Primary decomposition of ideals of lattice homomorphisms

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Abstract

For two given finite lattices L and M, we introduce the ideal of lattice homomorphism $J(L, M)$, whose minimal monomial generators correspond to lattice homomorphisms $\phi : L \to M$. We show that L is a distributive lattice if and only if the equidimensinal part of $J(L, M)$ is the same as the equidimensional part of the ideal of poset homomorphisms $I(L, M)$. Next, we study the minimal primary decomposition of $J(L, M)$ when L is a distributive lattice and $M = [2]$. We present some methods to check if a monomial prime ideal belongs to $\mathrm{ass}(J(L, [2]))$, and we give an upper bound in terms of combinatorial properties of L for the height of the minimal primes. We also show that if each minimal prime ideal of $J(L, [2])$ has height at most three, then L is a planar lattice and $\mathrm{width}(L) \leqslant 2$. Finally, we compute the minimal primary decomposition when $L = [m] \times [n]$ and $M = [2]$.

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1 Introduction

The study of monomial ideals and interaction between their algebraic and combinatorial properties is an important topic in combinatorial and computational commutative algebra. Such ideals serve as a useful tool for studying polynomial ideals and also have grown into an active research area.

Recently, some researchers have focused on some classes of monomial ideals and algebras associated to the ordered algebraic structures. Among them, we can point out to *Hibi rings* which is defined by Hibi in 1987 [11]. Corresponding to a distributive lattice L the *joint-meet ideal* I_L is defined in [3] as a monomial ideal in a specific polynomial ring which is closely related to Hibi rings. In 2005, Herzog and Hibi [9] associated to a poset P its so-called *Hibi ideal* which is again a monomial ideal. In 2011, Ene, Herzog and Mohammadi [4] considered *generalized Hibi ideals* and studied some of their algebraic properties. In 2014 Fløystad et al. [5] introduced a further generalization of such ideals corresponding to isotone maps between two posets. These ideals are called *ideals of poset homomorphisms* and further studied in [10] and [12]. Often, in the above researches, algebraic properties of mentioned monomial ideals are studied in terms of combinatorial properties of L or even of the underlying poset P.

In this paper, we consider lattice homomorphisms instead of poset homomorphisms and introduce the *ideals of lattice homomorphisms*. Our main goal is to study minimal primary decomposition of such ideals and carefully relate it to the combinatorial properties of the corresponding lattices.

Given two finite posets P and Q, a map $\phi : P \to Q$ is called *isotone* (or, *poset homomorphism*) if it is order preserving. In other words, $\phi : P \to Q$ is isotone if and only if $\phi(p_1) \leqslant \phi(p_2)$ for all $p_1, p_2 \in P$ with $p_1 \leqslant p_2$. The set of isotone maps $P \to Q$ is denoted by $\mathrm{Hom}_{\mathrm{Pos}}(P, Q)$. Given two finite lattices L and M, a map $\phi : L \to M$ is called a *lattice homomorphism* if for any $l_1, l_2 \in L$, $\phi(l_1 \vee l_2) = \phi(l_1) \vee \phi(l_2)$ and $\phi(l_1 \wedge l_2) = \phi(l_1) \wedge \phi(l_2)$. We denote by $\mathrm{Hom}_{\mathrm{Lat}}(L, M)$ the set of lattice homomorphism $L \to M$. It is clear that $\mathrm{Hom}_{\mathrm{Lat}}(L, M) \subseteq \mathrm{Hom}_{\mathrm{Pos}}(L, M)$.

Now, let S be the polynomial ring over a field $\mathbf{k}$ with variables $x_{l,m}$ where $l \in L$ and $m \in M$, i.e., $S = \mathbf{k}[x_{l,m}; l \in L, m \in M]$. As in [5] we associate to any $\phi \in \mathrm{Hom}_{\mathrm{Pos}}(L, M)$ the monomial

$$u_\phi = \prod_{l \in L} x_{l,\phi(l)}.$$

The ideal of poset homomorphisms associated to L and M (as defined in [5] for any posets P and Q) is the ideal of S whose minimal monomial generators correspond to poset homomorphisms, i.e.,

$$I(L, M) = (u_\phi \ ; \ \phi \in \mathrm{Hom}_{\mathrm{Pos}}(L, M)) \subset S.$$

We define the ideal of lattice homomorphisms, in a similar way, as

$$J(L, M) = (u_\phi \ ; \ \phi \in \mathrm{Hom}_{\mathrm{Lat}}(P, Q)) \subset S.$$

It is clear that $J(L,M) \subseteq I(L,M)$. Both ideals $J(L,M)$ and $I(L,M)$ are square-free monomial ideals and so are radical ideals. Thus $\operatorname{ass}(J(L,M)) = \min(J(L,M))$ and $\operatorname{ass}(I(L,M)) = \min(I(L,M))$, where by $\operatorname{ass}(J)$ we mean the set of associated prime ideals of the ideal J and by $\min(J)$ we mean the set of minimal prime ideals of it. As we have pointed out before, we are going to study the minimal primary decomposition of $J(L,M)$. Note that since $J(L,M)$ is radical, we have

$$J(L,M) = \bigcap_{\mathfrak{p} \in \operatorname{ass}(J(L,M))} \mathfrak{p}.$$

By [5, Proposition 1.5], the height of each $\mathfrak{p} \in \operatorname{ass}(I(L,M))$ is at least $|M|$ and the associated primes of $I(L,M)$ of the minimum height is of the form

$$\mathfrak{p} = \mathfrak{p}_\psi := (x_{\psi(m),m};\ m \in M), \quad \text{where } \psi \in \operatorname{Hom}_{\mathrm{Pos}}(M,L). \tag{1}$$

Proposition 2 shows that each $\mathfrak{p}_\psi$ described above belongs to $\operatorname{ass}(J(L,M))$ and Theorem 3 says that any associated prime of $J(L,M)$ of the minimum height is as (1) if and only if L is a distributive lattice. An immediate consequence of Theorem 3 is that the equidimensional part of $J(L,M)$ coincides with the equidimensional part of $I(L,M)$ if and only if L is a distributive lattice (Corollary 4). Moreover, by Theorem 3 and the main result of [10], we conclude that if L is a distributive lattice, then $J(L,M)$ is an unmixed ideal if and only if L is a chain and this is the case if and only if $I(L,M) = J(L,M)$ (Corollary 5).

