_0}$

$$\xi : \varphi^{-1}(H_{\omega_0}) \times (B \cdot H_{\omega_0}) \xrightarrow{\sim} \varphi^{-1}(B \cdot H_{\omega_0})$$

(such that $\xi(\,\cdot\,, H_{\omega_0}) = \mathrm{id}_{\varphi^{-1}(H_{\omega_0})}$) and an isomorphism

$$\zeta : \varphi^{-1}(H_{\omega_0}) \xrightarrow{\sim} X(\widetilde{d}, \widetilde{A}),\ (V_{i,j}) \mapsto (\widetilde{V}_{i,j}) \text{ with } \begin{cases} \widetilde{V}_{i,j} = V_{i,j} \text{ if } (i,j) \neq (p, j_0), \\ \widetilde{V}_{p,j_0} = H_{\omega_0}. \end{cases}$$

Letting $\widetilde{\varepsilon}_a := \varepsilon_{\sigma^{-1}(a)}$ (for $a = 1, \ldots, n$), with σ as in Lemma 10, we get a basis $(\widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_n)$ of A_q such that $\widetilde{A}_j = \langle \widetilde{\varepsilon}_a : 1 \leqslant a \leqslant \widetilde{d}_{p,j} \rangle_{\mathbb{C}}$ for all $j = 1, \ldots, q$. By induction hypothesis, we have a cell decomposition

$$X(\widetilde{d}, \widetilde{A}) = \bigsqcup_{\omega \in \mathcal{W}(\widetilde{d})} \widetilde{C}(\omega)$$

such that $\dim \widetilde{C}(\omega) = n_{\mathrm{inv}}(\omega)$ and $\widetilde{V}_\omega := (\langle \widetilde{\varepsilon}_a : a \in \omega_{i,j} \rangle_{\mathbb{C}})_{i,j} \in \widetilde{C}(\omega)$ for all $\omega \in \mathcal{W}(\widetilde{d})$. Considering the bijection $\phi^{-1}(\omega_0) \xrightarrow{\sim} \mathcal{W}(\widetilde{d})$, $\omega \mapsto \widetilde{\omega}$ defined in Lemma 10, we derive a cell decomposition of $\varphi^{-1}(B \cdot H_{\omega_0})$ parametrized by the set $\phi^{-1}(\omega_0)$, given by

$$\varphi^{-1}(B \cdot H_{\omega_0}) = \bigsqcup_{\omega \in \phi^{-1}(\omega_0)} \xi(\,\zeta^{-1}(\widetilde{C}(\widetilde{\omega})) \times (B \cdot H_{\omega_0})).$$

The cell $C(\omega) := \xi(\,\zeta^{-1}(\widetilde{C}(\widetilde{\omega})) \times (B \cdot H_{\omega_0})) \cong \widetilde{C}(\widetilde{\omega}) \times (B \cdot H_{\omega_0})$ satisfies (i)′ $V_\omega \in C(\omega)$; indeed we easily have $V_\omega = \zeta^{-1}(\widetilde{V}_{\widetilde{\omega}}) = \xi(\zeta^{-1}(\widetilde{V}_{\widetilde{\omega}}), H_{\omega_0})$.

(ii)′ $\dim C(\omega) = \dim \widetilde{C}(\widetilde{\omega}) + \dim B \cdot H_{\omega_0} = n_{\mathrm{inv}}(\widetilde{\omega}) + |J_0|$.

Invoking Lemma 10, we deduce that this cell decomposition satisfies conditions (i) and (ii) of Theorem 9 (b). This completes the proof of the theorem. □

From (5) and Theorems 1, 9, it follows:

Corollary 11. $\displaystyle\sum_{\omega\in\mathcal{W}(d)} x^{n_{\mathrm{inv}}(\omega)} = \prod_{\substack{2\leqslant i\leqslant p\\ 1\leqslant j\leqslant q}} \begin{bmatrix} d_{i-1,j-1} & d_{i-1,j} \\ & d_{i,j}\end{bmatrix}_x$ *(with* $d_{i,0} := 0$*).*

Example 12. (a) In the special case where $q = 1$, the chain of subspaces $A = (A_1)$ consists of a single space, say $A_1 = \mathbb{C}^n$, while the dimension matrix $d = (d_{i,1})_{i=1}^p$ consists of a single column. Then the variety $X(d, A)$ coincides with the variety of partial flags $F = (F_1 \subset \ldots \subset F_p = \mathbb{C}^n)$ such that $\dim F_i = d_{i,1}$, and Theorem 9 retrieves the properties of the decomposition of this partial flag variety into Schubert cells. Specifically, the map

$$w \in \mathfrak{S}_n \mapsto \omega(w) := (\{w_1, \ldots, w_{d_{i,1}}\})_{i=1}^p \in \mathcal{W}(d)$$

yields a bijection between the set $\mathcal{W}(d)$ and the quotient $\mathfrak{S}_n/\mathfrak{S}_d$ of the symmetric group by the parabolic subgroup $\mathfrak{S}_d := \{w \in \mathfrak{S}_n : w(\{1, \ldots, d_{i,1}\}) = \{1, \ldots, d_{i,1}\}\ \forall i = 1, \ldots, p\}$. In addition the inversion number $n_{\mathrm{inv}}(\omega(w))$ coincides with the Coxeter length of the representative of minimal length of the coset $w\mathfrak{S}_d$.

(b) Next let us consider the special case where $p = 2$, which means that the dimension matrix d consists of two rows; let $k := (d_{1,1} \leqslant \ldots \leqslant d_{1,q})$ be the entries in the first row; let $\ell := (d_{2,1} \leqslant \ldots \leqslant d_{2,q} = n)$ be the entries in the second row, i.e., the dimensions of the subspaces forming the (fixed) sequence $A = (A_1 \subset \ldots \subset A_q = \mathbb{C}^n)$. In this case $X(d, A)$ can be identified with the subvariety $Y \subset \mathrm{Fl}_k(\mathbb{C}^n)$ consisting of partial flags $F = (F_1 \subset \ldots \subset F_q \subset \mathbb{C}^n)$ such that $F_j \subset A_j$ for all $j = 1, \ldots, q$. Note that Y is P_A-stable, smooth, and irreducible (by Theorem 1), hence it is the closure of a P_A-orbit, i.e., a (smooth) Schubert variety of $\mathrm{Fl}_k(\mathbb{C}^n)$. Theorem 9 retrieves the properties of the decomposition of this Schubert variety into Schubert cells. Specifically, the map

$$\begin{array}{rcl} \{w \in \mathfrak{S}_n : w(\{1, \ldots, d_{1,j}\}) \subset \{1, \ldots, d_{2,j}\}\ \forall j = 1, \ldots, q\}/\mathfrak{S}_k & \to & \mathcal{W}(d) \\ w\mathfrak{S}_k & \mapsto & \omega = (\omega_{i,j}) \end{array}$$

(with $\mathfrak{S}_k := \{w \in \mathfrak{S}_n : w(\{1, \ldots, d_{1,j}\}) = \{1, \ldots, d_{1,j}\}\ \forall j = 1, \ldots, q\}$) given by

$$\omega_{1,j} := \{w_1, \ldots, w_{d_{1,j}}\}, \quad \omega_{2,j} := \{1, \ldots, d_{2,j}\} \quad \text{for all } j = 1, \ldots, q$$

is a bijection such that, for $\omega \in \mathcal{W}(d)$, the number $n_{\mathrm{inv}}(\omega)$ coincides with the length of the minimal representative of the corresponding coset $w\mathfrak{S}_k$.

4 Application to Spaltenstein varieties

In this section we fix the following data:

- $k = (0 = k_0 < k_1 < \ldots < k_p = n)$ is an increasing sequence of integers. As before, we denote by $\mathrm{Fl}_k(\mathbb{C}^n)$ the variety of partial flags $F = (0 = F_0 \subset F_1 \subset \ldots \subset F_p = \mathbb{C}^n)$ such that $\dim F_i = k_i$ for all $i = 1, \ldots, p$.

- $u : \mathbb{C}^n \to \mathbb{C}^n$ is a nilpotent endomorphism.

