Hamilton cycles in sparse robustly expanding digraphs

Allan Lo[*]

School of Mathematics
University of Birmingham
Birmingham, U.K.

s.a.lo@bham.ac.uk

Viresh Patel[†]

Korteweg de Vries Instituut voor Wiskunde
Universiteit van Amsterdam
1090 GE Amsterdam, The Netherlands

viresh.s.patel@gmail.com

Abstract

The notion of robust expansion has played a central role in the solution of several conjectures involving the packing of Hamilton cycles in graphs and directed graphs. These and other results usually rely on the fact that every robustly expanding (di)graph with suitably large minimum degree contains a Hamilton cycle. Previous proofs of this require Szemerédi's Regularity Lemma and so this fact can only be applied to dense, sufficiently large robust expanders. We give a proof that does not use the Regularity Lemma and, indeed, we can apply our result to sparser robustly expanding digraphs.

Mathematics Subject Classifications: 05C20, 05C35, 05C38, 05C45

1 Introduction

Throughout, we work with simple directed graphs (also called digraphs), i.e. directed graphs with no loops and with at most two edges between each pair of vertices (one in each direction). A Hamilton cycle in a (directed) graph is a (directed) cycle that passes through every vertex. Over the last several decades, there has been intense study in finding sufficient conditions for the existence of Hamilton cycles in graphs and digraphs. The seminal result in the case of graphs is Dirac's Theorem [5] and in the case of digraphs

is Ghouila-Houri's Theorem [7], each giving tight minimum degree conditions for the existence of Hamilton cycles.

This paper concerns Hamilton cycles in robust expanders. Below we define a robust expander and give some brief background.

Definition 1. For an n-vertex digraph $D = (V, E)$, $\nu \in (0, 1)$, and $S \subseteq V$, the *robust ν-outneighbourhood* of S, denoted $\mathrm{RN}_\nu^+(S)$, is the set of vertices that have at least νn inneighbours in S. Given $0 < \nu \leqslant \tau < 1$, we say D is a *robust (ν, τ)-outexpander* if

$$|\mathrm{RN}_\nu^+(S)| \geqslant |S| + \nu n$$

for every $S \subseteq V$ satisfying $\tau n \leqslant |S| \leqslant (1 - \tau)n$. The *robust ν-in-neighbourhood*, $\mathrm{RN}^-(S)$, and *robust (ν, τ)-inexpanders* are defined similarly. We refer to D as a robust (ν, τ)-expander if it is both a robust (ν, τ)-in and -outexpander.

Usually the parameters ν and τ are thought of as small constants as in Theorem 2 below, but we will also be interested in these parameters as functions of n. Note that robust expansion is a resilient property, i.e. if D is a robust outexpander, then D remains a robust outexpander (with slightly worse parameters) after removing a sparse subgraph.

Robust expansion has played a central role in the proofs of several conjectures about Hamilton cycles. The starting point of many of these proofs is the following result which says that a robust expander with linear minimum semi-degree contains a Hamilton cycle. The semi-degree $\delta^0(D)$ of a digraph D is given by $\delta^0(D) = \min(\delta^+(D), \delta^-(D))$ where $\delta^+(D)$ and $\delta^-(D)$ are respectively the minimum outdegree and minimum indegree of D.

Theorem 2 ([19]). *Let n_0 be a positive integer and γ, ν, τ be positive constants such that $1/n_0 \ll \nu \leqslant \tau \ll \gamma < 1$. Let D be a digraph on $n \geqslant n_0$ vertices with $\delta^0(D) \geqslant \gamma n$ which is a robust (ν, τ)-outexpander. Then D contains a Hamilton cycle.*

This result was first proved in [19] by Kühn, Osthus and Treglown. A simpler proof is given in [16] and an algorithmic version is given in [3]. The proofs of Theorem 2 presented in [19, 16, 3] all rely on the Regularity Lemma and so in particular one can only work with sufficiently large and dense digraphs.

Our main purpose in this paper is to give a proof of the above result that avoids the use of the Regularity Lemma, but uses instead the recent absorption technique developed by Rödl, Ruciński and Szemerédi [24] (with special forms appearing in earlier work e.g. [12]). We apply our technique to "sparse" robust expanders which have not been studied before but which we hope may find applications. In addition we consider cycles of different lengths. The most general form of our result is stated below.

Theorem 3. *Let $n \in \mathbb{N}$ and $\nu, \tau, \gamma \in (0, 1)$ satisfying $4 \sqrt[13]{\log^2 n / n} < \nu \leqslant \tau \leqslant \gamma/16 < 1/16$. Let D be an n-vertex digraph with $\delta^0(G) \geqslant \gamma n$ which is a robust (ν, τ)-outexpander. Then, for any $\nu n/2 \leqslant \ell \leqslant n$ and any vertex v of D, D contains a directed cycle of length ℓ through v.*

The result above is algorithmic. We believe some form of it should be true for much sparser graphs than we are able to prove it for.

Theorem 2 (and its undirected version) have been used as a black box in several papers including [20, 15, 11, 17, 22, 6]. Below we discuss results that require the Regularity Lemma only because they rely (directly or indirectly) on Theorem 2. For some such results, we can now replace Theorem 2 with Theorem 3 to give proofs that do not require the Regularity Lemma and consequently hold for much smaller values of n.

1.1 Hamiltonicity in oriented graphs

Robust expansion was first used to prove an approximate analogue of Dirac's Theorem for oriented graphs (an oriented graph is a directed graph in which there is at most one edge between each pair of vertices).

Theorem 4 ([10]). *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that if D is an oriented graph with $n > n_0$ vertices and $\delta^0(D) > \frac{3}{8}n + \varepsilon n$ then D contains a Hamilton cycle.*

Here the constant 3/8 cannot be improved due to examples given in [10]. The result above was proved using the Regularity Lemma and an exact version was proved later in [8] also using the Regularity Lemma. A consequence of Theorem 3 is that one can adapt the proof of Theorem 4 to avoid the use of the Regularity Lemma.

Corollary 5. *Let $n \in \mathbb{N}$ and $0 < \varepsilon < 1/64$ with $n > \varepsilon^{-40}$. If D is an n-vertex oriented graph with $\delta^0(D) > \frac{3}{8}n + \varepsilon n$ then D contains a Hamilton cycle.*

In fact, one can use Theorem 3 to adapt the proof of the exact version in [8] to avoid the use of the Regularity Lemma.

1.2 Hamiltonicity and degree sequences

In [19], Kühn, Osthus and Treglown give an approximate solution to a conjecture of Nash-Williams [21] about sufficient conditions on the degree sequence of a digraph to guarantee the existence of a Hamilton cycle. Their result uses the Regularity Lemma, but Theorem 3 can be used to adapt their proof to avoid the use of the Regularity Lemma and thus give a better approximation.

For a digraph D, consider its outdegree sequence $d_1^+ \leqslant \ldots \leqslant d_n^+$ and indegree sequence $d_1^- \leqslant \ldots \leqslant d_n^-$. Note that d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of D.

Theorem 6. *Let $n \in \mathbb{N}$ and $\gamma \in (0, 1/2)$ be such that $n \geqslant 2^{91}\gamma^{-27}$. Let D be an n-vertex digraph such that for all $i < n/2$,*

- $d_i^+ \geqslant i + \gamma n$ *or* $d_{n-i-\gamma n}^- \geqslant n - i$,

- $d_i^- \geqslant i + \gamma n$ *or* $d_{n-i-\gamma n}^+ \geqslant n - i$.

Then, for any $\nu n/2 \leqslant \ell \leqslant n$ and any vertex v of D, D contains a directed cycle of length ℓ through v.

1.3 Hamiltonicity in regular graphs

In [13, 14], Kühn, Osthus, Staden and the first author prove the one remaining case of a conjecture of Bollobás and Haggvist, making (indirect) use of the Regularity Lemma: they prove that there exits n_0 such that every 3-connected D-regular graph on $n \geqslant n_0$ vertices with $D \geqslant n/4$ is Hamiltonian. Replacing the use of Theorem 2 by Theorem 3 in [13, 14] gives a proof of the result avoiding the Regularity Lemma.

1.4 Outline

In the next section we collect some notation and in Section 3, we prove some simple facts about robustly expanding digraphs. Section 4 is devoted to describing and constructing an 'absorbing structure' H in a robustly expanding digraph D. Informally, one can think of H as a set of edges of D which have the property that (almost) any small collection of vertex-disjoint cycles of D can be connected together into a long cycle using the edges of H. In Section 5 we show that the vertices of any robustly expanding digraph can be covered by a small number of cycles. In Section 6 we combine these results to prove Theorem 3, and we give some concluding remarks in Section 7.

We mention here that during the course of various proofs, several straightforward calculations, which we feel detract from the main argument, are suppressed and can be found at the end of the paper.

2 Notation

The digraphs considered in this paper do not have loops and we allow up to two edges between any pair x, y of distinct vertices, at most one in each direction. Given a digraph $D = (V, E)$, we sometimes write $V(D) := V$ for its vertex set and $E(D) := E$ for its edge set and $|D|$ for the number of its vertices. We write xy for an edge directed from x to y.

We write $H \subseteq D$ to mean that H is a subdigraph of D, i.e. $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. Given $X \subseteq V(D)$, we write $D - X$ for the digraph obtained from D by deleting all vertices in X, and $D[X]$ for the subdigraph of D induced by X. Given $F \subseteq E(D)$, we write $D - F$ for the digraph obtained from D by deleting all edges in F. If H is a subdigraph of D, we write $D - H$ for $D - E(H)$. For two subdigraphs H_1 and H_2 of D, we write $H_1 \cup H_2$ for the subdigraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. For a set U, U^2 means the set of all ordered pairs of U, and $U^{[2]}$ means the set of all ordered pairs of U except pairs of the form (x, x).