The rest of the paper is devoted to the study of the minimal primes of $J(L,[2])$ when L is a distributive lattice. We should point out that in general, finding exactly all associated prime ideals of $J(L,M)$ seems to be pretty hard even if we restrict ourselves to the case that $M = [2]$. Given two nonempty subsets A,B of L, in Lemma 6, we observe that if $\bigwedge A \leqslant \bigvee B$ then

$$J(L,[2]) \subset \mathfrak{p} := \mathfrak{p}_{A,B} = (x_{a,1};\ a \in A) + (x_{b,2};\ b \in B). \tag{2}$$

Replacing the chain $[2]$ with an arbitrary chain $[n]$ with $n > 2$, we can not generalize Lemma 6 (see Remark 9). So, characterizing prime ideals that containing $J(L,M)$ is more complicated in the general situation. This is the reason that we assume that $M = [2]$. The next easy fact is that any associated prime ideal of $J(L,[2])$ has the form $\mathfrak{p}_{A,B}$ for some non-empty subsets A and B of L (see Lemma 7). Let $\varnothing \neq A \subset L$ and $\varnothing \neq B \subset L$, in Theorem 8, we prove that $\mathfrak{p}_{A,B}$ is an associated prime of $J(L,[2])$ if and only if the following statements hold:

(i) $\bigwedge A \leqslant \bigvee B$.

(ii) $\forall \varnothing \neq A_1 \subset A$ and $\forall \varnothing \neq B_1 \subset B$, $\bigwedge A_1 \nleqslant \bigvee B$ and $\bigwedge A \nleqslant \bigvee B_1$.

In the sequel, we give some interesting corollaries of Theorem 8. It is clear that by Theorem 3, for any distributive lattice L and any $\mathfrak{p} \in ass(J(L,[2]))$, $\operatorname{ht}(\mathfrak{p}) \geqslant 2$. It would be nice to find an upper bound for the height of such prime ideals. In Theorem 10 we prove that if $\mathfrak{p}$ is an associated prime ideal of $J(L,[2])$ then $\operatorname{ht}(\mathfrak{p}) \leqslant m(L) + M(L)$ (for the definition of $m(L)$ and $M(L)$ see the paragraph just before Theorem 10). While Theorem 3 shows that any minimal prime ideal of $J(L,[2])$ of height 2 is of the form $\mathfrak{p}_\psi$ for some $\psi \in \mathrm{Hom}_{\mathrm{Pos}}([2], L)$, in Corollary 11 we give another method to check when a prime ideal of height bigger than two belongs to $\mathrm{ass}(J(L,[2]))$. Indeed, we prove that if A and B are two non-empty subsets of L, and $|A|>1$ or $|B|>1$, then $\mathfrak{p}_{A,B} \in ass(J(L,[2]))$ if and only if the following statements hold:

(i) A and B are antichains.

(ii) $\forall a \in A$ and $\forall b \in B$, $a \nleq b$.

(iii) If a, a' are two arbitrary distinct elements of A and $A' = (A \setminus \{a, a'\}) \cup \{a \wedge a'\}$, then $\mathfrak{p}_{A',B} \in ass(J(L,[2]))$.

(iv) If b, b' are two arbitrary distinct elements of B and $B' = (B \setminus \{b, b'\}) \cup \{b \vee b'\}$, then $\mathfrak{p}_{A,B'} \in ass(J(L,[2]))$.

Next, we try to describe distributive lattices L in which any minimal prime of $J(L,[2])$ has height at most 3. As we see in Corollary 17, if any associated prime of $J(L,[2])$ has height at most 3 then L should be a planar lattice with $\mathrm{width}(L) \leqslant 2$, where by width we mean the maximum number of elements in an antichain contained in L.

Given the positive integers $1 < m \leqslant n$, in the last section, we completely find the minimal primary decomposition of $J([m]\times[n],[2])$ (see Theorem 19).

2 Preliminaries

2.1: Lattices In this section, we present some prerequisites related to the content of ordered algebraic structures. For more details we refer the reader to [6]. Throughout the text we assume that all lattices and partially ordered sets are finite.

We say that a partially ordered set P is *totally ordered*, or a *chain*, if all elements of P are comparable under $\leqslant$ (that is, $x \leqslant y$ or $y \leqslant x$ for all elements $x, y \in P$). We denote by $[n]$ the totally ordered poset $\{1, \dots, n\}$ with $1 < \cdots < n$. An *antichain* is a partially ordered set in which any two different elements are incomparable, that is, in which $x \leqslant y$ if and only if $x = y$. A *maximum antichain* of P is an antichain in P that has cardinality at least as large as every other antichain. The *width of* P, denoted by $\mathrm{width}(P)$ is the cardinality of a maximum antichain.

A lattice L is called *distributive* if, for all $a, b, c \in L$, the distributive laws $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ hold.

For every x, y of a partially ordered set P, we say y *covers* x or x *is covered* by y if $x < y$, but there does not exist a z such that $x < z < y$. We denote by $C(x)$ the set of all elements that cover x and by $C^*(x)$ the set of elements that are covered by x. Let L be a lattice. For $x \in L$, the join of all elements that cover x is denoted by x^* and the meet of all elements that are covered by x is denoted by x_*. A lattice is called *join-distributive* (resp. *meet-distributive*) if for any $x \in L$, the interval $[x, x^*]$ (resp. $[x_*, x]$) is a distributive lattice.

An element of L is called *meet-irreducible* (resp. *join-irreducible*) if exactly one element covers it (resp. exactly one element is covered by it).

A decomposition $y = q_1 \wedge \cdots \wedge q_m$ of an element $y \in L$ in to a meet of meet-irreducible elements is said to be *irredundant* if $\forall 1 \leqslant i \leqslant m, \quad y < \bigwedge_{j \neq i} q_j$. In this case, q_is are called the *meet-irreducible components* of y. The notion of *irredundant join-decomposition* is defined in a similar way. A well-known result of Dilworth says that every element of L has a unique irredundant meet-decomposition (resp. unique irredundant join-decomposition) if and only if L is a join-distributive lattice (resp. meet-distributive lattice), see [1] and also [7, Theorem 5-2.1]. It is clear that by the above result in each distributive lattice L every element has both the unique irredundant join-decomposition and the unique irredundant meet-decomposition.

The next corollary ([1, Corollary 1.3]) is useful in our investigation.

Corollary 1. *Let L be a join-distributive lattice. Then the number of meet-irreducible components of an element $x \in L$, is equal to the number of distinct elements that cover x.*

Finally we recall a property of distributive lattices that we use it several times in the sequel:

If L is a distributive lattice and $a, b \in L$ with $a < b$, then there exists a lattice homomorphism $\phi : L \to [2]$ such that $\phi(a) = 1$ and $\phi(b) = 2$ (see [6, Corollary 2.1.20]).

2.2: Monomial ideals For the concepts of primary decomposition of ideals and associated prime ideals of a given ideal in commutative rings we refer the reader to the standard texts of commutative algebra like [2]. Here, we recall some notions of monomial ideals. For more details see [8, Chapter 1].

