Lattice points and simultaneous core partitions

Paul Johnson
School of Mathematics and Statistics
University of Sheffield
Sheffield, United Kingdom

paul.johnson@sheffield.ac.uk

Submitted: Nov 24, 2015; Accepted: Mar 5, 2018; Published: Sep 21, 2018

Abstract

We apply lattice point techniques to the study of simultaneous core partitions. Our central observation is that for a and b relatively prime, the abacus construction identifies the set of simultaneous (a, b)-core partitions with lattice points in a rational simplex. We apply this result in two main ways: using Ehrhart theory, we reprove Anderson's theorem that there are $(a + b - 1)!/a!b!$ simultaneous (a, b)-cores; and using Euler-Maclaurin theory we prove Armstrong's conjecture that the average size of an (a, b)-core is $(a+b+1)(a-1)(b-1)/24$. Our methods also give new derivations of analogous formulas for the number and average size of self-conjugate (a, b)-cores.

Mathematics Subject Classifications: 11P81, 52B20, 05A17

1 Introduction

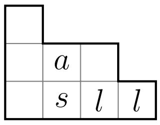
This paper introduces lattice point geometry to the study of simultaneous core partitions. We first establish some basic notation.

A *partition of* n is a nonincreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$ of positive integers such that $\sum \lambda_i = n$. We call n the *size* of λ and denote it by $|\lambda|$; we call k the *length* of λ and denote it by $\ell(\lambda)$.

We frequently identify λ with its Young diagram; there are many conventions for this. We draw λ in the first quadrant, with the parts of λ as the columns of a collection of boxes.

Definition 1. The *arm* $a(\square)$ of a cell $\square$ is the number of cells contained in λ and above $\square$, and the *leg* $l(\square)$ of a cell is the number of cells contained in λ and to the right of $\square$. The *hook length* $h(\square)$ of a cell is $a(\square) + l(\square) + 1$.

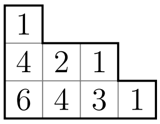
Example 2. The cell $(2, 1)$ of $\lambda = 3 + 2 + 2 + 1$ is marked s; the cells in the leg and arm of s are labeled a and l, respectively.



$a(s) = \#a = 1$
$l(s) = \#l = 2$
$h(s) = 4$

Definition 3. An *a-core* is a partition that has no hook lengths of size a. An *(a, b)-core* is a partition that is simultaneously an a-core and a b-core. We use $\mathcal{C}_{a,b}$ to denote the set of (a, b)-cores.

Example 4. We have labeled each cell $\square$ of $\lambda = 3 + 2 + 2 + 1$ with its hook length $h(\square)$.



We see that λ is *not* an a-core for $a \in \{1, 2, 3, 4, 6\}$; but *is* an a-core for all other a.

Partitions of n are closely related to the representation theory of the symmetric group (see [17]). For p a prime, p-core partitions of n are related to the p modular representation theory of S_n. This was the initiatial motivation for the study of a-cores, but they have taken on a combinatorial life of their own.

The study of simultaneous core partitions begins Anderson's result [2] that they are counted by the rational Catalan numbers, a natural generalization of Catalan numbers. Apart from their intrinsic combinatorial interest, rational Catalan numbers and their q and (q, t) analogs appear in the study of Hecke algebras [12], affine Springer varieties [18], and compactified Jacobians of singular curves [14, 15].

In this paper, we use lattice point techniques to reprove Anderson's result and to prove a conjecture of Armstrong about the average size of simultaneous core partitions. Our techniques also give new proofs about analogous results for self-conjugate simultaneous core partitions. A previous version of this paper contained extra sections that began to apply these ideas to q and (q, t) analogs, but this doubled the length of the paper for little gain, and we worried it diluted from our main point. This material is still available in the first version of his paper on the arXiv.

We reprove Anderson's theorem by identifying simultaneous core partitions with lattice points in a rational simplex. The connection between rational Catalan numbers and this simplex is not new; it appears for instance in [18, 13]. However, we are not aware of any work using this connection to apply lattice point techniques. After this identification is made, the other results follow quite naturally.

1.1 Statement of results

Recall that the Catalan number $\mathbf{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$. Catalan numbers count hundreds of different combinatorial objects; for example, the number of lattice paths from $(0, n)$ to $(n + 1, 0)$ that stay strictly below the line connecting these two points. Rational Catalan numbers are a natural two parameter generalization of $\mathbf{Cat}_n$.

Definition 5. For a, b relatively prime, the *rational Catalan number*, or (a,b) *Catalan number* $\mathbf{Cat}_{a,b}$ is

$$\mathbf{Cat}_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$$

The rational Catalan number $\mathbf{Cat}_{a,b}$ counts the lattice paths from $(0,a)$ to $(b,0)$ that stay beneath the line from $(0,a)$ to $(b,0)$. This is consistent with the specialization $\mathbf{Cat}_{n,n+1} = \mathbf{Cat}_n$.

Simultaneous cores and rational Catalan numbers are connected by:

Theorem 6 (Anderson [2]). *If a and b are relatively prime, then the number of simultaneous (a,b)-cores is the corresponding rational Catalan number: $|\mathcal{C}_{a,b}| = \mathbf{Cat}_{a,b}$.*

Our first result is a new proof of Theorem 6 using the geometry of lattice points in rational polyhedra. This framework easily extends to prove other results chief among them a proof of Armstrong's conjecture:

Theorem 7. *The average size of an (a,b)-core is $(a+b+1)(a-1)(b-1)/24$.*

Remark 8. Armstrong conjectured Theorem 7 in 2011; it first appeared in print in [4]. Stanley and Zanello [19] have proven the Catalan case $(a = b+1)$ of Armstrong's conjecture by different methods, and building on their work Aggarwal [1] has proven the case $a = mb + 1$.

Our two main tools are the abacus construction and Ehrhart theory. We briefly recall these ideas before giving a high-level overview of the proofs of Theorems 6 and 7.

