Crystals and Schur P-positive expansions

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Abstract

We give a new characterization of Littlewood–Richardson–Stembridge tableaux for Schur P-functions by using the theory of $\mathfrak{q}(n)$-crystals. We also give alternate proofs of the Schur P-expansion of a skew Schur function due to Ardila and Serrano, and the Schur expansion of a Schur P-function due to Stembridge using the associated crystal structures. Finally we introduce the notion of semistandard decomposition tableaux of a shifted skew shape and discuss its crystal structure.

Mathematics Subject Classifications: 17B37, 22E46, 05E10

1 Introduction

Let $\mathscr{P}^+$ be the set of strict partitions and let P_λ be the Schur P-function corresponding to $\lambda \in \mathscr{P}^+$ [12]. The set of Schur P-functions is an important class of symmetric functions, which is closely related with representation theory and algebraic geometry (see [10] and references therein). For example, the Schur P-polynomial $P_\lambda(x_1, \ldots, x_n)$ in n variables is the character of a finite-dimensional irreducible representation $V_n(\lambda)$ of the queer Lie superalgebra $\mathfrak{q}(n)$ with highest weight λ up to a power of 2 when the length $\ell(\lambda)$ of λ is no more than n [13].

The set of Schur P-functions forms a basis of a subring of the ring of symmetric functions, and the structure constants with respect to this basis are nonnegative integers, that is, given $\lambda, \mu, \nu \in \mathscr{P}^+$,

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda,$$

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for some nonnegative integers $f_{\mu\nu}^{\lambda}$. The first and the most well-known result on a combinatorial description of $f_{\mu\nu}^{\lambda}$ was obtained by Stembridge [16] using shifted Young tableaux, which is a combinatorial model for Schur P- or Q-functions [11, 17]. It is shown that $f_{\mu\nu}^{\lambda}$ is equal to the number of semistandard tableaux with entries in a $\mathbb{Z}_2$-graded set $\mathcal{N} = \{ 1' < 1 < 2' < 2 < \cdots \}$ of shifted skew shape λ/μ and weight ν such that (i) for each integer $k \geqslant 1$ the southwesternmost entry with value k is unprimed or of even degree and (ii) the reading words satisfy the *lattice property*. Here we say that the value $|x|$ is k when x is either k or k' in a tableau. Let us call these tableaux the *Littlewood–Richardson–Stembridge (LRS) tableaux* (Definitions 17 and 18).

Recently, two more descriptions of $f_{\mu\nu}^{\lambda}$ were obtained in terms of semistandard decomposition tableaux, which is another combinatorial model for Schur P-functions introduced by Serrano [14]. It is shown by Cho that $f_{\mu\nu}^{\lambda}$ is given by the number of semistandard decomposition tableaux of shifted shape μ and weight $w_0(\lambda-\nu)$ whose reading words satisfy the λ-*good property* (see [3, Corollary 5.14]). Here we assume that $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$, and w_0 denotes the longest element in the symmetric group $\mathfrak{S}_n$. Another description is given by Grantcharov, Jung, Kang, Kashiwara, and Kim [6] based on their crystal base theory for the quantized enveloping algebra of $\mathfrak{q}(n)$ [7]. They realize the crystal $\mathbf{B}_n(\lambda)$ associated to $V_n(\lambda)$ as the set of semistandard decomposition tableaux of shape λ with entries in $\{ 1 < 2 < \cdots < n \}$, and describe $f_{\mu\nu}^{\lambda}$ by characterizing the lowest weight vectors of weight $w_0\lambda$ in the tensor product $\mathbf{B}_n(\mu) \otimes \mathbf{B}_n(\nu)$. We also remark that bijections between the above mentioned combinatorial models for $f_{\mu\nu}^{\lambda}$ are studied in [4] using insertion schemes for semistandard decomposition tableaux.

The main result in this paper is to give another new description of $f_{\mu\nu}^{\lambda}$ using the theory of $\mathfrak{q}(n)$-crystals, and show that it is indeed equivalent to that of Stembridge. More precisely, we show that $f_{\mu\nu}^{\lambda}$ is equal to the number of semistandard tableaux with entries in $\mathcal{N}$ of shifted skew shape λ/μ and weight ν such that (i) for each integer $k \geqslant 1$ the southwesternmost entry with value k is unprimed or of even degree and (ii) the reading words satisfy the *hook lattice property* (see Definitions 12 and 13 and Theorem 14). It is obtained by semistandardizing the standard tableaux which parametrize the lowest weight vectors counting $f_{\mu\nu}^{\lambda}$ in [6], where the hook lattice property naturally arises from the configuration of entries in semistandard decomposition tableaux. We show that these tableaux for $f_{\mu\nu}^{\lambda}$ are equal to LRS tableaux (Theorem 20), and hence obtain a new characterization of LRS tableaux.

We study other Schur P- or Q-positive expansions and their combinatorial descriptions from a viewpoint of crystals. First we consider the Schur P-positive expansion of a skew Schur function

$$s_{\lambda/\delta_r} = \sum_{\nu \in \mathscr{P}^+} a_{\lambda/\delta_r\,\nu} P_\nu$$

for a skew diagram λ/δ_r contained in a rectangle $((r+1)^{r+1})$, where $\delta_r = (r, r-1, \ldots, 1)$ [1]. We give a combinatorial description of $a_{\lambda/\delta_r\,\nu}$ (Theorem 27) by considering a $\mathfrak{q}(n)$-crystal structure on the set of usual semistandard tableaux of shape λ/δ_r and characterizing the lowest weight vectors corresponding to each $\nu \in \mathscr{P}^+$. As a byproduct we also give a simple alternate proof of Ardila–Serrano's description of $a_{\lambda/\delta_r\,\nu}$ [1] (Theorem 31), which can be viewed as a standardization of our description.

We next consider the Schur expansion of a Schur P-function

$$P_\lambda = \sum_\mu g_{\lambda\mu} s_\mu$$

for $\lambda \in \mathscr{P}^+$. It is equivalent to the expansion of a symmetric function $S_\mu = S_\mu(x,x)$ in terms of Schur Q-functions $Q_\lambda = 2^{\ell(\lambda)}P_\lambda$, where $S_\mu(x,y)$ is a super Schur function in variables x and y. We give a simple and alternate proof of Stembridge's description of $g_{\lambda\mu}$ [16] (Theorem 33) by characterizing the type A lowest weight vectors of weight $w_0\mu$ in the $\mathfrak{q}(n)$-crystal $\mathbf{B}_n(\lambda)$ when $\ell(\lambda), \ell(\mu) \leqslant n$.

Finally, based on the characterization of semistandard decomposition tableaux in [6, Proposition 2.3], we introduce the notion of semistandard decomposition tableaux of a shifted skew shape λ/μ. The set of such tableaux, say $\mathbf{B}_n(\lambda/\mu)$, naturally admits a $\mathfrak{q}(n)$-crystal structure and we describe its decomposition into $\mathbf{B}_n(\nu)$'s generalizing the notion of hook lattice property. We remark that the character of $\mathbf{B}_n(\lambda/\mu)$ is not equal to the skew Schur P-function corresponding to λ/μ in general, and it would be interesting to have a more direct representation-theoretic interpretation of $\mathbf{B}_n(\lambda/\mu)$.

The paper is organized as follows. In Section 2, we review the notion of $\mathfrak{q}(n)$-crystals and related results. In Section 3, we describe a combinatorial description of $f^\lambda_{\mu\nu}$ and show that it is equivalent to that of Stembridge. In Sections 4 and 5, we discuss the Schur P-positive expansion of a skew Schur function and the Schur expansion of a Schur P-function, respectively. In Section 6, we discuss semistandard decomposition tableaux of shifted skew shape, and the Schur P-positive expansions of their characters.

2 Crystals for queer Lie superalgebras

2.1 Notation and terminology

In this subsection, we introduce necessary notation and terminologies. Let $\mathbb{Z}_+$ be the set of nonnegative integers. We fix a positive integer $n \geqslant 2$ throughout this paper.

Let $\mathscr{P} = \{\, \lambda = (\lambda_i)_{i\geqslant 1} \,|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_i \geqslant \lambda_{i+1}\, (i \geqslant 1),\ \sum_{i\geqslant 1}\lambda_i < \infty \,\}$ be the set of partitions, and let $\mathscr{P}^+ = \{\, \lambda = (\lambda_i)_{i\geqslant 1} \,|\, \lambda \in \mathscr{P},\ \lambda_i = \lambda_{i+1} \Rightarrow \lambda_i = 0\ (i \geqslant 1) \,\}$ be the set of strict partitions. For $\lambda \in \mathscr{P}$, let $\ell(\lambda)$ denote the length of λ, and $|\lambda| = \sum_{i\geqslant 1}\lambda_i$. Let $\mathscr{P}_n = \{\, \lambda \,|\, \ell(\lambda) \leqslant n \,\} \subseteq \mathscr{P}$ and $\mathscr{P}^+_n = \mathscr{P}^+ \cap \mathscr{P}_n$.

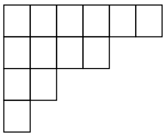
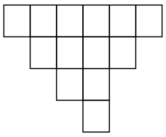
The (unshifted) diagram of $\lambda \in \mathscr{P}$ is defined to be the set

$$D_\lambda = \{\, (i,j) \in \mathbb{N}^2 \,:\, 1 \leqslant j \leqslant \lambda_i,\ 1 \leqslant i \leqslant \ell(\lambda) \,\},$$

and the shifted diagram of $\lambda \in \mathscr{P}^+$ is defined to be the set

$$D^+_\lambda = \{\, (i,j) \in \mathbb{N}^2 \,:\, i \leqslant j \leqslant \lambda_i + i - 1,\ 1 \leqslant i \leqslant \ell(\lambda) \,\}.$$

We identify D_λ and D^+_λ with diagrams where a box is placed at the i-th row from the top and the j-th column from the left for each $(i,j) \in D_\lambda$ and D^+_λ, respectively. For instance, if $\lambda = (6,4,2,1)$, then

$D_\lambda =$  and $D^+_\lambda =$  .

Let $\mathcal{A}$ be a linearly ordered set. We denote by $\mathcal{W}_\mathcal{A}$ the set of words of finite length with letters in $\mathcal{A}$. For $w \in \mathcal{W}_\mathcal{A}$ and $a \in \mathcal{A}$, let $c_a(w)$ be the number of occurrences of a in w.

For $\lambda, \mu \in \mathscr{P}$ with $D_\mu \subseteq D_\lambda$, a *tableau of shape* λ/μ means a filling on the skew diagram $D_\lambda \setminus D_\mu$ with entries in $\mathcal{A}$. For $\lambda, \mu \in \mathscr{P}^+$ with $D^+_\mu \subseteq D^+_\lambda$, a *tableau of shifted shape* λ/μ is defined in a similar way. For a tableau T of (shifted) shape λ/μ, let $w(T)$ be the word given by reading the entries of T row by row from top to bottom, and from right to left in each row. We denote by $T_{i,j}$ the j-th entry (from the left) of the i-th row of T from the top. For $1 \leqslant i \leqslant \ell(\lambda)$, let $T^{(i)} = T_{i,\lambda_i} \cdots T_{i,1}$ be the subword of $w(T)$ corresponding to the i-th row of T. Then we have $w(T) = T^{(1)} \cdots T^{(\ell(\lambda))}$. We also denote by $w_{\mathrm{rev}}(T)$ the word obtained by reading the entries of $w(T)$ from right to left. Note that $T_{i,j}$ is not the entry of T at the (i,j)-position of the (shifted) skew diagram of λ/μ, that is, $(i,j) \in D_\lambda \setminus D_\mu$ or $(i,j) \in D^+_\lambda \setminus D^+_\mu$. For $a \in \mathcal{A}$, let $c_a(T) = c_a(w(T))$ be the number of occurrences of a in T.

Suppose that $\mathcal{A}$ is a linearly ordered set with a $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. For $\lambda, \mu \in \mathscr{P}$ with $D_\mu \subseteq D_\lambda$, let $SST_\mathcal{A}(\lambda/\mu)$ be the set of tableaux of shape λ/μ with entries in $\mathcal{A}$ which is semistandard, that is, (i) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (ii) the entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row). Similarly, for $\lambda, \mu \in \mathscr{P}^+$ with $D^+_\mu \subseteq D^+_\lambda$, we define $SST^+_\mathcal{A}(\lambda/\mu)$ to be the set of semistandard tableaux of shifted shape λ/μ with entries in $\mathcal{A}$.