In the following, let $\mathbf{k}$ be a field and $R = \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbf{k}$. Any product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i \in \mathbf{Z}_{\geqslant 0}$ is called a *monomial* and an ideal $I \subseteq R$ which is generated by monomials is called a *monomial ideal*. We notice that each monomial ideal I of R has a unique minimal monomial set of generators (see [8, Proposition 1.1.6]) which is denoted by $G(I)$. If $\mathfrak{p}$ is an associated prime ideal of a monomial ideal I, then, by [8, Corollary 1.3.9], $\mathfrak{p}$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

A monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called *squarefree* if for each $1 \leqslant i \leqslant n$, $\alpha_i \in \{0, 1\}$. A monomial ideal I is called *squarefree monomial ideal* if it is generated by squarefree monomials. If $I \subset R$ is a squarefree monomial ideal, then, by [8, Corollary 1.2.5], I is a radical ideal and if $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ is the minimal primary decomposition of I, then the *Alexander dual* of I, denoted by $I^\vee$, is a squarefree monomial ideal generated by monomials $u_1, \ldots, u_r$, where each u_i is the product of the variables that generate the monomial prime ideal $\mathfrak{p}_i$. The ideals I and $I^\vee$ are closely related via the theory of simplicial complexes (see [8, Section 1.5]).

For a given ideal I of an arbitrary Noetherian ring R, if $I = Q_1 \cap \cdots \cap Q_r$ is a minimal primary decomposition of I, then the *equidimensional part of* I is defined as the intersection of all primary ideals Q_i, with $\dim(R/I) = \dim(R/Q_i)$, where by dim we mean the Krull dimension (for definition and basis properties of Krull dimension see [2, Part II Chapter 8]).

3 Equidimensional part of ideal of lattice homomorphisms

Let L and M be two lattices with $|L| > 1$ and $|M| > 1$. We are going to study $\operatorname{ass}(J(L,M))$ and relate it to $\operatorname{ass}(I(L,M))$. We remark that all pairs of posets P, Q for which $I(P,Q)$ is an unmixed ideal characterized in [10] (recall that an ideal is called unmixed if all of its associated prime ideals have the same height). For each $\psi \in \operatorname{Hom}_{\operatorname{Pos}}(M,L)$, we consider the prime ideal

$$\mathfrak{p}_\psi = (x_{\psi(m),m};\ m \in M) \subset S,$$

we also define monomial u_ψ^τ as

$$u_\psi^\tau := \prod_{m\in M} x_{\psi(m),m}.$$

By using [5, Proposition 1.5], we immediately get the first result.

Proposition 2. *Let L and M be two lattices. Then the following statements hold.*

(1) *If $\mathfrak{p} \in \operatorname{ass}(J(L,M))$ then $\operatorname{ht}(\mathfrak{p}) \geqslant |M|$.*

(2) *For each $\psi \in \operatorname{Hom}_{\operatorname{Pos}}(M,L)$, $\mathfrak{p}_\psi \in \operatorname{ass}(J(L,M))$.*

(3) *$I(M,L)^\tau \subseteq J(L,M)^\vee$, where $I(M,L)^\tau := (u_\psi^\tau;\ \psi \in \operatorname{Hom}_{\operatorname{Pos}}(M,L))$.*

(4) *$\operatorname{ht}(J(L,M)) = |M|$.*

Proof. (1). Let $\mathfrak{p} = (x_{l_1,m_1}, \ldots, x_{l_n,m_n}) \in \operatorname{ass}(J(L,M))$. For any $m \in M$, let $\phi_m : L \to M$ be the constant lattice homomorphism with $\phi_m(l) = m$ for each $l \in L$. It is clear that $u_{\phi_m} = \prod_{l\in L} x_{l,m} \in \mathfrak{p}$. So, for some $1 \leqslant i \leqslant n$, we should have $m_i = m$ which shows that $n \geqslant |M|$.

(2) and (3). Let $\psi \in \operatorname{Hom}_{\operatorname{Pos}}(M,L)$. By [5, Proposition 1.5], $J(L,M) \subseteq I(L,M) \subseteq \mathfrak{p}_\psi$. Now part (1), shows that $\mathfrak{p}_\psi \in \operatorname{ass}(J(L,M))$ and the conclusion follows.

(4). It is followed by parts (1) and (2). $\square$

In the next theorem, we detect when each associated prime ideal $\mathfrak{p}$ of $J(L,M)$ with $\operatorname{ht}(\mathfrak{p}) = |M|$ is of the form $\mathfrak{p}_\psi$ for some $\psi \in \operatorname{Hom}_{\operatorname{Pos}}(L,M)$.

Theorem 3. *Let L and M be two lattices. Then*

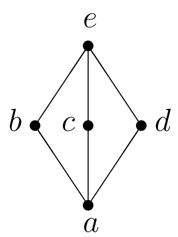
$$\{\mathfrak{p};\ \mathfrak{p} \in \operatorname{ass}(J(L,M)), \operatorname{ht}(\mathfrak{p}) \leqslant |M|\} = \{\mathfrak{p}_\psi;\ \psi \in \operatorname{Hom}_{\operatorname{Pos}}(M,L)\}$$

if and only if L is a distributive lattice.

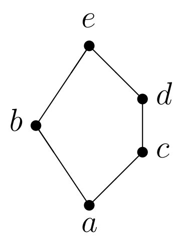
Proof. First, assume that L is a distributive lattice. Let $\mathfrak{p}$ be any minimal prime ideal of $J(L,M)$ with $\operatorname{ht}(\mathfrak{p}) = |M|$. Since $J(L,M) \subset \mathfrak{p}$, it follows that for each $m \in M$ there exists an element $\psi(m) \in L$ such that $x_{\psi(m),m} \in \mathfrak{p}$. Then $\mathfrak{p} = (x_{\psi(m),m};\ m \in M)$. It remains to be shown that $\psi : M \to L$ is isotone. Suppose this is not the case. Then there exist $m, m' \in M$ such that $m < m'$ and $\psi(m) \nleq \psi(m')$. So, $\psi(m') < \psi(m) \vee \psi(m')$.

Thus, there exists $\phi \in \mathrm{Hom}_{\mathrm{Lat}}(L, \{m < m'\}) \subseteq \mathrm{Hom}_{\mathrm{Lat}}(L, M)$ such that $\phi(\psi(m')) = m$ and $\phi(\psi(m) \vee \psi(m')) = m'$ (see the last paragraph in Section 2.1). This shows that $\phi(\psi(m')) = m$ and $\phi(\psi(m)) = m'$. So $u_\phi \notin \mathfrak{p}$ which is a contradiction.

Conversely, assume that L is not a distributive lattice. Then it contains a sublattice isomorphic to the Diamond lattice or the Pentagon lattice.