1.2 The abacus construction

The main tool used to study a-cores is the abacus construction; the canonical reference for the abacus is James and Kerber [17]. We review this construction in detail in Section 2. For now, we note that there are at least two variants of the abacus construction in the literature. The first construction, which we call the *positive abacus*, gives a bijection between a-cores and $\mathbb{N}^{a-1}$. Anderson's original proof used the positive abacus as part of a bijection between (a,b)-cores and (a,b)-Dyck paths, which were already known to be counted by $\mathbf{Cat}_{a,b}$. We use the second construction, which we call the *signed abacus*. The signed abacus is a bijection between a-core partitions and points in the $a-1$ dimensional lattice

$$\Lambda_a = \left\{ c_0, \ldots, c_a \in \mathbb{Z} \mid \sum c_{a-1} = 0 \right\}.$$

Key to our proof of Armstrong's conjecture is the following:

Theorem. Under the signed abacus bijection, the size of an a-core is given by the quadratic function

$$Q(c_1, \ldots, c_a) = \frac{a}{2}\sum c_i^2 + \sum i c_i$$

We are not sure where exactly where this theorem originates; a stronger version is used, seemingly independently, in [11] and [9] to prove generating functions counting certain partitions are modular forms. For completeness, proof this result as Theorem 22.

1.3 Ehrhart and Euler-Maclaurin

The abacus construction has long been used to study core partitions. The novelty of this paper is to bring lattice point techniques to bear on the subject.

The number of lattice points in a polytope can be viewed as a discrete version of the volume of a polytope; Ehrhart theory is the study of this analogy. A gentle introduction to Ehrhart theory may be found in [6]. Let V be an n dimensional real vector space, and $\Lambda \subset V$ an n dimensional lattice. A lattice polytope $P \subset V$ is a polytope all of whose vertices are points of Λ. For t a positive integer, let tP denote the polytope obtained by scaling P by t. For $t \geqslant 0$, define $L(P,t)$ to the number of lattice points in tP:

$$L(P,t) = \#\{\Lambda \cap tP\}$$

Clearly, the volume of tP is t^n times the volume of P. The central result in Ehrhart theory is that, parallel to this fact, $L(P,t)$ is a degree n polynomial in t.

Other than the fact that $L(P,t)$ is a polynomial of degree n, the one fact from Ehrhart theory we use is Ehrhart reciprocity. If we scale a polytope by $-t$, then keeping track of orientation the volume changes by $(-t)^n$. The polynomial $L(P,t)$ is not in general even or odd, and so $L(P,-t)$ cannot be $(-1)^n$ times the number of lattice points in $-P$. Ehrhart reciprocity states that instead

$$L(P,-t) = (-1)^n L(P^\circ, t)$$

where P° denotes the interior of P.

Euler-Maclaurin theory is the extension of Ehrhart theory to an analogy between integrating a polynomial over a polytope and summing it over the lattice points in the polytope. At the most basic level, this is the familiar "sum of the first n cubes" formulas. Specifically, if f is a polynomial of degree d on V, then $\int_{tP} f$ is a polynomial of degree $d+n$. Euler-Maclaurin theory says that the discrete analog

$$L(f,P,t) = \sum_{x \in \Lambda \cap tP} f(x)$$

is also a polynomial of degree $d+n$. Ehrhart reciprocity also extends:

$$L(f,P,-t) = (-1)^n L(f,-P^\circ,t)$$

Although unsurprising to experts, apparently this extension was first used (without proof) in [7]; a proof now appears in [3].

1.4 Overview of method

To explain the method used to prove Theorems 6 and 7, we begin with the following

False Hope. Fix a. Under the signed abacus construction, the set of (a,b)-cores are exactly those lattice points in bP, for some integral polytope $P \subset V_a := \Lambda_1 \otimes \mathbb{R}$.

If the false hope were true, Ehrhart theory would imply that, for b relatively prime to a fixed a, $|\mathcal{C}_{a,b}|$ would be a polynomial of degree $a-1$ in b. It is clear from the definition that this polynomiality property holds for $\mathbf{Cat}_{a,b}$. Thus, proving Anderson's theorem for a fixed a would reduce to showing that two polynomials are equal, which only requires checking finitely many values.

Furthermore, since the size of an a-core is a quadratic function Q on the charge lattice, we could apply Euler-Maclaurin theory to see that the total size of all (a,b) cores was a polynomial of degree $a+1$ in b, and we could exploit this polynomiality to prove Armstrong's conjecture.

Although the False Hope is not quite true, the strategy outlined above is essentially the one we follow. The set of b-cores inside the lattice of a-cores is a simplex, which we call $\mathbf{SC}_a(b)$ for *Simplex of Cores*. One minor tweak needed to the False Hope is that as we vary b $\mathbf{SC}_a(b)$ is not only scaled, but also changed by a linear transformation. These transformations preserve the number of lattice points and the quadratic function Q giving the size of the partitions, and so do not pose any real difficulties. More troubling is that the polytope $\mathbf{SC}_a(b)$ is not integral, but only rational. Recall that a polytope P is *rational* if there is some $k \in \mathbb{Z}$ such that kP is a lattice polytope.

Ehrhart and Euler/Maclaurin theory can be extended to rational polytopes at the cost of replacing polynomials by *quasipolynomials*.

Definition 9. A function $f : \mathbb{Z} \to \mathbb{C}$ is a quasipolynomial of degree d and period n if there exist n polynomials $p_0, \ldots, p_{n-1}$ of degree d, so that for $x \in k + n\mathbb{Z}$, we have $f(x) = p_k(x)$.

Example 10. Let P be the polytope $x, y \geqslant 0, 2x + y \leqslant 1$. Then

$$\#\{tP \cap \mathbb{Z}^2\} = \begin{cases} \frac{t^2+4t+4}{4} & t \text{ even} \\ \frac{t^2+4t+3}{4} & t \text{ odd} \end{cases}$$

Since $\mathbf{Cat}_{a,b}$ is defined only for a and b relatively prime, it fits nicely into the quasipolynomial framework. For a fixed, and b in a fixed residue class mod a, $\mathbf{Cat}_{a,b}$ is a polynomial. It just so happens that residue classes relatively prime to a have identical polynomials. Such "accidental" equalities between the polynomials for different residue classes happen frequently in Ehrhart theory, but are mysterious in general. Perhaps the most studied manifestation of this is *period collapse* (see [16] and references), where the quasipolynomial is in fact a polynomial. In our case, symmetry considerations give an elementary explanation of the "accidental" equalities between the polynomials for different residue classes.