Let $\mathcal{N} = \{\, 1' < 1 < 2' < 2 < \cdots \,\}$ be a linearly ordered set with a $\mathbb{Z}_2$-grading $\mathcal{N}_0 = \mathbb{N}$ and $\mathcal{N}_1 = \mathbb{N}' = \{1', 2', \cdots\}$. Put $[n] = \{\, 1, \ldots, n \,\}$ and $[n]' = \{\, 1', \ldots, n' \,\}$, where the $\mathbb{Z}_2$-grading and linear ordering are induced from $\mathcal{N}$. For $a \in \mathcal{N}$, we write $|a| = k$ when a is either k or k'.

2.2 Semistandard decomposition tableaux and Schur P-functions

Let us recall the notion of semistandard decomposition tableaux [6, 14], which is our main combinatorial object.

Definition 1.

(1) A word $u = u_1 \cdots u_s$ in $\mathcal{W}_\mathbb{N}$ is called a *hook word* if it satisfies $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_k < u_{k+1} < \cdots < u_s$ for some $1 \leqslant k \leqslant s$. In this case, let $u{\downarrow} = u_1 \cdots u_k$ be the weakly decreasing subword of maximal length and $u{\uparrow} = u_{k+1} \cdots u_s$ the remaining strictly increasing subword in u.

(2) For $\lambda \in \mathscr{P}^+$, let T be a tableau of shifted shape λ with entries in $\mathbb{N}$. Then T is called a *semistandard decomposition tableau* of shape λ if

(i) $T^{(i)}$ is a hook word of length λ_i for $1 \leqslant i \leqslant \ell(\lambda)$,

(ii) $T^{(i)}$ is a hook subword of maximal length in $T^{(i+1)}T^{(i)}$, the concatenation of $T^{(i+1)}$ and $T^{(i)}$, for $1 \leqslant i < \ell(\lambda)$.

For any hook word u, the decreasing part $u{\downarrow}$ is always nonempty by definition.

For $\lambda \in \mathscr{P}^+$, let $SSDT(\lambda)$ be the set of semistandard decomposition tableaux of shape λ. Let $x = \{x_1, x_2, \ldots\}$ be a set of formal commuting variables, and let $P_\lambda = P_\lambda(x)$ be the Schur P-function in x corresponding to $\lambda \in \mathscr{P}^+$ (see [10]). It is shown in [14] that P_λ is given by the weight generating function of $SSDT(\lambda)$:

$$P_\lambda = \sum_{T \in SSDT(\lambda)} x^T, \tag{1}$$

where $x^T = \prod_{i \geqslant 1} x_i^{c_i(T)}$.

Remark 2. Recall that the Schur P-function P_λ can be realized as the character of tableaux $T \in SST^+_{\mathcal{N}}(\lambda)$ with no primed entry or entry of odd degree on the main diagonal (cf. [10, 11, 17]). The notion of semistandard decomposition tableaux was introduced in [14] to give a plactic monoid model for Schur P-functions. In this paper, we follow its modified version (Definition 1) introduced in [6], by which it is easier to describe $\mathfrak{q}(n)$-crystals [6, Remark 2.6]. We also refer the reader to [4] for more details on relation between the combinatorics of these two models.

The following is a useful criterion for a tableau to be a semistandard decomposition one, which plays an important role in this paper.

Proposition 3. ([6, Proposition 2.3]) *For $\lambda \in \mathscr{P}^+$, let T be a tableau of shifted shape λ with entries in $\mathbb{N}$. Then $T \in SSDT(\lambda)$ if and only if $T^{(k)}$ is a hook word for $1 \leqslant k \leqslant \ell(\lambda)$, and none of the following conditions holds for each $1 \leqslant k < \ell(\lambda)$:*

(1) *$T_{k,1} \leqslant T_{k+1,i}$ for some $1 \leqslant i \leqslant \lambda_{k+1}$,*

(2) *$T_{k+1,i} \geqslant T_{k+1,j} \geqslant T_{k,i+1}$ for some $1 \leqslant i < j \leqslant \lambda_{k+1}$,*

(3) *$T_{k+1,j} < T_{k,i} < T_{k,j+1}$ for some $1 \leqslant i \leqslant j \leqslant \lambda_{k+1}$.*

Equivalently, $T \in SSDT(\lambda)$ if and only if $T^{(k)}$ is a hook word for $1 \leqslant k \leqslant \ell(\lambda)$, and the following conditions hold for $1 \leqslant k < \ell(\lambda)$:

(a) *if $T_{k,i} \leqslant T_{k+1,j}$ for $1 \leqslant i \leqslant j \leqslant \lambda_{k+1}$, then $i \neq 1$ and $T_{k+1,i-1} < T_{k+1,j}$,*

(b) *if $T_{k,i} > T_{k+1,j}$ for $1 \leqslant i \leqslant j \leqslant \lambda_{k+1}$, then $T_{k,i} \geqslant T_{k,j+1}$.*

For $\lambda \in \mathscr{P}^+$, let $SSDT_n(\lambda)$ be the set of tableaux $T \in SSDT(\lambda)$ with entries in $[n]$. By Proposition 3(1), we see that $SSDT_n(\lambda) \neq \varnothing$ if and only if $\lambda \in \mathscr{P}^+_n$. We denote by $P_\lambda(x_1, \ldots, x_n)$ the Schur P-polynomial in $x_1, \ldots, x_n$ given by specializing P_λ at $x_{n+1} = x_{n+2} = \cdots = 0$. Then we have $P_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSDT_n(\lambda)} x^T$.

For $\lambda \in \mathscr{P}_n^+$, let H_n^λ be the element in $SSDT_n(\lambda)$ where the subtableau with entry $\ell(\lambda) - i + 1$ is a connected border strip of size $\lambda_{\ell(\lambda)-i+1}$ starting at $(i,i) \in D_\lambda^+$ for each $i = 1, \dots, \ell(\lambda)$, and let L_n^λ be the one where the subtableau with entry $n - i + 1$ is a connected horizontal strip of size λ_i starting at $(i,i) \in D_\lambda^+$ for each $i = 1, \dots, \ell(\lambda)$. For example, when $n = 4$ and $\lambda = (4,3,1)$, we have

$$H_n^\lambda = \begin{array}{cccc} \boxed{3} & \boxed{2} & \boxed{2} & \boxed{1} \\ & \boxed{2} & \boxed{1} & \boxed{1} \\ & & \boxed{1} & \end{array} \qquad\qquad L_n^\lambda = \begin{array}{cccc} \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \\ & \boxed{3} & \boxed{3} & \boxed{3} \\ & & \boxed{2} & \end{array} .$$

Indeed, H_n^λ and L_n^λ are the unique tableaux in $SSDT_n(\lambda)$ such that

$$(c_1(H_n^\lambda), \dots, c_n(H_n^\lambda)) = \lambda, \quad (c_1(L_n^\lambda), \dots, c_n(L_n^\lambda)) = w_0\lambda.$$

Here we assume that $\mathscr{P}_n^+ \subset \mathbb{Z}_+^n$ and the symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Z}_+^n$ by permutation, where w_0 is the longest element in $\mathfrak{S}_n$.

2.3 Crystals

Let us first review the crystals for the general linear Lie algebra $\mathfrak{gl}(n)$ in [8, 9].

Let $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}e_i$ be the dual weight lattice and $P = \mathrm{Hom}_\mathbb{Z}(P^\vee, \mathbb{Z}) = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ the weight lattice with $\langle \epsilon_i, e_j \rangle = \delta_{ij}$ for $1 \leqslant i,j \leqslant n$. Define a symmetric bilinear form $(\,\cdot\,|\,\cdot\,)$ on P by $(\epsilon_i|\epsilon_j) = \delta_{ij}$ for $1 \leqslant i,j \leqslant n$. Let $\{\, \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i = 1, \dots, n-1) \,\}$ be the set of simple roots, and $\{\, h_i = e_i - e_{i+1}\ (i = 1, \dots, n-1) \,\}$ the set of simple coroots of $\mathfrak{gl}(n)$. Let $P^+ = \{\, \lambda \,|\, \lambda \in P,\ \langle \lambda, h_i \rangle \geqslant 0\ (i = 1, \dots, n-1) \,\}$ be the set of dominant integral weights.

A $\mathfrak{gl}(n)$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ for $i = 1, \dots, n-1$ satisfying the following conditions: for $b \in B$ and $i = 1, \dots, n-1$,

(1) $\varphi_i(b) = \langle \mathrm{wt}(b), h_i \rangle + \varepsilon_i(b)$,

(2) $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$, $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\widetilde{e}_i b \in B$,

(3) $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$, $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\widetilde{f}_i b \in B$,

(4) $\widetilde{f}_i b = b'$ if and only if $b = \widetilde{e}_i b'$ for $b' \in B$,

(5) $\widetilde{e}_i b = \widetilde{f}_i b = \mathbf{0}$ when $\varphi_i(b) = -\infty$.

Here $\mathbf{0}$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. For $\mu \in P$, let $B_\mu = \{\, b \in B \,|\, \mathrm{wt}(b) = \mu \,\}$. When B_μ is finite for all μ, we define the character of B by $\mathrm{ch}B = \sum_{\mu \in P} |B_\mu| e^\mu$, where e^μ is a basis element of the group algebra $\mathbb{Q}[P]$.

Let B_1 and B_2 be $\mathfrak{gl}(n)$-crystals. A tensor product $B_1 \otimes B_2$ is a $\mathfrak{gl}(n)$-crystal, which is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle\}, \\
\varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}, \\
\widetilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \widetilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
\widetilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \widetilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2), \end{cases}
\end{aligned} \tag{2}$$

for $i = 1, \ldots, n-1$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$.

For $\lambda \in \mathscr{P}_n$, let $B_n(\lambda)$ be the crystal associated to an irreducible $\mathfrak{gl}(n)$-module with highest weight λ, where we regard λ as $\sum_{i=1}^n \lambda_i \epsilon_i \in P^+$. We may regard $[n]$ as the set of vertices in $B_n(\epsilon_1)$, where $\mathrm{wt}(k) = \epsilon_k$ for $k \in [n]$, and hence $\mathcal{W}_{[n]}$ as a $\mathfrak{gl}(n)$-crystal where we identify $w = w_1 \ldots w_r$ with $w_1 \otimes \cdots \otimes w_r \in B_n(\epsilon_1)^{\otimes r}$. The crystal structure on $\mathcal{W}_{[n]}$ is easily described by the so-called signature rule (cf. [9, Section 2.1]). For $\lambda \in \mathscr{P}_n$, the set $SST_{[n]}(\lambda)$ becomes a $\mathfrak{gl}(n)$-crystal under the identification of T with $w(T) \in \mathcal{W}_{[n]}$, and it is isomorphic to $B_n(\lambda)$ [9]. In general, one can define a $\mathfrak{gl}(n)$-crystal structure on $SST_{[n]}(\lambda/\mu)$ for a skew diagram λ/μ. By abuse of notation, we set $B_n(\lambda/\mu) := SST_{[n]}(\lambda/\mu)$.

Next, let us review the notion of crystals associated to polynomial representations of the queer Lie superalgebra $\mathfrak{q}(n)$ developed in [6, 7].

Definition 4. A $\mathfrak{q}(n)$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ for $i \in I := \{\, 1, \ldots, n-1, \overline{1} \,\}$ satisfying the following conditions:

(1) B is a $\mathfrak{gl}(n)$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i, \widetilde{f}_i$ for $i = 1, \ldots, n-1$,

(2) $\mathrm{wt}(b) \in \bigoplus_{i \in [n]} \mathbb{Z}_+ \epsilon_i$ for $b \in B$,

(3) $\mathrm{wt}(\widetilde{e}_{\overline{1}} b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\widetilde{f}_{\overline{1}} b) = \mathrm{wt}(b) - \alpha_1$ for $b \in B$,

(4) $\widetilde{f}_{\overline{1}} b = b'$ if and only if $b = \widetilde{e}_{\overline{1}} b'$ for all $b, b' \in B$,

(5) for $3 \leqslant i \leqslant n-1$, we have

  (i) the operators $\widetilde{e}_{\overline{1}}$ and $\widetilde{f}_{\overline{1}}$ commute with $\widetilde{e}_i$, $\widetilde{f}_i$,

  (ii) if $\widetilde{e}_{\overline{1}} b \in B$, then $\varepsilon_i(\widetilde{e}_{\overline{1}} b) = \varepsilon_i(b)$ and $\varphi_i(\widetilde{e}_{\overline{1}} b) = \varphi_i(b)$.