If x is a vertex of a digraph D, then $N_D^+(x)$ denotes the *outneighbourhood* of x, i.e. the set of all those vertices y for which $xy \in E(D)$. Similarly, $N_D^-(x)$ denotes the *inneighbourhood* of x, i.e. the set of all those vertices y for which $yx \in E(D)$. We write $d_D^+(x) := |N_D^+(x)|$ for the *outdegree* of x and $d_D^-(x) := |N_D^-(x)|$ for its *indegree*. We denote the *minimum outdegree* of D by $\delta^+(D) := \min\{d_D^+(x) : x \in V(D)\}$ and the *minimum indegree* $\delta^-(D) := \min\{d_D^-(x) : x \in V(D)\}$. The *minimum semi-degree* of D is $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$.

Unless stated otherwise, when we refer to paths and cycles in digraphs, we mean directed paths and cycles, i.e. the edges on these paths and cycles are oriented consistently. We write $P = x_1 x_2 \cdots x_t$ to indicate that P is a path with edges $x_1 x_2, x_2 x_3, \ldots, x_{t-1} x_t$, where $x_1, \ldots, x_t$ are distinct vertices. We occasionally denote such a path P by $x_1 P x_t$ to indicate that it starts at x_1 and ends at x_t. We write $\mathring{P}$ for the interior of P, i.e. $\mathring{P} = x_2 \cdots x_{t-1}$. For two paths $P = a \cdots b$ and $Q = b \cdots c$, we write $aPbQc$ for the concatenation of the paths P and Q and this notation generalises to cycles in the obvious ways.

Throughout, logarithms are taken base e.

3   Preliminaries

In this section, we prove some basic properties of robust expanders. The following proposition follows immediately from the definition of a robust expander.

Proposition 7. *Suppose $D = (V, E)$ is a robust (ν, τ)-expander and $S \subseteq V$ with $|S| \leqslant \varepsilon n$. Then $D - S$ is a $(\nu - \varepsilon, \tau/(1 - \varepsilon))$-expander.*

The following observation of DeBiasio, which can be found in [25], says that robust inexpansion is essentially equivalent to robust outexpansion; thus we can and will restrict ourselves to digraphs that are robust (ν, τ)-expanders. We reproduce the proof explicitly quantifying the relationships between the various parameters.

Proposition 8 (DeBiasio)**.** *Suppose $D = (V, E)$ is an n-vertex robust (ν, τ)-outexpander with $\delta^0(D) \geqslant \gamma n$, where $\gamma > 2\tau$, $\tau\gamma > \nu^2/2$ and $\nu < 1/2$. Then D is a robust $(\nu^2/2, 2\tau)$-inexpander.*

Proof. Suppose that D is not a robust $(\nu^2/2, 2\tau)$-inexpander. Then there is a set $S \subseteq V$ with $2\tau n \leqslant |S| \leqslant (1 - 2\tau)n$ such that $|RN^-_{\nu^2/2}(S)| < |S| + \nu^2/2$. Let $T = V \setminus RN^-_{\nu^2/2}(S)$. Observe that

$$|S|\gamma n \leqslant e(V, S) \leqslant |RN^-_{\nu^2/2}(S)||S| + \nu^2 n^2/2,$$

so $|RN^-_{\nu^2/2}(S)| \geqslant \gamma n/2$, where we used that $|S|\gamma n/2 \geqslant \tau\gamma n^2 \geqslant \nu^2 n^2/2$. Therefore

$$\tau n < n - (1 - 2\tau + \nu^2/2)n < |T| \leqslant (1 - \gamma/2)n < (1 - \tau)n,$$

where the first and last inequalities follow from our choice of parameters. By the definition of T, we have that $e(T, S) < |T|\nu^2 n/2$ and so $|RN^+_\nu(T) \cap S| < |T|\nu/2 < \nu n/2$. Hence

$$\begin{aligned}
|RN^+_\nu(T)| &= |RN^+_\nu(T) \setminus S| + |RN^+_\nu(T) \cap S| \\
&< (n - |S|) + \nu n/2 \leqslant n - (|S| + \nu^2 n/2) + \nu n/4 \\
&< n - |RN^-_{\nu^2/2}(S)| + \nu n/4 \leqslant |T| + \nu n,
\end{aligned}$$

where we used that $\nu < \frac{1}{2}$ on the second line. Thus D is not a robust (ν, τ)-outexpander, a contradiction. $\square$

The next two lemmas show that robust expansion allows us to construct short paths between prescribed pairs of vertices.

Lemma 9. *Let $0 < \nu \leqslant \tau \leqslant \gamma/2 < 1/2$ and $n \in \mathbb{N}$ satisfying $n \geqslant 4\nu^{-2}$. Suppose that D is an n-vertex digraph which is a robust (ν, τ)-expander and $\delta^0(D) \geqslant \gamma n$. Given distinct vertices $u, v \in V(D)$, there exists a path $P = x_0 \cdots x_{t+1}$ in D where $x_0 = u$, $x_{t+1} = v$ and $t \leqslant \nu^{-1} - 1$. (Note that P consists of at most $\nu^{-1} + 1$ vertices.)*

Proof. Let $N_1 := N^+(u)$ and inductively define $N_{i+1} := RN_\nu^+(N_i)$. Note that $|N_1| \geqslant \gamma n > \tau n$, so for all $i \geqslant 1$ if $|N_i| < (1 - \tau)n$ then $|N_{i+1}| \geqslant |N_i| + \nu n$. Observe that, for some $t \leqslant \nu^{-1} - 1$, we have $|N_t| \geqslant (1 - \tau)n$. Moreover $N_{t+1} = V(D)$ as $\delta^0(D) \geqslant \gamma n \geqslant 2\tau n$.

Set $x_{t+1} = v$. For $i = t, t-1, \ldots, 1$, let x_i be a vertex in $(N^-(x_{i+1}) \cap N_i) \setminus \{u, x_{i+1}, \ldots, x_{t+1}\}$, which exists as $x_{i+1} \in RN_\nu^+(N_i)$ implies that $|N^-(x_{i+1}) \cap N_i| \geqslant \nu n \geqslant \nu^{-1} + 1 \geqslant t + 2$. By setting $x_0 = u$, we obtain a direct path $P = x_0 \cdots x_{t+1}$ in D. $\qquad\square$

Lemma 10. *Let $0 < \nu \leqslant \tau \leqslant \gamma/4 < 1/4$ and $n, r \in \mathbb{N}$ satisfying $n \geqslant (6r + 11)\nu^{-2}$. Suppose that D is an n-vertex digraph which is a robust (ν, τ)-expander and $\delta^0(D) \geqslant \gamma n$. Given distinct vertices $u_1 \ldots, u_r, v_1, \ldots, v_r \in V(D)$, there exists vertex-disjoint paths $P_1, \ldots, P_r$ in D where P_i is from u_i to v_i and $|P_i| \leqslant 2\nu^{-1} + 1$.*

Proof. By induction assume that we have constructed vertex-disjoint paths $P_1, \ldots, P_{k-1}$ in D for some $k < r$, where, for each $i = 1, \ldots, k-1$, P_i is from u_i to v_i and $|P_i| \leqslant 2\nu^{-1} + 1 \leqslant 3\nu^{-1}$ and $V(P_i) \cap \{u_{i+1}, \ldots, u_r, v_{i+1}, \ldots, v_r\} = \emptyset$. Let D_{k-1} be the digraph obtained from D by deleting all vertices in $P_1, \ldots, P_{k-1}$ and all vertices $u_{k+1}, \ldots, u_r, v_{k+1}, \ldots, v_r$. Note that D_{k-1} is obtained from D by deleting at most $3r\nu^{-1} \leqslant \frac{1}{2}\nu n$ vertices, so by Proposition 7, D_{k-1} is a robust $(\frac{1}{2}\nu, \frac{8}{7}\tau)$-expander with $\delta^0(D_{k-1}) \geqslant \frac{7}{8}\gamma n \geqslant \frac{7}{8}\gamma|D_{k-1}|$. Note that $|D_{k-1}| \geqslant n - 3r\nu^{-1} \geqslant n - r\nu^{-2} \geqslant 16\nu^{-2}$. Apply Lemma 9 to D_{k-1} giving a path P_k in D_{k-1} of length at most $2\nu^{-1} + 1$ from u_k to v_k. Thus P_k (as a path in D) is vertex-disjoint from $P_1, \ldots, P_{k-1}$ and $\{u_{k+1}, \ldots, u_r, v_{k+1}, \ldots, v_r\}$ as required. Thus by induction we can find the paths $P_1, \ldots, P_r$. $\qquad\square$

We give a simple inequality that we shall use several times.

Proposition 11. *Fix $k, a > 0$. Then $e^x > ax^k$ for all $x \geqslant \max(3k(\log k + 1) + 3\log a, 0)$. Similarly for $c, d > 0$ if $x > 3c(\log c + 1) + 3d$, then we have $x > c \log x + d$.*

Proof. We start by showing that for all $a > 0$ and $x \geqslant \max(3\log a + 3, 0)$, we have $e^x \geqslant ax$. This is clearly true if $0 < a \leqslant 1$. If $a > 1$, set $f(x) = e^x - ax$ and set $x_0 := 3\log a + 3 > 0$. We have $f(x_0) = e^3 a^3 - 3a\log a - 3a > 0$ and $f'(x) = e^x - a > 0$ for all $x \geqslant x_0$. Hence $f(x) > 0$ for all $x \geqslant x_0$ and so $e^x > ax$ for all $x \geqslant \max(3\log a + 3, 0)$.

Finally, making the transformation $X = kx$ and $A = a^k/k^k$, and assuming $A, k > 0$, the inequality above becomes $e^X \geqslant AX^k$ for all $X \geqslant \max(3k\log k + 3\log A + 3k, 0)$.

