Proper Distinguishing Colorings with
Few Colors for Graphs with Girth at Least 5

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Abstract

The distinguishing chromatic number, $\chi_D(G)$, of a graph G is the smallest number of colors in a proper coloring, φ, of G, such that the only automorphism of G that preserves all colors of φ is the identity map. Collins and Trenk conjectured that if G is connected with girth at least 5 and $G \neq C_6$, then $\chi_D(G) \leqslant \Delta + 1$. We prove this conjecture.

Mathematics Subject Classifications: 05C15, 05C25

1 Introduction

A *k-coloring* φ of a graph G is a map $\varphi : V(G) \to \{1, \ldots, k\}$. If $\varphi(v) \neq \varphi(w)$ whenever $vw \in E(G)$, then φ is *proper*. A vertex, w, is *fixed* by a coloring φ if every automorphism f that preserves all colors (that is $\varphi(f(v)) = \varphi(v)$ for all $v \in V(G)$) has $f(w) = w$. A coloring φ of a graph G is *distinguishing* if it fixes every vertex; equivalently, if the only automorphism f of G with $\varphi(f(v)) = \varphi(v)$ for every vertex v is the identity map. The *distinguishing chromatic number*, $\chi_D(G)$, is the smallest number of colors in a proper distinguishing coloring. In a breadth-first (search) spanning tree, *level i* is the set of vertices at distance i from the root. We write $\Delta(G)$ for the maximum degree of a graph G, and write Δ when G is clear from context.

The distinguishing chromatic number was introduced by Collins and Trenk [6]. They showed that $\chi_D(G) \leqslant 2\Delta$ for every connected graph G, with equality only if $G \in \{K_{\Delta,\Delta}, C_6\}$. Further, they conjectured that no connected graph has $\chi_D(G) = 2\Delta - 1$. Laflamme and Seyffarth [8] confirmed this conjecture for bipartite graphs, with the exception of $K_{\Delta,\Delta-1}$. For every tree T, Collins and Trenk proved $\chi_D(T) \leqslant \Delta + 1$. Collins, Hovey, and Trenk [5] studied $\chi_D(G)$ in terms of Aut(G), the automorphism group of G.

(margin notes: fixed; distinguishing; $\chi_D(G)$; level i)

Cavers and Seyffarth [2] characterized graphs G with $\chi_D(G) \in \{|V(G)| - 1, |V(G)| - 2\}$. Choi, Hartke, and Kaul [4] studied χ_D for cartesian products of graphs, and Cheng [3] studied it for interval graphs.

The goal of this note is to prove Theorem 1, which was conjectured[1] by Collins and Trenk [6], and proved for bipartite graphs by Alikhani and Soltani [1].

Theorem 1. *If G is connected with girth at least 5 and $G \neq C_6$, then $\chi_D(G) \leqslant \Delta + 1$.*

Theorem 1 is best possible in two ways. First, the star $K_{1,\Delta}$ requires $\Delta + 1$ colors. In fact, Collins and Trenk characterized the trees T for which $\chi_D(T) = \Delta(T) + 1$, and there are an infinite number of these for each Δ. Thus, we cannot improve the upper bound on χ_D, even if we require larger girth, or impose some requirement on Δ. Second, for each $\Delta \geqslant 3$, they constructed infinitely many graphs G with girth 4, maximum degree Δ, and $\chi_D(G) = 2\Delta - 2$. Thus, we cannot relax the girth bound.

Key Definition. Let G be a connected graph of girth at least 5, and $w \in V(G)$. Let T be a breadth-first spanning tree, rooted at w. Let σ denote the vertex order of T, and for each $v \in V(G)$, let σ_v denote the set of vertices that precede v in σ. When σ' is a prefix of σ, we often slightly abuse notation by writing σ' to mean the set of vertices in that order. Let φ' be a proper coloring of the subgraph induced by some σ'. We say *color greedily*, with respect to σ and φ', to mean that we extend φ' to a proper coloring φ by coloring each $v \in \sigma \setminus \sigma'$ in order, subject to the following two constraints. **(i)** If v has a neighbor in σ_v (other than its parent), then $\varphi(v)$ is the smallest color not used on $N(v) \cap \sigma_v$. **(ii)** If v has no neighbor in σ_v (other than its parent), then $\varphi(v)$ is the smallest color not used on $\sigma_v \cap S_v$, where S_v consists of the siblings and parent of v.