4.1 The Spaltenstein variety $\mathbf{Fl}_{k,u}$

The *Spaltenstein variety* $\mathrm{Fl}_{k,u}$ is the subvariety of $\mathrm{Fl}_k(\mathbb{C}^n)$ defined by

$$\mathrm{Fl}_{k,u} = \{F = (F_0, \ldots, F_p) : u(F_i) \subset F_{i-1}\ \forall i = 1, \ldots, p\}.$$

Thus $\mathrm{Fl}_{k,u}$ is a closed subvariety of $\mathrm{Fl}_k(\mathbb{C}^n)$, hence a projective variety – provided that it is nonempty.

Let $\lambda(u) = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ be the partition of n formed by the sizes of the Jordan blocks of u. This partition can be represented by a Young diagram (also denoted $\lambda(u)$) of rows of lengths $\lambda_1, \ldots, \lambda_r$. By $\lambda(u)^* = (\lambda_1^* \geqslant \ldots \geqslant \lambda_{\lambda_1}^*)$ we denote the dual partition of n, i.e., the lengths of the columns of $\lambda(u)$.

The dimension vector k yields a composition of n denoted $\mu(k) := (k_1 - k_0, \ldots, k_p - k_{p-1})$. By $\mu(k)^+$ we denote the partition of n obtained by putting the sequence $\mu(k)$ in nonincreasing order.

We emphasize the following properties of the Spaltenstein variety $\mathrm{Fl}_{k,u}$.

Proposition 13 ([12, 13]).

(a) $\mathrm{Fl}_{k,u}$ *is nonempty if and only if* $\mu(k)^+ \preceq \lambda(u)^*$, *where* $\preceq$ *stands for the dominance order.*

(b) *In this case,* $\mathrm{Fl}_{k,u}$ *is equidimensional of dimension* $\sum_{j=1}^{\lambda_1} \binom{\lambda_j^*}{2} - \sum_{i=1}^{p} \binom{k_i - k_{i-1}}{2}$.

(c) *Moreover, there is a bijection between the set of irreducible components of* $\mathrm{Fl}_{k,u}$ *and the set* $\mathrm{STab}_k(\lambda(u))$ *of semistandard tableaux of shape* $\lambda(u)$ *and weight* $\mu(k)$.

Recall that a semistandard tableau of shape $\lambda(u)$ and weight $\mu(k) = (\mu_1, \ldots, \mu_p)$ (with $\mu_i = k_i - k_{i-1}$) is a numbering of the boxes of the Young diagram $\lambda(u)$ by the integers $1, 2, \ldots, p$, comprising μ_i boxes of number i for all i, such that the entries in each row are increasing from left to right and the entries in each column are nondecreasing from top to bottom. The set $\mathrm{STab}_k(\lambda(u))$ of such semistandard tableaux is nonempty precisely when the condition $\mu(k)^+ \preceq \lambda(u)^*$ is fulfilled.

Example 14. For $k = (0, 2, 5, 8)$ and $\lambda(u) = (3, 2, 2, 1)$, we get $\mu(k) = (2, 3, 3)$, $\mu(k)^+ = (3, 3, 2) \preceq (4, 3, 1) = \lambda(u)^*$, $\dim \mathrm{Fl}_{k,u} = (6 + 3 + 0) - (1 + 3 + 3) = 2$, and $\mathrm{STab}_k(\lambda(u)) = \left\{ \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & \\ 2 & 3 & \\ 3 & & \end{array}, \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & \\ 2 & 3 & \\ 2 & & \end{array} \right\}$. Thus $\mathrm{Fl}_{k,u}$ is the union of two irreducible components of dimension 2.

In the next subsection, we recall from [12, 13] an explicit parametrization of the components of $\mathrm{Fl}_{k,u}$ by the semistandard tableaux of the set $\mathrm{STab}_k(\lambda(u))$.

4.2 The subvarieties $\mathbf{Fl}_{k,u,T}$ and $Y_T := (\mathbf{Fl}_{k,u,T})^S$ associated to semistandard tableaux

Given $F = (F_0, \ldots, F_p) \in \mathrm{Fl}_{k,u}$, for each i, we get by restriction a nilpotent endomorphism $u|_{F_i} : F_i \to (F_{i-1} \subset) F_i$, whose Jordan form can be encoded by a partition/a Young diagram $\lambda(u|_{F_i}) \vdash k_i$. This yields a chain of Young diagrams

$$\varnothing = \lambda(u|_{F_0}) \subset \lambda(u|_{F_1}) \subset \ldots \subset \lambda(u|_{F_p}) = \lambda(u).$$

Let T be the tableau of shape $\lambda(u)$ obtained by putting the number i in the boxes of $\lambda(u|_{F_i}) \setminus \lambda(u|_{F_{i-1}})$ for all $i = 1, \ldots, p$. The condition $u(F_i) \subset F_{i-1}$ implies that each row of $\lambda(u|_{F_i}) \setminus \lambda(u|_{F_{i-1}})$ contains at most one box, which guarantees that T is a semistandard tableau, in fact an element of $\mathrm{STab}_k(\lambda(u))$.

For every semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$, we define

$$\mathrm{Fl}_{k,u,T} = \{F = (F_0, \ldots, F_p) \in \mathrm{Fl}_{k,u} : \lambda(u|_{F_i}) = \text{shape of } T|_{\leqslant i}\ \forall i = 1, \ldots, p\}$$

where $T|_{\leqslant i}$ stands for the subtableau of T of entries $\leqslant i$. The above discussion shows that the Spaltenstein variety $\mathrm{Fl}_{k,u}$ is the disjoint union of the subsets $\mathrm{Fl}_{k,u,T}$ so-obtained. In fact, we have the following result:

Proposition 15 ([12, 13])**.**

(a) *For every $T \in \mathrm{STab}_k(\lambda(u))$, the subset $\mathrm{Fl}_{k,u,T} \subset \mathrm{Fl}_{k,u}$ is nonempty, locally closed, smooth, irreducible, of dimension equal to $\sum_{j=1}^{\lambda_1} \binom{\lambda_j^*}{2} - \sum_{i=1}^{p} \binom{k_i - k_{i-1}}{2}$.*

(b) *Therefore, the closures $\overline{\mathrm{Fl}_{k,u,T}}$, for $T \in \mathrm{STab}_k(\lambda(u))$, are exactly the irreducible components of $\mathrm{Fl}_{k,u}$.*

Note that parts (b) and (c) of Proposition 13 are consequences of this result. Proposition 15 (a) can be proved by induction on n. The fact that the subsets $\mathrm{Fl}_{k,u,T}$ are locally closed (and smooth) can also be shown as follows. The iterated kernels of u form an increasing sequence

$$\ker u \subset \ker u^2 \subset \ldots \subset \ker u^{\lambda_1} = \mathbb{C}^n,$$

i.e., a partial flag. The stabilizer $Q := \{g \in \mathrm{GL}_n(\mathbb{C}) : g(\ker u^j) = \ker u^j\ \forall j\}$ of this flag is a parabolic subgroup of $\mathrm{GL}_n(\mathbb{C})$. By definition the number of boxes in the first j columns of the Young diagram $\lambda(u|_{F_i})$ is equal to $\dim \ker(u|_{F_i})^j = \dim F_i \cap \ker u^j$. Therefore, we have

$$\mathrm{Fl}_{k,u,T} = \mathrm{Fl}_{k,u} \cap \{F \in \mathrm{Fl}_k(\mathbb{C}^n) : \dim F_i \cap \ker u^j = c_{\leqslant j}(T|_{\leqslant i})\ \forall i, j\} \tag{15}$$

where $c_{\leqslant j}(T|_{\leqslant i})$ stands here for the number of entries $\leqslant i$ in the first j columns of T. This description shows that the subsets $\mathrm{Fl}_{k,u,T}$ (for $T \in \mathrm{STab}_k(\lambda(u))$) coincide with the intersections between $\mathrm{Fl}_{k,u}$ and the Q-orbits of $\mathrm{Fl}_k(\mathbb{C}^n)$. Since every Q-orbit of $\mathrm{Fl}_k(\mathbb{C}^n)$ is locally closed, this guarantees that the subsets $\mathrm{Fl}_{k,u,T}$ are locally closed in $\mathrm{Fl}_{k,u}$.

Remark 16. The subspace $\mathfrak{n}_Q := \{y \in \mathrm{End}(\mathbb{C}^n) : y(\ker u^j) \subset \ker u^{j-1}\ \forall j \geqslant 1\}$ is the nilradical associated to the parabolic subgroup Q. The nilpotent endomorphism u is a *Richardson element* of Q, in the sense that the orbit $Q \cdot u := \{gug^{-1} : g \in Q\}$ is Zariski open in $\mathfrak{n}_Q$ (see [8, §3]).