For the other inequality, note that $x > c\log x + d$ if and only if $e^x > e^d x^c$, which holds if $x > \max(3c\log c + 3d + 3c, 0)$. $\qquad\square$

4 The absorbing structure

In this section, we describe what we mean by an *absorbing structure* and show how to find one in a robustly expanding digraph with large minimum in- and outdegree. We begin by informally describing the properties we desire our absorbing structure to have. Given a digraph D we shall seek a subdigraph $S \subseteq D$ with the properties that

- $|V(S)|$ is small;

- S contains a Hamilton cycle (on $V(S)$);

- In D, given a small number of any vertex-disjoint paths $P_1, \ldots, P_d$ that are also vertex-disjoint from S, we can use S to absorb $P_1, \ldots, P_d$ into C i.e. we can find a Hamilton cycle C' on $V(S) \cup (\bigcup_{i=1}^{d} V(P_i))$.

The sequence of definitions that follow will lead to a precise description of our absorbing structure. We start by defining an alternating path.

Definition 12. Let D be a digraph, and let $x_1, \ldots, x_t$ be distinct vertices of D with t even. An *alternating path* $P = [x_1 x_2 \cdots x_t]$ is a subgraph of D with vertex set $\{x_1, \ldots, x_t\}$ (where $x_1, \ldots, x_t$ are distinct vertices) and edge set

$$\{x_i x_{i+1} \mid i = 1, 3, 5, \ldots, t-1\} \cup \{x_{j+1} x_j \mid j = 2, 4, 6, \ldots, t-2\}.$$

We say P is an alternating path from x_1 to x_t.

An alternating path is thus a path where the directions of the edges alternate. It will be important for us that the number of vertices in an alternating path is even so that the first vertex has outdegree 1 and the last vertex has indegree 1.

As with paths, robust expansion allows us to construct alternating paths between prescribed vertices.

Lemma 13. *Let* $0 < \nu \leqslant \tau \leqslant \gamma/2 < 1/2$ *and* $n \in \mathbb{N}$ *satisfying* $n \geqslant 4\nu^{-2}$. *Suppose that* D *is an n-vertex digraph which is a robust (ν, τ)-expander and* $\delta^0(D) \geqslant \gamma n$. *Given distinct vertices* $u, v \in V(D)$, *there exists an alternating path* $P = [x_0 \cdots x_t x_t^* \cdots x_0^*]$ *in D where* $x_0 = u$, $x_0^* = v$ *and* $t \leqslant (\nu^{-1} + 4)/2$ *is even. (Thus P consists of at most $\nu^{-1} + 6$ vertices.)*

Proof. Let $N_1 := N^+(u)$ and inductively define

$$N_{i+1} := \begin{cases} \mathrm{RN}_\nu^+(N_i) & \text{if } i \text{ even}; \\ \mathrm{RN}_\nu^-(N_i) & \text{if } i \text{ odd}. \end{cases}$$

Note that $|N_1| \geqslant \gamma n > \tau n$, so for all $i \geqslant 1$ if $|N_i| < (1-\tau)n$ then $|N_{i+1}| \geqslant |N_i| + \nu n$. Set $r := \lceil \nu^{-1} \rceil$ and observe that $|N_r| \geqslant (1-\tau)n$. Note that $N_{r''} = V(D)$ for all $r'' > r$ as $\delta^0(D) \geqslant \gamma n \geqslant 2\nu n$. Choose r' to be the smallest integer that is greater than r and divisible by 4; thus $r' \leqslant \nu^{-1} + 4$ and $N_{r'} = V(D)$.

Set $y_{r'+1} = v$. For $i = r', r'-1, \ldots, 1$, let y_i be a vertex such that

$$y_i \in \begin{cases} N^-(y_{i+1}) \cap N_i \setminus \{u, y_{i+1}, \ldots, y_{r'+1}\} & \text{if } i \text{ even;} \\ N^+(y_{i+1}) \cap N_i \setminus \{u, y_{i+1}, \ldots, y_{r'+1}\} & \text{if } i \text{ odd.} \end{cases}$$

To see that such y_i exists, observe that since $y_{i+1} \in N_{i+1} = \mathrm{RN}_\nu^+(N_i)$, $|N^-(y_{i+1}) \cap N_i| \geqslant \nu n \geqslant 4\nu^{-1} \geqslant \nu^{-1} + 6 \geqslant r' + 2$ if i is even (and a similar inequality holds if i is odd).

Thus we obtain distinct vertices $y_1, \ldots, y_{r'}$ such that $y_{i+1}y_i, y_j y_{j+1} \in E(D)$ for $i = 1, 3, 5 \ldots, r'-1$ and $j = 2, 4, 6, \ldots, r'-2$. Then relabelling $y_1, \ldots, y_{r'}$ to $x_1, \ldots, x_t x_t^*, \ldots, x_1^*$ respectively and $x_0 := u, x_0^* := v$ gives the desired alternating path. Since r' is divisible by 4, we have that t is even as required. $\qquad\square$

Next we define ladders, which will be the key structures that allow us to absorb paths.

Definition 14. Let D be a digraph and let $u, v \in V(D)$ be distinct vertices. A *ladder* L from u to v is a subdigraph of D given by

$$L = Q \cup Q_1 \cup Q_3 \cup Q_5 \cup \cdots \cup Q_{t-1},$$

where

(i) $Q = [x_0 x_1 \cdots x_t x_t^* \cdots x_1^* x_0^*]$ is an alternating path (with t even) and $x_0 = u$ and $x_0^* = v$;

(ii) Q_i is a directed path from x_i to x_i^* for each $i = 1, 3, \ldots, t-1$; and

(iii) $Q_1, Q_3, \ldots, Q_{t-1}$ are vertex-disjoint paths and are each internally vertex-disjoint from Q.

We call Q the alternating path of L.

- For $i = 0, 2, 4, \ldots, t-2$, we define $R_i \subseteq L$ to be the path $R_i := x_i x_{i+1} Q_{i+1} x_{i+1}^* x_i^*$ and $R_t := x_t x_t^*$. We call these the *rung paths* of L.

- For $i = 2, 4, \ldots, t$, define $R_i' \subseteq L$ to be the path $R_i' := x_i x_{i-1} Q_{i-1} x_{i-1}^* x_i^*$. We call these the *alternative rung paths* of L.

We say the ladder L is *embedded* in the cycle C if $R_i \subseteq C$ for all $i = 0, 2, 4, \ldots, t$.

It is relatively easy to construct ladders in robust expanders. First we show how a ladder embedded in a cycle can be used to absorb a path into the cycle.

Lemma 15. *Let D be a digraph and let $u, v \in V(D)$ be distinct vertices. Let $L \subseteq D$ be a ladder from u to v embedded in a cycle $C \subseteq D$. For any path $P \subseteq D$ from u to v that is internally vertex-disjoint from C there exists a cycle $C' \subseteq D$ such that*

(i) $P \subseteq C'$,

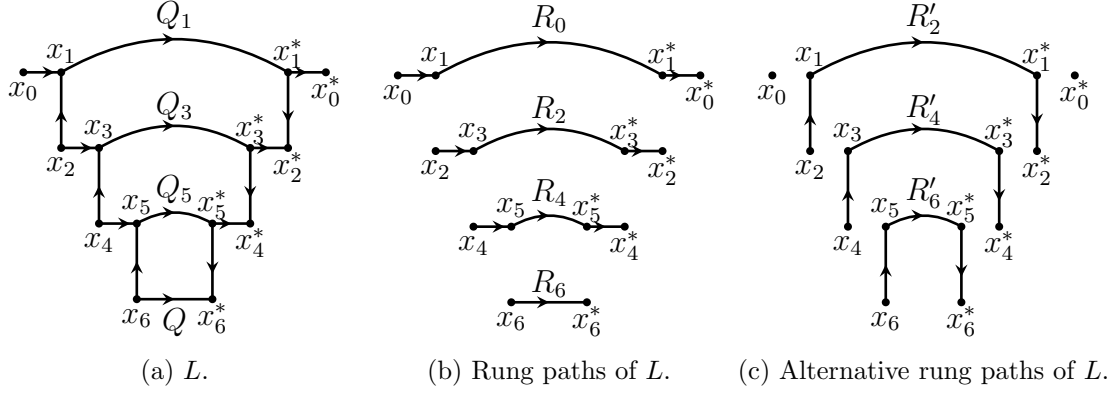(a) L. (b) Rung paths of L. (c) Alternative rung paths of L.

Figure 1: A Ladder $L = Q \cup Q_1 \cup Q_3 \cup Q_5$

(ii) $V(L) \subseteq V(C')$,

(iii) for any path $P' \subseteq C$ with $V(P') \cap V(L) = \emptyset$, we have $P' \subseteq C'$, and

(iv) if $x \in V(D)$ satisfies $x \notin V(C) \cup V(P)$, then $x \notin V(C')$.

In particular, (i), (ii), (iii), and (iv) immediately imply

(v) $V(C) \cup V(P) = V(C')$.

Proof. Let $Q = [x_0 \cdots x_t x_t^* \cdots x_0^*]$ be the alternating path of L, and let Q_i be the corresponding paths of L from x_i to x_i^* for $i = 1, 3, \ldots, t-1$. Let $R_0, R_2, \ldots, R_t$ be the rung paths of L and $R_2', R_4', \ldots, R_t'$ the alternative rung paths. Set $R_0' := P$. We simply replace R_i with R_i' in C one at a time to obtain the desired cycle C'. We spell out the details of the induction below.