We begin with the following easy lemma. Its proof draws on ideas from a similar result in [6], where G has no girth constraint.

Lemma 2. *Let G be a connected graph with girth at least 5, and $w \in V(G)$. Let T be a breadth-first spanning tree, rooted at w, with vertex order σ. Let φ be a greedy coloring of G, with respect to σ and a coloring φ' of some non-empty prefix σ' of σ. If φ fixes each vertex in σ', then φ is a proper distinguishing coloring.*

Proof. By construction, φ is proper, so we need only prove that φ fixes all vertices. We use induction on $|V(G)|$, at each step considering the next vertex, v, in σ that is not known to be fixed, along with all of its siblings. By assumption φ fixes every vertex in σ'. Consider the first vertex, v, that is not yet known to be fixed. By the induction hypothesis, the parent, x, of v is fixed. So, any automorphism must map $N(x)$ to $N(x)$. Suppose that some sibling y of v has a neighbor, z, that is already colored. Since z is already colored, it comes before v and y in T. So, by the induction hypothesis, z is fixed. Since x and z are fixed, and G has no 4-cycle, y is also fixed. So all siblings of v with a non-parent neighbor already colored are fixed. And by construction each sibling of v without such a neighbor already colored has a color different from v (or is fixed, since it is in σ'). So v and all its siblings are fixed, which completes the proof. $\square$

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[1]Through a minor oversight, the original conjecture did not exclude C_6.

Lemma 2 plays a key role in our proof of Theorem 1. To illustrate how we use it, we first prove a slightly weaker bound.

Proposition 3. *If G is connected with girth at least 5, then $\chi_D(G) \leqslant \Delta + 2$.*

Proof. Choose an arbitrary vertex w, and let T be a breadth-first spanning tree from w. Let $\varphi'(w) = \Delta + 2$, and let $\sigma' = \{w\}$. Now we color greedily, with respect to φ' and σ. Note that, except on w, this coloring only uses colors from $\{1, \ldots, \Delta + 1\}$, since each vertex v must avoid the colors either (i) on its neighbors, at most Δ, or (ii) on its parent and siblings, at most $1 + (\Delta - 1)$. Since w is the only vertex colored $\Delta + 2$, it is fixed. Now φ is a proper distinguishing coloring by Lemma 2. Thus, $\chi_D(G) \leqslant \Delta + 2$. $\qquad\square$

2   Main Result

Before proving our main result, we need an observation about the largest color possibly used in cases (i) and (ii) of a greedy coloring.

Observation 4. *Fix a connected graph G with girth at least 5. Choose an arbitrary $w \in V(G)$ and breadth-first spanning tree T, rooted at w. For any prefix σ' of σ, proper coloring φ' of the subgraph induced by σ', and greedy coloring φ, we have the following. Each vertex in $V(G) \setminus N(w)$ colored by (i) uses a color no larger than $\Delta + 1$ and uses $\Delta + 1$ only if all of its neighbors are already colored, and all use distinct colors. Each vertex $v \in V(G) \setminus N(w)$ colored by (ii) uses a color no larger than Δ.*

Proof. The first statement is obvious. The second holds since v has at most $\Delta - 2$ siblings. $\qquad\square$

Proof of Theorem 1. Assume the theorem is false, and G is a counterexample minimizing $|V(G)|$.

Claim 1. *G has $\Delta \geqslant 3$.*

We assume that $|V(G)| \geqslant 3$, since otherwise the theorem is trivial. Since G is connected, this implies that $\Delta \geqslant 2$. If G is a path, then we color one end of the path with 1, and the remaining vertices with 2 and 3 alternating. If G is a cycle, then we color its vertices (in cyclic order) as $1, 2, 3, 1, 2, \ldots$, where the vertices after the first five alternate colors 3 and 2. It is easy to check that these colorings are proper and distinguishing, unless $G = C_6$.