Any partial flag $F = (F_0, \ldots, F_p) \in \mathrm{Fl}_{k,u,T}$ gives rise to a parabolic nilradical $\mathfrak{n}(F) := \{y \in \mathrm{End}(\mathbb{C}^n) : y(F_i) \subset F_{i-1}\ \forall i = 1, \ldots, p\}$. Note that

$$\mathrm{Fl}_{k,u,T} = \mathrm{Fl}_{k,u} \cap (Q \cdot F) = \{g(F) : g \in \pi^{-1}((Q \cdot u) \cap \mathfrak{n}(F))\}$$

where $\pi : Q \to Q \cdot u$, $g \mapsto g^{-1}ug$. Then the smoothness of $\mathrm{Fl}_{k,u,T}$ (stated in Proposition 15) also follows from the fact that $(Q \cdot u) \cap \mathfrak{n}(F)$ is a smooth variety (since it is open in the space $\mathfrak{n}_Q \cap \mathfrak{n}(F)$) while $g \mapsto g(F)$ and π are smooth maps.

The variety Y_T

For deducing more facts on the structure of the subvariety $\mathrm{Fl}_{k,u,T}$, we need more notation. Since $\lambda(u) = (\lambda_1, \ldots, \lambda_r)$ is the Jordan form of u, there is a basis $(\varepsilon_{i,j} : 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant \lambda_i)$ of the space $\mathbb{C}^n$ such that

$$u(\varepsilon_{i,j}) = \begin{cases} \varepsilon_{i,j-1} & \text{if } j \geqslant 2 \\ 0 & \text{if } j = 1. \end{cases}$$

For $j \in \{1, \ldots, \lambda_1\}$, we set

$$K_j = \langle \varepsilon_{i,j} : 1 \leqslant i \leqslant r,\ \lambda_i \geqslant j \rangle_{\mathbb{C}}.$$

Thus we get a grading

$$\mathbb{C}^n = K_1 \oplus \ldots \oplus K_{\lambda_1}, \tag{16}$$

moreover the subspaces K_j satisfy

$$\ker u^j = K_1 \oplus \ldots \oplus K_j = \ker u^{j-1} \oplus K_j \quad \text{and} \quad u(K_j) \subset K_{j-1}$$

for all $j \in \{1, \ldots, \lambda_1\}$ (with $K_0 := 0$).

Let $S = \{h(t) : t \in \mathbb{C}^*\} \subset \mathrm{GL}_n(\mathbb{C})$ be the rank-one subtorus such that

$$h(t)v = t^{-2j}v \quad \text{for all } v \in K_j, \quad \text{for all } j = 1, \ldots, \lambda_1.$$

Thus $h(t)uh(t)^{-1} = t^2u$ for all $t \in \mathbb{C}^*$, hence each element of S normalizes u, and so stabilizes the kernels $\ker u^j$. This implies that S acts on the Spaltenstein variety $\mathrm{Fl}_{k,u}$ and preserves the subvariety $\mathrm{Fl}_{k,u,T}$ for every semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$. We can therefore define

$$Y_T := (\mathrm{Fl}_{k,u,T})^S = \{F \in \mathrm{Fl}_{k,u,T} : h(t)F = F\ \forall t \in \mathbb{C}^*\}.$$

In other words, Y_T is the subset of flags $F = (F_0, \ldots, F_p) \in \mathrm{Fl}_{k,u,T}$ whose subspaces F_i are homogeneous with respect to the grading of (16) in the sense that

$$F_i = F_i \cap K_1 \oplus \ldots \oplus F_i \cap K_{\lambda_1} \quad \text{for all } i = 1, \ldots, p.$$

The notation Y_T is not ambiguous since, up to isomorphism, the variety Y_T only depends on the semistandard tableau T.

Proposition 17. *Y_T is a smooth, projective, and irreducible variety. Moreover, the map $\varphi_T : \mathrm{Fl}_{k,u,T} \to Y_T$, $F \mapsto \lim_{t\to 0} h(t)F$ is an algebraic affine bundle.*

Proof. Our aim is to apply [7, Proposition 2]. The torus $S = \{h(t) : t \in \mathbb{C}^*\}$ acts by conjugation on the Lie algebra $\mathfrak{gl}_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$, and this action induces a grading

$$\mathfrak{gl}_n(\mathbb{C}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i) \quad \text{where} \quad \mathfrak{g}(i) = \{y \in \mathfrak{gl}_n(\mathbb{C}) : h(t)yh(t)^{-1} = t^i y \ \forall t \in \mathbb{C}^*\}.$$

Note that:

- $u \in \mathfrak{g}(2)$;
- the Lie subalgebra $\mathfrak{g}(\geqslant 0) := \bigoplus_{i\geqslant 0} \mathfrak{g}(i)$ consists of the endomorphisms $y : \mathbb{C}^n \to \mathbb{C}^n$ which preserve the kernels $\ker u^j$ ($j = 1, \ldots, \lambda_1$), in particular every endomorphism which commutes with u belongs to $\mathfrak{g}(\geqslant 0)$.

These observations mean that the grading $\mathfrak{gl}_n(\mathbb{C}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ is good for u in the sense of [7, Proposition 2]. Moreover the second observation means that the parabolic subgroup $Q \subset \mathrm{GL}_n(\mathbb{C})$ formed by the elements which preserve the kernels $\ker u^j$ is corresponding to the cocharacter $t \mapsto h(t)$ in the sense of [7, Section 2.1.3]. As shown above, $\mathrm{Fl}_{k,u,T}$ is the intersection between the Spaltenstein variety $\mathrm{Fl}_{k,u}$ and a Q-orbit of the partial flag variety $\mathrm{Fl}_k(\mathbb{C}^n)$. We are now in position to apply [7, Proposition 2], which shows that Y_T is smooth, projective, and that the map φ_T is an algebraic affine bundle over each connected component of Y_T. Therefore the proof of the proposition is complete once we know that Y_T is also an irreducible variety. This fact is shown in Section 4.3 below (it follows from Theorem 1 and the claim made in the title of Section 4.3 below). ☐

Remark 18. The reasoning made in [9, Section 11.16] shows that, if $C \subset Y_T$ is a locally closed subset isomorphic to an affine space, then so is its inverse image $\varphi_T^{-1}(C) \subset \mathrm{Fl}_{k,u,T} \subset \mathrm{Fl}_{k,u}$ (and the codimension of $\varphi_T^{-1}(C)$ in $\mathrm{Fl}_{k,u}$ coincides with the codimension of C in Y_T). In Section 4.4, we show that the variety Y_T has a cell decomposition for all semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$. By collecting the inverse images of these cells by the various maps φ_T, we therefore obtain a cell decomposition of the whole Spaltenstein variety $\mathrm{Fl}_{k,u}$.

4.3 The variety Y_T is isomorphic to a variety of the form $X(d, A)$

As in Section 4.2, we consider a nilpotent endomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ of Jordan form $\lambda(u) = (\lambda_1, \ldots, \lambda_r) \vdash n$. Thus $q := \lambda_1$ is the nilpotency order of u, i.e., $u^q = 0$, $u^{q-1} \neq 0$. As in Section 4.2, we consider a grading

$$\mathbb{C}^n = K_1 \oplus \ldots \oplus K_q$$

such that $\ker u^j = K_1 \oplus \ldots \oplus K_j$ and $u(K_j) \subset K_{j-1}$ for all $j \in \{1, \ldots, q\}$ (with $K_0 := 0$). We fix a semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$ and focus on the variety $Y_T = (\mathrm{Fl}_{k,u,T})^S$

of partial flags $F = (F_0, \dots, F_p)$ which both belong to the subvariety $\mathrm{Fl}_{k,u,T} \subset \mathrm{Fl}_{k,u}$ and are homogeneous with respect to the grading $\mathbb{C}^n = \bigoplus_{j=1}^q K_j$, i.e.,

$$F_i = F_i \cap K_1 \oplus \dots \oplus F_i \cap K_q \quad \text{for all } i = 1, \dots, p. \tag{17}$$

The tableau T has q columns and its entries belong to $\{1, \dots, p\}$. For $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, we denote by $c_j(T|_{\leqslant i})$ (resp., $c_{\leqslant j}(T|_{\leqslant i})$) the number of boxes in the j-th column (resp., in the first j columns) of the subtableau $T|_{\leqslant i}$; i.e., the number of entries $\leqslant i$ in the j-th column (resp., in the first j columns) of T.