We define cycles $C_{-2}, C_0, C_2, C_4, \ldots, C_t$ as follows. Set $C_{-2} := C$. By induction, we assume that C_{i-2} is a cycle with $R_0', \ldots, R_{i-2}', R_i, \ldots, R_t \subseteq C_{i-2}$ (implicitly noting these paths are vertex-disjoint) and that $\mathring{R}_i' = Q_{i-1}$ is vertex-disjoint from C_{i-2}. We obtain C_i by deleting R_i from C_{i-2} and replacing it with R_i'. Since R_i and R_i' are internally vertex-disjoint and both are paths from x_i to x_i^*, then C_i is a cycle. Clearly we have $R_0', \ldots, R_i', R_{i+2}, \ldots, R_t \subseteq C_i$. Since $\mathring{R}_i = \mathring{R}_{i+2}' = Q_{i+1}$ is vertex-disjoint from C_i (since we deleted $\mathring{R}_i$), then $\mathring{R}_{i+2}'$ is vertex-disjoint from C_i.

Thus by induction, we have that $C' := C_t$ is a cycle with $R_0', \ldots, R_t' \subseteq C'$. Therefore $P = R_0' \subseteq C'$ proving (i). Furthermore, since

$$V\left(\bigcup_{\substack{i=0 \\ i \text{ even}}}^{t} R_i \right) = V\left(\bigcup_{\substack{i=2 \\ i \text{ even}}}^{t} R_i' \right) = V(L)$$

then $V(L) \subseteq V(C')$ proving (ii). In the above induction, we note that if $P' \subseteq C_{i-2}$ is a path vertex-disjoint from L, then $P' \subseteq C_i$, so by induction if $P' \subseteq C = C_0$ is a path

vertex-disjoint from L, then $P' \subseteq C_t = C'$ proving (iii). Finally, we note that, in the above induction, for any vertex $x \in V(D) \setminus (V(P) \cup V(L))$, if $x \notin C_{i-2}$ then $x \notin C_i$, proving (iv) and completing the proof. $\qquad \square$

From the previous lemma, we now see that embedding several carefully chosen ladders into a cycle can give us the absorbing structure we desire. The next definition makes precise what we mean by 'carefully' in the previous sentence.

Definition 16. Given a digraph D and distinct vertices $x, y, u, v \in V(D)$, we say that the ordered pair $(u, v) \in V(D)^{[2]}$ *covers* $(x, y) \in V(D)^2$ if $ux, yv \in E(D)$. Given $K \subseteq V(D)^{[2]}$ and $U \subseteq V(D)$, we say that K *d-covers* U if for every $(x, y) \in U^2$ there exist d distinct elements of K each of which covers (x, y). We say K is vertex-disjoint if no two elements of K share a vertex.

Our motivation for this definition is the following. Suppose L is a ladder from u to v embedded in a cycle C and P is a path from x to y that is vertex-disjoint from C, and suppose further that (u, v) covers (x, y). Then we can extend P to the path $uxPyv$ and use the previous lemma to absorb P into C. For a digraph D, if we can find a small set $K \subseteq V(D)^{[2]}$ which d-covers $V(D)$, then we might hope to construct vertex-disjoint ladders from u to v for each $(u, v) \in K$ and embed all those ladders into a cycle C. This structure would then have the property that any d vertex-disjoint paths of D (that are also vertex-disjoint from C) could be absorbed into C. This will be our absorbing structure.

Definition 17. Given a digraph D and $d \in \mathbb{N}$, a *d-absorber* S of D is a triple $S = (K, \mathcal{L}, C)$, where

- $K \subseteq V(D)^{[2]}$ is a set of vertex-disjoint pairs which d-covers $V(D)$,

- $\mathcal{L}$ is a set of vertex-disjoint ladders such that for each $(u, v) \in K$, we have a ladder $L \in \mathcal{L}$ from u to v,

- $C \subseteq D$ is a cycle such that each $L \in \mathcal{L}$ is embedded in C.

We sometimes abuse notation by also writing S for the subgraph $(\cup_{L \in \mathcal{L}} L) \cup C$ of D. Note that $V(C) = V(S)$.

It follows from Lemma 15 that a d-absorber can absorb d vertex-disjoint paths into its cycle.

Corollary 18. *Let D be a digraph and let $S \subseteq D$ be a d-absorber. Suppose $P_1, \ldots, P_r$ are vertex-disjoint paths in D that are also vertex-disjoint from $V(S)$ and $r \leqslant d$. Then there exists a cycle C^* in D such that $V(C^*) = V(S) \cup V(P_1) \cup \cdots \cup V(P_r)$.*

Proof. Let x_i and y_i be such that P_i is a path from x_i to y_i for $i = 1, \ldots, r$ and let $S = (K, \mathcal{L}, C)$. Since S is a d-absorber, for each $i = 1, \ldots, r$, there exists $(u_i, v_i) \in K$ and $L_i \in \mathcal{L}$ such that (u_i, v_i) covers (x_i, y_i) and L_i is a ladder from u_i to v_i, and where

$u_1, \ldots, u_r, v_1, \ldots, v_r$ are distinct vertices. For each i, observe that $Q_i := u_i x_i P_i y_i v_i$ is a path in D and that $Q_1, \ldots, Q_r$ are vertex-disjoint.

Set $C_0 := C$ and assume by induction that there is a cycle $C_{i-1} \subseteq D$ with the property that $V(C_{i-1}) = V(C_0) \cup V(Q_1) \cup \cdots \cup V(Q_{i-1})$ and where $L_i, \ldots, L_r$ are embedded in C_{i-1}. Since L_i is a ladder from u_i to v_i embedded in C_{i-1} and Q_i is a path from u_i to v_i internally vertex-disjoint from C_{i-1}, Lemma 15 implies that there exists a cycle C_i such that $V(C_i) = V(C_{i-1}) \cup V(Q_i) = V(C) \cup V(Q_1) \cup \cdots \cup V(Q_i)$. Furthermore, by Lemma 15, since $L_{i+1}, \ldots, L_r$ are vertex-disjoint from L_i, and are embedded in C_{i-1}, so they are embedded in C_i.

This completes the induction step and so we obtain a cycle $C^* := C_r$ of D where $V(C_r) = V(C) \cup V(Q_1) \cup \cdots \cup V(Q_r) = V(S) \cup V(P_1) \cup \cdots \cup V(P_r)$. $\qquad\square$

The sequence of lemmas that follow show how to build a d-absorber in a robust expander. The first lemma shows how to find a d-cover in a digraph.

Lemma 19. *Let* $\gamma \in (0, 1)$ *and* $n, d \in \mathbb{N}$ *with* $d \geqslant 8$ *and*

$$n > 10^5 d^2 \gamma^{-4} \log^2(100 d \gamma^{-2}).$$

If D *is an* n-*vertex digraph with* $\delta^0(D) \geqslant \gamma n$ *and* $U \subseteq V(D)$, *then there exists a vertex-disjoint* $K \subseteq V(D)^{[2]}$ *with* $|K| = \lceil 24\gamma^{-2}(d \log(24d\gamma^{-2} + 2\log n) \rceil$ *which* d-*covers* U.

Proof. Set $m := \lceil 24\gamma^{-2}(d \log(24d\gamma^{-2}) + 2\log n) \rceil$ and construct K^* randomly by taking a set of m elements, each picked independently and uniformly at random, from $V(D)^{[2]}$; thus K^* may not be vertex-disjoint. We have that

$$
\begin{aligned}
\mathbb{P}(K^* \text{ is vertex-disjoint}) &= \prod_{i=0}^{m-1} \left(\binom{n-2i}{2} \bigg/ \left(\binom{n}{2} - i \right) \right) \\
&\geqslant \prod_{i=0}^{m-1} \left(\binom{n-2i}{2} \bigg/ (n^2/2) \right) = \prod_{i=0}^{2m-1} \left(1 - \frac{i}{n} \right) \\
&\geqslant 1 - \sum_{i=1}^{2m-1} \frac{i}{n} \geqslant 1 - \frac{2m^2}{n} > \frac{1}{2},
\end{aligned}
$$

our choice of m and n and applying Proposition 11 [1]

For $(x, y) \in U^2$, let $\mathrm{cov}(x, y)$ be the set of elements in $V(D)^{[2]}$ that cover (x, y). For a uniformly random element (u, v) of $V(D)^{[2]}$, set

$$p := \mathbb{P}((u, v) \in \mathrm{cov}(x, y)) \geqslant \frac{\gamma n(\gamma n - 1)}{n(n-1)} \geqslant \frac{\gamma^2}{2},$$

where the last inequality follows by our choice of n. Let $E_{x,y}$ be the number of distinct

elements of K^* that cover (x, y) so that $E_{x,y} \sim \text{bin}(m, p)$. In particular,

$$
\begin{aligned}
\mathbb{P}(E_{x,y} < d) = \sum_{i=0}^{d-1} \binom{m}{i} p^i (1-p)^{m-i} &\leqslant (1-p)^{m-d} \sum_{i=0}^{d-1} \frac{m^i}{i!} \\
&\leqslant em^d (1-p)^{m-d} \leqslant em^d (1-\gamma^2/2)^{m-d} \\
&\leqslant m^d \exp(-\gamma^2(m-d)/2) \leqslant m^d \exp(-\gamma^2 m/8) \leqslant n^{-2}/2.
\end{aligned}
$$

by our choices of m and applying Proposition 11[2]. Let X be the number of elements of U^2 not d-covered by K^*. Then

$$
\mathbb{P}(X \geqslant 0) \leqslant \mathbb{E}(X) = \sum_{(x,y) \in U^2} \mathbb{P}(E_{x,y} < d) \leqslant 1/2.
$$

Therefore $\mathbb{P}(X = 0 \text{ and } K^* \text{ is vertex-disjoint}) > 0$. $\qquad\square$

Next we show how to build a ladder in a robust expander.