Claim 2. *G is Δ-regular.*

Suppose, to the contrary, that G has a vertex, w, with degree at most $\Delta - 1$. Let T be [w, T] a breadth-first spanning tree rooted at w. Color w with $\Delta + 1$, then color $\sigma \setminus \{w\}$ greedily; call the resulting coloring φ. Note that no other vertex with degree at most $\Delta - 1$ uses [φ] color $\Delta + 1$. Thus, w is fixed by φ. Now φ is a proper distinguishing coloring of G by Lemma 2. Thus, $\chi_D(G) \leqslant \Delta + 1$, a contradiction.
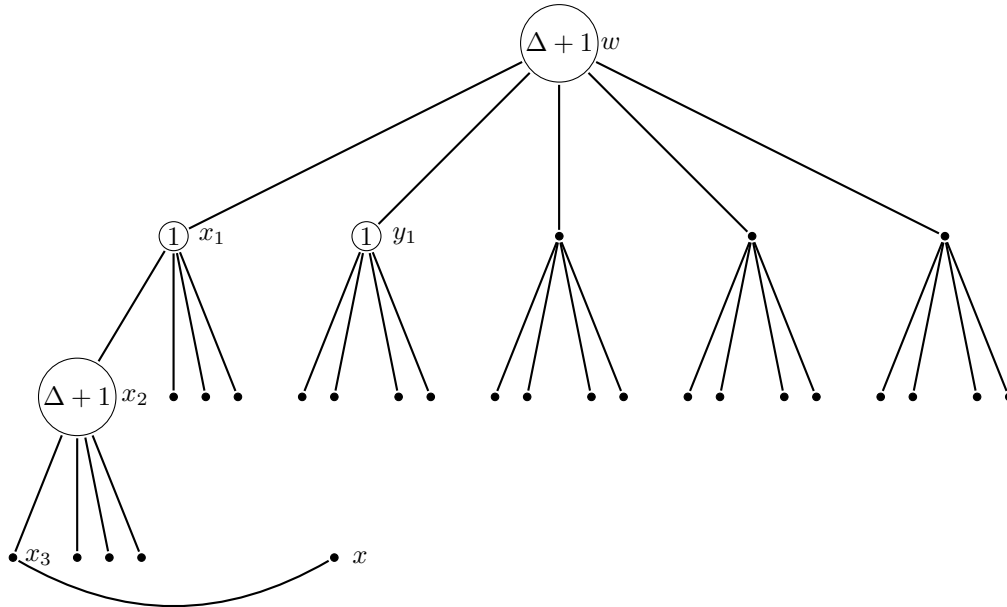
Figure 1: A partial coloring, constructed in Claim 3.

Claim 3. *G has no vertices w, x, x_3, with x and x_3 adjacent, such that $dist(w, x) \geqslant 3$ and $dist(w, x_3) = 3$. In particular, G has diameter at most 3.*

If G has a pair of vertices w and x with $dist(w, x) = 4$, then let x_3 be a neighbor of x on a shortest path from x to w. Thus, the second statement follows from the first. Now we prove the first. Assume the contrary. Let $wx_1x_2x_3x$ be a path with $dist(w, x_i) = i$ $\quad w, x_i, x$ for each $i \in \{1, 2, 3\}$ and $dist(w, x) \geqslant 3$; see Figure 1. Let T be a breadth-first spanning $\quad T$ tree rooted at w, such that each x_i is the first vertex at level i of T. Let y_1 be the second $\quad y_1$ child of w. Color T greedily except for the following modifications. Color w and x_2 with $\Delta + 1$, color x_1 and y_1 with 1, and color x_3 after all of its siblings so that the multiset of colors appearing on $N(x_2)$ differs from that appearing on $N(w)$. This final step is possible because x is uncolored at the time we color x_3, so we have at least two options for x_3. Call the resulting coloring φ. $\quad \varphi$