Lemma 19. *Let $F = (F_0, \dots, F_p) \in \mathrm{Fl}_k(\mathbb{C}^n)$ be a partial flag homogeneous with respect to the grading (i.e., such that (17) holds). The following conditions are equivalent:*

(i) *F belongs to the variety Y_T;*

(ii) *We have*

$$\dim K_j \cap F_i = c_j(T|_{\leqslant i}) \quad \textit{and} \quad u(K_j \cap F_i) \subset K_{j-1} \cap F_{i-1}$$

for all $j = 1, \dots, q$, all $i = 1, \dots, p$.

Proof. Assume that (i) holds. Hence F belongs to $\mathrm{Fl}_{k,u,T}$. In particular F belongs to $\mathrm{Fl}_{k,u}$, which implies that $u(F_i) \subset F_{i-1}$ for all $i \in \{1, \dots, p\}$. Since $u(K_j) \subset K_{j-1}$, we deduce that $u(K_j \cap F_i) \subset K_{j-1} \cap F_{i-1}$. In addition by (15) and the homogeneity of F, we have

$$\dim F_i \cap K_j = \dim F_i \cap \ker u^j - \dim F_i \cap \ker u^{j-1} = c_j(T|_{\leqslant i}).$$

This shows (ii). Conversely assume that (ii) holds. Since F is already assumed to be homogeneous, we just need to show that F belongs to $\mathrm{Fl}_{k,u,T}$. Again the homogeneity of F, combined with the assumption in (ii), implies that

$$\dim F_i \cap \ker u^j = \sum_{j'=1}^{j} \dim F_i \cap K_{j'} = \sum_{j'=1}^{j} c_{j'}(T|_{\leqslant i}) = c_{\leqslant j}(T|_{\leqslant i})$$

and

$$u(F_i) = u\Big(\bigoplus_{j=1}^{q} F_i \cap K_j\Big) \subset \sum_{j=1}^{q} u(F_i \cap K_j) \subset \sum_{j=1}^{q} F_{i-1} \cap K_{j-1} \subset F_{i-1}.$$

By (15), we conclude that $F \in \mathrm{Fl}_{k,u,T}$. $\square$

Notation. When $A = (a_{i,j})$ is a $p \times q$ matrix (whose coefficients $a_{i,j}$ are numbers, linear spaces, or sets), we define its shifting $A^\sharp$ as the $(p+q-1) \times q$ matrix whose j-th column has the following content:

$$(\underbrace{a_{1,j}\,,\ \dots\ ,\ a_{1,j}}_{j \text{ terms}}\,,\ a_{2,j}\,,\ \dots\ ,\ a_{p-1,j}\,,\ \underbrace{a_{p,j}\,,\ \dots\ ,\ a_{p,j}}_{q+1-j \text{ terms}}).$$

For instance, $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}^\sharp = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 4 & 7 \\ 3 & 5 & 7 \\ 3 & 6 & 8 \\ 3 & 6 & 9 \end{pmatrix}$.

Theorem 20. *We consider a semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$.*

- *Let $d_T = (d_{i,j})$ be the $p \times q$ matrix of nonnegative integers given by*

$$d_{i,j} = c_{q+1-j}(T|_{\leqslant i}) \quad (= \textit{the number of entries} \leqslant i \textit{ in the } (q+1-j)\textit{-th column of } T\,).$$

- *Let $A = (u^{q-1}(\ker u^q) \subset \ldots \subset u(\ker u^2) \subset \ker u)$.*

Then, there is an isomorphism of varieties $\Phi_T : Y_T \to X(d_T^\sharp, A)$ given by

$$F = (F_0, \ldots, F_p) \mapsto V^\sharp$$

where $V = (V_{i,j})$ is the $p \times q$ matrix of linear spaces such that

$$V_{i,j} = u^{q-j}(F_i \cap \ker u^{q+1-j}) \quad \textit{for all } i = 1, \ldots, p, \textit{ all } j = 1, \ldots, q.$$

Combining Theorems 1 and 20, we obtain in particular a closed formula for the Poincaré polynomial of the variety Y_T:

Corollary 21. *Let $d_T = (d_{i,j})$ be the $p \times q$ matrix of Theorem 20. Set by convention $d_{i,0} := 0$. Then:*

(a) $\dim Y_T = \sum_{\substack{2 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} (d_{i,j} - d_{i-1,j})(d_{i-1,j} - d_{i,j-1})$;

(b) $P(Y_T)(x) := \sum_{m \geqslant 0} \dim H^{2m}(Y_T, \mathbb{Q}) x^m = \prod_{\substack{2 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} \begin{bmatrix} d_{i,j-1} & d_{i-1,j} \\ & d_{i,j} \end{bmatrix}_x.$

Example 22. For $T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 2 \\ \cline{1-2} 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 & 3 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$ (the two tableaux of Example 14), we get

$$d_{T_1}^\sharp = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}^\sharp = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}, \quad d_{T_2}^\sharp = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{pmatrix}^\sharp = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix},$$

and Corollary 21 yields $\dim Y_{T_1} = 1$, $\dim Y_{T_2} = 2$,

$$P(Y_{T_1})(x) = \begin{bmatrix} 1 & 2 \\ & 3 \end{bmatrix}_x = 1 + x, \quad \text{and} \quad P(Y_{T_2})(x) = \begin{bmatrix} 1 & 2 \\ & 4 \end{bmatrix}_x = 1 + x + x^2.$$

Proof of Theorem 20. We first check that the variety $X(d_T^\sharp, A)$ is well defined. The tableau T being semistandard, every box of entry $\leqslant i$ contained in the j-th column of T ($j \geqslant 2$) is on the right of a box of entry $\leqslant i-1$ (contained in the $(j-1)$-th column). Thus we must have

$$c_j(T|_{\leqslant i}) \leqslant c_{j-1}(T_{\leqslant i-1}) \quad \text{for all } i = 1, \ldots, p, \text{ all } j = 2, \ldots, q.$$

In view of the definition of the matrix $d_T = (d_{i,j})$, this yields

$$d_{i,j} \leqslant d_{i-1,j+1} \quad \text{for all } i = 1, \dots, p, \text{ all } j = 1, \dots, q-1 \tag{18}$$

(with $d_{0,j} := 0$). Moreover, it is clear that $c_j(T|_{\leqslant i}) \leqslant c_j(T|_{\leqslant i+1})$, hence

$$d_{i,j} \leqslant d_{i+1,j} \text{ for all } i = 1, \dots, p-1, \text{ all } j = 1, \dots, q. \tag{19}$$

Relations (18) and (19) ensure that the shifted matrix $d_T^\sharp$ has nondecreasing rows and columns. In addition the j-th subspace $A_j := u^{q-j}(\ker u^{q+1-j})$ of the sequence A has dimension

$$\dim A_j = \dim \ker u^{q+1-j} - \dim \ker u^{q-j} = c_{q+1-j}(T|_{\leqslant p}) = d_{p,j}$$

which coincides with the last coefficient of the j-th column of $d_T^\sharp$. These observations ensure that the variety $X(d_T^\sharp, A)$ is well defined.

Let $\hat{u}^{j-1} : K_j \to A_{q-j+1} = u^{j-1}(\ker u^j)$ denote the restriction of u^{j-1}. We note that $\hat{u}^{j-1}$ is a linear isomorphism. Whenever $M \subset \mathbb{C}^n$ is a subspace homogeneous with respect to the grading $\mathbb{C}^n = \bigoplus_{j=1}^q K_j$, we note that $u^{j-1}(M \cap \ker u^j) = \hat{u}^{j-1}(M \cap K_j)$. Both observations are used throughout the rest of the proof.