Let $\mathbf{B}_n$ be a $\mathfrak{q}(n)$-crystal which is the $\mathfrak{gl}(n)$-crystal $B_n(\epsilon_1)$ together with $\widetilde{f}_{\overline{1}}\ \boxed{1} = \boxed{2}$ (in dashed arrow):

$$\begin{tikzcd} \boxed{1} \arrow[r, shift left, "1"] \arrow[r, shift right, dashed, "\overline{1}"'] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n} \end{tikzcd}.$$

Here we write $b \xrightarrow{i} b'$ if $\widetilde{f}_i b = b'$ for $b, b' \in B$ and $i \in I \setminus \{\overline{1}\}$ as usual, and $b \overset{\overline{1}}{\dashrightarrow} b'$ if $\widetilde{f}_{\overline{1}} b = b'$.

For $\mathfrak{q}(n)$-crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ is the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ where the actions of $\widetilde{e}_{\overline{1}}$ and $\widetilde{f}_{\overline{1}}$ are given by

$$\begin{aligned} \widetilde{e}_{\overline{1}}(b_1 \otimes b_2) &= \begin{cases} \widetilde{e}_{\overline{1}} b_1 \otimes b_2, & \text{if } \langle \epsilon_1, \mathrm{wt}(b_2) \rangle = \langle \epsilon_2, \mathrm{wt}(b_2) \rangle = 0, \\ b_1 \otimes \widetilde{e}_{\overline{1}} b_2, & \text{otherwise,} \end{cases} \\ \widetilde{f}_{\overline{1}}(b_1 \otimes b_2) &= \begin{cases} \widetilde{f}_{\overline{1}} b_1 \otimes b_2, & \text{if } \langle \epsilon_1, \mathrm{wt}(b_2) \rangle = \langle \epsilon_2, \mathrm{wt}(b_2) \rangle = 0, \\ b_1 \otimes \widetilde{f}_{\overline{1}} b_2, & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

Then it is easy to see that $B_1 \otimes B_2$ is a $\mathfrak{q}(n)$-crystal. In particular, $\mathcal{W}_{[n]}$ is also a $\mathfrak{q}(n)$-crystal.

Let B be a $\mathfrak{q}(n)$-crystal. Suppose that B is a regular $\mathfrak{gl}(n)$-crystal, that is, each connected component in B is isomorphic to $B_n(\lambda)$ for some $\lambda \in \mathscr{P}_n$. Let $W = \mathfrak{S}_n$ be the Weyl group of $\mathfrak{gl}(n)$ which is generated by the simple reflection r_i corresponding to α_i for $i = 1, \ldots, n-1$. We have a group action of W on B denoted by S such that

$$S_{r_i}(b) = \begin{cases} \widetilde{f}_i^{\langle \mathrm{wt}(b), h_i \rangle} b, & \text{if } \langle \mathrm{wt}(b), h_i \rangle \geqslant 0, \\ \widetilde{e}_i^{-\langle \mathrm{wt}(b), h_i \rangle} b, & \text{if } \langle \mathrm{wt}(b), h_i \rangle \leqslant 0, \end{cases}$$

for $b \in B$ and $i = 1, \ldots, n-1$. For $2 \leqslant i \leqslant n-1$, let $w_i \in W$ be such that $w_i(\alpha_i) = \alpha_1$, and let

$$\widetilde{e}_{\overline{i}} = S_{w_i^{-1}} \widetilde{e}_{\overline{1}} S_{w_i}, \quad \widetilde{f}_{\overline{i}} = S_{w_i^{-1}} \widetilde{f}_{\overline{1}} S_{w_i}. \tag{4}$$

For $b \in B$, we say that b is a $\mathfrak{q}(n)$*-highest weight vector* if $\widetilde{e}_i b = \widetilde{e}_{\overline{i}} b = \mathbf{0}$ for $1 \leqslant i \leqslant n-1$, and b is a $\mathfrak{q}(n)$*-lowest weight vector* if $S_{w_0} b$ is a $\mathfrak{q}(n)$-highest weight vector.

For $\lambda \in \mathscr{P}^+$, let $\mathbf{B}_n(\lambda) = SSDT_n(\lambda)$, and consider an injective map

$$\begin{tikzcd} \mathbf{B}_n(\lambda) \arrow[r, hook] & \mathcal{W}_{[n]} \\ T \arrow[r, mapsto] & w_{\mathrm{rev}}(T). \end{tikzcd} \tag{5}$$

Then we have the following.

Theorem 5. ([6, Theorem 2.5]) *Let $\lambda \in \mathscr{P}_n^+$ be given.*

(a) *The image of $\mathbf{B}_n(\lambda)$ in (5) together with $\{\mathbf{0}\}$ is invariant under the action of $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$, and hence $\mathbf{B}_n(\lambda)$ is a $\mathfrak{q}(n)$-crystal.*

(b) *The $\mathfrak{q}(n)$-crystal $\mathbf{B}_n(\lambda)$ is connected where H_n^λ is a unique $\mathfrak{q}(n)$-highest weight vector and L_n^λ is a unique $\mathfrak{q}(n)$-lowest weight vector.*

Remark 6. In [7], a semisimple tensor category over the quantum superalgebra $U_q(\mathfrak{q}(n))$ is introduced, and it is shown that each irreducible highest weight module $V_n(\lambda)$ in this category, parametrized by $\lambda \in \mathscr{P}_n^+$, has a crystal base. Furthermore, it is shown in [6, Theorem 2.5(c)] that the crystal of $V_n(\lambda)$ is isomorphic to $\mathbf{B}_n(\lambda)$.

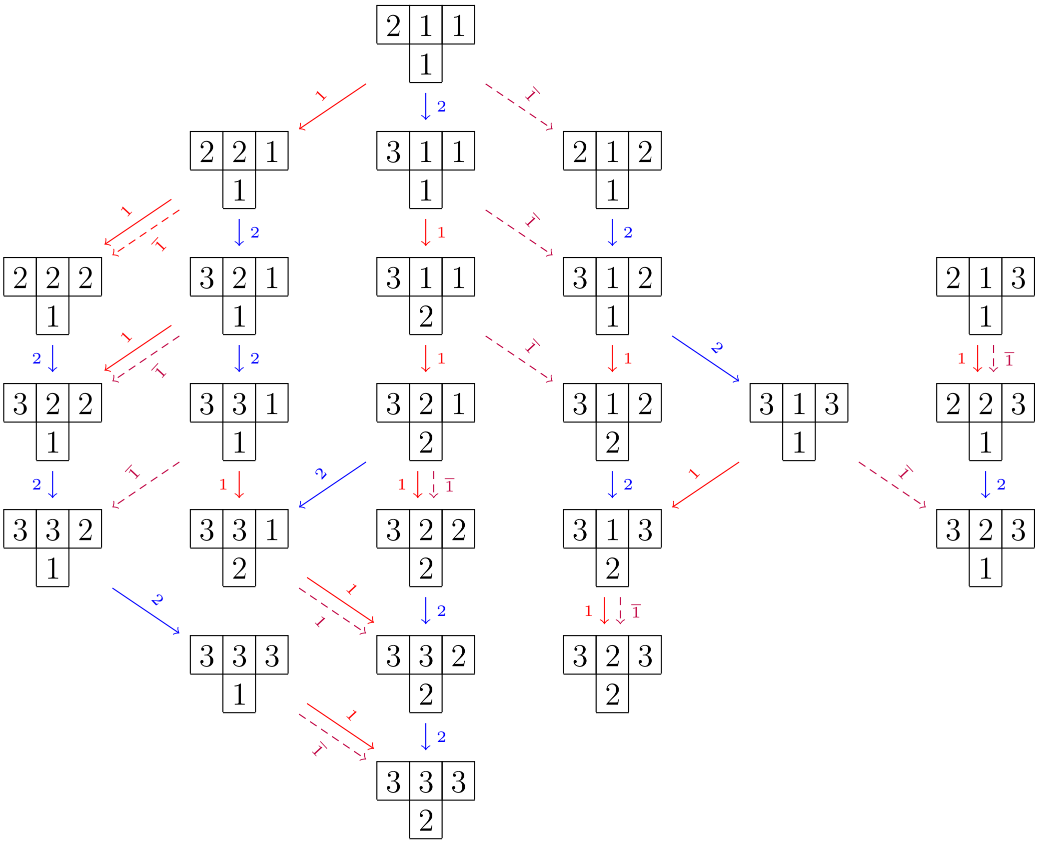


Figure 1: The $\mathfrak{q}(3)$-crystal $\mathbf{B}_3(3,1)$

Let B_1 and B_2 be $\mathfrak{q}(n)$-crystals. For $b_1 \in B_1$ and $b_2 \in B_2$, let us say that b_1 and b_2 are equivalent and write $b_1 \equiv b_2$ if there exists an isomorphism of $\mathfrak{q}(n)$-crystals $\psi : C(b_1) \longrightarrow C(b_2)$ such that $\psi(b_1) = b_2$ where $C(b_i)$ denotes the connected component of $b_i \in B_i$ $(i = 1, 2)$ as a $\mathfrak{q}(n)$-crystal.

By [7, Theorem 4.6], each connected component in $\mathbf{B}_n^{\otimes N}$ $(N \geqslant 1)$ is isomorphic to $\mathbf{B}_n(\lambda)$ for some $\lambda \in \mathscr{P}_n^+$ with $|\lambda| = N$. Indeed, for $b = b_1 \otimes \cdots \otimes b_N \in \mathbf{B}_n^{\otimes N}$, there exists a unique $\lambda \in \mathscr{P}_n^+$ and $T \in \mathbf{B}_n(\lambda)$ such that $b \equiv T$. In particular, b is a $\mathfrak{q}(n)$-lowest (resp. $\mathfrak{q}(n)$-highest) weight vector if and only if $b \equiv L_n^\lambda$ (resp. H_n^λ).

The following lemma plays a crucial role in characterization of $\mathfrak{q}(n)$-lowest weight vectors in $\mathbf{B}_n^{\otimes N}$ and hence describing the decompositions of $\mathbf{B}_n^{\otimes N}$ and $\mathbf{B}_n(\mu) \otimes \mathbf{B}_n(\nu)$ $(\mu, \nu \in \mathscr{P}_n^+)$ into connected components in [6].

Lemma 7. ([6, Lemma 1.15, Corollary 1.16]) *For $b = b_1 \otimes \cdots \otimes b_N \in \mathbf{B}_n^{\otimes N}$, the following are equivalent:*

(1) *b is a $\mathfrak{q}(n)$-lowest weight vector,*

(2) *$b' = b_2 \otimes \cdots \otimes b_N$ is a $\mathfrak{q}(n)$-lowest weight vector and $\epsilon_{b_1} + \mathrm{wt}(b') \in w_0\mathscr{P}_n^+$,*

(3) $\mathrm{wt}(b_M \otimes \cdots \otimes b_N) \in w_0 \mathscr{P}_n^+$ *for all* $1 \leqslant M \leqslant N$.

Hence, we have the following immediately by Lemma 7.

Corollary 8. *For* $\lambda^{(1)}, \ldots, \lambda^{(s)} \in \mathscr{P}_n^+$ *and* $T_1 \otimes \cdots \otimes T_s \in \mathbf{B}_n(\lambda^{(1)}) \otimes \cdots \otimes \mathbf{B}_n(\lambda^{(s)})$*, the following are equivalent:*

(1) $T_1 \otimes \cdots \otimes T_s$ *is a* $\mathfrak{q}(n)$*-lowest weight vector,*

(2) $T_r \otimes \cdots \otimes T_s \in \mathbf{B}_n(\lambda^{(s)}) \otimes \cdots \otimes \mathbf{B}_n(\lambda^{(r)})$ *is a* $\mathfrak{q}(n)$*-lowest weight vector for all* $1 \leqslant r \leqslant s$.

Note that we do not have an analogue of Lemma 7 for $\mathfrak{q}(n)$-highest weight vectors.