Clearly, φ is proper and uses at most $\Delta + 1$ colors. We must verify that φ fixes all vertices. The proof uses Lemma 2, so we must show that φ fixes σ'. As in Observation 4, no vertex other than w and (possibly) x_2 uses color $\Delta + 1$ and also has a repeated color in its neighborhood. By construction, the multisets of colors used on $N(x_2)$ and $N(w)$ differ. Thus, w is fixed. The only children of w that use a common color are x_1 and y_1; so all of its other children are fixed. Consider a child of y_1, call it y_2. If y_2 is colored by (ii), then $\varphi(y_2) \leqslant \Delta$, by Observation 4. If y_2 is colored by (i), then y_2 has at most one colored neighbor (other than y_1), since G has girth at least 5. Thus, $\varphi(y_2) \leqslant 3 \leqslant \Delta$. So x_1 has a neighbor (other than w) colored $\Delta + 1$, but y_1 has no such neighbor. Hence, x_1 and y_1 are fixed. Let σ' be the prefix of σ ending with x_2 and φ' the restriction of φ to the subgraph induced by σ'. Now Lemma 2 implies that all vertices are fixed, so φ is a proper distinguishing coloring using at most $\Delta + 1$ colors, a contradiction.

Claim 4. *G has diameter 2 or $\Delta = 3$.*

Suppose, to the contrary, that $\Delta \geqslant 4$ and G has diameter at least 3. Claim 3 implies that G has diameter at most 3, so assume G has diameter 3. Choose $w, z_1, z_2, z_3 \in V(G)$ such that $wz_1z_2z_3$ is a path and $\mathrm{dist}(w, z_i) = i$ for each $i \in \{1, 2, 3\}$. Let T be a breadth-first spanning tree rooted at w. Each z_i is at level i in T. Further, by Claim 3, every neighbor of z_3 is at level 2 in T. Choose $x_1, x_2, y_1, y_2 \in V(G)$ such that $wx_1x_2z_3$ and $wy_1y_2z_3$ are paths, and $x_1, x_2, y_1, y_2, z_1, z_2$ are distinct; see Figure 2. After possibly reordering some siblings, we can assume that x_1, y_1, and z_1 are (respectively) the first, second, and third children of w. We can also assume that z_2 is the first child of z_1 and that z_3 is the last child of z_2 (even though z_3 would naturally be a child of x_2). Further, we assume x_2 and y_2 are the second children, respectively, of x_1 and y_1. Finally, by reordering children of y_1 if needed, we assume that z_2 is not adjacent to the first child of y_1. Since $\Delta \geqslant 4$ (and G has girth at least 5), y_1 has at least 3 children; one of these is y_2 and at most one is adjacent to z_2, since G has no 4-cycle. Similarly, we assume that y_2 is not adjacent to the first child of x_1.