Diamond lattice
$\boldsymbol{M_5}$



Pentagon lattice
$\boldsymbol{N_5}$

Case 1: L has a sublattice isomorphic to the Diamond lattice. Let m_1 be the least element of M and $m_2 \in M$ covers m_1. Let

$$\mathfrak{p} = (x_{e,m};\ m \in M \setminus \{m_2\}) + (x_{d,m_2}).$$

It is clear that $\mathrm{ht}(\mathfrak{p}) = |M|$. We show that $\mathfrak{p} \in \mathrm{ass}(J(L, M))$. To see it, we prove that $J(L, M) \subset \mathfrak{p}$. Let $\phi \in \mathrm{Hom}_{\mathrm{Lat}}(L, M)$. If $\phi(e) \neq m_2$, then $x_{e,\phi(e)}|u_\phi$ and $x_{e,\phi(e)} \in \mathfrak{p}$. So $u_\phi \in \mathfrak{p}$. So we just need to check the case that $\phi(e) = m_2$. In this situation we discuss about $\phi(d)$. We claim that $\phi(d) = m_2$. If this is not the case, then $\phi(d) = m_1$. Therefore, by the fact that $\phi(e) = \phi(c) \vee \phi(d) = \phi(b) \vee \phi(d)$, we have $\phi(c) = \phi(b) = m_2$. But we should also have $\phi(a) = \phi(b) \wedge \phi(c) = \phi(c) \wedge \phi(d)$ that implies that $m_1 = m_2$ which is a contradiction. So $\phi(d) = m_2$ and since $x_{d,m_2} \in \mathfrak{p}$, we conclude $u_\phi \in \mathfrak{p}$. Now to complete the argument, it is enough to note that the map $\psi : M \to L$ corresponding to $\mathfrak{p}$ is not an isotone map.

Case 2: L has a sublattice isomorphic to the Pentagon lattice. Let m_1 and m_2 be as in the first case. Let

$$\mathfrak{p} = (x_{d,m};\ m \in M \setminus \{m_2\}) + (x_{c,m_2}).$$

Again we prove that $J(L, M) \subset \mathfrak{p}$. Let $\phi \in \mathrm{Hom}_{\mathrm{Lat}}(L, M)$. In order to show that $u_\phi \in \mathfrak{p}$, as the previous paragraph, we just need to discuss the case that $\phi(d) = m_2$. We claim that in this situation $\phi(c) = m_2$. If it is not the case, then $\phi(c) = \phi(a) = m_1$. So $\phi(e) = \phi(b) \vee \phi(c) = \phi(b)$. So, $\phi(d) = m_2 \leqslant \phi(e) = \phi(b)$. By this equality we have

$$m_1 = \phi(c) \wedge \phi(b) = \phi(a) = \phi(d) \wedge \phi(b) = m_2$$

which is a contradiction. So $\phi(c) = m_2$ and $J(L, M) \subseteq \mathfrak{p}$. Again, the map $\psi : M \to L$ corresponding to $\mathfrak{p}$ is not an isotone map. $\square$

Theorem 3 yields the following.

Corollary 4. *Let L and M be two lattices. Then $I(L, M)$ and $J(L, M)$ have the same equidimensional part if and only if L is a distributive lattice.*

Now, the question of when for a distributive lattice L, $J(L, M)$ is an unmixed ideal is easy to answer.

Corollary 5. *For a given distributive lattice L and a lattice M the following conditions are equivalent.*

(1) $J(L, M) = I(L, M)$.

(2) L *is a chain.*

(3) $J(L, M)$ *is unmixed.*

Proof. (1) $\Rightarrow$ (2). Assume that L is not a chain. So, there exist two incomparable elements $l_1, l_2 \in L$. Let m_1, m_2 be the unique minimal element and the unique maximal element of M respectively. We define $\phi : L \to M$ by $\phi(l) = m_1$ if $l \leqslant l_1 \wedge l_2$ and $\phi(l) = m_2$ if $l \nleqslant l_1 \wedge l_2$. One can easily see that

$$\phi \in \mathrm{Hom}_{\mathrm{Pos}}(L, M) \setminus \mathrm{Hom}_{\mathrm{Lat}}(L, M).$$

So $J(L, M) \subsetneq I(L, M)$.

(2) $\Rightarrow$ (3). If L is a chain, it is clear that $\mathrm{Hom}_{\mathrm{Pos}}(L, M) = \mathrm{Hom}_{\mathrm{Lat}}(L, M)$. So $J(L, M) = I(L, M)$ and, by [10, Corollary 1.5], it should be an unmixed ideal.

(3) $\Rightarrow$ (1). Now assume that $J(L, M)$ is unmixed. So, by Theorem 3,

$$J(L, M) = \bigcap_{\psi \in \mathrm{Hom}_{\mathrm{Pos}}(M, L)} \mathfrak{p}_\psi.$$

Therefore, $J(L, M) = I(L, M)$. □

4 Primary decomposition of $J(L, [2])$

In this section, we assume that $M = [2]$ and we study $\mathrm{ass}(J(L, [2]))$ when L is a distributive lattice. Note that by the results of the previous section if $\mathfrak{p} \in \mathrm{ass}(J(L, [2]))$ and $\mathrm{ht}(\mathfrak{p}) = 2$ then $\mathfrak{p} = \mathfrak{p}_\psi$ for some $\psi \in \mathrm{Hom}_{\mathrm{Pos}}([2], L)$. So, we are going to determine $\mathfrak{p} \in \mathrm{ass}(J(L, [2]))$ with $\mathrm{ht}(\mathfrak{p}) > 2$. We start with the following easy lemma.

Lemma 6. *Let L be a lattice. If A and B are nonempty subsets of L such that $\bigwedge A \leqslant \bigvee B$, then*

$$J(L, [2]) \subseteq \mathfrak{p}_{A,B} = (x_{a,1};\ a \in A) + (x_{b,2};\ b \in B).$$

Proof. Consider $\phi \in \mathrm{Hom}_{\mathrm{Lat}}(L, [2])$. If $\phi(\bigvee B) = 2$, then there exists $b \in B$ such that $\phi(b) = 2$, which implies that

$$x_{b,2} = x_{b,\phi(b)} | u_\phi.$$

Hence, $u_\phi \in \mathfrak{p}_{A,B}$. If $\phi(\bigvee B) = 1$, then $\phi(\bigwedge A) = 1$, which follows that there exists $a \in A$ such that $\phi(a) = 1$. Thus,

$$x_{a,1} = x_{a,\phi(a)} | u_\phi,$$

it implies that $u_\phi \in \mathfrak{p}_{A,B}$. Therefore, $J(L, [2]) \subseteq \mathfrak{p}_{A,B}$. $\square$

Lemma 7. *Let L be a lattice, and A, B be subsets of L. If $\mathfrak{p}_{A,B} \in \operatorname{ass}(J(L, [2]))$, then $A \neq \varnothing$ and $B \neq \varnothing$.*

Proof. It is enough to follow the proof of Proposition 2. $\square$

Now, let L be an arbitrary distributive lattice. It is clear that each minimal prime ideal of $J(L, [2])$ is of the form $\mathfrak{p}_{A,B}$ for some $A \subseteq L$ and $B \subseteq L$. Which ones of the prime ideals presented in this form are the minimal prime ideals of $J(L, M)$? The following results help to find them.