Lemma 20. *Let $0 < \nu \leqslant \tau < \gamma/8 < 1/8$ and $n \in \mathbb{N}$ satisfying $n \geqslant 57\nu^{-3}$. Let D be a robust (ν, τ)-expander on n vertices with $\delta^0(D) \geqslant \gamma n$ and let u, v be distinct vertices of D. Then there exists a ladder L from u to v with $|L| \leqslant 3\nu^{-2}$ and where the alternating path of L has at most $2\nu^{-1}$ vertices.*

Proof. By Lemma 13, we can find an alternating path $Q = [x_0 \cdots x_t x_t^* \cdots x_0^*]$, where $x_0 = u$, $x_0^* = v$, and $t \leqslant (\nu^{-1} + 4)/2$ is even (so this alternating path has at most $\nu^{-1} + 6 \leqslant 2\nu^{-1}$ vertices). Next, as in the definition of ladders, we construct vertex-disjoint paths $Q_1, Q_3, \ldots, Q_{t-1}$, where Q_i is from x_i to x_i^* and is vertex-disjoint from P (except at its end points). We do this using Lemma 10.

Let D' be the digraph obtained from D by deleting x_i and x_i^* for each even value $i = 0, \ldots, t$; thus we delete $t + 2 \leqslant (\nu^{-1} + 8)/2 \leqslant \nu^{-1}$ vertices and by our choice of large n, Proposition 7 implies[3] that D' is a robust $(\frac{1}{2}\nu, \frac{16}{15}\tau)$-expander with $\delta^0(D') \geqslant \frac{15}{16}\gamma n$. By our choice of parameters and sufficiently large n, we can apply Lemma 10[4] with $r = t/2$ to obtain vertex-disjoint paths $Q_1, Q_3, \ldots, Q_{t-1}$ in D' with each Q_i from x_i to x_i^* and of length at most $4\nu^{-1} + 3$. As paths in D, these paths are also vertex-disjoint from Q except at their end-points.

Thus the union of the alternating path Q with the paths $Q_1, Q_3, \ldots, Q_{t-1}$ gives a ladder L from u to v. We have $|Q| \leqslant \nu^{-1} + 6 \leqslant \nu^{-2}/2$, $|Q_i| \leqslant 4\nu^{-1} + 3 \leqslant 9\nu^{-1}/2$ for each odd $i < t \leqslant \nu^{-1}$. Thus $|L| \leqslant 9\nu^{-2}/4 + |Q| \leqslant 3\nu^{-2}$. $\qquad\square$

Next we show that we can build several ladders (between prescribed vertices) in a robustly expanding digraph (for a suitable choice of parameters).

Lemma 21. *Let $0 < \nu \leqslant \tau \leqslant \gamma/16 < 1/16$ and $n, k \in \mathbb{N}$ satisfying $n \geqslant 460k\nu^{-3}$. Let D be a robust (ν, τ)-expander on n vertices with $\delta^0(D) \geqslant \gamma n$ and let $u_1, \ldots, u_k, v_1, \ldots, v_k$ be distinct vertices of D. Then we can construct vertex-disjoint ladders $L_1, \ldots, L_k$ from u_i to v_i such that $|L_i| \leqslant 12\nu^{-2}$ and $|P_i| \leqslant 8\nu^{-1}$, where P_i is the alternating path of L_i.*

Proof. By induction, suppose we have constructed vertex-disjoint ladders $L_1, \ldots, L_{i-1}$ for some $i \leqslant k$ where L_j is from u_j to v_j and $|L_j| \leqslant 12\nu^{-2}$ for all $j < i$, where the alternating path P_j of L_j satisfies $|P_j| \leqslant 8\nu^{-1}$ for all $j < i$, and where $L_1, \ldots, L_{i-1}$ are disjoint from $S_i := \{u_i, \ldots, u_k, v_i, \ldots, v_k\}$. Let D_i be obtained from D by deleting all the vertices of $L_1, \ldots, L_{i-1}$ and $S_i \setminus \{u_i, v_i\}$ (so $u_i, v_i \in V(D_i)$); thus the number of vertices deleted is at most

$$12\nu^{-2}(i-1) + 2(k-i) \leqslant 12k\nu^{-2} \leqslant \nu n/2,$$

where the last inequality follows from our choice of n. The inequality above together with Proposition 7 implies that D_i is a robust $(\frac{1}{2}\nu, \frac{32}{31}\tau)$-expander with $\delta(D_i) \geqslant \frac{31}{32}\gamma n$. By our choice of parameters and n, we can apply[5] Lemma 20 to obtain a ladder L_i from u_i to v_i in D_i with $|L_i| \leqslant 12\nu^{-2}$. and with alternating path P_i satisfying $|P_i| \leqslant 8\nu^{-1}$. By our choice of D_i, we see that $L_1, \ldots, L_i$ are vertex-disjoint ladders disjoint from S_{i+1}, where L_i is from u_i to v_i, completing the induction step and the proof. $\qquad\square$

Finally we combine our various constructions to show how to build a d-absorber in a robustly expanding digraph.

Theorem 22. *Let $0 < \nu \leqslant \tau \leqslant \gamma/16 < 1/16$ and $n, d \in \mathbb{N}$. Suppose $d \geqslant 8$ and*

$$n > \max\left(10^4 d^2 \gamma^{-5} \log^2(100 d\gamma^{-2}), 10^5 d\gamma^{-2}\nu^{-3} \log(1500 d\gamma^{-2}\nu^{-1})\right).$$

If D is a robust (ν, τ)-expander on n vertices with $\delta^0(D) \geqslant \gamma n$ then we can find a d-absorber S in D such that $|V(S)| \leqslant 1600\nu^{-2}\gamma^{-2}(d\log(d\gamma^{-2}) + \log n)$.

Proof. For our choice of γ, d, n, we can apply Lemma 19 to D to obtain a vertex-disjoint $K \subseteq V(D)^{[2]}$ which d-covers $V(D)$, and moreover $m := |K| = \lceil 24\gamma^{-2}(d\log(24d\gamma^{-2}) + 2\log n)\rceil$.

Next, by our choice of n, we can apply Lemma 21 [6] (taking $k = m$) to construct a ladder from a to b for every $(a, b) \in K$ such that each ladder has at most $12\nu^{-2}$ vertices, the alternating path of each ladder has length at most $8\nu^{-1}$, and the ladders are vertex-disjoint.

Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be the set of constructed ladders and let $R_1, \ldots, R_s$ be the collection of all rung paths of all the ladders constructed; thus $s \leqslant 4\nu^{-1}m$. Let x_i and y_i be the initial and final vertices of R_i and let D' be the digraph obtained from D by deleting all internal vertices of $R_1, \ldots, R_s$. So we have deleted at most $12\nu^{-2}m \leqslant \nu n/2$ vertices[7]. Then D' is a $(\frac{1}{2}\nu, \frac{32}{31}\tau)$-expander by Proposition 7. By our choice[8] of parameters and n, we can apply Lemma 10 (with $n = |D'|$, $r = s$ and ν, τ, γ replaced by $\frac{1}{2}\nu, \frac{32}{31}\tau, \frac{31}{32}\gamma$) to find paths U_i from y_i to x_{i+1} for each $i = 1, \ldots, s$, where indices are understood to be modulo s and each path has length at most $4\nu^{-1} + 1$. Then $C = x_1 R_1 U_1 \cdots R_s U_s x_1$ is a cycle in which all the ladders $L_1, \ldots, L_m$ are embedded. Thus $S = (K, \mathcal{L}, C)$ is a d-absorber of D. Also $|V(S)| \leqslant 12\nu^{-2}m + s(4\nu^{-1} + 1) \leqslant 32\nu^{-2}m$. Recall that $m = \lceil 24\gamma^{-2}(d\log(24d\gamma^{-2}) + 2\log n)\rceil \leqslant 25d\gamma^{-2}\log(24d\gamma^{-2}) + 48\gamma^{-2}\log n$ as $d\gamma^{-2}\log(24d\gamma^{-2}) > 1$. Therefore $|V(S)| \leqslant 1600\nu^{-2}\gamma^{-2}(d\log(d\gamma^{-2}) + \log n)$ as required. $\qquad\square$

5 Rotation-extension: 1-factors with few cycles

Let D be a digraph. Throughout this section, a *factor* U of D refers to a *1-factor* of D, i.e. a spanning subgraph of D in which every vertex has in- and outdegree 1. Thus a factor consists of a collection of vertex-disjoint cycles. We shall think of U interchangeably as both a set of vertex-disjoint cycles $U = \{C_1, \ldots, C_k\}$ and as the corresponding subgraph $U = C_1 \cup \cdots \cup C_k$ of D. The purpose of this section is to show that any robustly expanding digraph with sufficiently high minimum in- and outdegree contains a factor with few cycles: our main tool is an interesting variation of the rotation-extension technique of Pósa [23]. The first lemma shows that any robustly expanding digraph with large enough minimum in- and outdegree has a factor.