Begin by coloring w with 1 and coloring x_1 and z_1 with $\Delta + 1$. We greedily color the rest of T except for the following modifications; none of these change the order of the vertices, only our choice of color for a vertex when we reach it in the order. No child of x_1 uses color 1 (so its children use colors $2, \ldots, \Delta$). Vertex z_2 uses 1 and no other child of z_1 uses 1. (If such a child, say z_2', has no colored neighbor other than z_1, then this happens naturally as a result of a greedy coloring, since z_2 uses 1. If z_2' does have a colored neighbor other than z_1, then it has at most two: one child of x_1 and one child of y_1, since G has no 4-cycles. Since $\Delta \geqslant 4$, z_2' has some available color from among $\{2, 3, 4\}$ that is
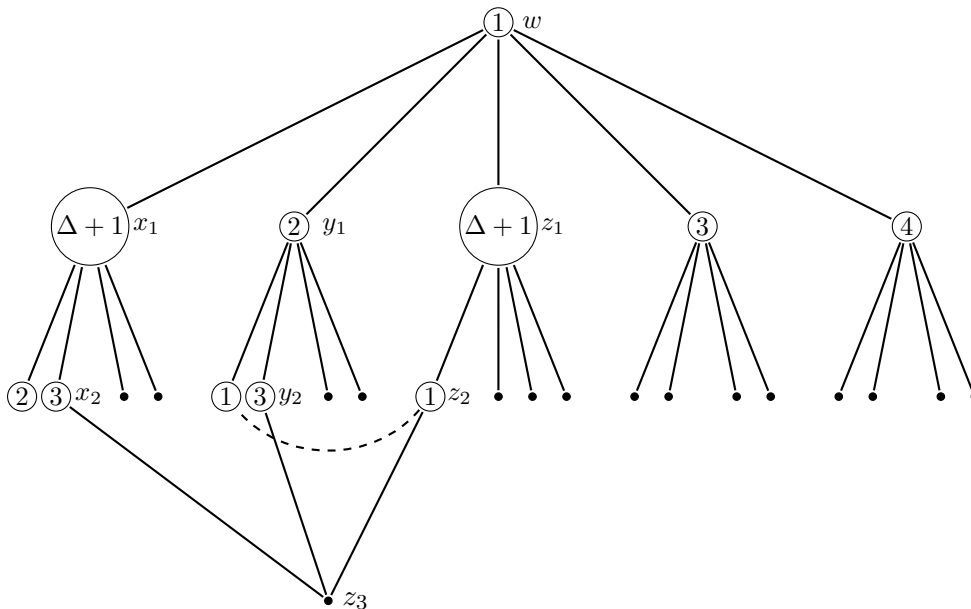

Figure 2: A partial coloring, constructed in Claim 4.

not already used on any of its neighbors, so z_2' can avoid 1; recall that z_1 uses $\Delta + 1 > 4$, so z_1 does not forbid a color in $\{2, 3, 4\}$.) Note that x_2 and y_2 both use 3. When we color z_3 (as the last child of z_2), we do so to ensure that the multisets of colors used on $N(w)$ and $N(z_2)$ differ; this is possible because x_2 and y_2 use the same color, so z_3 has at least two available colors. Further, if possible, we also require that z_3 not use color $\Delta + 1$; so z_3 uses $\Delta + 1$ only if all of its neighbors use distinct colors, except for x_2 and y_2, which both use 3. Call the resulting coloring φ. φ

Now we show that φ fixes all vertices. Since $\varphi(z_2) = 1$, vertex z_1 is the unique vertex colored $\Delta + 1$ with at least two neighbors colored 1, by Observation 4. Thus z_1 is fixed. By design, no child of z_1 other than z_2 uses color 1. By our choice of color for z_3, the multisets of colors on $N(w)$ and $N(z_2)$ differ. Thus, w is fixed and z_2 is fixed. Since we already know z_1 is fixed, each neighbor of w is fixed. Now, as in the proof of Lemma 2, by induction, we show for each vertex in $N(w)$ that its children are fixed. Finally, consider a vertex, u, at level 3 of T. By Claim 3, each neighbor of u is at level 2. Thus, since G has no 4-cycle and u has at least 2 fixed neighbors, u is fixed. So $\chi_D(G) \leqslant \Delta + 1$, a contradiction.

Claim 5. *G has $\Delta = 3$.*

Suppose that $\Delta \geqslant 4$. By Claim 4, G has diameter 2. Since G is Δ-regular with diameter 2 and girth 5, we know that $|V(G)| = \Delta^2 + 1$. Choose an arbitrary vertex w, w, G' and let $G' = G \setminus \{w \cup N(w)\}$. Note that G' is regular, with degree $\Delta(G) - 1$. Since G has diameter 2, for each $v_1, v_2 \in V(G')$, if v_1 and v_2 have no common neighbor (in G) among $N(w)$, then $\text{dist}_{G'}(v_1, v_2) \leqslant 2$. This implies that G' is connected. Since $\Delta(G) - 1 \geqslant 3$, by the minimality of G we know that $\chi_D(G') \leqslant \Delta(G') + 1 = \Delta(G)$; let φ' be a coloring of φ' G' showing this. To extend φ' to G, use color $\Delta(G) + 1$ on each vertex in $N(w)$ and use an arbitrary color (other than $\Delta(G) + 1$) on w; we call this coloring φ. Clearly w is fixed φ by φ, since each other vertex has at most one neighbor colored $\Delta(G) + 1$. Further, each vertex of G' is fixed by φ', so we get that each vertex of G' is fixed by φ. Finally, each vertex $v \in N(w)$ has all of its neighbors fixed by φ. Since G has no 4-cycles, v is also fixed by φ. Thus, $\chi_D(G) \leqslant \Delta(G) + 1$, a contradiction.