In Lemma 27 we show that the the polytope $\mathbf{SC}_a(b)$ is isomorphic to a rational simplex we call $\mathbf{TD}_a(b)$ (for *Trivial Determinant*) that we now describe. Let L_k be the one dimensional representation of $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ where $1 \in \mathbb{Z}_a$ acts as $\exp(2\pi ik/a)$. Then any b dimensional representation V of $\mathbb{Z}_a$ may be written as

$$V = \bigoplus_{k=0}^{a-1} L_k^{\oplus z_k}$$

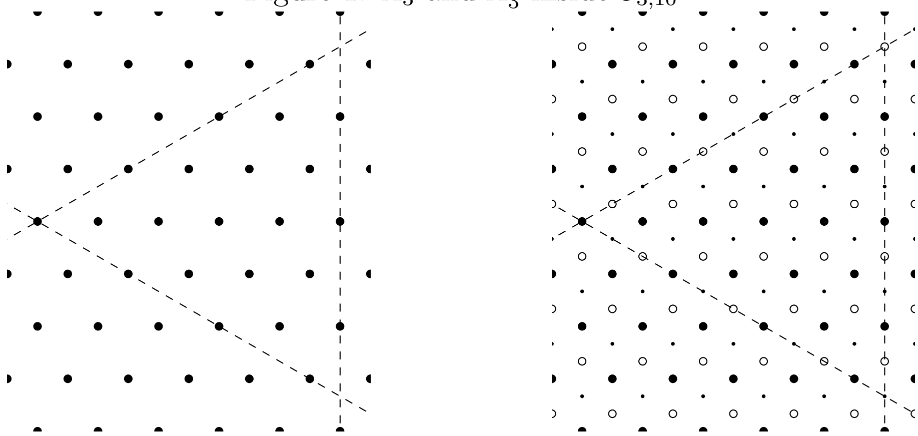
for nonnegative integers z_i satisfying $\sum z_i = b$. Thus, there is a bijection between the set of b dimensional representations of $\mathbb{Z}_a$ and lattice points in the standard simplex $b\Delta_{a-1}$, which has $\binom{a-1+b}{b}$ lattice points.

The simplex $\mathbf{TD}_a(b)$ is obtained by considering only those representations that have trivial determinant (i.e., $\wedge^b V \cong L_0$). This is equivalent to restricting to the index a sublattice given by $\sum iz_i = 0 \pmod{a}$.

More generally, consider the set of representations with determinant isomorphic to L_k for any k. Tensoring V by L_1 corresponds to the cyclic permutation of coordinates $z_k \mapsto z_{k+1}$, and changes the determinant of V by tensoring by $L_1^{\otimes b} = L_b$ (where we are using periodic indices). Thus, the dual $\mathbb{Z}_a$ acts on the set of all b dimensional representations of $\mathbb{Z}_a$, and when b is relatively prime to a this action is free, and each orbit contains exactly one representation with trivial determinant. Hence, the number of points in $\mathbf{TD}_a(b)$ is exactly one ath of the number of points in $b\Delta_a$, namely $\binom{a-1+b}{b}/a = \mathbf{Cat}_{a,b}$, reproving Anderson's theore. Armstrong's conjecture follows with a little more work.

The situation is illustrated in Figure 1. The left hand picture shows $\mathbf{TD}_3(10) \cong \mathbf{SC}_3(10)$, while the right hand picture shows the standard simplex $10\Delta_2$. The large black dots are the representations with trivial determinant, while the small dots and circles are those representations with determinant L_1 and L_2, respectively. Rotating about the center of the triangle by 120 degrees corresponds to tensoring by L_1 and permutes the different style dots.

Figure 1: Λ_3 and Λ_3' inside $\mathcal{C}_{3,10}$



1.5 Self-conjugate simultaneous cores

The lattice point technique easily adapts to treat the case of self-conjugate simultaneous cores. Recall that the conjugate of a partition is obtained by reflecting it about the line $y = x$; a partition is self-conjugate if it is equal to its conjugate.

Ford, Mai and Sze have shown [10] that self-conjugate (a, b)-core partitions are counted by

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor}$$

Armstrong conjectured, and Chen, Huang and Wang recently proved [8], that the average size of self-conjugate (a, b)-core partitions is the same as the average size of all (a, b)-core partitions, namely $(a + b + 1)(a - 1)(b - 1)/24$.

In Section 3.3 we give new proofs of both of these results. The key idea is that the action of conjugation on $\mathbf{SC}_a(b)$ corresponds to the action of taking dual representations on $\mathbf{TD}_a(b)$.

This paper initially included sections applying lattice point ideas to q and (q, t) enumerations of simultaneous core partitions. This material greatly increased the length of the paper but the results in these later sections were weaker and more speculative, and so were cut to make the key point and simplicity of the results clearer. This material is still available on the version of this paper found on the arXiv: arXiv:1502.07934.

2 Abaci and Electrons

This section is a review of the fermionic viewpoint of partitions and the abacus model of a-cores. It contains no original material. The main results are that a-cores are in bijection with points on the "charge lattice" Λ_a, and that the size of a given a-core is given by a quadratic function on the lattice .

2.1 Fermions

We begin with a motivating fairy tale. It should not be mistaken for an attempt at accurate physics or accurate history.

According to quantum mechanics, the possible energies levels of an electron are quantized – they can only be half integers (perhaps a better name is odd half integers), i.e. elements of

$$\mathbb{Z}_{1/2} = \{a + 1/2 \mid a \in \mathbb{Z}\}.$$

In particular, basic quantum mechanics predicts electrons with negative energy. Physically, it makes no sense to have negative energy electrons.

Dirac's electron sea solves the problem of negative energy electrons by redefining the vacuum state **vac**. The Pauli exclusion principle states that each possible energy level can have at most one electron in it; thus, we can view any set of electrons as a subset $S \subset \mathbb{Z}_{1/2}$. Intuitively, the vacuum state **vac** should consist of empty space with no electrons at all, and hence correspond to the empty set $\emptyset \subset \mathbb{Z}_{1/2}$.

Dirac suggested instead to take **vac** to be an infinite "sea" of negative energy electrons. Specifically, in Dirac's vacuum state every negative energy level is filled with an electron, but none of the positive energy states are filled. Then by Pauli's exclusion principle we cannot add a negative energy electron to **vac**, but positive energy electrons can be added as usual. Thus, Dirac's electron sea solves the problem of negative energy electrons.

As an added benefit, Dirac's electron sea predicts the positron, a particle that has the same energy levels as an electron, but positive charge. Namely, a positron corresponds to a "hole" in the electron sea – a negative energy level *not* filled with an electron. Removing a negative energy electron results in adding positive charge and positive energy, and hence can be interpreted as adding a positron.

Our fairy tale leads us to the following definitions:

Definition 11. Let $\mathbb{Z}^{\pm}_{1/2}$ denote the set of all positive/negative half integers, respectively.