Next, we check that the map Φ_T is well defined. So let $F = (F_0, \dots, F_p) \in Y_T$ and let $V = (V_{i,j})$ be as in the statement. By Lemma 19, we have

$$\begin{array}{rcl} V_{i,j} & = & u^{q-j}(F_i \cap \ker u^{q+1-j}) = u^{q-j}(F_i \cap K_{q+1-j}) \\ & \subset & u^{q-j-1}(F_{i-1} \cap K_{q-j}) = u^{q-(j+1)}(F_{i-1} \cap \ker u^{q+1-(j+1)}) = V_{i-1,j+1} \end{array}$$

for all $i = 2, \dots, p$, all $j = 1, \dots, q-1$. Next, it is clear that

$$V_{i,j} = u^{q-j}(F_i \cap \ker u^{q+1-j}) \subset u^{q-j}(F_{i+1} \cap \ker u^{q+1-j}) = V_{i+1,j} \subset V_{p,j} = A_j$$

whenever $i = 1, \dots, p-1$, $j = 1, \dots, q$. Finally, invoking again Lemma 19, we get

$$\dim V_{i,j} = \dim u^{q-j}(F_i \cap \ker u^{q+1-j}) = \dim F_i \cap K_{q+1-j} = c_{q+1-j}(T|_{\leqslant i}) = d_{i,j}$$

for all i, j. These observations guarantee that the shifted matrix of spaces $V^\sharp$ belongs to the variety $X(d_T^\sharp, A)$.

The well-defined map $\Phi_T : Y_T \to X(d_T^\sharp, A)$ so-obtained is clearly algebraic. Assume that we know that Φ_T is bijective. Then, since Y_T and $X(d_T^\sharp, A)$ are projective varieties, it is also bicontinuous. Since $X(d_T^\sharp, A)$ is irreducible (by Theorem 1), we deduce that Y_T is irreducible (which, by the way, completes the proof of Proposition 17). Since Y_T and $X(d_T^\sharp, A)$ are smooth varieties, by Zariski's main theorem (see, e.g., [1, §AG.18.2]), Φ_T is in fact an isomorphism. Thus, the proof of the theorem is complete once we check that Φ_T is bijective.

Let us check that Φ_T is injective. So let $F = (F_i)_{i=0}^p, F' = (F'_i)_{i=0}^p \in Y_T$ such that $\Phi_T(F) = \Phi_T(F')$. In view of the definition of Φ_T, this implies that $u^{j-1}(F_i \cap \ker u^j) = u^{j-1}(F'_i \cap \ker u^j)$ for all i, j, i.e., $\hat{u}^{j-1}(F_i \cap K_j) = \hat{u}^{j-1}(F'_i \cap K_j)$ with the notation $\hat{u}^{j-1}$

introduced above. Since $\hat{u}^{j-1}$ is injective, we derive $F_i \cap K_j = F'_i \cap K_j$ for all i, j. Whence $F_i = F'_i$ for all i (since F, F' satisfy (17)).

It remains to show that Φ_T is surjective. Let $V' = (V'_{i,j}) \in X(d_T^\sharp, A)$. So V' is a $(p+q-1) \times q$ matrix of subspaces. The subspaces in the j-th column of V' form a chain

$$V'_{1,j} \subset \ldots \subset V'_{j,j} \subset V'_{j+1,j} \subset \ldots \subset V'_{j+p-2,j} \subset V'_{j+p-1,j} \subset \ldots \subset V'_{p+q-1,j} (= A_j)$$

and their respective dimensions are the corresponding coefficients of the matrix $d_T^\sharp$, i.e.,

$$d_{1,j} = \ldots = d_{1,j} \leqslant d_{2,j} \leqslant \ldots \leqslant d_{p-1,j} \leqslant d_{p,j} = \ldots = d_{p,j} (= \dim A_j)$$

whence $V'_{1,j} = \ldots = V'_{j,j} =: V_{1,j}$ and $V_{p,j} := V'_{j+p,j} = \ldots = V'_{p+q-1,j} (= A_j)$. For $1 < i < p$, let $V_{i,j} := V'_{j+i-1,j}$. Altogether, we obtain a $p \times q$ matrix $V = (V_{i,j})$ such that $V' = V^\sharp$. Moreover, the subspaces $V_{i,j}$ satisfy

$$V_{1,j} \subset \ldots \subset V_{p,j} = A_j = u^{q-j}(\ker u^{q+1-j}) \ \text{ for all } j = 1, \ldots, q, \tag{20}$$

$$\dim V_{i,j} = d_{i,j} = c_{q+1-j}(T|_{\leqslant i}) \ \text{ for all } i, j, \tag{21}$$

and (since V' has increasing rows with respect to inclusion)

$$V_{i,j} \subset V_{i-1,j+1} \ \text{ for all } i = 2, \ldots, p, \text{ all } j = 1, \ldots, q-1. \tag{22}$$

For $i \in \{1, \ldots, p\}$, we set

$$F_i = \bigoplus_{j=1}^{q} (\hat{u}^{j-1})^{-1}(V_{i,q+1-j})$$

where $\hat{u}^{j-1} : K_j \to A_{q+1-j}$ is the linear isomorphism obtained by restriction of u^{j-1}. By construction, the subspace F_i is homogeneous with respect to $\mathbb{C}^n = \bigoplus_{j=1}^q K_j$ and, by (21),

$$F_i \cap K_j = (\hat{u}^{j-1})^{-1}(V_{i,q+1-j}) \text{ has dimension } d_{i,q+1-j} = c_j(T|_{\leqslant i}). \tag{23}$$

In particular $\dim F_i$ is the number of boxes in the subtableau $T|_{\leqslant i}$, hence $\dim F_i = k_i$. By (20), we have $F_1 \subset \ldots \subset F_p$. Hence $F := (F_0 = 0, F_1, \ldots, F_p)$ belongs to $\mathrm{Fl}_k(\mathbb{C}^n)$ and is homogeneous. On the basis of (22), we have

$$\begin{aligned} u(K_j \cap F_i) &= u((\hat{u}^{j-1})^{-1}(V_{i,q+1-j})) = u(\{v \in K_j : u^{j-1}(v) \in V_{i,q+1-j}\}) \\ &\subset u(\{v \in K_j : u^{j-1}(v) \in V_{i-1,q+2-j}\}) \\ &\subset \{v' \in K_{j-1} : u^{j-2}(v') \in V_{i-1,q+2-j}\} \\ &= (\hat{u}^{j-2})^{-1}(V_{i-1,q+2-j}) \\ &= K_{j-1} \cap F_{i-1} \ \text{ for all } i = 2, \ldots, p, \text{ all } j = 2, \ldots, q. \end{aligned}$$

We further note that $u(K_1 \cap F_i) = 0$ (since $K_1 \subset \ker u$) and $K_j \cap F_1 = 0$ if $j \geqslant 2$ (by (23) and the fact that the entry 1 appears only in the first column of the semistandard tableau T). Whence, finally,

$$u(K_j \cap F_i) \subset K_{j-1} \cap F_{i-1} \ \text{ for all } i = 1, \ldots, p, \text{ all } j = 1, \ldots, q. \tag{24}$$

By (23) and (24), F satisfies the conditions of Lemma 19 (ii). Therefore Lemma 19 guarantees that $F \in Y_T$. Finally, by the first equality in (23), we have

$$V_{i,j} = u^{q-j}(F_i \cap K_{q+1-j}) = u^{q-j}(F_i \cap \ker u^{q+1-j})$$

for all i, j. Hence $V' = V^\sharp = \Phi_T(F)$. The surjectivity of Φ_T is established, the proof is complete. $\square$

4.4 Cell decomposition of Y_T

Recall that $k = (k_0 = 0 \leqslant k_1 \leqslant \ldots \leqslant k_p = n)$. By $\lambda(u) = (\lambda_1 \geqslant \ldots \geqslant \lambda_r) \vdash n$ we denote the Jordan form of the nilpotent endomorphism $u \in \mathrm{End}(\mathbb{C}^n)$, seen as a Young diagram.

Definition 23.

(a) Let $\mathrm{RTab}_k(\lambda(u))$ be the set of tableaux τ of shape $\lambda(u)$ and entries $1, \ldots, p$, such that τ contains $k_i - k_{i-1}$ entries equal to i for all i, and the entries in each row of τ are (strictly) increasing from left to right. Note that $\mathrm{STab}_k(\lambda(u))$ is a subset of $\mathrm{RTab}_k(\lambda(u))$.