Lemma 23. *Let $0 < \nu \leqslant \tau < \gamma < 1$ and $n \in \mathbb{N}$. If $D = (V, E)$ is an n-vertex robust (ν, τ)-expander with $\delta^0(D) \geqslant \gamma n$ then D has a 1-factor.*

Proof. Let $V = \{v_1, \ldots, v_n\}$. Consider the bipartite (undirected) graph G whose vertex set is $X \cup Y$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ and $x_i y_j$ is an edge of G if and only if $v_i v_j \in E$. Note that D contains a factor if and only if G has a perfect matching, so it is sufficient for us to verify Hall's condition for G. Indeed suppose $S \subseteq X$. If $|S| \leqslant \tau n$, then $|N_G(S)| = |N_D^+(S)| \geqslant \gamma n > \tau n \geqslant |S|$. If $|S| \geqslant (1 - \tau)n$ then since every vertex in Y has degree at least $\gamma n > \tau n$ (since $\delta^-(D) \geqslant \gamma n$) then $|N_G(S)| = |Y| \geqslant |S|$. If $\tau n \leqslant |S| \leqslant (1 - \tau)n$, then $|N_G(S)| = |N_D^+(S)| \geqslant |\mathrm{RN}_\nu^+(S)| \geqslant |S| + \nu n > |S|$. Hence by Hall's Theorem (see e.g. [2]) G has a perfect matching and hence D has a factor. $\qquad\square$

We now introduce various notions we shall need. We say F is a *prefactor* of D if F can be obtained from a factor of D by deleting one edge. Thus F' consists of a collection of cycles $C_1, \ldots, C_{k-1}$ together with a path P. We interchangeably think of F as the set $F = \{C_1, \ldots, C_{k-1}, P\}$ and as the subgraph $F = C_1 \cup \cdots \cup C_{k-1} \cup P$ of D. If P is a path from a vertex x to a vertex y, we say x is the *origin* of F and y is the *terminus* of F written $x = \mathrm{ori}(F)$ and $y = \mathrm{ter}(F)$ respectively. Every vertex v of D except $\mathrm{ori}(F)$ has a unique inneighbour in F which we denote by $F^-(v)$.

An *extension* of F (in D) is a prefactor F' of D obtained from F as follows. Assuming $F = \{C_1, \ldots, C_{k-1}, P\}$, $x = \mathrm{ori}(F)$ and $y = \mathrm{ter}(F)$, we pick any vertex $z \in N_D^+(y) \setminus \{x\}$:

(i) if $z \in V(P)$ we set $F' = \{C_1, \ldots, C_{k-1}, C', P'\}$, where $C' = zPyz$ and $P' = xPz^-$, where $z^- := F^-(z)$ is the predecessor of z on P;

(ii) if $z \in V(C_i)$ for some i then set $F' = \{C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_{k-1}, P'\}$, where $P' = xPyzC_i z^-$ and $z^- := F^-(z)$ is the predecessor of z in C_i.

We say F' is an extension of F along the edge yz. Notice that F and F' differ only in their path and in that one or the other contains an additional cycle. For case (i), we say F' is a *cycle-creating* extension of F and for case (ii) we say F' is a *cycle-destroying* extension of F. Notice also that for any extension F' of F, we have $\mathrm{ori}(F) = \mathrm{ori}(F')$ and that F' is uniquely determined from F by specifying the terminus of F'.

Here is the main step in obtaining a factor with few cycles.

Lemma 24. *Let $n \in \mathbb{N}$ and $\nu, \tau, \gamma, \xi \in (0,1)$ satisfying $\nu \leqslant \tau$, $\gamma > 2\tau + \xi$, $\xi < \frac{1}{4}\nu^2$ and $n > 32\nu^{-3}$. Suppose $D = (V, E)$ is an n-vertex robust (ν, τ)-expander with $\delta^0(D) \geqslant \gamma n$ and suppose that for each prefactor F of D, we have an associated set $B(F) \subseteq V$ of 'forbidden' vertices satisfying $\mathrm{ori}(F) \in B(F)$ and $|B(F)| \leqslant \xi n$. Fix any prefactor F^* of D. Then for all but at most τn vertices $y \in V$, there exists a sequence of prefactors $F_0 = F^*, F_1, \ldots, F_t$ where $y = \mathrm{ter}(F_t)$ and for each $i = 1, \ldots, t$ we have that F_i is an extension of F_{i-1} and $\mathrm{ter}(F_i) \notin B(F_{i-1})$.*

Proof. Let $x = \mathrm{ori}(F_0) = \mathrm{ori}(F^*)$. For each $r \in \mathbb{N}$, we define S_r to be the set of vertices that are *reachable* from F_0 by a sequence of at most r successive extensions while avoiding forbidden sets. More precisely, $y \in S_r$ if and only if there exists a sequence $F^* = F_0, F_1, \ldots, F_{r'}$ with $r' \leqslant r$ such that $y = \mathrm{ter}(F_{r'})$, and for all $i = 1, \ldots, r'$, F_i is an extension of F_{i-1} and $\mathrm{ter}(F_i) \notin B(F_{i-1})$. For each $y \in S_r$, we set $F_y^{(r)} := F_{r'}$ (if there are many choices of $F_{r'}$, we pick one arbitrarily). In particular $y \in \mathrm{ter}(F_y^{(r)})$.

In order to prove the lemma, it is sufficient to show that $|S_t| \geqslant (1 - \tau)n$ for some t. Let us begin by noting that $|S_1| \geqslant (\gamma - \xi)n - 1 \geqslant 2\tau n - 1 \geqslant \tau n$, where the last two inequalities follow by our choice of parameters and n. To see the first inequality note that each distinct outneighbour w of $\mathrm{ter}(F_0)$ (except possibly x) gives an extension of F_0 with a distinct terminus $w^- := F_0^-(w)$, and each such w^- is in S_1 unless $w^- \in B(F_0)$.

We shall show that S_{r+1} contains most vertices in $\{F_0^-(w) : w \in \mathrm{RN}_\nu^+(S_r)\}$. Fix $r \geqslant 1$. For each $w \in \mathrm{RN}_\nu^+(S_r) \setminus \{x\}$, we say w is *good* if there exists $v \in S_r$ such that $w \in N^+(v)$, $F_v^{(r)-}(w) = F_0^-(w)$, and $F_0^-(w) \notin B(F_v^{(r)})$. Otherwise we say w is *bad*. Note that if $w \in \mathrm{RN}_\nu^+(S_r) \setminus \{x\}$ is good, then $F_0^-(w) \in S_{r+1}$. Indeed, let $F_0, \ldots, F_{r'} = F_v^{(r)}$ be a sequence of extensions that show $v \in S_r$. Then extending $F_{r'} = F_v^{(r)}$ along the edge vw gives an extension F' whose terminus is $F_v^{(r)-}(w) = F_0^-(w) \notin B(F_v^{(r)})$. Thus the sequence $F_0, \ldots, F_{r'}, F'$ shows that $F_0^-(w) \in S_{r+1}$.

Since the function $w \mapsto F_0^-(w)$ is injective, each $w \in \mathrm{RN}_\nu^+(S_r) \setminus \{x\}$ that is good corresponds to a distinct vertex of S_{r+1}. Thus, assuming $|S_r| \leqslant (1 - \tau)n$, we have

$$|S_{r+1}| \geqslant |\mathrm{RN}_\nu^+(S_r)| - b - 1 \geqslant |S_r| + \nu n - b - 1,$$

where b is the number of bad vertices, which we now bound from above.

Let

$$A := \{(v, w) : v \in S_r, \ w \in \mathrm{RN}_\nu^+(S_r) \cap N^+(v), \ F_0^-(w) \in B(F_v^{(r)})\}$$
$$B := \{(v, w) : v \in S_r, \ w \in \mathrm{RN}_\nu^+(S_r) \cap N^+(v), \ F_v^{(r)-}(w) \neq F_0^-(w)\}.$$

We have that $|A \cup B| \geqslant b\nu n$. To see this note that each bad vertex $w \in \mathrm{RN}_\nu^+(S_r)$ has at least νn inneighbours $v \in S_r$, and each such pair (v, w) belongs to $A \cup B$. On the other hand, we have $|A| \leqslant \sum_{v \in S_r} |B(F_v^{(r)})| \leqslant |S_r|\xi n$ and $|B| \leqslant |S_r|r$. The first inequality is clear while second inequality follows from the following claim:

Claim: For each $v \in S_r$, there are at most r vertices w for which $F_v^{(r)-}(w) \neq F_0^-(w)$.

Proof. (of Claim) If F' is any extension of F then there is exactly one vertex w for which $F'^{-}(w) \neq F^{-}(w)$. Therefore if F' is obtained from F by a sequence of at most r successive extensions, then there are at most r vertices w for which $F'^{-}(w) \neq F^{-}(w)$, and the claim follows. $\qquad\square$

Thus we have that $b\nu n \leqslant (\xi n + r)|S_r|$, whence $b \leqslant \nu^{-1}\xi n + \nu^{-1}r$. For each $r \leqslant 2\nu^{-1}$, if $|S_r| \leqslant (1 - \tau)n$ then we have

$$|S_{r+1}| \geqslant |S_r| + \nu n - b - 1 \geqslant |S_r| + \nu n - \nu^{-1}\xi n - 2\nu^{-2} - 1 \geqslant |S_r| + \frac{1}{2}\nu n,$$

where the last inequality follows[9] by our choice of parameters and n. Thus for some $t \leqslant 2\nu^{-1}$, we have $|S_t| \geqslant (1 - \tau)n$, as required. $\qquad\square$

We give one piece of notation before proving the existence of factors with few cycles in robustly expanding digraphs. If P and Q are paths in a directed graph D, we write $Q \subseteq P$ if Q is an initial segment of P, i.e. P and Q have the same initial vertex and $P[V(Q)] = Q$. If $Q \subseteq P$ but $Q \neq P$, we write $Q \subset P$.

Theorem 25. *Let $n \in \mathbb{N}$ and $\nu, \tau, \gamma, \xi \in (0, 1)$ satisfying $\nu \leqslant \tau$, $\gamma > 2\tau + \xi$, $\xi < \frac{1}{4}\nu^2$, and $n > 32\nu^{-3}$. If $D = (V, E)$ is an n-vertex robust (ν, τ)-expander with $\delta^0(D) \geqslant \gamma n$ then there exists a factor U^* of D which consists of at most $2\xi^{-1}$ cycles.*

Proof. By Lemma 23, D contains a factor U_0. Suppose U is any factor in which all cycles have length at least s for some $s < \frac{1}{2}\xi n$ and where exactly $\ell \geqslant 1$ cycles have length s. We claim that, using Lemma 24, we can obtain a factor U' from U in which all cycles have length at least s and at most $\ell - 1$ cycles have length s. Applying this claim iteratively, we eventually obtain a factor U^* of D in which every cycle has length at least $\frac{1}{2}\xi n$ and so this factor has at most $2\xi^{-1}$ cycles, proving the theorem.