Theorem 8. *Let L be a distributive lattice, and A, B be nonempty subsets of L. Then $\mathfrak{p}_{A,B} \in \operatorname{ass}(J(L, [2]))$ if and only if the following statements hold:*

(1) $\bigwedge A \leqslant \bigvee B$.

(2) *For every $\varnothing \neq A_1 \subset A$ and for every $\varnothing \neq B_1 \subset B$, $\bigwedge A_1 \nleqslant \bigvee B$ and $\bigwedge A \nleqslant \bigvee B_1$.*

Proof. First assume that (1) and (2) hold. By Lemma 6, $J(L, [2]) \subseteq \mathfrak{p}_{A,B}$. So, there exists a prime ideal $\mathfrak{p}_1 \in \operatorname{ass}(J(L, [2]))$ such that $J(L, [2]) \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_{A,B}$. Hence, by Lemma 7, we can assume that there exist $\varnothing \neq A_1 \subseteq A$ and $\varnothing \neq B_1 \subseteq B$ such that $\mathfrak{p}_1 = \mathfrak{p}_{A_1,B_1}$. If $A_1 = A$ and $B_1 = B$, then $\mathfrak{p}_{A,B} = \mathfrak{p}_{A_1,B_1} \in \operatorname{ass}(J(L, [2]))$ and the proof is now complete. Now assume that $A_1 \subset A$ or $B_1 \subset B$. By (2), $\bigwedge A_1 \nleqslant \bigvee B_1$, which follows that $\bigvee B_1 < \bigwedge A_1 \vee \bigvee B_1$. Thus, there exists a lattice homomorphism $\phi : L \to [2]$ such that $\phi(\bigvee B_1) = 1$ and $\phi(\bigwedge A_1 \vee \bigvee B_1) = 2$. It implies that

$$\forall a \in A_1, \phi(a) = 2 \text{ and } \forall b \in B_1, \phi(b) = 1,$$

which follows that $x_{a,1} \nmid u_\phi$ for every $a \in A_1$ and $x_{b,2} \nmid u_\phi$ for every $b \in B_1$. Therefore, $u_\phi \notin \mathfrak{p}_{A_1,B_1}$ and this is a contradiction, which proves $A = A_1$ and $B = B_1$.

Conversely, suppose that $\mathfrak{p}_{A,B} \in \operatorname{ass}(J(L, [2]))$. If $\bigwedge A \nleqslant \bigvee B$, then $\bigvee B < \bigwedge A \vee \bigvee B$. Thus, there exists a lattice homomorphism $\phi : L \to [2]$ such that $\phi(\bigvee B) = 1$ and $\phi(\bigwedge A \vee \bigvee A) = 2$. It implies that

$$\forall a \in A, \phi(a) = 2 \text{ and } \forall b \in B, \phi(b) = 1,$$

which follows that $x_{a,1} \nmid u_\phi$ for every $a \in A$ and $x_{b,2} \nmid u_\phi$ for every $b \in B$. Therefore, $u_\phi \notin \mathfrak{p}_{A,B}$ and again we get a contradiction. Thus, the statement (1) holds.

Now, suppose that $\varnothing \neq A_1 \subseteq A$ and $\bigwedge A_1 \leqslant \bigvee B$. By Lemma 6,

$$J(L, [2]) \subseteq \mathfrak{p}_{A_1,B} \subseteq \mathfrak{p}_{A,B} \in \operatorname{ass}(J(L, [2])) = \min(J(L, [2])),$$

and so, $A = A_1$. By a similar argument, we can show that if $\varnothing \neq B_1 \subseteq B$ and $\bigwedge A \leqslant \bigvee B_1$ then $B = B_1$. Thus, the statement (2) holds. $\square$

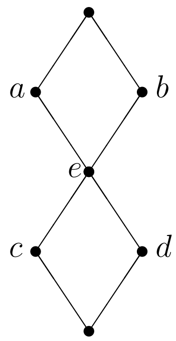
Remark 9. Let L be a distributive lattice, $n > 2$ and $\mathfrak{p} \subset \mathbf{k}[x_{l,m};\ l \in L, m \in [n]]$ be a monomial prime ideal. If $J(L,[n]) \subset \mathfrak{p}$ then, by the proof of Proposition 2, there exist nonempty subsets $A_1, \ldots, A_n$ of L such that

$$\mathfrak{p} = \mathfrak{p}_{A_1,\ldots,A_n} := \sum_{i=1}^{n} (x_{a,i};\ a \in A_i).$$

By a similar argument as the proof of Theorem 8, one can see that if $J(L,[n]) \subset \mathfrak{p}_{A_1,\ldots,A_n}$ for some nonempty subsets $A_1, \ldots A_n$ of L then

$$\forall 1 \leqslant i < j \leqslant n,\ \bigwedge A_i \leqslant \bigvee A_j \tag{3}$$

But it may happen that (3) holds and $J(L,[n]) \nsubseteq \mathfrak{p}_{A_1,\ldots A_n}$. For example, let $n = 3$ and L be the following lattice:



L

Put $A_1 = \{e\}$, $A_2 = \{a, c\}$ and $A_3 = \{b\}$. Then (3) holds for A_1, A_2, A_3 but $J(L,[3]) \nsubseteq \mathfrak{p}_{A_1,A_2,A_3}$. Because if $\phi : L \to [3]$ is the lattice homomorphism with $\phi(d) = \phi(e) = \phi(b) = 2$, $\phi(a) = 3$ and $\phi(c) = 1$, then $u_\phi \notin \mathfrak{p}_{A_1,A_2,A_3} = (x_{e,1}, x_{a,2}, x_{c,2}, x_{b,3})$.

We are going to apply Theorem 8 and find an upper bound for the height of minimal prime ideals of $J(L,[2])$. In the following theorem

$$m(L) = \max\{|C(x)|;\ x \in L\}$$

and

$$M(L) = \max\{|C^*(x)|;\ x \in L\}.$$

Theorem 10. *Let L be a distributive lattice and $\mathfrak{p} \in \operatorname{ass}(J(L,[2]))$. Then $\operatorname{ht}(\mathfrak{p}) \leqslant m(L) + M(L)$.*

Proof. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_t\}$ be two subsets of L and $\mathfrak{p}_{A,B} \in \operatorname{ass}(J(L,[2]))$. Consider

$$a = \bigwedge A = \bigwedge_{i=1}^{n} a_i. \tag{4}$$

Assume that for each $1 \leqslant \ell \leqslant n$, $a_\ell = \bigwedge_{k=1}^{m_\ell} a_{\ell k}$ is the unique irredundant meet-decomposition of a_ℓ. If in Equation (4) we replace each a_ℓ with its irredundant meet-decomposition, we get a decomposition of a as a meet of meet-irreducible elements. We can refine this decomposition to produce the unique one in which no terms is redundant. So each meet-irreducible component of a is one of the $a_{\ell k}$s.