(b) Given a tableau $\tau \in \mathrm{RTab}_k(\lambda(u))$, define its rectification $\mathrm{Rect}(\tau)$ to be the tableau of shape $\lambda(u)$ obtained from τ by reordering the entries of each column in non-decreasing order from top to bottom. In fact, the tableau $\mathrm{Rect}(\tau)$ so-obtained is semistandard, hence belongs to $\mathrm{STab}_k(\lambda(u))$ (see [4]). Given a semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$, we define

$$\mathrm{RTab}(T) = \{\tau \in \mathrm{RTab}_k(\lambda(u)) : \mathrm{Rect}(\tau) = T\}.$$

(c) In [4], P. Drube introduces a notion of inversion number $n_{\mathrm{inv}}(\tau)$ which measures how far a row-increasing tableau $\tau \in \mathrm{RTab}_k(\lambda(u))$ is from being semistandard. We summarize this definition here: an inversion pair of τ consists of two entries $a < b$ in the same column of τ such that one of the following conditions holds – by $(a_1, a_2, \ldots, a_\ell)$, resp., $(b_1, b_2, \ldots, b_m)$, we denote the (possibly empty) list of entries directly to the right of a, resp. b, in τ, read from left to right:

- a is below b (in particular $\ell \leqslant m$) and $a_j = b_j$ for all $j = 1, \ldots, \ell$; or
- there is $j_0 \leqslant \min\{\ell, m\}$ such that $a_j = b_j$ for $1 \leqslant j < j_0$ and $a_{j_0} > b_{j_0}$.

Then $n_{\mathrm{inv}}(\tau)$ denotes the total number of inversion pairs of τ. We have $n_{\mathrm{inv}}(\tau) = 0$ if and only if τ is semistandard ([4]).

Example 24. Let $k = (0, 2, 3, 5, 7, 8, 10)$ and $\lambda(u) = (4, 4, 2)$. For $\tau = \begin{array}{|c|c|c|c|}\hline 1&4&5&6\\\hline 2&3&4&6\\\hline 1&3\\\cline{1-2}\end{array} \in \mathrm{RTab}_k(\lambda(u))$, we get $\mathrm{Rect}(\tau) = \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array} \in \mathrm{STab}_k(\lambda(u))$. The inversion pairs of τ are the couples $(1_1, 2)^1$, $(1_2, 2)^1$, $(3_2, 4)^2$, $(4, 5)^3$, thus $n_{\mathrm{inv}}(\tau) = 4$. In that list of inversion pairs (following the convention used in [4, 5]), the superscript indicates the column which the pair belongs to, while the notation a_i means the i-th entry of value a in the corresponding column.

Recall that we have fixed a Jordan basis $(\varepsilon_{\ell,j})$ of u parametrized by the couples (ℓ, j) with $1 \leqslant \ell \leqslant r$ and $1 \leqslant j \leqslant \lambda_\ell$ (those couples correspond to the various positions of the boxes of the Young diagram $\lambda(u)$). The basis is such that

$$u(\varepsilon_{\ell,j}) = \varepsilon_{\ell,j-1} \text{ if } j \geqslant 2 \quad \text{and} \quad u(\varepsilon_{\ell,1}) = 0.$$

Definition 25. For a row-increasing tableau $\tau \in \mathrm{RTab}_k(\lambda(u))$, we define a partial flag $F_\tau = (F_0, F_1, \ldots, F_p)$ by letting

$$F_i = \langle \varepsilon_{\ell,j} : \text{the } (\ell, j) \text{ entry of } \tau \text{ is} \leqslant i\rangle_{\mathbb{C}} \text{ for all } i,$$

where by (ℓ, j) entry we mean the entry situated in the ℓ-th row, j-th column. Clearly, $F_\tau = F_{\tau'}$ iff $\tau = \tau'$.

Lemma 26.

(a) *The partial flag F_τ belongs to the variety Y_T for $T = \mathrm{Rect}(\tau)$.*

(b) *Let $g : \mathbb{C}^n \to \mathbb{C}^n$ be a linear isomorphism which is diagonal in the basis $(\varepsilon_{\ell,j})$, with n pairwise distinct eigenvalues, and such that $gug^{-1} \in \mathbb{C}^*u$. Such a g exists. Then g acts on Y_T in a natural way and*

$$(Y_T)^g := \{F \in Y_T : g(F) = F\} = \{F_\tau : \tau \in \mathrm{RTab}(T)\}.$$

Proof. (a) By construction, $\dim F_i$ is equal to the number of entries $\leqslant i$ in τ. Since τ belongs to the set $\mathrm{RTab}_k(\lambda(u))$, this number is equal to k_i. Hence $F \in \mathrm{Fl}_k(\mathbb{C}^n)$. By construction, each subspace F_i is spanned by vectors which belong to $\bigcup_{j=1}^q K_j$, hence the flag F is homogeneous. Moreover for all i, j we have

$$F_i \cap K_j = \langle \varepsilon_{\ell,j} : \text{the } \ell\text{-th box of the } j\text{-th column of } \tau \text{ is} \leqslant i\rangle_{\mathbb{C}}. \tag{25}$$

On the one hand this implies that

$$\dim F_i \cap K_j = (\text{number of entries} \leqslant i \text{ in the } j\text{-th column of } \tau) = c_j(T|_{\leqslant i})$$

since the j-th columns of τ and $T = \mathrm{Rect}(\tau)$ have the same content. On the other hand, for $j \geqslant 2$, using that the rows of τ are increasing, we get

$$\begin{aligned} u(F_i \cap K_j) &= \langle \varepsilon_{\ell,j-1} : \text{the } \ell\text{-th box of the } j\text{-th column of } \tau \text{ is} \leqslant i\rangle_{\mathbb{C}} \\ &\subset \langle \varepsilon_{\ell,j-1} : \text{the } \ell\text{-th box of the } (j-1)\text{-th column of } \tau \text{ is} \leqslant i-1\rangle_{\mathbb{C}} \\ &= F_{i-1} \cap K_{j-1} \end{aligned}$$

while for $j = 1$ the inclusion $u(F_i \cap K_1) \subset u(K_1) = 0 = F_{i-1} \cap K_0$ holds. By Lemma 19, these observations imply that F_τ belongs to Y_T.

(b) The linear map

$$g_0 : \mathbb{C}^n \to \mathbb{C}^n,\ \varepsilon_{\ell,j} \mapsto 2^{\ell-jr}\varepsilon_{\ell,j} \text{ for } 1 \leqslant \ell \leqslant r,\ 1 \leqslant j \leqslant \lambda_\ell$$

is an example of map g which fulfills the conditions; it satisfies $g_0 u g_0^{-1} = 2^r u$. The fact that g normalizes u and preserves each subspace K_j guarantees that its natural action on partial flags stabilizes the variety Y_T (see Lemma 19). The inclusion $\{F_\tau : \tau \in \mathrm{RTab}(T)\} \subset (Y_T)^g$ is clear. Conversely, if $F = (F_0, \ldots, F_p) \in Y_T$ is g-fixed, then each subspace F_i is spanned by a family of eigenvectors $(\varepsilon_{\ell,j} : (\ell, j) \in I_i)$ for some subset I_i of cardinality k_i. Let σ be the tableau of shape $\lambda(u)$ obtained by putting the number i in the (ℓ, j) box of $\lambda(u)$ whenever $(\ell, j) \in I_i \backslash I_{i-1}$. Since $u(F_i) \subset F_{i-1}$, the implication $(\ell, j) \in I_i \Rightarrow (\ell, j-1) \in I_{i-1}$ holds whenever $i \geqslant 1$, $j \geqslant 2$. This guarantees that the rows of σ are increasing, therefore the tableau σ belongs to $\mathrm{RTab}_k(\lambda(u))$, and $F = F_\sigma$ is the partial flag corresponding to this tableau in the sense of Definition 25. Finally, part (a) guarantees that $\mathrm{Rect}(\sigma) = T$, i.e., $\sigma \in \mathrm{RTab}(T)$. The proof is complete. □

The main result of this section states as follows.

Theorem 27. *We consider a semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$ and the corresponding variety Y_T. There is a cell decomposition $Y_T = \bigsqcup Y(\tau)$ parametrized by the row-increasing tableaux $\tau \in \mathrm{RTab}(T)$ (of rectification T), which satisfies the following conditions:*

(a) $F_\tau \in Y(\tau)$;

(b) $\dim Y(\tau) = n_{\mathrm{inv}}(\tau)$.