Remark 9. Let $m \geqslant n$ be a positive integer, and put $t = m - n$. For $N \geqslant 1$, let $\psi_t : \mathbf{B}_n^{\otimes N} \longrightarrow \mathbf{B}_m^{\otimes N}$ be the map given by $\psi_t(u_1 \otimes \cdots \otimes u_N) = (u_1 + t) \otimes \cdots \otimes (u_N + t)$. Then for $\lambda \in \mathscr{P}_n^+$ and $u \in \mathbf{B}_n^{\otimes N}$ we have $u \equiv L_n^\lambda$ if and only if $\psi_t(u) \equiv L_m^\lambda$. This implies that the multiplicity of $\mathbf{B}_n(\lambda)$ in $\mathbf{B}_n^{\otimes N}$ is equal to that of $\mathbf{B}_m(\lambda)$ in $\mathbf{B}_m^{\otimes N}$ for $\lambda \in \mathscr{P}_n^+$.

3 Littlewood–Richardson rule for Schur *P*-functions

For $\lambda, \mu, \nu \in \mathscr{P}^+$, the *shifted Littlewood–Richardson coefficients* $f_{\mu\nu}^\lambda$ are the coefficients given by

$$P_\mu P_\nu = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda. \tag{6}$$

In this section we give a new combinatorial description of $f_{\mu\nu}^\lambda$ using the theory of $\mathfrak{q}(n)$-crystals. We also show that our description of $f_{\mu\nu}^\lambda$ is equivalent to Stembridge's description [16].

3.1 Shifted Littlewood–Richardson rule

Definition 10. Let $w = w_1 \cdots w_N$ be a word in $\mathcal{W}_{\mathbf{N}}$. Let $m_k = c_k(w) + c_{k'}(w)$ for $k \geqslant 1$. We define $w^* = w_1^* \cdots w_N^*$ to be the word obtained from w after applying the following steps for each $k \geqslant 1$:

(1) Consider the letters w_i's with $|w_i| = k$. Label them with $1, 2, \ldots, m_k$ (as subscripts), first enumerating the w_p's with $w_p = k$ from left to right, and then the w_q's with $w_q = k'$ from right to left.

(2) After the step (1), remove all $'$ in each labeled letter k'_j, that is, replace any k'_j by k_j for $c_k(w) < j \leqslant m_k$.

Example 11.

$$w = 11'11'1 \quad \longrightarrow \quad 1_1 1'_5 1_2 1'_4 1_3 \quad \longrightarrow \quad w^* = 1_1 1_5 1_2 1_4 1_3$$

$$w = 21'12'2'121 \quad \longrightarrow \quad 2_1 1'_4 1_1 2'_4 2'_3 1_2 2_2 1_3 \quad \longrightarrow \quad w^* = 2_1 1_4 1_1 2_4 2_3 1_2 2_2 1_3$$

Definition 12. Let $w = w_1 \cdots w_N \in \mathcal{W}_{\mathbb{N}}$ be given. We say that w satisfies the *hook lattice property* if the word $w^* = w_1^* \cdots w_N^*$ associated to w given in Definition 10 satisfies the following for $k \geqslant 1$:

(L1) if $w_i^* = k_1$, then no $k+1_j$ for $j \geqslant 1$ occurs in $w_1^* \cdots w_{i-1}^*$,

(L2) if $(w_s^*, w_t^*) = (k+1_i, k_{i+1})$ for some $s < t$ and $i \geqslant 1$, then no $k+1_j$ for $i < j$ occurs in $w_s^* \cdots w_t^*$,

(L3) if $(w_s^*, w_t^*) = (k_{j+1}, k+1_j)$ for some $s < t$ and $j \geqslant 1$, then no k_i for $i \leqslant j$ occurs in $w_s^* \cdots w_t^*$.

Definition 13. For $\lambda, \mu, \nu \in \mathscr{P}^+$, let $\mathtt{F}^{\lambda}_{\mu\nu}$ be the set of tableaux Q such that

(1) $Q \in SST^+_{\mathbb{N}}(\lambda/\mu)$ with $c_k(Q) + c_{k'}(Q) = \nu_k$ for $k \geqslant 1$,

(2) for $k \geqslant 1$, if x is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,

(3) $w(Q)$ satisfies the hook lattice property in Definition 12.

Then we have the following characterization of $f^{\lambda}_{\mu\nu}$.

Theorem 14. *For $\lambda, \mu, \nu \in \mathscr{P}^+$, we have*

$$f^{\lambda}_{\mu\nu} = \left|\mathtt{F}^{\lambda}_{\mu\nu}\right|,$$

that is, the shifted LR coefficient $f^{\lambda}_{\mu\nu}$ is equal to the number of tableaux in $\mathtt{F}^{\lambda}_{\mu\nu}$.

Proof. Choose n such that $\lambda, \mu, \nu \in \mathscr{P}^+_n$. Put

$$\mathtt{L}^{\lambda}_{\mu\nu} = \{\, T \mid T \in \mathbf{B}_n(\nu),\ T \otimes L^{\mu}_n \equiv L^{\lambda}_n \,\}. \tag{7}$$

By Corollary 8, we have

$$\mathbf{B}_n(\nu) \otimes \mathbf{B}_n(\mu) \cong \bigsqcup_{\lambda \in \mathscr{P}^+_n} \mathbf{B}_n(\lambda)^{\oplus |\mathtt{L}^{\lambda}_{\mu\nu}|}. \tag{8}$$

Hence we have $|\mathtt{L}^{\lambda}_{\mu\nu}| = f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$ from (1) and the linear independence of Schur P-polynomials $P_\lambda(x_1, \ldots, x_n)$'s.

Let us prove $f^{\lambda}_{\mu\nu} = |\mathtt{F}^{\lambda}_{\mu\nu}|$ by constructing a bijection

$$\begin{array}{ccc} \mathtt{L}^{\lambda}_{\mu\nu} & \longrightarrow & \mathtt{F}^{\lambda}_{\mu\nu} \\ T & \longmapsto & Q_T. \end{array} \tag{9}$$

Let $T \in \mathtt{L}^{\lambda}_{\mu\nu}$ be given. Assume that $w_{\mathrm{rev}}(T) = u_1 \cdots u_N$ where $N = |\nu|$. By Lemma 7, there exists $\mu^{(m)} \in \mathscr{P}^+_n$ for $1 \leqslant m \leqslant N$ such that

(i) $(u_{N-m+1} \cdots u_N) \otimes L^{\mu}_n \equiv L^{\mu^{(m)}}_n$ and $\mu^{(N)} = \lambda$,

(ii) $\mu^{(m)}$ is obtained by adding a box in the $(n - u_m + 1)$-st row of $\mu^{(m-1)}$.

Here we assume that $\mu^{(0)} = \mu$. Recall that

$$w_{\mathrm{rev}}(T) = T^{(\ell(\nu))} \cdots T^{(1)},$$

where $T^{(k)} = T_{k,1} \cdots T_{k,\lambda_k}$ is a hook word for $1 \leqslant k \leqslant \ell(\nu)$. We define Q_T to be a tableau of shifted shape λ/μ with entries in $\mathcal{N}$, where $\mu^{(m)}/\mu^{(m-1)}$ is filled with

$$\begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)}\uparrow, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)}\downarrow, \end{cases} \tag{10}$$

for some $1 \leqslant k \leqslant \ell(\nu)$. In other words, the boxes in Q_T corresponding to $T^{(k)}\uparrow$ are filled with k' from right to left as a vertical strip and then those corresponding to $T^{(k)}\downarrow$ are filled with k from left to right as a horizontal strip.

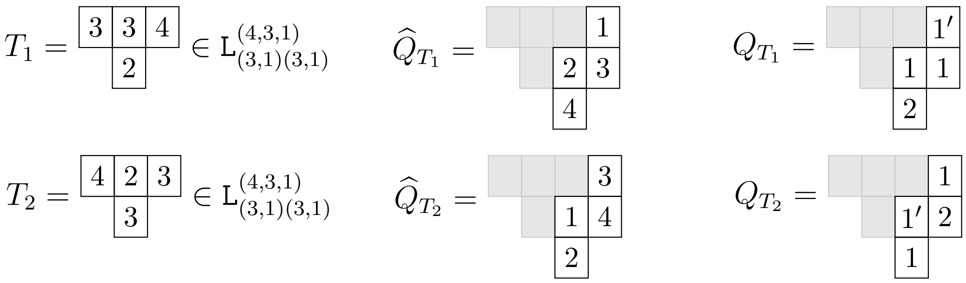
By construction, it is clear that $Q_T \in SST^+_{\mathcal{N}}(\lambda/\mu)$ with $c_{k'}(Q_T) + c_k(Q_T) = \nu_k$ for $1 \leqslant k \leqslant \ell(\nu)$. Let $w(Q_T) = w_1 \cdots w_N$. Since $T^{(k)}$ is a hook word for each k and the rightmost letter, say u_m, in $T^{(k)}\downarrow$ is strictly smaller than the leftmost letter u_{m+1} in $T^{(k)}\uparrow$, the entry k in Q_T corresponding to u_m is located to the southeast of all k''s in Q_T. So the conditions Definition 13(1) and (2) are satisfied.

It remains to check that $w(Q_T)$ satisfies the hook lattice property. Note that if we label k and k' in (10) as k_j and k'_j, respectively when $u_m = T_{k,j}$, then it coincides with the labeling on the letters in $w(Q_T)$ given in Definition 10(1). Now it is not difficult to see that the conditions Proposition 3(1), (2), and (3) on T implies the conditions Definition 12 (L1), (L2), and (L3), respectively. Therefore, $Q_T \in \mathtt{F}^{\lambda}_{\mu\nu}$.

If $T, T' \in \mathtt{L}^{\lambda}_{\mu\nu}$ with $T^{(i)} \neq T'^{(i)}$ for some $i \geqslant 1$, then it follows from (10) that $Q_T \neq Q_{T'}$, so the correspondence $T \mapsto Q_T$ is injective. Moreover, this correspondence is reversible, and hence the map (9) is a bijection. This completes the proof. $\square$

Remark 15. We see from Remark 9 that $|\mathtt{L}^{\lambda}_{\mu\nu}|$ does not depend on n for all sufficiently large n. Hence (8) also implies the Schur P-positivity of the product $P_\mu P_\nu$.

Remark 16. For $T \in \mathtt{L}^{\lambda}_{\mu\nu}$, let $\widehat{Q}_T$ be the tableau of shifted shape λ/μ, which is defined in the same way as Q_T in the proof of Theorem 14 except that we fill $\mu^{(m)}/\mu^{(m-1)}$ with m in (10) for $1 \leqslant m \leqslant N$. Then the set $\{\, \widehat{Q}_T \,|\, T \in \mathtt{L}^{\lambda}_{\mu\nu} \,\}$ is equal to the one given in [6, Theorem 4.13] to describe $f^{\lambda}_{\mu\nu}$. For example,



3.2 Stembridge's description of $f^{\lambda}_{\mu\nu}$

Definition 17. Let $w = w_1 \cdots w_N$ be a word in $\mathcal{W}_{\mathbb{N}}$ and w_{rev} be the reverse word of w. Let $\widehat{w}$ be the word obtained from w by replacing k by $(k+1)'$ and k' by k for each $k \geqslant 1$. Suppose that $w\widehat{w}_{\text{rev}} = a_1 \cdots a_{2N}$, and let $m_k(i) = c_k(a_1 \cdots a_i)$ for $k \geqslant 1$ and $0 \leqslant i \leqslant 2N$. Then we say that w satisfies the *lattice property* if

$$m_{k+1}(i) = m_k(i) \text{ implies } |a_{i+1}| \neq k+1 \text{ for } k \geqslant 1 \text{ and } i \geqslant 0. \tag{11}$$

Here we assume that $m_k(0) = 0$.

Definition 18. For $\lambda, \mu, \nu \in \mathscr{P}^+$, let $\texttt{LRS}^{\lambda}_{\mu\nu}$ be the set of tableaux Q such that

(1) $Q \in SST^+_{\mathbb{N}}(\lambda/\mu)$ with $c_k(Q) + c_{k'}(Q) = \nu_k$ for $k \geqslant 1$,

(2) for $k \geqslant 1$, if x is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,

(3) $w(Q)$ satisfies the lattice property in Definition 17.

We call $\texttt{LRS}^{\lambda}_{\mu\nu}$ the set of *Littlewood–Richardson–Stembridge tableaux.*

Theorem 19. ([16, Theorem 8.3]) *For $\lambda, \mu, \nu \in \mathscr{P}^+$, we have*

$$f^{\lambda}_{\mu\nu} = \left|\texttt{LRS}^{\lambda}_{\mu\nu}\right|,$$

that is, the shifted LR coefficient $f^{\lambda}_{\mu\nu}$ is equal to the number of tableaux in $\texttt{LRS}^{\lambda}_{\mu\nu}$.