The vacuum $\mathbf{vac} \subset \mathbb{Z}_{1/2}$ is the set $\mathbb{Z}^{-}_{1/2}$.

A *state* S is a set $S \subset \mathbb{Z} + 1/2$ such that the symmetric difference

$$S \triangle \mathbf{vac} = (S \cap \mathbb{Z}^{+}_{1/2}) \cup (S^c \cap \mathbb{Z}^{-}_{1/2})$$

is finite. States should be interpreted as a finite collection of electrons – the elements of $S \cap \mathbb{Z}^{+}_{1/2}$, which we will denote by S^+ – and a finite collection of positrons – the elements of $S^c \cap \mathbb{Z}^{-}_{1/2}$, which we will denote by S^-.

The *charge* $c(S)$ of a state S is the number of positrons minus the number of electrons:

$$c(S) = |S^-| - |S^+|$$

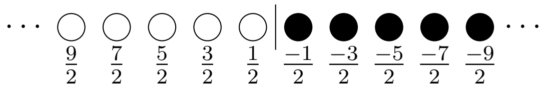
The *energy* $e(S)$ of a state S is the sum of all the energies of the positrons and the electrons:

$$e(S) = \sum_{k \in S^+} k + \sum_{k \in S^-} -k$$

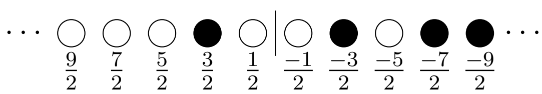
It is convenient to represent a state S as a Maya diagram.

Definition 12. The *Maya diagram* of S is an infinite sequence of circles on the x-axis, one circle centered at each element of $\mathbb{Z}_{1/2}$, with the positive circles extending to the left and the negative direction to the right. A black "stone" is placed on the circle corresponding to $k \in \mathbb{Z}_{1/2}$ if and only if $k \in S$.

Example 13. The Maya diagram corresponding to the vacuum vector **vac** is shown below.



Example 14. The following Maya diagram illustrates the state S consisting of an electron with energy 3/2, and two positrons, with energy 1/2 and 5/2.

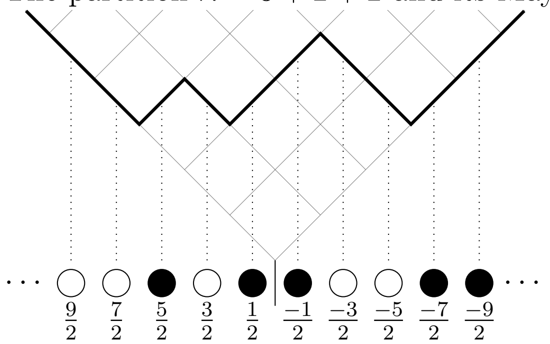


2.2 Boundary Paths

We now describe a bijection from the set of partitions $\mathcal{P}$ to the set of charge 0 states, that sends a partition $\lambda \in \mathcal{P}_n$ of size n to a state S_λ with energy $e(S_\lambda) = n$. This bijection can be understood in two ways: as recording the boundary path of λ, or recording the modified Frobenius coordinates of λ.

We draw partitions in "Russian notation" – rotated $\pi/4$ radians counterclockwise and scaled up by a factor of $\sqrt{2}$, so that each segment of the border path of λ is centered above a half integer on the x-axis. We traverse the boundary path of Λ from left to right. For each segment of the border path, we place an electron in the corresponding energy level if that segment of the border slopes up, and we leave the energy state empty if that segment of border path slopes down.

Figure 2: The partition $\lambda = 3 + 2 + 2$ and its Maya diagram



Example 15. Figure 2 illustrates the bijection in the case of $\lambda = 3 + 2 + 2$. The corresponding state S_λ consists of two electrons with energy 5/2 and 1/2, and two positrons with energy 3/2 and 5/2.

The energies of the electrons and the positrons of λ are sometimes called the *modified Frobenius coordinates*, and can also be seen as follows. The y-axis dissects the partition λ into two pieces. The left side of λ consists of c rows, where c is the number of electrons. The number of boxes in the ith row is the energy of the ith electron. The right half of λ also consists of c rows, withthe number of boxes in a given row the energy of the corresponding positron.

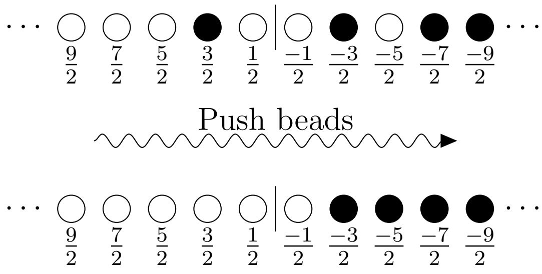
Example 16. Return again to Figure 2. If the y-axis was drawn in, left of the y-axis would be two rows, the bottom row having 2.5 boxes and the top row 0.5 boxes, which are the energies of the electrons in S. Similarly, the right hand side has two rows with 2.5 and 1.5 boxes, the energies of the positrons in S.

The bijection between partitions and states of charge zero may be modified to give a bijection between partitions and states of charge c for any $c \in \mathbb{Z}$, simply translate the partition to the right by c.

2.3 The Abacus construction

Rather than view the Maya diagram as a series of stones in a line, we now view it as beads on the runner of an abacus. Sliding the beads to be right justified allows the charge of a state to be read off, as it is easy to see how many electrons have been added or are missing from the vacuum state. In what follows, we mix our metaphors and talk about electrons and positrons on runners of an abacus.

Example 17. We revisit the Maya diagram in Example 14, which had two positrons and an electron. Pushing the beads to be right justified, we see the first bead is one step to the right of zero, and hence the original state had charge 1. This is illustrated below.



To understand how to read the hooklengths of λ off the Maya diagram for λ, we introduce the notion of *inversions*. The cells $\square \in \lambda$ are in bijection with the *inversions* of the boundary path; that is, with pairs of segments $(\text{step}_1, \text{step}_2)$, where step_1 occurs before step_2, but step_1 is traveling NE and step_2 is traveling SE. The bijection sends $\square$ to the segments at the end of its arm and leg.

Translating to the fermionic viewpoint, cells of λ are in bijection with pairs

$$\big\{(e, e-k) \mid e \in \mathbb{Z}_{1/2}, k > 0\big\}$$

of a filled energy level e and an empty energy level $e-k$ of lower energy; we call such a pair an *inversion*. We call the first entry the *hand* of the inversion, and the second entry the *foot* of the inversion, as they are the portions of the boundary path that lie at the end of the arm and the leg of the corresponding cell, respectively. The hook length $h(\square)$ of the corresponding cell is k.