Claim 6. *G is either bipartite or vertex-transitive.*

Suppose there exist vertices w, x_1, y_1 with $x_1, y_1 \in N(w)$ and no automorphism maps w, x_1, y_1 x_1 to y_1. Let T be a breadth-first spanning tree, rooted at w, with x_1 and y_1 as the first T children of w in T. Color w with $\Delta + 1$, color x_1 and y_1 with 1, and color the rest of T greedily. Now w is fixed, since it is the only vertex colored $\Delta + 1$ with two neighbors colored 1. Vertices x_1 and y_1 are fixed, since no automorphism maps one to the other, and all other neighbors of w are fixed, since they receive distinct colors. All remaining vertices are fixed by Lemma 2. So $\chi_D(G) \leqslant \Delta + 1$, a contradiction.

Instead, assume that for every vertex w and $x_1, y_1 \in N(w)$ some automorphism maps w, x_1, y_1 x_1 to y_1. Thus, for each pair of vertices x_1, y_1 joined by a walk of even length, some automorphism maps x_1 to y_1 (the proof is by induction on the length of the walk). If G is bipartite, we are done; so assume it is not. Let C be an odd cycle in G. Since G C

is connected, for every pair of vertices $x_1, y_1 \in V(G)$, there exists an x_1, y_1-walk of even length. Consider a walk from x_1 to y_1 that visits C. If the walk has odd length, then we extend it by going around C once. Thus, G is vertex-transitive, as desired.

Claim 7. *G is either the Petersen graph or the Heawood graph; see Figure 3.*

Recall that G is 3-regular, by Claims 2 and 5. Choose an arbitrary vertex w, and let T be a breadth-first spanning tree rooted at w. Since G has girth at least 5, the number $\quad$ w, T of vertices at level 2 is $3(3-1) = 6$. By Claim 3, G has diameter at most 3, so any additional vertices of G are at level 3. Further, by Claim 3, no pair of vertices at level 3 are adjacent. So each vertex at level 3 has all three of its neighbors at level 2. Since each of the 6 vertices at level 2 has at most two neighbors at level 3, the number of vertices at level 3 is at most $\frac{2}{3}(6) = 4$. Thus, $|V(G)| \leqslant 1 + 3 + 6 + 4 = 14$. If G has girth at least 6, then G must be the Heawood graph, since it is the unique 3-regular graph with girth at least 6 and at most 14 vertices. Otherwise, G has a 5-cycle. So Claim 6 implies that G is vertex-transitive.

To finish, we simply check that the only 3-regular vertex-transitive graphs with girth at least 5 and at most 14 vertices are the Petersen graph and the Heawood graph. A convenient reference to verify this is [9], which catalogues all vertex-transitive graphs on at most 19 vertices.

Claim 8. *G does not exist.*

Consider the 4-colorings in Figure 3 of the Petersen graph (left) and Heawood graph (right). Clearly, both colorings are proper. In the coloring of the Petersen graph, every two vertices with the same color have distinct multisets of colors on their neighborhoods. Thus, the coloring is distinguishing.