Theorem 20. *For $\lambda, \mu, \nu \in \mathscr{P}^+$, we have*

$$\texttt{F}^{\lambda}_{\mu\nu} = \texttt{LRS}^{\lambda}_{\mu\nu}.$$

Proof. Since Definition 13(1) and (2) are the same as Definition 18(1) and (2), respectively, it suffices to show that for any $Q \in SST^+_{\mathbb{N}}(\lambda/\mu)$, $w := w(Q)$ satisfies the hook lattice property in Definition 12 if and only if w satisfies the lattice property in Definition 17. We assume that $N = |\nu|$, $w = w_1 \cdots w_N$, $w^* = w^*_1 \cdots w^*_N$, and $w\widehat{w}_{\text{rev}} = a_1 \cdots a_{2N}$.

Suppose that w satisfies the hook lattice property in Definition 12. We use induction on $1 \leqslant i \leqslant 2N$ to show that $a_1 \cdots a_{2N}$ satisfies (11). We first observe from (L1) that $a_1 = 1$ or $1'$, and a_1 satisfies (11) since $m_k(0) = 0$ for all $k \geqslant 1$.

We now assume that $a_1 \cdots a_i$ for some $1 \leqslant i < 2N$ satisfies (11). Suppose for the sake of contradiction that $m_{k+1}(i) = m_k(i) = m$ and $|a_{i+1}| = k+1$ for some $k \geqslant 1$. Here $m > 0$ by (L1). By induction hypothesis, there exist $s < t \leqslant i$ such that $a_s = k$ with $m_k(s) = m$ and $a_t = k+1$ with $m_{k+1}(t) = m$. Note that for each $k \geqslant 1$

$$c_k(w) + c_{k'}(w) > c_{k+1}(w) + c_{(k+1)'}(w), \tag{12}$$

which implies that the number of k's in $w\widehat{w}_{\text{rev}}$ is greater than the number of $k+1$'s in $w\widehat{w}_{\text{rev}}$. So we can choose an integer $u > i+1$ such that $a_u = k$ and $m_k(u) = m+1$. We now consider the following four cases:

Case 1. Let $1 \leqslant s < t < i+1 \leqslant N$. In this case $w^*_s = k_m$, $w^*_t = k+1_m$ and $w^*_{i+1} = k+1_M$ for some $M \geqslant m+1$. (i) If $u \leqslant N$, then we have $(w^*_t, w^*_{i+1}, w^*_u) = (k+1_m, k+1_M, k_{m+1})$, which contradicts (L2). (ii) If $N < u < 2N - s + 1$, then $a_{2N-u+1} = w_{2N-u+1} = k'$ $(s < 2N - u + 1 \leqslant N)$ but no k occurs in $w_{s+1} \cdots w_N$ which contradicts Definition 13(2). (iii) If $2N - s + 1 < u \leqslant 2N$, then we have $(w^*_{2N-u+1}, w^*_s, w^*_t) = (k_{m+1}, k_m, k+1_m)$, which contradicts (L3).

Case 2. Let $1 \leqslant s < t \leqslant N < i+1 \leqslant 2N$. In this case $w^*_s = k_m$ and $w^*_t = k+1_m$. Since $w_{2N-u+1} = k'$ $(2N-u+1 < N)$, we have $s \neq 2N-u+1$. (i) If $s < 2N-u+1$, then we have $w_{2N-u+1} = k'$ but no k in $w_{s+1} \cdots w_n$ since $m_k(i) = m$, which contradicts Definition 13(2). (ii) If $2N - u + 1 < s$, then we have $(w^*_{2N-u+1}, w^*_s, w^*_t) = (k_{m+1}, k_m, k + 1_m)$, which contradicts (L3).

Case 3. Let $1 \leqslant s \leqslant N < t < i+1 \leqslant 2N$. In this case $w^*_s = k_m$, $w^*_{2N-t+1} = k+1_m$. If $a_{i+1} = (k+1)'$, then $a_{2N-i} = k$ but it is impossible from the assumption $m_k(i) = m$. So $a_{i+1} = k+1$ and $w^*_{2N-i} = k+1_{m+1}$. (i) If $s < 2N-t+1$, then we have $w_{2N-t+1} = (k+1)'$ $(2N - t + 1 \leqslant N)$ but no $k+1$ in $w_{2N-t+2} \cdots w_N$ since $m_{k+1}(i) = m$, which contradicts Definition 13(2). (ii) If $2N-t+1 < s$, then by (12) there is an integer $v > u$ such that $a_v = k$ and $m_k(v) = m+2$. So we have $(w^*_{2N-v+1}, w^*_{2N-u+1}, w^*_{2N-i}) = (k_{m+2}, k_{m+1}, k+1_{m+1})$, which contradicts (L3).

Case 4. Let $N < s < t < i+1 \leqslant 2N$. In this case $w^*_{2N-s+1} = k_m$ and $w^*_{2N-t+1} = k+1_m$. (i) If $a_{i+1} = k+1$, then $w_{2N-i} = (k+1)'$ and $w^*_{2N-i} = k+1_{m+1}$. By (12) there is an integer $v > u$ such that $a_v = k$ and $m_k(v) = m+2$. So we have $(w^*_{2N-v+1}, w^*_{2N-u+1}, w^*_{2N-i}) = (k_{m+2}, k_{m+1}, k+1_{m+1})$, which contradicts (L3). (ii) If $a_{i+1} = (k+1)'$, then $w_{2N-i} = k$ and $w^*_{2N-i} = k_M$ for some $M < m$. So we have $(w^*_{2N-u+1}, w^*_{2N-i}, w^*_{2N-t+1}) = (k_{m+1}, k_M, k+1_m)$, which contradicts (L3).

Conversely, we assume that w satisfies the lattice property in Definition 17. We first claim that w satisfies (L1). Given $k \geqslant 1$, let $w^*_i = k_1$ for some $1 \leqslant i \leqslant N$. If $w^*_j = k+1_1$ for some $1 \leqslant j < i$, then it follows that $m_k(j-1) = m_{k+1}(j-1) = 0$ and $a_j = k+1$, which contradicts (11). Hence w satisfies (L1).

Next, we claim that w satisfies (L2). Suppose that there is a triple $(w^*_s, w^*_u, w^*_t) = (k+1_i, k+1_j, k_{i+1})$ for some $k \geqslant 1$, $i < j$, and $1 \leqslant s < u < t \leqslant N$. We may assume that $j = i+1$. Since $w^*_s = k+1_i$ is placed to the left of $w^*_u = k+1_{i+1}$, it follows from Definition 10 that $a_s = k+1$, and from Definition 18(2) that $a_u = k+1$. Since w satisfies the lattice property, there is a positive integer $v < s$ such that $w^*_v = k_i$, i.e., $a_v = k$ and $m_k(v) = i$ for some $v < s$. We have $m_{k+1}(u-1) = m_k(u-1) = i$ and $a_u = k+1$, a contradiction. So w satisfies (L2).

Finally, we claim that w satisfies (L3). Suppose for the sake of contradiction that $(w^*_s, w^*_u, w^*_t) = (k_{j+1}, k_i, k+1_j)$ for some $k \geqslant 1$, $i \leqslant j$, and $1 \leqslant s < u < t \leqslant N$. We may assume that $i = j$. Since $w^*_s = k_{j+1}$ is placed to the left of $w^*_u = k_j$, it follows that $a_s = k'$. We consider four cases depending whether a_u and a_t are primed or not as follows:

Case 1. Let $a_u = k'$ and $a_t = (k+1)'$. It follows that $a_{2N-u+1} = k$ $(m_k(2N-u+1) = j)$ and $a_{2N-t+1} = k+1$ $(m_k(2N-u+1) = j)$. So we have $m_{k+1}(2N-t) = m_k(2N-t) = j-1$ and $a_{2N-t+1} = k+1$, as desired.

Case 2. Let $a_u = k'$ and $a_t = k+1$. It follows that $a_{2N-u+1} = k$, $m_k(2N-u+1) = j$ and $m_{k+1}(t) = j$. Since $t < 2N-u+1$, we have $m_k(t) < m_{k+1}(t)$. So there is an integer $0 \leqslant \hat{t} < t$ such that $m_k(\hat{t}) = m_{k+1}(\hat{t}) < j$ and $a_{\hat{t}+1} = k+1$, as desired.

Case 3. Let $a_u = k$ and $a_t = (k+1)'$. It follows that $a_{2N-t+1} = k+1$ and $m_{k+1}(2N-t+1) = j$. From $m_k(2N-s+1) = j+1$ and $2N-t+1 < 2N-s+1$ we have $m_k(2N-t+1) = m_{k+1}(2N-t+1) = j$. If there is another $k+1$ between w_t^* and w_u^*, then we obtain the desired contradiction. Otherwise, $m_k(2N-u) = m_{k+1}(2N-u)$ and $a_{2N-u+1} = (k+1)'$, as desired.

Case 4. Let $a_u = k$ and $a_t = k+1$. From $a_s = k'$ ($w_s^* = k_{j+1}$) it follows that $m_k(2N-u) = j$. If $m_{k+1}(2N-u) = j$, from $a_{2N-u+1} = (k+1)'$ we get a contradiction. If $m_{k+1}(2N-u) > j$, by choosing the smallest integer $\hat{t} > t$ such that $m_{k+1}(\hat{t}) = j+1$ this leads to a contradiction. □

Indeed, we have shown in the proof of Theorem 20 that

Corollary 21. *Let $w \in \mathcal{W}_{\mathrm{N}}$ be such that*

(1) $(c_k(Q) + c_{k'}(Q))_{k \geqslant 1} \in \mathscr{P}^+$,

(2) for $k \geqslant 1$, if x is the rightmost letter in w with $|x| = k$, then $x = k$.

Then w satisfies the hook lattice property in Definition 12 if and only if w satisfies the lattice property in Definition 17.

Remark 22. A bijection from $\mathtt{LRS}_{\mu\nu}^{\lambda}$ to $\mathtt{L}_{\mu\nu}^{\lambda}$ is also given in [4, Theorem 4.7], which coincides with the inverse of the map $T \mapsto Q_T$ in (9) (see also the remarks in [4, p.82]). The proof of [4, Theorem 4.7] use insertion schemes for two versions of semistandard decomposition tableaux and another combinatorial model for $f_{\mu\nu}^{\lambda}$ by Cho [3] as an intermediate object between $\mathtt{LRS}_{\mu\nu}^{\lambda}$ and $\mathtt{L}_{\mu\nu}^{\lambda}$.

On the other hand, we prove more directly that the map $T \mapsto Q_T$ in (9) is a bijection from $\mathtt{L}_{\mu\nu}^{\lambda}$ to $\mathtt{LRS}_{\mu\nu}^{\lambda}$ by using a new characterization of the lattice property in Theorem 20.

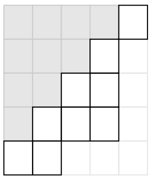
4 Schur P-expansions of skew Schur functions

4.1 The Schur P-expansion of s_{λ/δ_r}

For $r \geqslant 0$, let us denote by δ_r the partition $(r, r-1, \ldots, 1)$ if $r \geqslant 1$, and (0) if $r = 0$. We fix a nonnegative integer r.

Let $\lambda \in \mathscr{P}$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$. Here $((r+1)^{r+1})$ means the rectangular partition $(r+1, \ldots, r+1)$ with length $r+1$. For instance, the diagram

$$D_{(5,4,4,4,2)/\delta_4} =$$



is contained in $D_{(5^5)}$.

It is shown in [1, 5] that the skew Schur function s_{λ/δ_k} has a nonnegative integral expansion in terms of Schur P-functions

$$s_{\lambda/\delta_r} = \sum_{\nu \in \mathscr{P}^+} a_{\lambda/\delta_r\,\nu} P_\nu, \tag{13}$$

together with a combinatorial description of $a_{\lambda/\delta_r\,\nu}$. Moreover it is shown that these skew Schur functions are the only ones (up to rotation of shape by 180°), which have Schur P-positivity. In this section, we give a new simple description of $a_{\lambda/\delta_r\,\nu}$ using $\mathfrak{q}(n)$-crystals.

First we consider a $\mathfrak{q}(n)$-crystal structure on $B_n(\lambda/\delta_r)$.