It remains to prove the claim. Suppose $U = \{C_1, \ldots, C_k\}$ where $C_1, \ldots, C_k$ are the cycles of U in increasing order of length with $|C_1| = s < \frac{1}{2}\xi n$. Delete any edge of C_1 to form a path P and let $F_0 = \{C_2, \ldots, C_k, P\}$ be the resulting prefactor of D, and let x be its origin.

For each prefactor F of D, let $B(F)$ denote the set of the first and last $\frac{1}{2}\xi n$ vertices on the path in F (if the path has at most ξn vertices then $B(F)$ is the set of all vertices on the path). Note that for the prefactor F_0, $|P| = |C_1| < \frac{1}{2}\xi n$ and so $B(F_0) = V(P)$. By Lemma 24, for at least $(1 - \tau)n$ vertices $y \in V$, there exists a sequence of extensions $F_0, F_1, \ldots, F_t$ such that F_i is an extension of F_{i-1}, $\mathrm{ter}(F_i) \notin B(F_{i-1})$, and $\mathrm{ter}(F_t) = y$. Since $|N^-(x) \setminus B(F_0)| \geqslant \gamma n - \xi n > \tau n$, we can choose y to be in $N^-(x) \setminus B(F_0)$.

Writing P_i for the path in the prefactor F_i, by our choice of $B(\cdot)$, it is straightforward to show by induction that $P = P_0 \subset P_i$ for all $i = 1, \ldots, t$. Indeed, since $B(F_0) = V(P)$, F_1 must be a cycle-destroying extension of F_0, and so $P = P_0 \subset P_1$. Suppose $P \subset P_{i-1}$ for some $i > 1$ and let P'_{i-1} be the subpath of P_{i-1} consisting of the first $\frac{1}{2}\xi n$ vertices; in particular $P \subset P'_{i-1}$. If F_i is a cycle-creating extension of F_{i-1}, then since $V(P'_{i-1}) \subseteq B(F_{i-1})$, we must have $P_i \supseteq P'_{i-1} \supset P$. If F is a cycle-destroying extension of F_i the $P_i \supset P_{i-1} \supset P_0$.

Our choice of $B(\cdot)$ also ensures that if F_i is a cycle-creating extension of F_{i-1}, then the new cycle has length at least $\frac{1}{2}\xi n$.

Let $F_t = \{C'_1, \ldots, C'_{k'}, P_t\}$, where $C'_1 \ldots, C'_{k'}$ are cycles and we know P_t is a path from x to y of length more than $|P| = |C_1|$. Since $y \in N_D^-(x)$, we can turn P_t into a cycle C^* and form a factor $U' = \{C'_1 \ldots, C'_{k'}, C^*\}$ of D. We have $|C^*| = |P_t| > |C_1| = s$.

Every cycle in U' that was created in the sequence of extensions $F_0, \ldots, F_t$ has length at least $\frac{1}{2}\xi n > s$ and $|C^*| > |C_1| = s$. Every other cycle of U' was also a cycle of U. Hence every cycle in U' has length at least s and the number of cycles of length exactly s has been reduced by at least one. This proves the claim and the theorem. $\qquad\square$

6 Hamiltonicity

We now combine Theorem 22, Corollary 18 and Theorem 25 to give the following result from which we deduce Theorem 3.

Theorem 26. *Let $0 < \nu \leqslant \tau \leqslant \gamma/16 < 1/16$ and let $n \in \mathbb{N}$. Assume*

$$n > \max\{10^8\gamma^{-5}\nu^{-4}\log^2(10^4\gamma^{-2}\nu^{-2}), 10^7\gamma^{-2}\nu^{-5}\log(150000\gamma^{-2}\nu^{-3})\}.$$

If D is an n-vertex robust (ν, τ)-expander with $\delta^0(D) \geqslant \gamma n$, then for any $\nu n/2 \leqslant \ell \leqslant n$ and any $v \in V(D)$, D contains a cycle of length ℓ containing v.

Proof. Let $\xi := \nu^2/32$ and $d := \lceil 2\xi^{-1} \rceil \geqslant 8$. We begin by applying Theorem 22 to D to find a d-absorber S, where

$$|V(S)| \leqslant 1600\nu^{-2}\gamma^{-2}(d\log(d\gamma^{-2}) + \log n).$$

One can check that the conditions on the parameters and n are met[10]

Set $D' := D - V(S)$. By our choice[11] of n, we have $|V(S)| < \nu n/2$ and so by Proposition 7 D' is a robust $(\frac{1}{2}\nu, \frac{32}{31}\tau)$-expander with $\delta^0(D') > \frac{31}{32}\gamma n$. By our choice of ξ and n, we can apply Theorem 25[12] to D' to obtain a factor in D' with at most $2\xi^{-1} \leqslant d$ cycles. By removing one edge from each of the cycles let $P_1, \ldots, P_r$ be the resulting paths with $r \leqslant d$. Consider any $\nu n/2 \leqslant \ell \leqslant n$ and any $v \in V(D)$. Note D' contains vertex-disjoint paths $P'_1, \ldots, P'_{r'}$ such that $r' \leqslant d$ and $|P'_1| + \ldots + |P'_{r'}| = \ell - |V(S)|$ and $v \in V(S) \cup \bigcup_{i \in r'} V(P'_i)$ (by removing appropriate vertices of $P_1, \ldots, P_r$ if necessary). Applying Corollary 18, to these paths and the d-absorber S, we obtain a cycle C of length ℓ in D with $v \in V(C)$. $\qquad\square$

Finally we can prove Theorem 3.

Proof of Theorem 3. Given that D is an n-vertex robust (ν, τ)-outexpander with $\delta^0(D) \geqslant \gamma n$, by Proposition 8, D is a robust (ν', τ)-expander where $\nu' = \nu^2/2$ (our choice of parameters ensures the conditions of Proposition 8 are met). By our choice [13] of n we can apply Theorem 26 to D to obtain a Hamilton cycle in D. $\qquad\square$

We deduce Corollary 5 from Theorem 3, but first we need the following leema from [17, Lemma 13.1].

Lemma 27. *Let $n \in \mathbb{N}$ and $\nu, \tau, \varepsilon \in (0,1)$ satisfy $\nu \leqslant \frac{1}{8}\tau^2$ and $\tau \leqslant \frac{1}{2}\varepsilon$. If D is an oriented graph on n vertices with $\delta^+(D) + \delta^-(D) + \delta(D) \geqslant 3n/2 + \varepsilon n$ then G is a robust (ν, τ)-outexpander.*

The explicit dependence between the parameters was not given in [17], but we have computed them and included them in the statement above.

Proof of Corollary 5. Let $\nu = \varepsilon^2/2$, $\tau := 2\varepsilon$ and $\gamma = 3/8$ so $4\sqrt[13]{\log^2 n/n} \leqslant \nu \leqslant \tau \leqslant \gamma/16 < 1/16$. Given an n-vertex oriented graph D with $\delta^0(D) \geqslant 3n/8 + \varepsilon n$, we have that $\delta(D) + \delta^+(D) + \delta^-(D) \geqslant 3n/2 + 4\varepsilon n$. So by Lemma 27, D is a robust (ν, τ)-outexpander. Finally, we apply Theorem 3 to obtain a Hamilton cycle. $\square$

To prove Theorem 6, we need the following lemma from [19, Lemma 13].

Lemma 28. *Let $n \in \mathbb{N}$ and $\tau, \gamma \in (0,1)$ satisfy $2\tau + 4\tau^2 \leqslant \gamma \leqslant 1/2$ and $n \geqslant \gamma^{-2}$. Let D be an n-vertex digraph such that for all $i < n/2$,*

- $d_i^+ \geqslant i + \gamma n$ *or* $d_{n-i-\gamma n}^- \geqslant n - i$,

- $d_i^- \geqslant i + \gamma n$ *or* $d_{n-i-\gamma n}^+ \geqslant n - i$,

Then G is a robust (τ^2, τ)-outexpander and $\delta^0(D) \geqslant \gamma n$.

Proof of Theorem 6. Let $\tau := \gamma/16$. By Lemma 28, D is a robust (τ^2, τ)-outexpander and $\delta^0(D) \geqslant \gamma n$. Finally, we apply Theorem 3 to obtain a Hamilton cycle. $\square$

7 Concluding remarks and an open problem

It would be interesting to know for which choices of parameters $\nu = \nu(n)$ and $\tau = \tau(n)$ an n-vertex robust (ν, τ)-expander is guaranteed to be Hamiltonian. We believe the true values of ν and τ for which this holds should be much smaller than what we have proved.

References

[1] N. Alon and J. Spencer. The probabilistic method,. volume 57, Wiley-Interscience, 2000.

[2] J.A. Bondy and U.S.R. Murty. Graph theory. Graduate texts in Mathematics **244**, Springer, New York, 2008.

[3] D. Christofides, P. Keevash, D. Kühn, and D. Osthus. Finding Hamilton cycles in robustly expanding digraphs. *J. Graph Algorithms Appl.* 16:337–360, 2012.

[4] B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown. Proof of the 1-factorization and Hamilton decomposition conjectures. *Mem. Amer. Math. Soc.* 224, monograph 1154, 2016.

[5] G. A. Dirac. Some theorems on abstract graphs. *Proc. London. Math. Soc.* 2:69-81, 1952.

[6] A. Ferber, M. Krivelevich and B. Sudakov. Counting and packing Hamilton cycles in dense graphs and oriented graphs. *J. Combin. Theory Ser. B* 122:196–220, 2017.

[7] A. Ghouila-Houri. Une condition suffisante d'existence d'un circuit hamiltonien. *C.R. Acad. Sci. Paris* 25:495–497, 1960.