Let $1 \leqslant \ell \leqslant n$. By Theorem 8, $a < \bigwedge_{i \neq \ell} a_i$. So, one can see that there exists $1 \leqslant k \leqslant m_\ell$ such that $a_{\ell k}$ is a meet-irreducible component of both a_ℓ and a while it is not a meet-irreducible component of any of the remaining a_is. So the number of meet-irreducible components of a is at least n. Therefore, by Corollary 1, $|C(a)| \geqslant n$.

By a similar method, we can decompose $b = \bigvee B = \bigvee_{j=1}^{t} b_j$ as a join of join-irreducible elements and see that $C^*(b) \geqslant t$. Now, we have

$$\mathrm{ht}(\mathfrak{p}) = n + t \leqslant C(a) + C^*(b) \leqslant m(L) + M(L). \qquad \square$$

Note that in some distributive lattices, the bound proposed in Theorem 10 is sharp and in some others is not. For example if $L = [2] \times [n]$, $n > 1$, this bound is not sharp. In this case, the minimal prime ideals of $J(L, [2])$ are of height 2 or 3 while $m(L)+M(L) = 4$. But, if we consider the lattices described in Lemma 18 the given bound is sharp.

The next corollary is followed by Theorem 8. It suggests a recursive method to check if a monomial prime ideal belongs to $\mathrm{ass}(J(L, [2]))$ or not.

Corollary 11. *Let L be a distributive lattice and A, B be two nonempty subsets of L where $|A| > 1$ or $|B| > 1$. Then $\mathfrak{p}_{A,B} \in \mathrm{ass}(J(L, [2]))$ if and only if the following statements hold.*

(1) *Both A and B are antichains in L.*

(2) *for every $a \in A$ and every $b \in B$, $a \nleqslant b$.*

(3) *If a, a' are two arbitrary distinct elements of A and $A' = (A \setminus \{a, a'\}) \cup \{a \wedge a'\}$, then $\mathfrak{p}_{A',B} \in ass(J(L, [2]))$.*

(4) *If b, b' are two arbitrary distinct elements of B and $B' = (B \setminus \{b, b'\}) \cup \{b \vee b'\}$, then $\mathfrak{p}_{A,B'} \in ass(J(L, [2]))$.*

Proof. If $\mathfrak{p}_{A,B} \in \mathrm{ass}(J(L, [2]))$, it is easy to see that (1),(2),(3) and (4) hold.

Conversely, assume that the monomial prime ideal $\mathfrak{p}_{A,B}$ satisfies the given conditions. We show that the necessary conditions of Theorem 8 hold for A and B. We suppose that $|A| > 1$ (the case $|B| > 1$ can be discussed similarly). Choose two distinct elements $a, a' \in A$ and let $A' = A \setminus \{a, a'\}$. By the statement (3), $\mathfrak{p}_{A',B} \in \mathrm{ass}(J(L, [2]))$. So, by applying Theorem 8 for $\mathfrak{p}_{A',B}$,

$$\bigwedge A = \bigwedge A' \leqslant \bigvee B,$$

and for every $\varnothing \neq B_1 \subset B$ we have $\bigwedge A \nleqslant \bigvee B_1$.

We prove that for every $\varnothing \neq A_1 \subset A$, $\bigwedge A_1 \nleqslant \bigvee B$. By contrary assume that there exists $A_1 \subset A$ with $|A_1| = |A|-1$ such that $\bigwedge A_1 \leqslant \bigvee B$. We conclude from the statement (2) that $|A| > 2$ or $|B| > 1$.

Assume that $|A| > 2$. Choose two distinct elements $a_1, a_2 \in A_1$. Let $A' = (A \setminus \{a_1, a_2\}) \cup \{a_1 \wedge a_2\}$ and $A'_1 = (A_1 \setminus \{a_1, a_2\}) \cup \{a_1 \wedge a_2\}$. It is clear that $\bigwedge A_1 = \bigwedge A'_1 \leqslant \bigvee B$. Now by Lemma 6 and condition (3) we have:

$$J(L, [2]) \subseteq \mathfrak{p}_{A'_1, B} \subseteq \mathfrak{p}_{A', B} \in \operatorname{ass}(J(L, [2])) = \min(J(L, [2])$$

which shows that $\mathfrak{p}_{A'_1, B} = \mathfrak{p}_{A', B}$ and so $A' = A'_1$. But, by statement (1), $A'_1 \subset A'$ so we get a contradiction.

Assume that $|B| > 1$. Choose two distinct elements $b, b' \in B$ and let $B' = (B \setminus \{b, b'\}) \cup \{b \vee b'\}$. By the statement (4), $\mathfrak{p}_{A,B'} \in \operatorname{ass}(J(L, [2]))$. Since $\bigwedge A_1 \leqslant \bigvee B = \bigvee B'$, by applying Theorem 8 for $\mathfrak{p}_{A,B'}$, we get a contradiction. $\square$

The next result is an immediate consequence of Corollary 11 and shows that the Alexander dual of $J(L, [2])$ is generated in successive degrees.

Corollary 12. *Let L be a distributive lattice and*

$$s = \max\{\deg(u);\ u \in G(J(L, [2]))^\vee\}.$$

Then for each $2 \leqslant i \leqslant s$, $G(J(L, [2]))^\vee$ has an element of degree i.

In the following we are trying to detect when each element of $\operatorname{ass}(J(L, [2]))$ has height at most 3. First, note that by Theorem 8, we have:

Corollary 13. *Let L be a distributive lattice. A monomial prime ideal $\mathfrak{p} \subset S$ of height 3 is an associated prime of $J(L, [2])$ if and only if $\mathfrak{p}$ has one of the following shapes:*

$$\mathfrak{p} = (x_{a,1}, x_{b_1,2}, x_{b_2,2}),\ \textit{where } a \nleqslant b_1, a \nleqslant b_2, a \leqslant b_1 \vee b_2 \textit{ and } \{b_1, b_2\} \textit{ is an antichain,}$$

or

$$\mathfrak{p} = (x_{a_1,1}, x_{a_2,1}, x_{b,2}),\ \textit{where } a_1 \nleqslant b, a_2 \nleqslant b, a_1 \wedge a_2 \leqslant b \textit{ and } \{a_1, a_2\} \textit{ is an antichain.}$$

To proceed our goal we see that if L has an antichain with 3 elements, then $J(L, [2])$ has an associated prime of height 4. For this, we need the following lemmas.