Proposition 23. *Let $\lambda \in \mathscr{P}_n$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{(r+1)^{r+1}}$. Then the $\mathfrak{gl}(n)$-crystal $B_n(\lambda/\delta_r)$, regarded as a subset of $\mathcal{W}_{[n]}$ together with $\mathbf{0}$ is invariant under $\widetilde{e}_{\overline{1}}$ and $\widetilde{f}_{\overline{1}}$. Hence $B_n(\lambda/\delta_r)$ is a $\mathfrak{q}(n)$-crystal.*

Proof. Let $N = |\lambda| - |\delta_r|$. For $T \in B_n(\lambda/\delta_r)$, let $w(T) = w_1 \cdots w_N$. Recall that T is identified with $w(T)$ in $\mathcal{W}_{[n]}$. Here we call the box in D_{λ/δ_r} containing w_i the w_i-box, and call the set of boxes $(x, r-x+2) \in D_{\lambda/\delta_r}$ for $1 \leqslant x \leqslant r+1$ the main anti-diagonal of D_{λ/δ_r}.

Suppose that $\widetilde{f}_{\overline{1}} w(T) \neq 0$. There exists $1 \leqslant i \leqslant N-1$ such that $w_i = 1$ and $w_j \neq 1, 2$ for all $i < j \leqslant N$, and

$$\widetilde{f}_{\overline{1}}(w_1 \cdots w_{i-1}\, 1\, w_{i+1} \cdots w_N) = w_1 \cdots w_{i-1}\ 2\ w_{i+1} \cdots w_N,$$

by the tensor product rule (3). We first observe that the entry 1 in T can be placed only on the main anti-diagonal in D_{λ/δ_r}. If there is a box in D_{λ/δ_r} below the w_i-box, then it corresponds to w_j for some $j > i$, and hence its entry is greater than 2. Moreover, if there is a box in D_{λ/δ_r} to the right of the w_i-box, then its entry is greater than 1 since it is not on the main anti-diagonal. So we conclude that there exists $T' \in SST_{[n]}(\lambda/\delta_r)$ such that $w(T') = \widetilde{f}_{\overline{1}} w(T)$.

Suppose that $\widetilde{e}_{\overline{1}} w(T) \neq 0$. There exists $1 \leqslant i \leqslant N-1$ such that $w_i = 2$ and $w_j \neq 1, 2$ for all $i < j \leqslant N$, and

$$\widetilde{e}_{\overline{1}}(w_1 \cdots w_{i-1}\, 2\, w_{i+1} \cdots w_N) = w_1 \cdots w_{i-1}\ 1\ w_{i+1} \cdots w_N, \tag{14}$$

by the tensor product rule (3). If the w_i-box is not on the main anti-diagonal, then the w_{i+1}-box is placed to the left of the w_i-box. Then the w_{i+1}-box is filled with 1 or 2, which contradicts (14). So the w_i-box is on the main anti-diagonal, and thus $\widetilde{f}_{\overline{1}} w(T) = w(T')$ for some $T' \in B_n(\lambda/\delta_r)$. This completes the proof. □

Remark 24. The above proposition is a slight generalization of [7, Example 2.10(d)], which considers only the set of semistandard tableaux of shape Y_λ with entries in $[n]$ for a strict partition $\lambda \in \mathscr{P}^+$. Here Y_λ is the skew diagram having λ_1 boxes on the main anti-diagonal, λ_2 boxes on the second one, etc.

Corollary 25. *Under the above hypothesis, the skew Schur function* s_{λ/δ_r} *is Schur* P*-positive.*

Proof. Since $B_n(\lambda/\delta_r)$ is a $\mathfrak{q}(n)$-crystal, the skew Schur polynomial $s_{\lambda/\delta_r}(x_1,\dots,x_n)$ is a nonnegative integral linear combination of $P_\nu(x_1,\dots,x_n)$. From the fact that $B_n(\lambda/\delta_r)$ lies inside $B(\epsilon_1)^N$ for $N=|\lambda|-|\delta_r|$ we then apply Remark 9. $\square$

Definition 26. Let $\lambda\in\mathscr{P}$ be such that $D_{\delta_r}\subseteq D_\lambda\subseteq D_{((r+1)^{r+1})}$ and $\nu\in\mathscr{P}^+$. Let $\mathtt{A}_{\lambda/\delta_r\,\nu}$ be the set of tableaux Q such that

(1) $Q\in SST^+_{[r+1]}(\nu)$ with $c_k(Q)=\lambda_{r-k+2}-k+1$ for $1\leqslant k\leqslant r+1$,

(2) for $1\leqslant k\leqslant r$ and $1\leqslant i\leqslant N$,

$$m_k(i)\leqslant m_{k+1}(i)+1,$$

where $w_{\rm rev}(Q)=w_1\cdots w_N$ and $m_k(i)=c_k(w_1\cdots w_i)$.

Then we have the following combinatorial description of $a_{\lambda/\delta_r\,\nu}$.

Theorem 27. *For* $\lambda\in\mathscr{P}$ *with* $D_{\delta_r}\subseteq D_\lambda\subseteq D_{((r+1)^{r+1})}$ *and* $\nu\in\mathscr{P}^+$*, we have*

$$a_{\lambda/\delta_r\,\nu}=\left|\mathtt{A}_{\lambda/\delta_r\,\nu}\right|.$$

Proof. Choose n such that $\lambda,\nu\in\mathscr{P}^+_n$. We may assume that $\lambda_1=\ell(\lambda)=r+1$. Let

$$\mathtt{L}_{\lambda/\delta_r\,\nu}=\{\,T\in B_n(\lambda/\delta_r)\,|\,T\equiv L^\nu\,\}. \tag{15}$$

By Proposition 23, we have

$$B_n(\lambda/\delta_r)\cong\bigsqcup_{\nu\in\mathscr{P}^+_n}\mathbf{B}_n(\nu)^{\oplus|\mathtt{L}_{\lambda/\delta_r\,\nu}|}. \tag{16}$$

By linear independence of $P_\nu(x_1,\dots,x_n)$'s for $\nu\in\mathscr{P}^+_n$, we have $a_{\lambda/\delta_r\,\nu}=|\mathtt{L}_{\lambda/\delta_r\,\nu}|$.

Let us construct a bijection

$$\begin{array}{ccc}\mathtt{L}_{\lambda/\delta_r\,\nu} & \longrightarrow & \mathtt{A}_{\lambda/\delta_r\,\nu}\\ T & \longmapsto & Q_T\end{array} \tag{17}$$

as follows. Let $T\in\mathtt{L}_{\lambda/\delta_r\,\nu}$ be given. Suppose that $w(T)=u_1\cdots u_N$, where $N=|\nu|$. By Lemma 7, there exists $\nu^{(m)}\in\mathscr{P}^+_n$ for $1\leqslant m\leqslant N$ such that $u_{N-m+1}\cdots u_N\equiv L^{\nu^{(m)}}$, where $\nu^{(1)}=(1)$, $\nu^{(N)}=\nu$, and $\nu^{(m)}$ is obtained by adding a box in the $(n-u_m+1)$-st row of $\nu^{(m-1)}$ for $1\leqslant m\leqslant N$ with $\nu^{(0)}=\varnothing$.

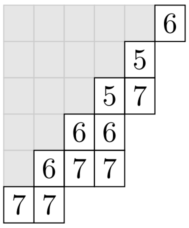
Note that $w_{\rm rev}(T)=T^{(r+1)}\cdots T^{(1)}$, where $T^{(l)}=T_{l,1}\cdots T_{l,\lambda_l-r-1+l}$ is a weakly increasing word corresponding to the l-th row of T for $1\leqslant l\leqslant r+1$. Let Q_T be a tableau of shifted shape ν with entries in $\mathbb{N}$, where $\nu^{(m)}/\nu^{(m-1)}$ is filled with $r+2-l$ if u_m occurs in $T^{(l)}$, for some $1 \leqslant l \leqslant r+1$. Note that the boxes in Q_T corresponding to $T^{(l)}$ are filled with $r+2-l$ as a horizontal strip. So Q_T satisfies the condition Definition 26(1).

For each $k \geqslant 1$, let us enumerate the letter k's in Q_T from southwest to northeast by $k_1, k_2, \ldots$. Since $T \in SST_n(\lambda/\delta_r)$, we see that the entry k_i in Q_T corresponds to $T_{l,i}$ for $i \geqslant 1$, where $l = r+2-k$, and moreover $(k+1)_i$ is located in the southwest of k_{i+1} for $i \geqslant 2$. This implies the condition Definition 26(2), and hence $Q_T \in \mathtt{A}_{\lambda/\delta_r\,\nu}$.

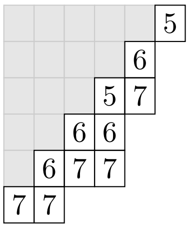
Finally, one can check that correspondence $T \mapsto Q_T$ is a bijection. $\square$

Example 28. Let $\lambda = (6,5,5,4,4,2)$ with $D_\lambda \subseteq D_{(6^6)}$ and $n = 7$. For $\nu = (5,4,2)$, we have $\mathtt{L}_{\lambda/\delta_5\,\nu} = \{\, T_1, T_2 \,\}$ and $\mathtt{A}_{\lambda/\delta_5\,\nu} = \{\, Q_{T_1}, Q_{T_2} \,\}$ as follows.

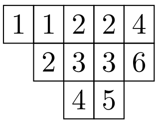
$T_1 =$



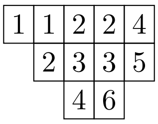
$T_2 =$



$Q_{T_1} =$



$Q_{T_2} =$



Moreover, we have

$$s_{(6,5,5,4,4,2)/\delta_5} = 2P_{(5,3,2,1)} + 2P_{(5,4,2)} + P_{(6,3,2)} + P_{(6,4,1)}.$$

Remark 29. For $\lambda \in \mathscr{P}$ with $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$, let $\lambda^\natural = (\lambda^\natural_1, \ldots, \lambda^\natural_{r+1})$ such that $\lambda^\natural_i$ is the number of boxes on the i-th anti-diagonal from the main anti-diagonal, which is obviously a strict partition. One can see that there is a unique tableau of shape λ/δ_r with weight $\lambda^\natural$ satisfying the condition in Lemma 7. Moreover, if $a_{\lambda/\delta_r\,\nu} \neq 0$, then ν is less than or equal to $\lambda^\natural$ with respect to the dominance ordering. So we have

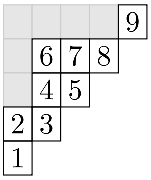
$$s_{\lambda/\delta_r} = P_{\lambda^\natural} + \sum_{\nu < \lambda^\natural} a_{\lambda/\delta_r,\nu} P_\nu.$$

4.2 Ardila–Serrano's expansion of $s_{\delta_{r+1}/\mu}$

We fix a nonnegative integer r. For $\mu \in \mathscr{P}$ with $D_\mu \subseteq D_{\delta_{r+1}}$, let us recall the result on the Schur P-expansion of he skew Schur function $s_{\delta_{r+1}/\mu}$ by Ardila and Serrano [1].

Let $N = |\delta_{r+1}| - |\mu|$, and let $T_{\delta_{r+1}/\mu}$ be the tableau obtained by filling δ_{r+1}/μ with $1, 2, \ldots, N$ subsequently, starting from the bottom row to top, and from left to right in each row. For instance,

$T_{\delta_5/(4,1,1)} =$



.

For $\nu \in \mathscr{P}^+$ with $|\nu| = N$, let $\mathtt{B}_{\delta_{r+1}/\mu\,\nu}$ be the set of tableaux Q such that

(1) $Q \in SST^{+}_{[N]}(\nu)$ where each entry $i \in [N]$ occurs exactly once,

(2) if j is directly above i in $T_{\delta_{r+1}/\mu}$, then j is placed strictly to the right of i in Q,

(3) if $i+1$ is placed to the right of i in $T_{\delta_{r+1}/\mu}$, then $i+1$ is strictly below i in Q.

Theorem 30. ([1, Theorem 4.10]) *For $\mu \in \mathscr{P}$ with $D_\mu \subseteq D_{\delta_{r+1}}$, the skew Schur function $s_{\delta_{r+1}/\mu}$ is given by a nonnegative integral linear combination of Schur P-functions*

$$s_{\delta_{r+1}/\mu} = \sum_{\nu \in \mathscr{P}^+} b_{\delta_{r+1}/\mu\,\nu} P_\nu,$$

where $b_{\delta_{r+1}/\mu\,\nu} = |\mathtt{B}_{\delta_{r+1}/\mu\,\nu}|$.

Now we show that Theorem 27 (after a little modification of its proof) implies Theorem 30. Let $\lambda \in \mathscr{P}$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$.