Now consider the coloring of the Heawood graph. The vertices colored 4 are fixed, since the multisets on their neighborhoods differ. The same is true for the vertices colored 2, except for two with the multiset $\{1, 1, 3\}$. But for exactly one of these vertices the neighbor colored 3 has a neighbor colored 4. Thus, the vertices colored 2 are fixed. Now every other vertex has at least two neighbors fixed. Since G has no 4-cycles, every vertex is fixed. So the coloring is distinguishing. This proves claim 8, which finishes the proof of Theorem 1. $\qquad \square$
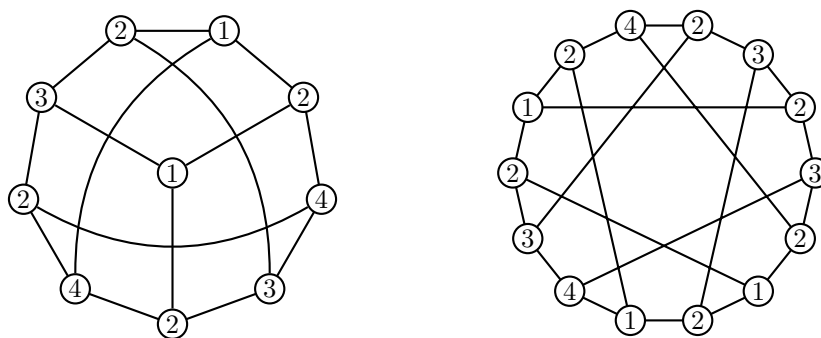

Figure 3: Proper distinguishing 4-colorings of the Petersen graph and Heawood graph.

We conclude with a few remarks. The *distinguishing list chromatic number*, $\chi_D^\ell(G)$, is the smallest k such that given any list assignment L with $|L(v)| \geqslant k$ for all $v \in V(G)$, there exists a proper distinguishing coloring φ of G such that $\varphi(v) \in L(v)$ for all $v \in V(G)$. In general, $\chi_D^\ell(G)$ can be much larger than $\chi_D(G)$. For example, form a graph G from $K_{1000,1000}$ by adding a pendant edge at each vertex and subdividing these edges so the resulting paths have distinct lengths. The only automorphism of G is the identity, so $\chi_D(G) = \chi(G) = 2$. However, $\chi_D^\ell(G)$ is at least the list-chromatic number, which is greater than 2 (and grows arbitrarily large on complete bipartite graphs). In contrast, we note that the bound on χ_D in Proposition 3 also holds for χ_D^ℓ.

$\chi_D^\ell(G)$

Proposition 5. *If G is connected with girth at least 5, then $\chi_D^\ell(G) \leqslant \Delta + 2$.*

Proof. The proof is nearly identical to that of Proposition 3. We choose an arbitrary vertex w, choose $\alpha \in L(w)$, color w with α and let $L'(v) = L(v) - \alpha$ for every other vertex v. Now find a breadth-first spanning tree, T, rooted at w, and color greedily from L' with respect to σ. It is straightforward to check that the proof of Lemma 2 still holds in this more general setting. $\square$

Observation 6. *All proofs in this paper are constructive, and imply algorithms to find the colorings, and these algorithms can be easily implemented to run in polynomial time.*

Proof. We can easily check that a graph is regular, and compute a breadth-first spanning tree. To determine diameter, we use an algorithm for all-pairs shortest path. In general, computing the automorphisms of a graph can be difficult. However, we only need to do this for graphs on at most 14 vertices, so we can finish in constant time, simply considering every possible map from $V(G)$ to $V(G)$, and checking whether it is an automorphism. $\square$

Question 9. When G is connected with girth at least 5, is $\chi_D^\ell(G) \leqslant \Delta + 1$?

Two extensions of list chromatic number are online list chromatic number, denoted χ^{OL}, and correspondence chromatic number, denoted χ^{corr}; see [10] and [7] for the definitions. When we require also that our colorings be distinguishing, we get the parameters χ_D^{OL} and χ_D^{corr}. It is natural to study the maximum values of these parameters on certains classes of graphs. In particular, what are the maximum values on connected graphs with girth at least 5? Does the bound of Proposition 5, or even Question 9, still hold?

Acknowledgments

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