Lemma 14. *Assume that $\{a, b, c\}$ is an antichain contained in a distributive lattice L. Then the following statements hold.*

1. *If $a \wedge b = a \wedge c$ then $\{a \vee b, b \vee c, a \vee c\}$ is an antichain.*
2. *If $a \vee b = a \vee c$ then $\{a \wedge b, b \wedge c, a \wedge c\}$ is an antichain.*

Proof. (1). From $a \wedge b = a \wedge c$, we infer that

$$b = b \vee (a \wedge b) = b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c)$$

Now, if $b \vee a$ and $b \vee c$ are comparable then $b = b \vee a$ or $b = b \vee c$, which shows that b, c are comparable or b, a are comparable which is a contradiction. So $b \vee a$ and $b \vee c$ are incomparable. Similarly, we can see that $a \vee c$ and $b \vee c$ are incomparable. Now, if $a \vee b \leqslant a \vee c$ then $a \vee b \vee c = a \vee c$, which shows that $b \vee c \leqslant a \vee c$ which is again a contradiction. So $\{a \vee b, b \vee c, a \vee b\}$ is an antichain.

The proof of (2) is similar. $\square$

Lemma 15. *Assume that $\{a,b,c\}$ is an antichain contained in a distributive lattice L. If $a\wedge b$, $a\wedge c$ and $b\wedge c$ are three different elements of L and $a\wedge b<a\wedge c$ then the following statements hold.*

(1) *$a\wedge c$ and $b\wedge c$ are incomparable and $a\wedge b<b\wedge c$.*

(2) *$\{a\vee b, a\vee c, b\vee c\}$ is an antichain or, $a\vee c$ and $b\vee c$ are incomparable, $a\vee b>a\vee c$ and $a\vee b>b\vee c$.*

Proof. (1). From $a\wedge b<a\wedge c$, we conclude that $a\wedge b\wedge c=a\wedge b$, which implies that $a\wedge b<b\wedge c$. On the other hand, if for example we have $b\wedge c<a\wedge c$ then $a\wedge b\wedge c=b\wedge c$, which follows that $a\wedge b=b\wedge c$ and this is a contradiction.

(2). Note that, by Lemma 14(2), $a\vee c$, $b\vee c$ and $a\vee b$ are three different elements of L. Now, if $\{a\vee b, a\vee c, b\vee c\}$ is not an antichain, then similar to the proof of (1), one can see that one of the $a\vee b, a\vee c, b\vee c$ is bigger than the others and the other two elements are incomparable. We show that the case that $a\vee c>b\vee c$, $a\vee c>a\vee b$ and, $b\vee c, a\vee b$ are incomparable does not happen (by a similar argument one can show that the case that $b\vee c>a\vee c, b\vee c>a\vee b$ and, $a\vee c$ and $a\vee b$ are incomparable does not happen).

Assume that $a\vee c>b\vee c$ and $a\vee c>a\vee b$. Since

$$b\geqslant (a\vee c)\wedge b\geqslant (b\vee c)\wedge b=b,$$

we conclude from (1) that

$$b=(a\wedge b)\vee(c\wedge b)=c\wedge b\leqslant c,$$

which is a contradiction. □

Using the above lemmas, next we show that if $\mathrm{width}(L)>2$ then $\mathrm{ass}(J(L,[2]))$ has an element of height 4.

Theorem 16. *Assume that $\{a,b,c\}$ is an antichain contained in a distributive lattice L. Then $J(L,[2])$ has a minimal prime of height 4.*

Proof. Case 1: Assume that $\{a\wedge b, a\wedge c, b\wedge c\}$ is an antichain. If we let $A=\{a,b,c\}$ and $B=\{a\wedge b\wedge c\}$, then by Theorem 8, $\mathfrak{p}_{A,B}\in\mathrm{ass}(J(L,[2]))$.

Case 2: Assume that $a\wedge b=a\wedge c$. Then, by Lemma 14(1), $\{a\vee b, a\vee c, b\vee c\}$ is an antichain. If we let $A=\{a\vee b\vee c\}$ and $B=\{a,b,c\}$, then by Theorem 8, $\mathfrak{p}_{A,B}\in\mathrm{ass}(J(L,[2]))$.

Case 3: Assume that $a\wedge b$, $a\wedge c$ and $b\wedge c$ are three different elements of L and $a\wedge b<a\wedge c$. If $\{a\vee b, a\vee c, b\vee c\}$ is an antichain, then, if we define A, B as the case 2, we get $\mathfrak{p}_{A,B}\in\mathrm{ass}(J(L,[2]))$. If $\{a\vee b, a\vee c, b\vee c\}$ is not an antichain, then, by Lemma 15(1), $a\wedge c$ and $b\wedge c$ are incomparable and $a\wedge b<b\wedge c$. Also, by Lemma 15(2), $a\vee b>a\vee c$, $a\vee b>b\vee c$ and $a\vee c, b\vee c$ are incomparable. So we have

$$(a\wedge c)\vee(b\wedge c)=(a\vee b)\wedge c=(a\vee b\vee c)\wedge c=c$$

and

$$(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c = (a \wedge b \wedge c) \vee c = c.$$

Thus, if we put $A = \{a \vee c, b \vee c\}$ and $B = \{a \wedge c, b \wedge c\}$, then by Theorem 8, $\mathfrak{p}_{A,B} \in \mathrm{ass}(J(L,[2]))$. $\square$

We recall that a distributive lattice is *planar* if and only if it is a sublattice of a direct product of two chains if and only if no element covers more than two elements (See [13, page 3]). The next corollary is an immediate consequence of Theorem 16 and shows that if $J(L,[2])^\vee$ does not have any generator of degree 4, then L must be a planar lattice of width at most 2.

Corollary 17. *If L is a distributive lattice and any associated prime of $J(L,[2])$ is of height at most 3, then L is a planar lattice and* $\mathrm{width}(L) \leqslant 2$.

Proof. First note that if L has an antichain with 3 elements then, by Theorem 16, $\mathrm{ass}(J(L,[2]))$ has an element of the height 4. So $\mathrm{width}(L) \leqslant 2$. It is clear that in this case, no element of L covers more than two other elements. So L should be a planar lattice. $\square$

Note that the converse of Corollary 17 does not hold. For example if L is the lattice of Remark 9, then $\mathrm{width}(L) = 2$ and $(x_{a,1}, x_{b,1}, x_{c,2}, x_{d,2}) \in \mathrm{ass}(J(L,[2]))$.

5 The case $J([m] \times [n], [2])$

In this section, we are going to study $J([m] \times [n], [2])$ more carefully. We assume that $m \leqslant n$ and describe $\mathrm{ass}(J([m] \times [n], [2]))$. We first need the following lemma.