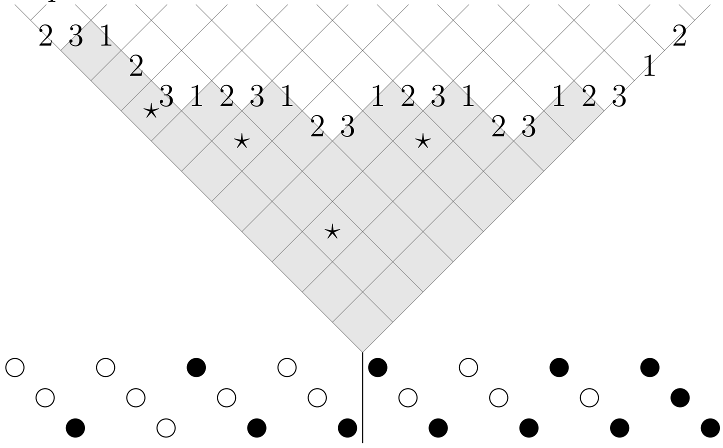
If $(e, e-k)$ is such a pair, reducing the energy of the electron from e to $e-k$ changes λ by removing the rim hook corresponding to the cell $\square$. This rim-hook has length k.

Example 18. Consider again the partition $\lambda = 3+2+2$ shown in Figure 2. We have $h(\square) = 3$, and the cell corresponds to the electron in energy state $1/2$ and the empty energy level $-5/2$; which are three apart.

To connect to cores and quotients, rather than place the electrons corresponding to λ on one runner, we place them on a different runners labelled from 0 to $a-1$, putting the energy levels $ka - i - 1/2$ on runner i.

If the hook length $h(\square) = ka$ is divisible by a then the two energy levels of inversion($\square$) lie on the same runner. Similarly, any inversion of energy states on the same runner corresponds to a cell with hook length divisible by a.

Figure 3: The partition $\lambda = 10 + 7 + 6 + 4 + 4 + 3 + 1 + 1$ and its 3-abacus.



Example 19. Figure 3 shows a partition drawn in Russian notation above its 3-abacus. The numbers on the boundary path illustrate which runner of the abacus that step belongs to. The cells marked with a $\star$ are those with hook lengths divisible by three; they can be seen to correspond to inversions whose hand and foot are on the same runner.

Thus, λ is an a-core if and only if the beads on each runner of the a-abacus are right justified. Sliding the beads of an a-abacus of a partition to be right justified corresponds to "taking the a-core" of the partition, and the abacus construction was originally introduce to show that this process was well defined.

More important to us, the abacus construction gives another description of a-core partitions in terms of a lattice. Although the total charge of all the runners in an a-abacus must be zero, the charge need not be evenly divided among the runners. Let c_i be the charge on the ith runner; then we have $\sum c_i = 0$, and the c_i determine λ.

Similarly, given any $\mathbf{c} = (c_0, \ldots, c_{a-1}) \in \mathbb{Z}^a$ with $\sum c_i = 0$, there is a unique right justified abacus with charge c_i on the ith runner. The corresponding partition is an a-core which we denote $\mathbf{core}_a(\mathbf{c})$.

We have shown:

Lemma 20. *There is a bijection*

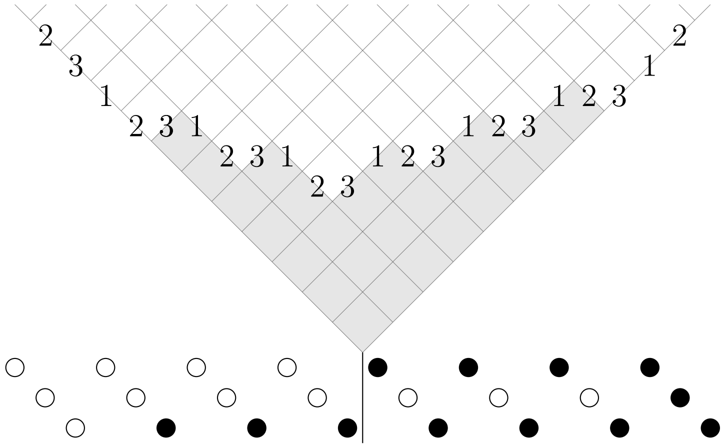
$$\mathbf{core}_a : \{(c_0, \ldots, c_{a-1} \mid c_i \in \mathbb{Z}, \sum c_i = 0\} \to \{\lambda \mid \lambda \textit{ is in a-core}\}$$

Example 21. Looking at Figure 4, we see that $\mathbf{core}_3(0, 3, -3) = 7+5+3+3+2+2+1+1$; note that this is the 3-core of the partition from Figure 3.

Theorem 22.

$$|\mathbf{core}_a(\mathbf{c})| = \frac{a}{2} \sum_{k=0}^{a-1} c_k^2 + kc_k$$

Figure 4: The 3-core $7+5+3+3+2+2+1+1$ its 3-abacus



Proof. If $c_k > 0$ the kth runner has c_k positrons, with energies

$$\begin{aligned}&(k+1/2),\\&(k+1/2)+a,\\&(k+1/2)+2a,\\&\quad\vdots\quad\vdots\\&(k+1/2)+(c_k-1)a\end{aligned}$$

and so the particles on the kth runner have total energy

$$\frac{a}{2}(c_k^2 - c_k) + (k+1/2)c_k.$$

If $c_k < 0$, the kth runner has $-c_k$ electrons, and a similar calculation shows they have a total energy of

$$\frac{a}{2}(c_k^2 + c_k) - c_k(a-k-1/2) = \frac{a}{2}(c_k^2 - c_k) + (k+1/2)c_k.$$

Since $\sum c_k = 0$, the total energy of all particles simplifies to $\frac{a}{2}\sum(c_k^2 + kc_k)$. □

3 Simultaneous Cores

We now turn to studying the set of b-cores within the lattice Λ_a of a-cores.

3.1 Under the abacus construction, (a,b)-cores form a simplex

First, some notation and conventions.

Let $r_a(x)$ be the remainder when x is divided by a, and $q_a(x)$ be the integer part of x/a, so that $x = aq_a(x) + r_a(x)$ for all x. Furthermore, we use cyclic indexing for $\mathbf{c} \in \Lambda_a$; that is, for $k \in \mathbb{Z}$, we set $c_k = c_{r_a(k)}$.