Let $\nu \in \mathscr{P}^+$ with $|\nu| = N = |\lambda| - |\delta_r|$, and let $\mathtt{L}_{\lambda/\delta_r\,\nu}$ be as in (15). Then $|\mathtt{L}_{\lambda/\delta_r\,\nu}| = a_{\lambda/\delta_r\,\nu}$ by (16). Let $T \in \mathtt{L}_{\lambda/\delta_r\,\nu}$ be given with $w(T) = u_1 \cdots u_N$. Recall by Lemma 7 that there exists a sequence of strict partitions $\nu^{(m)} \in \mathscr{P}^+_n$ for $1 \leqslant m \leqslant N$ such that $u_{N-m+1} \cdots u_N \equiv L^{\nu^{(m)}}$, where $\nu^{(1)} = (1)$, $\nu^{(N)} = \nu$, and $\nu^{(m)}$ is obtained by adding a box in the $(n - u_m + 1)$-st row of $\nu^{(m-1)}$ with $\nu^{(0)} = \varnothing$.

We define Q'_T to be the tableau of shifted shape ν such that $\nu^{(m)}/\nu^{(m-1)}$ is filled with m for $1 \leqslant m \leqslant N$. Then we have the following.

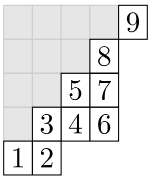
Theorem 31. *Let $\lambda \in \mathscr{P}$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$ and $\nu \in \mathscr{P}^+$. Then we have a bijection*

$$\begin{array}{ccc} \mathtt{L}_{\lambda/\delta_r\,\nu} & \longrightarrow & \mathtt{B}_{\delta_{r+1}/(\lambda^{\mathsf{c}})'\,\nu} \\ T & \longmapsto & Q'_T \end{array}$$

where $\lambda^{\mathsf{c}} := (r+1-\lambda_{r+1}, r+1-\lambda_r, \ldots, r+1-\lambda_1)$ is the complement of λ in $((r+1)^{r+1})$.

Proof. Let T'_{λ/δ_r} be the tableau obtained by filling λ/δ_r with $1, 2, \ldots, N$ subsequently, starting from the leftmost column to rightmost, and from bottom to top in each column. For instance, when $\lambda = (5,4,4,4,2)$ and $r = 4$, we have

$$T'_{\lambda/\delta_r} =$$



.

By definition of Q'_T, we can check that

(1) $Q'_T \in SST^{+}_{[N]}(\nu)$ where each entry $i \in [N]$ occurs exactly once,

(2) if j is directly above i in T'_{λ/δ_r}, then then j is strictly below i in Q'_T,

(3) if $i+1$ is placed to the right of i in T'_{λ/δ_r}, then $i+1$ is placed strictly to the right of i in Q'_T.

We see that $T_{\delta_{r+1}/(\lambda^c)'}$ is obtained from T'_{λ/δ_r} by flipping with respect to the main anti-diagonal. This implies that $Q'_T \in \mathtt{B}_{\delta_{r+1}/(\lambda^c)'\,\nu}$. Since the correspondence $T \mapsto Q'_T$ is reversible, it is a bijection. $\square$

Corollary 32. *Under the above hypothesis, we have a bijection*

$$\begin{tikzcd} \mathtt{A}_{\lambda/\delta_r\,\nu} \arrow[r] & \mathtt{B}_{\delta_{r+1}/(\lambda^c)'\,\nu} \\ Q_T \arrow[r, mapsto] & Q'_T \end{tikzcd}$$

for $T \in \mathtt{L}_{\lambda/\delta_r\,\nu}$.

Recall that for a skew shape η/ζ, we have $s_{\eta/\zeta} = s_{(\eta/\zeta)^\pi}$, where $(\eta/\zeta)^\pi$ is the (skew) diagram obtained from η/ζ by rotating 180 degree (which can be seen for example by reversing the linear ordering on $\mathbb{N}$ in [2]). Also if $s_{\eta/\zeta}$ has a Schur P-expansion, then we have $s_{\eta/\zeta} = s_{\eta'/\zeta'}$ by applying the involution ω on the ring symmetric function sending s_η to $s_{\eta'}$ since $\omega(P_\nu) = P_\nu$ for $\nu \in \mathscr{P}^+$(see [10, p. 259, Exercise 3.(a)]).

Hence we have

$$s_{\lambda/\delta_r} = s_{\delta_{r+1}/\lambda^c} = s_{\delta_{r+1}/(\lambda^c)'},$$

for $\lambda \in \mathscr{P}$ such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$. This implies that

$$a_{\lambda/\delta_r\,\nu} = b_{\delta_{r+1}/\lambda^c\,\nu} = b_{\delta_{r+1}/(\lambda^c)'\,\nu}, \tag{18}$$

for $\nu \in \mathscr{P}^+$, where $a_{\lambda/\delta_r\,\nu}$ are given in (13). Equivalently, we have

$$a_{(\mu^c)'/\delta_r\,\nu} = b_{\delta_{r+1}/\mu'\,\nu} = b_{\delta_{r+1}/\mu\,\nu}, \tag{19}$$

for $\mu \in \mathscr{P}$ with $D_\mu \subseteq D_{\delta_{r+1}}$. Therefore Theorem 30 follows from Theorem 27, Corollary 32, and (18) (or (19)).

5 Schur expansion of Schur *P*-function

For $\lambda \in \mathscr{P}^+$ and $\mu \in \mathscr{P}$, let $g_{\lambda\mu}$ be the coefficient of s_μ in the Schur expansion of P_λ, that is,

$$P_\lambda = \sum_\mu g_{\lambda\mu} s_\mu. \tag{20}$$

The purpose of this section is to give an alternate proof of the following combinatorial description of $g_{\lambda\mu}$ due to Stembridge.

Theorem 33. ([16, Theorem 9.3]) *For* $\lambda \in \mathscr{P}^+$ *and* $\mu \in \mathscr{P}$, *we have*

$$g_{\lambda\mu} = |\mathtt{G}_{\lambda\mu}|,$$

where $\mathtt{G}_{\lambda\mu}$ *is the set of tableaux* Q *such that*

(1) *$Q \in SST_{\mathcal{N}}(\mu)$ with $c_k(Q) + c_{k'}(Q) = \lambda_k$ for $k \geqslant 1$,*

(2) *for $k \geqslant 1$, if x is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,*

(3) *$w(Q)$ satisfies the lattice property.*

Proof. The proof is similar to that of Theorem 14. Choose n such that $\lambda \in \mathscr{P}_n^+$ and $\mu \in \mathscr{P}_n$. Let

$$\mathbf{L}_{\lambda\mu} = \{\, T \,|\, T \in \mathbf{B}_n(\lambda),\ \widetilde{f}_i T = \mathbf{0}\ (1 \leqslant i \leqslant n-1),\ \mathrm{wt}(T) = w_0\mu \,\}.$$

Then we have as a $\mathfrak{gl}(n)$-crystal

$$\mathbf{B}_n(\lambda) \cong \bigsqcup_{\mu} B_n(\mu)^{\oplus|\mathbf{L}_{\lambda\mu}|}, \tag{21}$$

and hence $g_{\lambda\mu} = |\mathbf{L}_{\lambda\mu}|$ by linear independence of Schur polynomials. Let us define a map

$$\begin{array}{ccc} \mathbf{L}_{\lambda\mu} & \longrightarrow & \mathtt{G}_{\lambda\mu} \\ T & \longmapsto & Q_T \end{array}$$

as follows. Let $T \in \mathbf{L}_{\lambda\mu}$ be given. Assume that $w_{\mathrm{rev}}(T) = u_1 \cdots u_N$ where $N = |\lambda|$. Since T is a $\mathfrak{gl}(n)$-lowest weight vector, we have by (2) that $u_{N-m+1} \otimes \cdots \otimes u_N \in \mathbf{B}_n^{\otimes m}$ is a $\mathfrak{gl}(n)$-lowest weight element for $1 \leqslant m \leqslant N$. This implies that there exists $\mu^{(m)} \in \mathscr{P}_n$ for $1 \leqslant m \leqslant N$ such that $u_{N-m+1} \cdots u_N$ is equivalent as an element of $\mathfrak{gl}(n)$-crystal to a $\mathfrak{gl}(n)$-lowest weight element in $B_n(\mu^{(m)})$, where $\mu^{(N)} = \mu$ and $\mu^{(m)}$ is obtained by adding a box in the $(n - u_m + 1)$-st row of $\mu^{(m-1)}$ with $\mu^{(0)} = \varnothing$.

We define Q_T to be a tableau of shape μ with entries in $\mathcal{N}$, where $\mu^{(m)}/\mu^{(m-1)}$ is filled with

$$\begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)}\uparrow, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)}\downarrow, \end{cases}$$

for some $1 \leqslant k \leqslant \ell(\lambda)$. By almost the same arguments as in the proof of Theorem 14, we see that Q_T satisfies the conditions (1) and (2) for $\mathtt{G}_{\lambda\mu}$, and $w(Q_T)$ satisfies the hook lattice property, which implies that it satisfies the lattice property by Corollary 21. (We leave the details to the reader.) Finally the correspondence $T \mapsto Q_T$ is a well-defined bijection. $\square$

Example 34. Let $\lambda = (3, 1)$. From Figure 1 we get three $\mathfrak{gl}(3)$-lowest weight vectors in $\mathbf{B}_3(\lambda)$

$$\begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 3 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 3 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \; .$$

By applying the mapping $T \mapsto Q_T$ in the proof of Theorem 33 to these tableaux we have

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1' & 1 \\ \hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1' & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} .$$

Thus $P_{(3,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$.

Remark 35. Let $\lambda \in \mathscr{P}^+$ be such that $D^+_\lambda \subseteq D^+_{\delta_{r+1}}$ for some $r \geqslant 0$. Let $\lambda^{\mathsf{c}+}$ be a strict partition obtained by counting complementary boxes $D^+_{\delta_{r+1}} \setminus D^+_\lambda$ in each column from right to left. It is shown in [5] that

$$s_{\delta_{r+1}/\lambda} = \sum_{\substack{\nu \in \mathscr{P}^+ \\ |\nu| = |\lambda|}} g_{\nu\lambda} P_{\nu^{\mathsf{c}+}}.$$

By (18) or (19), we have $g_{\nu\,\lambda} = a_{\lambda^{\mathsf{c}}/\delta_r\,(\nu^{\mathsf{c}+})'}$. One may expect that there is a natural bijection between $\mathtt{G}_{\nu\,\lambda}$ and $\mathtt{A}_{\lambda^{\mathsf{c}}/\delta_r\,(\nu^{\mathsf{c}+})'}$ that we have not yet make explicit.

6 Semistandard decomposition tableaux of skew shapes

Let λ/μ be a shifted skew diagram for $\lambda, \mu \in \mathscr{P}^+$ with $D^+_\mu \subseteq D^+_\lambda$. Without loss of generality, we assume in this section that $\lambda_1 > \mu_1$ and $\ell(\lambda) > \ell(\mu)$.

Let T be a tableau of shifted skew shape λ/μ. For $p, q \geqslant 1$, let $T(p,q)$ denote the entry of T at the p-th row and the q-th diagonal from the main diagonal in D^+_λ (that is, $\{\,(i,i) \,|\, i \geqslant 1\} \cap D^+_\lambda$) whenever it is defined. Note that $T(p,q)$ is not necessarily equal to $T_{p,q}$ if μ is nonempty.

For example, when $\lambda/\mu = (5,4,2)/(3,1)$, we have

$$\begin{array}{|c|c|c|c|c|}\hline & & & T(1,4) & T(1,5) \\ \hline \multicolumn{1}{c|}{} & & T(2,2) & T(2,3) & T(2,4) \\ \cline{2-5} \multicolumn{2}{c|}{} & T(3,1) & T(3,2) & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array} \quad = \quad \begin{array}{|c|c|c|c|c|}\hline \phantom{T_{1,1}} & \phantom{T_{1,1}} & \phantom{T_{1,1}} & T_{1,1} & T_{1,2} \\ \hline \multicolumn{1}{c|}{} & & T_{2,1} & T_{2,2} & T_{2,3} \\ \cline{2-5} \multicolumn{2}{c|}{} & T_{3,1} & T_{3,2} & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array}$$

Definition 36. For $\lambda, \mu \in \mathscr{P}^+$ with $D^+_\mu \subseteq D^+_\lambda$, a *skew semistandard decomposition tableau* T of shape λ/μ is a tableau of shifted shape λ/μ with entries in $\mathbb{N}$ such that $T^{(k)}$ is a hook word for $1 \leqslant k \leqslant \ell(\lambda)$ and the following holds for $1 \leqslant k < \ell(\lambda)$ and $1 \leqslant i \leqslant j \leqslant \lambda_{k+1}$:

(S1) if $T(k,i) \leqslant T(k+1,j)$, then $i \neq 1$ and $T(k+1,i-1) < T(k+1,j)$,

(S2) if $T(k,i) > T(k+1,j)$, then $T(k,i) \geqslant T(k,j+1)$,

where we assume that $T(p,q)$ for $p, q \geqslant 1$ is empty if it is not defined.