This result is a consequence of Theorems 9, 20 above and Proposition 28 below. Before stating the proposition, we review some notation. We fix a semistandard tableau $T \in \mathrm{STab}_k(\lambda(u))$. We consider the $p \times q$ matrix $d_T = (d_{i,j})$ and the chain of subspaces $A = (A_1, \ldots, A_q) := (u^{q-1}(\ker u^q) \subset \ldots \subset u(\ker u^2) \subset \ker u)$ introduced in Theorem 20. The vectors $\varepsilon_a := \varepsilon_{a,1}$ $(1 \leqslant a \leqslant r)$ form a basis of $A_q = \ker u$. Moreover

$$A_j = \langle \varepsilon_a : 1 \leqslant a \leqslant d_{p,j} \rangle_{\mathbb{C}} \quad \text{for all } j = 1, \ldots, q,$$

since $A_j = u^{q-j}(K_{q+1-j})$ and the vectors $\varepsilon_{a,q+1-j}$ $(a = 1, \ldots, d_{p,j})$ generate K_{q+1-j}. Finally, recall the set $\mathcal{W}(d_T^\sharp)$ considered in Section 3. Every $\omega \in \mathcal{W}(d_T^\sharp)$ is a $(p+q-1) \times q$ matrix of sets and gives rise to an element $V_\omega \in X(d_T^\sharp, A)$ (see Definition 8). Recall the inversion number $n_{\mathrm{inv}}(\omega)$ defined in Definition 6.

Proposition 28. *For a row-increasing tableau $\tau \in \mathrm{RTab}(T)$, let $\omega(\tau) = (\omega_{i,j})$ be the $p \times q$ matrix of sets given by*

$$\omega_{i,j} = \{\ell : \textit{the } \ell\textit{-th entry of the } (q+1-j)\textit{-th column of } \tau \textit{ is} \leqslant i\}.$$

Then the map

$$\Xi_T : \mathrm{RTab}(T) \to \mathcal{W}(d_T^\sharp), \ \tau \mapsto \omega(\tau)^\sharp$$

is a well-defined bijection. Moreover, this bijection satisfies

$$\Phi_T(F_\tau) = V_{\Xi_T(\tau)} \quad \textit{and} \quad n_{\mathrm{inv}}(\tau) = n_{\mathrm{inv}}(\Xi_T(\tau)) \quad \textit{for all } \tau \in \mathrm{RTab}(T),$$

where $\Phi_T : Y_T \to X(d_T^\sharp, A)$ is the isomorphism of Theorem 20.

Proof. Let $\omega(\tau) = (\omega_{i,j})$ be as in the statement. Thus $|\omega_{i,j}|$ is equal to the number of entries $\leqslant i$ in the $(q+1-j)$-th column of τ, so $|\omega_{i,j}| = d_{i,j}$. The inclusion $\omega_{i,j} \subset \omega_{i+1,j}$ is immediate for all i, j, moreover $\omega_{i,j} \subset \omega_{p,j} = \{1, \ldots, d_{p,j}\}$. Finally, since the tableau τ is row increasing, we also have the inclusion $\omega_{i,j} \subset \omega_{i-1,j+1}$ for all $i = 2, \ldots, p$, all $j = 1, \ldots, q-1$. Altogether these observations show that the shifted matrix of sets $\omega(\tau)^\sharp$ belongs to the set $\mathcal{W}(d_T^\sharp)$. Hence the map Ξ_T is well defined.

Let us check the equality $\Phi_T(F_\tau) = V_{\Xi_T(\tau)}$. Clearly $V_{\Xi_T(\tau)} = V^\sharp$ where $V = (V_{i,j})$ is the $p \times q$ matrix of spaces given by $V_{i,j} = \langle \varepsilon_a : a \in \omega_{i,j} \rangle_{\mathbb{C}}$ (with $\omega(\tau) = (\omega_{i,j})$ as above). On the other hand, by (25), the flag $F_\tau = (F_0, \ldots, F_p)$ satisfies

$$u^{q-j}(F_i \cap K_{q+1-j}) = u^{q-j}(\langle \varepsilon_{a,q+1-j} : a \in \omega_{i,j} \rangle_{\mathbb{C}}) = \langle \varepsilon_{a,1} : a \in \omega_{i,j} \rangle_{\mathbb{C}},$$

so $u^{q-j}(F_i \cap K_{q+1-j}) = V_{i,j}$ for all i, j. Whence the equality.

Let us show that the map Ξ_T is bijective. If $\Xi_T(\tau) = \Xi_T(\tau')$ then $F_\tau = F_{\tau'}$ (by the equality just shown) and so $\tau = \tau'$; hence the map is injective. Let $g : \mathbb{C}^n \to \mathbb{C}^n$ be as in Lemma 26 (b) and let $\overline{g} := g|_{\ker u} : \ker u \to \ker u$, thus $\overline{g} \in S_A$ where S_A is as in Theorem 9. Clearly the isomorphism $\Phi_T : Y_T \to X(d_T^\sharp, A)$ is g-equivariant in the sense that $\Phi_T(gF) = \overline{g}\Phi_T(F)$ for all $F \in Y_T$. This fact, combined with Theorem 9 (a) and Lemma 26 (b), yields

$$|\mathrm{RTab}(T)| = |(Y_T)^g| = |(X(d_T^\sharp, A))^{\overline{g}}| \geqslant |(X(d_T^\sharp, A))^{S_A}| = |\mathcal{W}(d_T^\sharp)|.$$

Therefore, $\Xi_T : \mathrm{RTab}(T) \to \mathcal{W}(d_T^\sharp)$ is bijective.

It remains to show that $n_{\mathrm{inv}}(\tau) = n_{\mathrm{inv}}(\Xi_T(\tau))$. Let $j \in \{1, \ldots, q\}$. Recall that $d_{p,j}$ coincides with the number of boxes in the $(q+1-j)$-th column of τ. For $a \in \{1, \ldots, d_{p,j}\}$, let i_a denote the $(a, q+1-j)$ entry of τ. The j-th column of the matrix $\Xi_T(\tau) = \omega(\tau)^\sharp$ consists of the chain of subsets

$$\omega_{1,j} = \ldots = \omega_{1,j} \subset \omega_{2,j} \subset \ldots \subset \omega_{p-1,j} \subset \omega_{p,j} = \ldots = \omega_{p,j} = \{1, \ldots, d_{p,j}\}$$

and we have $i_a = \min\{i = 1, \ldots, p : a \in \omega_{i,j}\}$. For $a, b \in \{1, \ldots, d_{p,j}\}$, we deduce the equivalences $a <_j b \Leftrightarrow i_a < i_b$ and $a \sim_j b \Leftrightarrow i_a = i_b$. Let $1 \leqslant a < b \leqslant d_{p,j}$ and let us show the equivalence:

$$(a, b) \in \mathrm{Inv}_j(\omega(\tau)^\sharp) \Leftrightarrow (i_a, i_b) \text{ or } (i_b, i_a) \text{ is an inversion pair for } \tau \tag{26}$$

(depending on whether $i_a < i_b$ or $i_b < i_a$); the desired formula $n_{\mathrm{inv}}(\tau) = n_{\mathrm{inv}}(\Xi_T(\tau))$ is clearly guaranteed once we show (26). Let $((i_a)_1, \ldots, (i_a)_\ell)$ (resp. $((i_b)_1, \ldots, (i_b)_m)$ be the list of entries directly to the right of i_a (resp. i_b) in τ. Since i_b is below i_a in τ, we have $m \leqslant \ell$. As above,

$$(i_a)_s = \min\{i : a \in \omega_{i,j-s}\},\ (i_b)_s = \min\{i : b \in \omega_{i,j-s}\} \ \text{ for all } s = 1, \ldots, m$$

and m is the minimal number such that $b > d_{p,j-(m+1)}$ (using the convention $d_{p,0} = 0$), so

$$b > \max\{d_{p,j-(m+1)}, a\}, \quad \text{whence} \quad a <_{j-(m+1)} b. \tag{27}$$

Case 1: Assume that $i_a < i_b$, i.e., $a <_j b$.

In this case, the couple (i_a, i_b) is an inversion of τ if and only if there is $s_0 \in \{1, \ldots, m\}$ such that $(i_a)_s = (i_b)_s$ whenever $1 \leqslant s < s_0$ and $(i_a)_{s_0} > (i_b)_{s_0}$. Equivalently (taking also (27) into account), there is $s_0 \in \{1, \ldots, j\}$ such that $a \sim_{j-s} b$ whenever $1 \leqslant s < s_0$ and $a >_{j-s_0} b$, which means that $(a, b) \in \mathrm{Inv}_j(\omega(\tau)^\sharp)$.