Lemma 23. *Within the lattice of a cores, the set of b cores are the lattice points satisfying the inequalities*

$$c_{i+b} - c_i \leqslant q_a(b+i)$$

for $i \in \{0, \dots, a-1\}$.

Proof. Fix $\mathbf{c} \in \Lambda_a$, let $\lambda = \mathbf{core}_a(\mathbf{c})$ be the corresponding a-core, and consider the a-abacus of λ.

Let e_i denote the highest energy level of an electron on the ith runner. We claim that $\mathbf{core}_a(\mathbf{c})$ is a b-core if an only if for each i, the energy state $e_i - b$ is filled.

Certainly this condition is necessary. To see that it is sufficient, suppose that λ is an a-core, and that for each i, the energy level $e_i - b$ is filled. To see λ is a b core, we must show that for any filled energy level L, the energy level $L - b$ is also filled.

If L is on the ith runner, then $L = e_i - aw$ for some $w \geqslant 0$, and so $L - b = (e_i - b) - aw$. But by supposition $e_i - b$ is a filled state, and $e_i - b - aw$ is to the right of it and on the same runner, and so it must be filled since λ is an a-core. Thus, we have seen that $\mathbf{core}_a(\mathbf{c})$ is also a b-core.

Using this, we now derivative the inequalities governing the set of b-cores within the lattice of a-cores. The energy level $e_i - b$ is on runner $r_a(i+b)$, and so λ is b-core if and only if $e_i - b \leqslant e_{i+b}$ (recall that we are using cyclic indexing).

Substituting $e_k = -ac_k - r(k) - 1/2$ and simplifying gives that our inequality is equivalent to

$$a(c_{i+b} - c_i) \leqslant b + i - r_a(i+b)$$

and hence to

$$c_{i+b} - c_i \leqslant q_a(b+i). \qquad \square$$

We have a hyperplanes in an $a-1$ dimensional space; they either form a simplex or an unbounded polytope. It is easy to see the form a simplex when a and b are relatively prime, and an unbounded space when a, b have a common factor.

Remark 24. Generating functions of (a,b)-cores when a and b are not relatively prime have been studied in [5], where they are shown to be modular forms. We briefly indicate how our viewpoint of (a,b)-cores sheds light on this result.

First, it is easy to see there are infinitely many (a,b) cores, as if $d = \gcd(a,b)$, then any d-core is also an (a,b)-core.

The inequalities given for $\mathbf{SC}_a(b)$ still describe the space of (a,b)-cores when a, b are no longer relatively prime, but these inequalities no longer describe a simplex. The inequalities no longer relate all the c_i to each other; rather, they decouple into d sets of a/d of variables

$$S_0 = \{c_0, c_d, c_{2d}, \dots, c_{a-d}\}$$

$$S_1 = \{c_1, c_{d+1}, \dots, c_{a-d+1}\}$$
$$\dots$$
$$S_{d-1} = \{c_{d-1}, c_{2d-1}, \dots, c_{a-1}\}$$

If $\mathbf{core}_a(\mathbf{c})$ is also a b-core, then the inequalities described force the charges c_i in a given group to be close together. However, for any vector $(v_0, \dots, v_{d-1})$ with $\sum v_i = 0$, we may shift each element of S_i by v_i and all inequalities will still be satisfied.

These shifts generate a lattice isomorphic to the lattice of d-cores for $d = \gcd(a, b)$. The remaining choices of the c_i in each group form simplex. So we see the set of (a, b) cores is a finite number of translates of a lattice. The sum over Q over the points in a lattice will be a theta function, and so we see the generating function of (a, b) cores will be a finite sum of theta functions, and hence modular.

In the charge coordinates $\mathbf{c}$, neither the hyperplanes defining the set of b cores nor the quadratic form Q are symmetrical about the origin. We shift coordinates to remedy this.

Definition 25. Define $\mathbf{s} = (s_0, \dots, s_a) \in V_{a-1}$ by

$$s_i = \frac{i}{a} - \frac{a-1}{2a}$$

The i/a term ensures $s_{i+1} - s_i = 1/a$; subtracting $\frac{a-1}{2a}$ ensures that $\mathbf{s} \in V_a$, i.e. $\sum s_i = 0$.

Lemma 26. *In the shifted charge coordinates*

$$x_i = c_i + s_i$$

the inequalities defining the set of b cores become

$$x_{i+b} - x_i \leqslant b/a$$

and the size of an a-core is given by

$$Q(\mathbf{x}) = -\frac{a^2-1}{24} + \frac{a}{2}\sum_{i=0}^{a-1} x_i^2$$

Proof. That the linear term of Q vanishes in the $\mathbf{x}$ coordinates follows immediately from the definition of $\mathbf{s}$. The constant term of Q in the $\mathbf{x}$ coordinates is $-\frac{a}{2}\sum_{i=0}^{a-1} s_i^2$, which a short computation shows is $-\frac{a^2-1}{24}$.

The statement about the set of b-cores follows from the computation

$$\begin{aligned} x_{i+b} - x_i &= c_{i+b} - c_i + s_{i+b} - s_i \\ &\leqslant q_a(i+b) + r_a(i+b)/a - i/a \\ &= (b+i)/a - i/a \\ &= b/a \end{aligned}$$

□

Although we often use the x coordinates, to show that the simplex of $\mathbf{SC}_a(b)$ is isomorphic to the simplex $\mathbf{TD}_a(b)$ of trivial determinant representations another change of variables is needed.

Lemma 27. *Let a and b be relatively prime, and let*

$$k = -\frac{b+1}{2} \pmod{a}$$

Then the change of variables

$$z_i = x_{ib+k} - x_{(i+1)b+k} + b/a$$

gives an isomorphism between the rational simplices $\mathbf{SC}_a(b)$ and $\mathbf{TD}_a(b)$.

Proof. It is immediate that the z_i satisfy $\sum z_i = b$ and $z_i \geqslant 0$. The integrality of the z_i follows from the fact that the fractional part of $x_i - x_j$ is $(i-j)/a$. We must show $\sum iz_i = 0 \mod a$.

One computes:

$$\sum_{i=0}^{a-1} iz_i = -ax_k + \sum_{i=0}^{a-1} x_i + \frac{b}{a}\sum_{i=0}^{a-1} i$$

Since the fractional part of x_k is $s_k = k/a - (a-1)/2a$, plugging in the definition of k gives that $ax_k = -b/2 \pmod{a}$. Since $\sum x_i = 0$ and $\sum i = (a-1)a/2$, we see $\sum iz_i = 0 \pmod{a}$.