Lemma 18. *Let $L = [m_1] \times \cdots \times [m_\ell]$, where $\ell \geqslant 2$ and each m_i is at least 3. Then*

$$\max\{\deg(u);\ u \in G(J(L,[2])^\vee)\} = 2\ell.$$

Proof. For each $1 \leqslant i \leqslant \ell$, let $\mathbf{a_i} = (a_{i1}, \cdots, a_{i\ell})$, where $a_{ij} = \begin{cases} 3, & \text{if } j \neq i \\ 2, & \text{if } j = i \end{cases}$ and let $\mathbf{b_i} = (b_{i1}, \cdots, b_{i\ell})$, where $b_{ij} = \begin{cases} 1, & \text{if } j \neq i \\ 2, & \text{if } j = i \end{cases}$. Using Theorem 8, one can easily see that $\mathfrak{p}_{A,B} \in \mathrm{ass}(J(L,[2]))$ where $A = \{\mathbf{a_1}, \ldots, \mathbf{a_\ell}\}$ and $B = \{\mathbf{b_1}, \ldots, \mathbf{b_\ell}\}$. Note that $\mathrm{ht}(\mathfrak{p}_{A,B}) = 2\ell$.

On the other hand, if $\mathbf{a} = (a_1, \ldots, a_\ell), \mathbf{b} = (b_1, \ldots, b_\ell) \in L$, then $\mathbf{a}$ covers $\mathbf{b}$ if and only if there exists $1 \leqslant i \leqslant \ell$ such that $a_i = b_i + 1$ and for each $j \neq i$, $a_j = b_j$. This shows that $m(L) + M(L) = 2\ell$ and the conclusion follows by Theorem 10. $\square$

Note that by Lemma 18, each minimal prime of $J([m] \times [n], [2])$ has height at most 4.

Theorem 19. *Let $L = [m] \times [n]$ and $\mathfrak{p}$ is a monomial prime ideal of S. Then $\mathfrak{p} \in \mathrm{ass}(J(L,[2]))$ if and only if one of the following conditions hold:*

(1) $\mathfrak{p} = \mathfrak{p}_\psi$ *for some* $\psi \in \mathrm{Hom}_{\mathrm{Pos}}([2], [m] \times [n])$.

(2) $\mathfrak{p} = (x_{(i_1,j_1),1}, x_{(i_2,j_2),2}, x_{(i_3,j_3),2})$, *where*

$$1 \leqslant i_2 < i_1 \leqslant i_3 \leqslant m \text{ and } 1 \leqslant j_3 < j_1 \leqslant j_2 \leqslant n.$$

(3) $\mathfrak{p} = (x_{(i_1,j_1),1}, x_{(i_2,j_2),1}, x_{(i_3,j_3),2})$, *where*

$$1 \leqslant i_1 \leqslant i_3 < i_2 \leqslant m \text{ and } 1 \leqslant j_2 \leqslant j_3 < j_1 \leqslant n.$$

(4) $\mathfrak{p} = (x_{(i_1,j_1),1}, x_{(i_2,j_2),1}, x_{(i_3,j_3),2}, x_{(i_4,j_4),2})$, *where*

$$1 \leqslant i_3 < i_1 \leqslant i_4 < i_2 \leqslant m \text{ and } 1 \leqslant j_4 < j_2 \leqslant j_3 < j_1 \leqslant n.$$

Proof. First note that if $\mathfrak{p} \in \mathrm{ass}(J(L, [2]))$, then $2 \leqslant \mathrm{ht}(\mathfrak{p}) \leqslant 4$. If $\mathrm{ht}(\mathfrak{p}) = 2$ then, by Theorem 3, $\mathfrak{p} \in \mathrm{ass}(J([m] \times [n], [2]))$ if and only if $\mathfrak{p} = \mathfrak{p}_\psi$ for some $\psi \in \mathrm{Hom}_{\mathrm{Pos}}([2], [m] \times [n])$.

If $\mathrm{ht}(\mathfrak{p}) = 3$ or 4, and $\mathfrak{p}$ is of the form described in the parts 2, 3 or 4 of theorem, then one can check that by Theorem 8, $\mathfrak{p} \in \mathrm{ass}(J(L, [2]))$.

Now, assume that $\mathfrak{p} \in \mathrm{ass}(J(L, [2]))$ and $\mathrm{ht}(\mathfrak{p}) = 3$. Then

$$\mathfrak{p} = (x_{(i_1,j_1),1}, x_{(i_2,j_2),2}, x_{(i_3,j_3),2})$$

or

$$\mathfrak{p} = (x_{(i_1,j_1),1}, x_{(i_2,j_2),1}, x_{(i_3,j_3),2})$$

for some $(i_1, j_1), (i_2, j_2), (i_3, j_3) \in [m] \times [n]$. Actually, in the first case, $\mathfrak{p} = \mathfrak{p}_{A,B}$ for $A = \{(i_1, j_1)\}$ and $B = \{(i_2, j_2), (i_3, j_3)\}$. By Corollary 13, B is an antichain. So, without loss of generality, we can assume that $i_2 < i_3$ and $j_3 < j_2$. Again, by Corollary 13, $(i_1, j_1) \leqslant (i_2, j_2) \vee (i_3, j_3) = (i_3, j_2)$, $(i_1, j_1) \nleqslant (i_2, j_2)$ and $(i_1, j_1) \nleqslant (i_3, j_3)$. So $1 \leqslant i_2 < i_1 \leqslant i_3 \leqslant m$ and $1 \leqslant j_3 < j_1 \leqslant j_2 \leqslant n$ and so $\mathfrak{p}$ satisfies the condition (2). By a similar argument we can see that in the second case $\mathfrak{p}$ satisfies the condition (3).

If $\mathrm{ht}(\mathfrak{p}) = 4$, then $\mathfrak{p}$ has the following shape:

$$\mathfrak{p} = (x_{(i_1,j_1),1}, x_{(i_2,j_2),1}, x_{(i_3,j_3),2}, x_{(i_4,j_4),2})$$

for some $(i_1, j_1), (i_2, j_2), (i_3, j_3), (i_4, j_4) \in [m] \times [n]$, i.e.,

$$\mathfrak{p} = \mathfrak{p}_{A,B} \text{ for some } A = \{(i_1, j_1), (i_2, j_2)\} \text{ and } B = \{(i_3, j_3), (i_4, j_4)\}.$$

Because otherwise one of A or B has three elements and we immediately get a contradiction with statement 2 of Theorem 8.

Since, by Corollary 11, A and B are two antichains, without loss of generality, we can assume that $i_1 < i_2$, $j_2 < j_1$, $i_3 < i_4$ and $j_4 < j_3$. Also, again by Theorem 8,

$$(i_1, j_2) = (i_1, j_1) \wedge (i_2, j_2) \leqslant (i_3, j_3) \vee (i_4, j_4) = (i_4, j_3).$$

So $i_1 \leqslant i_4$ and $j_2 \leqslant j_3$. Now, by Theorem 8, we see that $1 \leqslant i_3 < i_1 \leqslant i_4 < i_2 \leqslant m$ and $1 \leqslant j_4 < j_2 \leqslant j_3 < j_1 \leqslant n$. ☐

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