[8] P. Keevash, D. Kühn and D. Osthus. An exact minimum degree condition for Hamilton cycles in oriented graphs. *J. London Math. Soc.* 79:144-166, 2009.

[9] L. Kelly. Arbitrary Orientations of Hamilton Cycles in Oriented Graphs. *Electr. J. Comb.* 18(1):#P186, 2011.

[10] L. Kelly, D. Kühn and D. Osthus. A Dirac type result on Hamilton cycles in oriented graphs. *Combin. Probab. Comput.* 17:689–709, 2008.

[11] F. Knox and A. Treglown. Embedding spanning bipartite graphs of small bandwidth. *Combin. Probab. Comput.* 22:71–96, 2013.

[12] M. Krivelevich. Triangle factors in random graphs. *Combin. Probab. Comput.* 6:337–347, 1997.

[13] D. Kühn, A. Lo, D. Osthus, and K. Staden. The robust component structure of dense regular graphs and applications. *Proc. London. Math. Soc.* 110:19–56, 2015.

[14] D. Kühn, A. Lo, D. Osthus and K. Staden. Solution to a problem of Bollobas and Häggkvist on Hamilton cycles in regular graphs. *J. Combin. Theory Ser. B* 121:85–145, 2016.

[15] D. Kühn, R. Mycroft and D. Osthus. An approximate version of Sumner's universal tournament conjecture. *J. Combin. Theory Ser. B* 101:415–447, 2011.

[16] D. Kühn and D. Osthus. A survey on Hamilton cycles in directed graphs. *European J. Combinatorics* 33:750–766, 2012.

[17] D. Kühn and D. Osthus. Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments. *Adv. Math.* 237:62–146, 2013.

[18] D. Kühn and D. Osthus. Hamilton decompositions of regular expanders: applications. *J. Combin. Theory Ser. B* 104:1–27, 2014.

[19] D. Kühn, D. Osthus and A. Treglown. Hamiltonian degree sequences in digraphs. *J. Combin. Theory Ser. B* 100:367–380, 2010.

[20] D. Kühn, D. Osthus and A. Treglown. Hamilton decompositions of regular tournaments. *Proc. London. Math. Soc.* 101:303–335, 2010.

[21] C.St.J.A. Nash-Williams. Hamiltonian circuits. *Studies in Math.* 12:301–360, 1975.

[22] D. Osthus and K. Staden. Approximate Hamilton decompositions of robustly expanding regular digraphs. *SIAM J. Discrete Math.* 27:1372–1409, 2013.

[23] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Math.* 14:359–364, 1976.

[24] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *Combin. Probab. Comput.* 15:229-251, 2006.

[25] A. Taylor, The regularity method for graphs and digraphs. MSci thesis, School of Mathematics, University of Birmingham, arXiv:1406.6531, (2013).

Notes

[1] Let $c' = 100\gamma^{-2}$ and $d' = 50d\gamma^{-2}\log(24d\gamma^{-2})$. Since $d \geqslant 8$, we have

$$\sqrt{n} \geqslant 300d\gamma^{-2}\log(100d\gamma^{-2}) \geqslant 300\gamma^{-2}(\log(100d\gamma^{-2}) + 1) + 150d\gamma^{-2}\log(24d\gamma^{-2}) \geqslant 3c'(\log c' + 1) + 3d'.$$

By Proposition 11, we have

$$\sqrt{n} \geqslant c'\log\sqrt{n} + d' \geqslant 50\gamma^{-2}\log n + 50d\gamma^{-2}\log(24d\gamma^{-2}) > 2m.$$

[2] Note $n^2 m^d \exp(-\gamma^2 m/8) \leqslant 1/2$ holds if $e^m \geqslant (2m^d n^2)^{8\gamma^{-2}}$, which holds by Proposition 11 if $m \geqslant 24\gamma^{-2}(d\log(8d\gamma^{-2}) + d + \log(2n^2))$, which holds if $m \geqslant 24\gamma^{-2}(d\log(24d\gamma^{-2}) + 2\log n)$ as $d\log 3 \geqslant d + \log 2$.

[3] The choice of n implies $\nu^{-1} \leqslant \frac{1}{2}\nu n \leqslant \frac{1}{16}\gamma n \leqslant \frac{1}{16}n$.

[4] Note that

$$n - (t + 2) \geqslant n - \nu^{-1} \geqslant 56\nu^{-3} \geqslant 4(12\nu^{-1} + 11)\nu^{-2}$$

Thus conditions of Lemma 10 hold (with $r = t/2 \leqslant \nu^{-1}/2$ and ν, τ, γ, n replaced by $\frac{1}{2}\nu, \frac{16}{15}\tau, \frac{15}{16}\gamma, n - (t + 2)$).

[5] We check the conditions of Lemma 20 with ν, τ, γ, n replaced by $\frac{1}{2}\nu, \frac{32}{31}\tau, \frac{31}{32}\gamma, |D_i|$. Note that $|D_i| \geqslant n - 12k\nu^{-2} \geqslant 459\nu^{-3} \geqslant 57(\nu/2)^{-3}$.

[6] Let $c' = 24000\gamma^{-2}\nu^{-3}$ and $d' = 12000\gamma^{-2}\nu^{-3}d\log(24d\gamma^{-2})$. Since $d \geqslant 8$, we have

$$n/2 \geqslant 5 \cdot 10^4\gamma^{-2}\nu^{-3}d\log(1500d\gamma^{-2}\nu^{-1}) \geqslant 4 \cdot 10^5\gamma^{-2}\nu^{-3}\log(1500\gamma^{-1/2}\nu^{-3/4})$$
$$\geqslant 10^5\gamma^{-2}\nu^{-3}\log(24000\gamma^{-2}\nu^{-3}) \geqslant 4c'\log c' \geqslant 3c'(\log c' + 1).$$

Also, we have

$$n/2 \geqslant 5 \cdot 10^4\gamma^{-2}\nu^{-3}d\log(1500d\gamma^{-2}\nu^{-1}) \geqslant 36000\gamma^{-2}\nu^{-3}\log(24d\gamma^{-2}) = 3d'.$$

Proposition 11 implies that

$$n \geqslant c'\log n + d' \geqslant 12000\gamma^{-2}\nu^{-3}(2\log n + d\log(24d\gamma^{-2}))$$
$$\geqslant 460\nu^{-3} \cdot 25\gamma^{-2}(2\log n + d\log(24d\gamma^{-2})) \geqslant 460m\nu^{-3}.$$

[7] We need $n \geqslant 24\nu^{-3}$, which is true by the previous note.

[8] Note that $11 \leqslant s \leqslant 4\nu^{-1}m$ and $n > 460m\nu^{-3}$ by Note 6. Thus, $|D'| \geqslant n - 12\nu^{-2}m \geqslant 28 \cdot 4\nu^{-3}m \geqslant 4(6s + 11)\nu^{-2}$.

[9] We note $\nu^{-1}\xi n \leqslant \frac{1}{4}\nu n$ and $1 \leqslant 2\nu^{-2} \leqslant \frac{1}{8}\nu n$.

[10] Note that $d := \lceil 2\xi^{-1} \rceil \leqslant 3\xi^{-1} \leqslant 100\nu^{-2}$ and $\nu^{-1} \geqslant 16$. Hence

$$10^4 d^2\gamma^{-5}\log^2(100d\gamma^{-2}) \leqslant 10^8\gamma^{-5}\nu^{-4}\log^2(10^4\gamma^{-2}\nu^{-2}) \leqslant n$$

and

$$10^5 d\gamma^{-2}\nu^{-3}\log(1500 d\gamma^{-2}\nu^{-1}) \leqslant 10^7\gamma^{-2}\nu^{-5}\log(150000\gamma^{-2}\nu^{-3}) \leqslant n.$$

[11] Need $n > 2\nu^{-1}|V(S)|$ so sufficient that $n > 3200\gamma^{-2}\nu^{-3}(d\log(d\gamma^{-2}) + \log n)$. By Proposition 11 this holds if

$$n > 9600\gamma^{-2}\nu^{-3}\left(\log(3200\gamma^{-2}\nu^{-3}) + 1\right) + 9600\gamma^{-2}\nu^{-3}d\log(d\gamma^{-2}).$$

Recall that $d \leqslant 100\nu^{-2}$. The inequality above holds if $n > 10^7\gamma^{-2}\nu^{-5}\log(10^5\gamma^{-2}\nu^{-4})$, which holds if $n > 3 \cdot 10^7\gamma^{-2}\nu^{-5}\log(\gamma^{-2}\nu^{-4})$.

[12] We check that $\frac{31}{32}\gamma > \frac{64}{31}\tau + \xi$, which holds (using $\gamma > 16\tau$ and $\xi \leqslant \nu \leqslant \tau$). We check that $\xi < \frac{1}{4}(\frac{1}{2}\nu)^2 = \frac{1}{16}\nu^2$, which holds. We check $n - |V(S)| > \max(32(\frac{1}{2}\nu)^{-3}, \tau^{-1})$. Since $|V(S)| \leqslant \frac{1}{2}\nu n \leqslant \frac{1}{2}n$, it is sufficient that $n > 512\nu^{-3}$. This is clearly implied by our choice of n.

[13] Recall that $\nu \leqslant \gamma/16$. so

$$\max\{10^8\gamma^{-5}(\nu')^{-4}\log^2(10^4\gamma^{-2}(\nu')^{-2}), 10^7\gamma^{-2}(\nu')^{-5}\log(15000\gamma^{-2}(\nu')^{-3})\}$$
$$\leqslant 40000\nu^{-13}\log^2 250\nu^{-8} \leqslant 40000\nu^{-13}\log^2\nu^{-10} \leqslant (4\nu^{-1})^{13}\log^2\nu^{-1} \leqslant n.$$