A further computation shows this change of variables is invertible. □

Corollary 28 (Anderson [2])**.** *The number of simultaneous (a,b)-cores is $\mathbf{Cat}_{a,b}$.*

Proof. This follows quickly from Lemma 27, and was outlined in the introduction; we reproduce it here for completeness.

The scaled simplex $b\Delta_a$ has $\binom{a+b-1}{a-1}$ usual lattice points. Cyclically permuting the variables preserves $b\Delta_a$ and the standard lattice, and when b is relatively prime to a it cyclical permutes the a cosets of the charge lattice.

Thus the standard lattice points in $b\Delta_a$ are equidistributed among the a-cosets of the charge lattice, and hence each one contains $\frac{1}{a}\binom{a+b-1}{a-1} = \mathbf{Cat}_{a,b}$. □

3.2 The size of simultaneous cores

We now have all the ingredients needed to prove Armstrong's conjecture. We derive it as a consequence of:

Theorem 29. *For fixed a, and b relatively prime to a, the average size of an (a,b)-core is a polynomial of degree 2 in b.*

Proof. For fixed a, the number of (a, b)-cores is $1/a$ times the number of lattice points in $b\Delta_{a-1}$, which is a polynomial $F_a(b)$ of degree $a-1$. In the $\mathbf{x}$-coordinates $Q(\mathbf{x}) = |\mathbf{core}_a(\mathbf{x})|$ is invariant under S_a, and in particular rotation, so the sum of the sizes of all (a, b)-cores is $1/a$ times the sum of Q over the lattice points in $b\Delta_{a-1}$. By Euler-Maclaurin theory, the sum of a quadratic function over the lattice points in $b\Delta_{a-1}$ is a polynomial $G_a(b)$ of degree $a + 1$.

Thus, the average value of an (a, b)-core is $G_a(b)/F_a(b)$, the quotient of a polynomial of degree $a + 1$ in b by a polynomial of degree $a - 1$ in b. To show this is a polynomial of degree two in b, we need to show that every root of F_a is a root of G_a.

Corollary 28 says that the roots of F_a are $-1, -2, \ldots, -(a - 1)$. We now give another derivation of this fact, using Ehrhart reciprocity, that easily adapts to shown these are also roots of G_a.

Ehrhart reciprocity says that $F_a(-x)$ is, up to a sign, the number of points in the *interior* of $x\Delta_{a-1}$. The interior consists of the points in $x\Delta_{a-1}$ none of whose coordinates are zero, and so the first interior point in $x\Delta_{a-1}$ is $(1, 1, \ldots, 1) \in a\Delta_{a-1}$. Thus, $F_a(b)$ vanishes at $b = -1, \ldots, -(a - 1)$, and as it has degree $a - 1$ it has no other roots.

Ehrhart reciprocity extends to Euler-Maclaurin theory, to say that up to a sign $Q_a(-x)$ is the sum of F of the interior points of $x\Delta_{a-1}$. Thus $Q_a(-x)$ also vanishes at $b = -1, \ldots, -(a - 1)$, and so P_a/Q_a is a polynomial of degree 2. ☐

Corollary 30. *When (a, b) are relatively prime, the average size of an (a, b) core is $(a + b + 1)(a - 1)(b - 1)/24$.*

Proof. Fix a, and let $P_a(b) = G_a(b)/F_a(b)$ be the degree two polynomial that gives the average value of the (a, b)-cores when a and b are relatively prime. As we know $P_a(b)$ is a polynomial of degree 2, we can determine it by computing only three values.

First, we find the two roots of $P_a(b)$. As the only 1 core is the empty partition, we have $F_a(1) = 1$ and $G_a(1) = 0$, and so $P_a(1) = 0$.

Ehrhart reciprocity gives that $G_a(-a-b)$ is, up to a sign, the sum of Q over the lattice points in the interior of $(a+b)\Delta_a$, which are just the lattice points contained in $b\Delta_a$, and hence equal to $G_a(b)$. In particular, $P_a(-a - 1) = 0$.

Finally, we compute $P_a(0)$. It is clear that $\mathcal{S}_a(0) = \{0\}$. Although this is not a point of Λ_a, it is in Λ'_a, and so $P_a(0) = Q(0) = -(a^2 - 1)/24$. ☐

3.3 Self-conjugate (a, b)-cores

In Lemma 31, we show that under the bijection between (a, b)-cores and b-dimensional representations of $\mathbb{Z}_a$ with trivial determinant, conjugating a partition corresponds to sending a representation V to its dual V^*. In the lattice point of view, this is a linear map T, and hence the self-dual (a, b)-cores correspond to the lattice points in the fixed point locus of T.

We show in Lemma 32, that the T-fixed lattice points in $\mathbf{SC}_a(b)$ are the lattice points in the $\lfloor a/2 \rfloor$ dimensional simplex $\lfloor b/2 \rfloor \Delta_{\lfloor a/2 \rfloor}$, hence rederiving the count of simultaneous (a, b)-core partitions.

Once we have done this, an analogous application of Euler-Maclaurin theory reproves the statement about the average value.

Let $T : V_a \to V_a$ be the linear map given by

$$T(c_i) = -c_{-1-i}$$

It is easy to check that when translated to core partitions, T corresponds to taking the conjugate, that is:

$$\mathbf{core}_a(c)^T = \mathbf{core}_a(T(c))$$

Thus the set of self-conjugate (a, b)-cores is the T fixed locus of $\mathbf{SC}_a(b)$.

Since $T(s) = s$, the same formula holds in the shifted coordinates $\mathbf{x}$.

Lemma 31. *Under the isomorphism between $\mathbf{SC}_a(b)$ and $\mathbf{TD}_a(b)$ established in Lemma 27, the map T sending a partition to its conjugate corresponds to taking the dual $\mathbb{Z}_a$ representation.*

Proof. We want to show $T(z_i) = z_{-i}$. We compute:

$$\begin{aligned} T(z_i) &= T(x_{ib+k} - x_{(i+1)b+k}) \\ &= -x_{-ib-k-1} + x_{-ib-b-k-1} \\ &= x_{-ib+k-(b+1+2k)} - x_{(-i+1)b+k-(b+1+2k)} \end{aligned}$$

And so we need $b+1+2k = 0 \pmod a$, but this is exactly the definition of k in Lemma 27. ☐

Lemma 32. *The number of b-dimensional, self-dual $\mathbb{Z}_a$ representations with trivial determinant is*

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor}$$

Proof. Let $a = 2k$ or $2k+1$. We give a bijection between the representations in question and k-tuples of non-negative integers $(z_1, \ldots, z_k)$ with $2\sum z_i \leqslant b$. The set of such z_i are the lattice points in $\lfloor b/2 \rfloor \Delta_{\lfloor a/2 \rfloor}$, which are counted by the given binomial coefficient.