Case 2: Assume that $i_a > i_b$, i.e., $a >_j b$.

Here, the couple (i_b, i_a) is an inversion of τ if and only if one of the following conditions holds

- $(i_a)_s = (i_b)_s$ whenever $1 \leqslant s \leqslant m$; or
- there is $s_0 \in \{1, \ldots, m\}$ such that $(i_a)_s = (i_b)_s$ whenever $1 \leqslant s < s_0$ and $(i_a)_{s_0} < (i_b)_{s_0}$.

In view of (27), this is equivalent to the single condition:

- there is $s_0 \in \{1, \ldots, j\}$ such that $a \sim_{j-s} b$ whenever $1 \leqslant s < s_0$ and $a <_{j-s_0} b$,

which means that $(a, b) \in \mathrm{Inv}_j(\omega(\tau)^\sharp)$.

In both cases we have shown (26). The proof is complete. $\square$

Theorem 27 (combined with (5)) and Corollary 21 yield the following corollary, which is in fact a reformulation of [5, Theorem 2.4]:

Corollary 29. $\displaystyle\sum_{\tau \in \mathrm{RTab}(T)} x^{n_{\mathrm{inv}}(\tau)} = \prod_{\substack{2 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} \begin{bmatrix} d_{i,j-1} \; d_{i-1,j} \\ d_{i,j} \end{bmatrix}_x$, *where $d_T = (d_{i,j})$ is the $p \times q$ matrix of Theorem 20 (with $d_{i,0} := 0$).*

Remark 30. (a) In view of Theorem 27 (and (5)), the generating function for inversion number on row-increasing tableaux $\chi^T(x) := \sum_{\tau \in \mathrm{RTab}(T)} x^{n_{\mathrm{inv}}(\tau)}$ is therefore realized as the Poincaré polynomial of the smooth, irreducible, projective variety Y_T. The fact that $\chi^T(x)$ is unimodal and palindromic (pointed out in [5, Corollaries 2.8–2.9]) can then be viewed as a consequence of the Lefschetz theorem.

(b) By Theorem 27, the maximal inversion number of an element $\tau \in \mathrm{RTab}(T)$ is $\dim Y_T$, and it is attained for a unique tableau τ_0 (see also [5, Corollary 2.7]). The equality

$$\dim H^{2m}(Y_T, \mathbb{Q}) = \dim H^{2(\dim Y_T - m)}(Y_T, \mathbb{Q}) \quad \text{for all } m = 0, \ldots, \dim Y_T$$

(which is due to the Lefschetz theorem, or to the fact that $\chi^T(x)$ is palindromic) implies that there is an involution $\mathrm{RTab}(T) \to \mathrm{RTab}(T)$, $\tau \mapsto \tau^*$ such that

$$n_{\mathrm{inv}}(\tau^*) = \dim Y_T - n_{\mathrm{inv}}(\tau) \quad \text{for all } \tau \in \mathrm{RTab}(T).$$

In particular this involution must verify $(\tau_0)^* = T$. For arbitrary τ, we have no explicit formula for τ^*.

(c) Part (b) of the remark yields a "dual" cell decomposition

$$Y_T = \bigsqcup_{\tau \in \mathrm{RTab}(T)} Y'(\tau) \quad \text{such that} \quad \dim Y'(\tau) = \dim Y_T - n_{\mathrm{inv}}(\tau).$$

In view of Remark 18, this yields a cell decomposition of each subvariety $\mathrm{Fl}_{k,u,T} \subset \mathrm{Fl}_{k,u}$, and finally of the whole Spaltenstein variety $\mathrm{Fl}_{k,u}$:

$$\mathrm{Fl}_{k,u,T} = \bigsqcup_{\tau \in \mathrm{RTab}(T)} \widetilde{Y}'(\tau), \quad \mathrm{Fl}_{k,u} = \bigsqcup_{\tau \in \mathrm{RTab}_k(\lambda(u))} \widetilde{Y}'(\tau),$$
$$\text{with } \dim \widetilde{Y}'(\tau) = \dim \mathrm{Fl}_{k,u} - n_{\mathrm{inv}}(\tau),$$

such that $\widetilde{Y}'(\tau) := (\varphi_T)^{-1}(Y'(\tau))$ whenever $\tau \in \mathrm{RTab}(T)$, where $\varphi_T : \mathrm{Fl}_{k,u,T} \to Y_T$ is the affine bundle of Proposition 17. We deduce the following equality of Poincaré polynomials

$$\sum_{m=0}^{m_0} \dim H^{2(m_0-m)}(\mathrm{Fl}_{k,u}, \mathbb{Q}) x^m = \sum_{\tau \in \mathrm{RTab}_k(\lambda(u))} x^{n_{\mathrm{inv}}(\tau)} = \sum_{T \in \mathrm{STab}_k(\lambda(u))} P(Y_T)(x),$$

where $m_0 := \dim \mathrm{Fl}_{k,u}$. Since the Spaltenstein variety $\mathrm{Fl}_{k,u}$ is connected, we know that $\dim H^0(\mathrm{Fl}_{k,u}, \mathbb{Q}) = 1$, hence m_0 is the maximal inversion number for the elements of $\mathrm{RTab}_k(\lambda(u))$ and it is attained for a unique tableau $\tau_{\max}$. This tableau and its rectification $T_{\max} := \mathrm{Rect}(\tau_{\max})$ are explicitly described in [4, §2.1]. For this tableau we have $\dim Y_{T_{\max}} = n_{\mathrm{inv}}(\tau_{\max}) = \dim \mathrm{Fl}_{k,u} = \dim \mathrm{Fl}_{k,u,T_{\max}}$, which means that the affine bundle $\varphi_{T_{\max}} : \mathrm{Fl}_{k,u,T_{\max}} \to Y_{T_{\max}}$ must be an isomorphism. This implies that $\mathrm{Fl}_{k,u,T_{\max}}$ is a projective (hence closed) subvariety of $\mathrm{Fl}_{k,u}$. Hence it is actually an irreducible component of $\mathrm{Fl}_{k,u}$ which is smooth and isomorphic to the variety $X(d^\sharp_{T_{\max}}, A)$ of Theorem 20.

In particular, every Spaltenstein variety contains at least one smooth irreducible component, which is isomorphic to a variety of the form $X(d, A)$.

References

[1] A. Borel, *Linear algebraic groups*. Second edition. Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[2] M. Brion, *Lectures on the geometry of flag varieties*. In: Topics in cohomological studies of algebraic varieties, pp. 33–85, Trends Math., Birkhäuser, Basel, 2005.

[3] J. Brundan and V. Ostrik, *Cohomology of Spaltenstein varieties*, Transform. Groups **16** (2011), 619–648.

[4] P. Drube, *Inversions of semistandard Young tableaux*, Electron. J. Combin. **23** (2016), no. 1, #P1.43.

[5] P. Drube, *Generating functions for inverted semistandard Young tableaux and generalized ballot numbers*, Electron. J. Combin. **24** (2017), no. 2, #P2.26.

[6] L. Fresse, *Betti numbers of Springer fibers in type A*, J. Algebra **322** (2009), 2566–2579.

[7] L. Fresse, *Existence of affine pavings for varieties of partial flags associated to nilpotent elements*, Int. Math. Res. Not. IMRN **2016** (2016), 418–472.

[8] W.H. Hesselink, *Polarizations in the classical groups*, Math. Z. **160** (1978), 217–234.

[9] J.C. Jantzen, *Nilpotent orbits in representation theory.* In: Lie theory, pp. 1–211, Progr. Math., vol. 228, Birkhäuser Boston, Boston, MA, 2004.

[10] P. Magyar, *Schubert polynomials and Bott-Samelson varieties*, Comment. Math. Helv. **73** (1998), no. 4, 603–636.

[11] H. Nakajima, *Homology of moduli spaces of instantons on ALE spaces. I*, J. Differential Geom. **40** (1994), 105–127.

[12] N. Spaltenstein, *The fixed point set of a unipotent transformation on the flag manifold*, Indag. Math. **38** (1976), 452–456.

[13] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Mathematics, vol. 946, Springer-Verlag, Berlin-New York, 1982.