Let $SSDT(\lambda/\mu)$ be the set consisting of skew semistandard decomposition tableaux of shape λ/μ. Note that when μ is empty, the set $SSDT(\lambda/\mu)$ is equal to $SSDT(\lambda)$ by Proposition 3.

Suppose that $\ell(\lambda) \leqslant n$. Let $\mathbf{B}_n(\lambda/\mu)$ be the set of $T \in SSDT(\lambda/\mu)$ with entries in $[n]$. As in (5), consider the injective map

$$\begin{tikzcd} \mathbf{B}_n(\lambda/\mu) \arrow[r, hook] & \mathcal{W}_{[n]} \\ T \arrow[r, mapsto] & w_{\mathrm{rev}}(T). \end{tikzcd} \tag{22}$$

Proposition 37. *Under the above hypothesis, the image of* $\mathbf{B}_n(\lambda/\mu)$ *in* (22) *together with* $\{\mathbf{0}\}$ *is invariant under the action of* $\widetilde{e}_i$ *and* $\widetilde{f}_i$ *for* $i \in I$, *and hence* $\mathbf{B}_n(\lambda/\mu)$ *is a* $\mathfrak{q}(n)$*-crystal.*

Proof. Choose a sufficiently large M such that all the entries in L^{μ}_M are greater than n. For a tableau T of shifted shape λ/μ with entries in $[n]$, let $\widetilde{T} := L^{\mu}_M * T$ be the tableau of shifted shape λ, that is, the subtableau of shape shifted μ in $\widetilde{T}$ is L^{μ}_M and its complement in $\widetilde{T}$ is T. By definition of $SSDT(\lambda/\mu)$ and Proposition 3, we have

$$T \in \mathbf{B}_n(\lambda/\mu) \ \text{ if and only if } \ \widetilde{T} \in \mathbf{B}_M(\lambda). \tag{23}$$

Let $T \in \mathbf{B}_n(\lambda/\mu)$ and $i \in I$ be given. If $\tilde{x}_i\widetilde{T} \neq \mathbf{0}$ $(x = e, f)$, then we have by (23) that $\tilde{x}_i\widetilde{T} = L^{\mu}_M * T'$ for some $T' \in \mathbf{B}_n(\lambda/\mu)$. This implies that $\tilde{x}_i w_{\mathrm{rev}}(T) = w_{\mathrm{rev}}(T')$. Therefore, the image of $\mathbf{B}_n(\lambda/\mu)$ in (22) together with $\{\mathbf{0}\}$ is invariant under the action of $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$. ☐

Since $\mathbf{B}_n(\lambda/\mu)$ is a subcrystal of $\mathbf{B}_n^{\otimes N}$ with $N = |\lambda| - |\mu|$, we have

$$\mathbf{B}_n(\lambda/\mu) \cong \bigsqcup_{\substack{\nu \in \mathscr{P}^+_n \\ |\nu| = N}} \mathbf{B}_n(\nu)^{\oplus f^{\lambda/\mu}_{\nu}(n)} \tag{24}$$

for some $f^{\lambda/\mu}_{\nu}(n) \in \mathbb{Z}_+$. Moreover by Remark 9, we have

$$f^{\lambda/\mu}_{\nu} := f^{\lambda/\mu}_{\nu}(m) = f^{\lambda/\mu}_{\nu}(n) \quad (m \geqslant n). \tag{25}$$

If we put

$$P^{\circ}_{\lambda/\mu} = \sum_{T \in SSDT(\lambda/\mu)} x^T,$$

then we have from (24) and (25)

$$P^{\circ}_{\lambda/\mu} = \sum_{\nu \in \mathscr{P}^+} f^{\lambda/\mu}_{\nu} P_{\nu}. \tag{26}$$

Example 38. For $\eta \in \mathscr{P}_n^+$ with $\ell(\eta) = \ell$, let $\lambda = \eta + L\delta_\ell$ and $\mu = L\delta_\ell \in \mathscr{P}_n^+$, where $L \geqslant \eta_1$. Since each column in λ/μ has at most one box, we have

$$\mathbf{B}_n(\lambda/\mu) \cong \mathbf{B}_n(\eta_1) \otimes \cdots \otimes \mathbf{B}_n(\eta_\ell).$$

By applying Theorem 14 repeatedly, we see that $f_\nu^{\lambda/\mu}$ for $\nu \in \mathscr{P}_n^+$ in this case is equal to the number of tableaux Q such that

(1) $Q \in SST_{\mathbb{N}}^+(\nu)$ with $c_k(Q) + c_{k'}(Q) = \eta_k$ for $k \geqslant 1$,

(2) for each $k \geqslant 1$, if x is the rightmost in $w(Q)$ with $|x| = k$, then $x = k$.

One can generalize the notion of hook lattice property in Definition 12 to describe the coefficient $f_\nu^{\lambda/\mu}$.

Definition 39. Let $w = w_1 \cdots w_N \in \mathcal{W}_{\mathbb{N}}$ be given and let $w^* = w_1^* \cdots w_N^*$ be the word associated to w given in Definition 10. For $\mu \in \mathscr{P}^+$, we say that w satisfies the *hook μ-lattice property* if w^* satisfies the following for each $k \geqslant 1$:

(L1) if $k > \ell(\mu)$ and $w_i^* = k_1$, then no $k+1_j$ for $j \geqslant 1$ occurs in $w_1^* \cdots w_{i-1}^*$,

(L2) if $(w_s^*, w_t^*) = (k+1_i, k_{i+1-\alpha_k})$ for some $s < t$ and $\alpha_k < i$, then no $k+1_j$ for $i < j$ occurs in $w_s^* \cdots w_t^*$,

(L3) if $(w_s^*, w_t^*) = (k_{j+1-\alpha_k}, k+1_j)$ for some $s < t$ and $\alpha_k < j$, then no k_i for $i \leqslant j - \alpha_k$ occurs in $w_s^* \cdots w_t^*$,

where $\alpha_k = \mu_k - \mu_{k+1}$.

Theorem 40. *For $\lambda, \mu, \nu \in \mathscr{P}^+$, we have*

$$f_\nu^{\lambda/\mu} = \left|\mathtt{F}_\nu^{\lambda/\mu}\right|,$$

where $\mathtt{F}_\nu^{\lambda/\mu}$ is the set of tableaux Q such that

(1) *$Q \in SST_{\mathbb{N}}^+(\nu)$ with $c_k(Q) + c_{k'}(Q) = \lambda_k - \mu_k$ for $k \geqslant 1$,*

(2) *for $k \geqslant 1$, if x is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,*

(3) *$w(Q)$ satisfies the hook μ-lattice property.*

Proof. The proof is similar to that of Theorem 14. Choose n such that $\lambda, \mu, \nu \in \mathscr{P}_n^+$. Put

$$\mathbf{L}_\nu^{\lambda/\mu} = \{\, T \mid T \in \mathbf{B}_n(\lambda/\mu),\ T \equiv L^\nu \,\}.$$

From (24) and (25), we have $|\mathbf{L}_\nu^{\lambda/\mu}| = f_\nu^{\lambda/\mu}$. Let us define a map

$$\begin{array}{ccc} \mathbf{L}_\nu^{\lambda/\mu} & \longrightarrow & \mathtt{F}_\nu^{\lambda/\mu} \\ T & \longmapsto & Q_T \end{array}$$

as follows. Let $N = |\lambda| - |\mu|$. Suppose that $T \in \mathtt{L}_\nu^{\lambda/\mu}$ is given with $w_{\rm rev}(T) = u_1 \cdots u_N$. By Lemma 7 there exists $\nu^{(m)} \in \mathscr{P}_n^+$ for $1 \leqslant m \leqslant N$ such that $u_{N-m+1} \cdots u_N \equiv L^{\nu^{(m)}}$ where $\nu^{(N)} = \nu$ and $\nu^{(m)}$ is obtained by adding a box in the $(n - u_m + 1)$-st row of $\nu^{(m-1)}$ with $\nu^{(0)} = \varnothing$.

Note that $w_{\rm rev}(T) = T^{(\ell(\lambda))} \cdots T^{(1)}$, where $T^{(k)}$ is a hook word for $1 \leqslant k \leqslant \ell(\lambda)$. Then we define Q_T to be a tableau of shifted shape ν with entries in $\mathcal{N}$, where $\nu^{(m)}/\nu^{(m-1)}$ is filled with

$$\begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)}\uparrow, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)}\downarrow, \end{cases} \tag{27}$$

for some $1 \leqslant k \leqslant \ell(\lambda)$.

First, by the same argument as in the proof of Theorem 14, we see that Q_T satisfies the condition (2) for $\mathtt{F}_\nu^{\lambda/\mu}$ by the same argument as in the proof of Theorem 14.

Let us check that $w(Q_T)$ satisfies the hook μ-lattice property. If we label k and k' in (27) as k_j and k'_j, respectively, when $u_m = T_{k,j}$, then it coincides with the labeling on the letters in $w(Q_T)$ given in Definition 10(1).

Choose a sufficiently large M such that all the entries in L_M^μ are greater than n. Let $S = L_M^\mu * T$ (see the proof of Proposition 37). Since $S \in \mathbf{B}_M(\lambda)$, the conditions Proposition 3(1), (2), and (3) on S and hence on T (cf. (23)) imply the conditions Definition 39(L1), (L2), and (L3), respectively. Therefore, $Q_T \in \mathtt{F}_\nu^{\lambda/\mu}$.

Finally the correspondence $T \mapsto Q_T$ is injective and also reversible. Hence it is a bijection. $\square$

Example 41. Let $\lambda/\mu = (3,1)/(1)$. Then

$$P^\circ_{(3,1)/(1)} = P_{(3)} + P_{(2,1)},$$

since we have for $\nu \in \{(3),(2,1)\}$ and $n = 4$

$$T_1 = \begin{ytableau} *(lightgray) & 4 & 4 \\ \none & 4 \end{ytableau} \in \mathtt{L}_\nu^{\lambda/\mu} \qquad Q_{T_1} = \begin{ytableau} 1 & 1 & 2 \end{ytableau} \in \mathtt{F}_\nu^{\lambda/\mu}$$

$$T_2 = \begin{ytableau} *(lightgray) & 4 & 4 \\ \none & 3 \end{ytableau} \in \mathtt{L}_\nu^{\lambda/\mu} \qquad Q_{T_2} = \begin{ytableau} 1 & 1 \\ \none & 2 \end{ytableau} \in \mathtt{F}_\nu^{\lambda/\mu} \quad .$$

Remark 42. Recall that for $\lambda, \mu \in \mathscr{P}^+$ with $D_\mu^+ \subseteq D_\lambda^+$, the skew Schur P-function $P_{\lambda/\mu}$ is given by the weight generating function

$$P_{\lambda/\mu} = \sum_T x^T,$$

where the sum is over all tableaux in $SST_{\mathcal{N}}^+(\lambda/\mu)$ with no primed entry or entry of odd degree on the main diagonal (cf. [10, 11, 17]). Then it is well-known that

$$P_{\lambda/\mu} = \sum_{\nu \in \mathscr{P}^+} 2^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} f_{\mu\nu}^\lambda P_\nu.$$

We should remark that $P^{\circ}_{\lambda/\mu}$ is not in general equal to $P_{\lambda/\mu}$, that is, $f_{\nu}^{\lambda/\mu}$ is not necessarily equal to $2^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} f_{\mu\nu}^{\lambda}$. For example, we have

$$P_{(3,1)/(1)} = P_{(3)} + 2P_{(2,1)},$$

which is not equal to $P^{\circ}_{(3,1)/(1)}$ in Example 41. It would be also interesting to have a more direct representation-theoretic interpretation of $\mathbf{B}_n(\lambda/\mu)$.

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