First, suppose that $a = 2k+1$. Then the only irreducible self-conjugate representation is the identity, and T has a k dimensional fixed point set consisting of points of the form $(u_0, u_1, \ldots, u_k, u_k, \ldots, u_1)$. Thus, $\sum_{i=1}^k 2u_k \leqslant b$, and value of u_0 is fixed by $u_0 + 2\sum_{i=1}^k u_i = b$.

When $a = 2k$, there are two irreducible self-conjugate representations, the identity and the sign representation induced by the surjection $\mathbb{Z}_a \to \mathbb{Z}_2$. Again, T has a k dimensional fixed point set, this time consisting of points of the form $(u_0, u_1, \ldots u_{k-1}, w_k, u_{k-1}, \ldots, u_1)$. Now for such a representation, having trivial determinant is equivalent to w_k being even, say $w_k = 2u_k$. Then again we have $\sum_{i=1}^k 2u_k \leqslant b$, with u_0 being determined by $u_0 + 2\sum_{i=1}^k u_i = b$. ☐

Proposition 33. *Let a and b be relatively prime. Then the average size of a self-conjugate (a, b)-core is $(a-1)(b-1)(a+b+1)/24$.*

Proof. Since a and b are relatively prime, at most one is even, so we may assume a is odd.

The proof is essentially the same as that for all (a, b)-cores. One complication is that it seems we must treat odd and even values of b separately. In each each case, an argument identical to Lemma 29 gives that the average size is a polynomial of degree 2 in b. A priori, we may have different polynomials for b odd and b even; however, the symmetry $(a, b) \leftrightarrow (a, -a-b)$ coming from Ehrhart reciprocity still holds and interchanges odd and even values of b, and so if we can compute three values of either polynomial (that don't get identified by this symmetry), we identify both polynomials.

All 1 and 2 cores are self conjugate, and thus if b is 1 or 2, the average value is the same. The arguments made in Corollary 30 for the value of the polynomial at $b = 0$ holds for self-conjugate partitions as well, giving a third value. □

Acknowledgements

I learned about Armstrong's conjecture over dinner after speaking in the MIT combinatorics seminar. I would like to thank Jon Novak for the invitation, Fabrizio Zanello for telling me about the conjecture, and funding bodies everywhere for supporting seminar dinners.

The anonymous referee gave many comments that improved the exposition.

References

[1] Amol Aggarwal. Armstrong's conjecture for $(k, mk+1)$-core partitions. *European J. Combin.*, 47:54–67, 2015. arXiv:1407.5134v1.

[2] Jaclyn Anderson. Partitions which are simultaneously t_1- and t_2-core. *Discrete Math.*, 248(1-3):237–243, 2002.

[3] Federico Ardila and Erwan Brugallé. The double Gromov-Witten invariants of Hirzebruch surfaces are piecewise polynomial. *Int. Math. Res. Not. IMRN*, (2):614–641, 2017. arXiv:1412.4563.

[4] Drew Armstrong, Christopher R. H. Hanusa, and Brant C. Jones. Results and conjectures on simultaneous core partitions. *European J. Combin.*, 41:205–220, 2014. arXiv:1308.0572.

[5] David Aukerman, Ben Kane, and Lawrence Sze. On simultaneous s-cores/t-cores. *Discrete Math.*, 309(9):2712–2720, 2009.

[6] Matthias Beck and Sinai Robins. *Computing the continuous discretely.* Undergraduate Texts in Mathematics. Springer, New York, 2007. http://math.sfsu.edu/beck/ccd.html.

[7] Renzo Cavalieri, Paul Johnson, and Hannah Markwig. Wall crossings for double Hurwitz numbers. *Adv. Math.*, 228(4):1894–1937, 2011. arXiv:1003.1805.

[8] William Y. C. Chen, Harry H. Y. Huang, and Larry X. W. Wang. Average size of a self-conjugate (s, t)-core partition. *Proc. Amer. Math. Soc.*, 144(4):1391–1399, 2016. arXiv:1405.2175.

[9] Robbert Dijkgraaf and Piotr Sułkowski. Instantons on ALE spaces and orbifold partitions. *J. High Energy Phys.*, (3):013, 24, 2008. arXiv:0712.1427.

[10] Ben Ford, Hoàng Mai, and Lawrence Sze. Self-conjugate simultaneous p- and q-core partitions and blocks of A_n. *J. Number Theory*, 129(4):858–865, 2009.

[11] Frank Garvan, Dongsu Kim, and Dennis Stanton. Cranks and t-cores. *Invent. Math.*, 101(1):1–17, 1990.

[12] Iain G. Gordon and Stephen Griffeth. Catalan numbers for complex reflection groups. *Amer. J. Math.*, 134(6):1491–1502, 2012. arXiv:0912.1578.

[13] Eugene Gorsky, Mikhail Mazin, and Monica Vazirani. Affine permutations and rational slope parking functions. *Trans. Amer. Math. Soc.*, 368(12):8403–8445, 2016. arXiv:1403.0303.

[14] Evgeny Gorsky and Mikhail Mazin. Compactified Jacobians and q,t-Catalan numbers, I. *J. Combin. Theory Ser. A*, 120(1):49–63, 2013. arXiv:1105.1151.

[15] Evgeny Gorsky and Mikhail Mazin. Compactified Jacobians and q,t-Catalan numbers, II. *J. Algebraic Combin.*, 39(1):153–186, 2014. arXiv:1204.5448.

[16] Christian Haase and Tyrrell B. McAllister. Quasi-period collapse and $\mathrm{GL}_n(\mathbb{Z})$-scissors congruence in rational polytopes. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 115–122. Amer. Math. Soc., Providence, RI, 2008. arXiv:0709.4070.

[17] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[18] G. Lusztig and J. M. Smelt. Fixed point varieties on the space of lattices. *Bull. London Math. Soc.*, 23(3):213–218, 1991.

[19] R. Stanley and P. Zanello. The Catalan case of Armstrong's conjecture on simultaneous core partitions. *SIAM J. Discrete Math.*, 29(1), 658–666, 2015.