On arc-transitive metacyclic covers of graphs with order twice a prime

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Abstract

Quite a lot of attention has been paid recently to the characterization and construction of edge- or arc-transitive abelian (mostly cyclic or elementary abelian) covers of symmetric graphs, but there are rare results for nonabelian covers since the voltage graph techniques are generally not easy to be used in this case. In this paper, we will classify certain metacyclic arc-transitive covers of all non-complete symmetric graphs with prime valency and twice a prime order $2p$ (involving the complete bipartite graph $\mathsf{K}_{p,p}$, the Petersen graph, the Heawood graph, the Hadamard design on 22 points and an infinite family of prime-valent arc-regular graphs of dihedral groups). A few previous results are extended.

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1 Introduction

Throughout the paper, by a graph, we mean a connected, simple and undirected graph with valency at least three.

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For a graph Γ and an automorphism group $X \leqslant \mathsf{Aut}\,\Gamma$, Γ is called X-*vertex-transitive*, X-*edge-transitive* or X-*arc-transitive*, if X is transitive on its vertex set, edge set or arc set, respectively. For a positive integer s, an s-*arc* of Γ is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_s$ of $s+1$ vertices such that α_{i-1}, α_i are adjacent for $1 \leqslant i \leqslant s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leqslant i \leqslant s-1$. Then Γ is called (X, s)-*arc-transitive* or (X, s)-*arc-regular* if X is transitive or regular on the set of s-arcs of Γ, respectively. In particular, if $\mathsf{Aut}\,\Gamma$ is regular on the set of s-arcs of Γ, then Γ is simply called s-*arc-regular*.

An essential method for studying edge- or arc-transitive graphs is taking normal quotient graphs. Let Γ be a graph with vertex set $V\Gamma$, and suppose that $X \leqslant \mathsf{Aut}\,\Gamma$ acts edge- or arc-transitively on Γ and has an intransitive normal subgroup N. Denote by $V\Gamma_N$ the set of N-orbits on $V\Gamma$. Then the *normal quotient graph* of Γ induced by N, denoted by Γ_N, is defined with vertex set $V\Gamma_N$ and two vertices $B, C \in V\Gamma_N$ are adjacent if and only if some vertex in B is adjacent in Γ to some vertex in C. If further, for each adjacent B and C, the induced subgraph $[B, C] \cong n\mathsf{K}_2$ is a complete matching, where $n = |B| = |C|$, then Γ is called a *normal cover* (or *regular cover*, or *cover* for short) of Γ_N. In some cases, to emphasize the groups X and N, Γ is called an X-edge-transitive or X-arc-transitive N-cover of Γ_N respectively. If N is a cyclic, abelian or nonabelian group, then Γ is called a *cyclic cover*, *abelian cover* or *nonabelian cover* of Γ_N, respectively. We remark that 'the cover of graph' can also be defined by using notions of fibre-preserving group and covering transformation group, refer to [9, 25, 26].

From the above definition, an important strategy for studying transitive graphs naturally arises, involving the following two steps. Step 1 would be concerned to obtain a characterization of 'basic' transitive graphs (that is, the graphs which are not covers of their normal quotient graphs). Step 2 would then consists in approaching all transitive covers of basic graphs.

Characterizing covers of graphs is thus often a key step for studying edge- or arc-transitive graphs. By using voltage graph techniques (which generally are powerful for finding cyclic and elementary abelian covers, refer to [21, 25, 26, 36]), a lot of classifications of transitive cyclic or elementary abelian covers of symmetric graphs have been obtained, for example, see [16, 26, 27, 28] and references therein for edge- or arc-transitive cyclic and elementary abelian covers of certain small graphs, and see [11, 12, 13, 39, 41] for 2-arc-transitive cyclic and certain elementary abelian covers of K_n (the compete graph) and $\mathsf{K}_{n,n} - n\mathsf{K}_2$ (the bipartite complete graph with a 1-factor removed), see also [40] for 2-arc-transitive metacyclic covers of K_n. Quite recently, a new approach was founded by investigating the action of universal group of basic graph and then used to find all arc-transitive abelian covers of a few small cubic symmetric graphs (see [5, 6]). However, since the voltage graph techniques are generally difficult for treating nonabelian covers, the results of nonabelian covers are rare.

The main purpose of this paper is to determine all arc-transitive K-covers of graph Σ, where Σ is a non-complete graph with prime valency r and twice a prime order $2p$, and $K \cong \mathbb{Z}_m : \mathbb{Z}_q$ is a split metacyclic but not cyclic group (the arc-transitive cyclic covers of Σ have been determined by [32]), with q a prime less than r. We notice that certain special cases have been investigated in the literature: see [14] for K cyclic and $\Sigma = \mathsf{K}_{3,3}$

($p=3$ and $r=3$); [15] for K cyclic and $\Sigma=\mathsf{O}_2$ the Petersen graph ($p=5$ and $r=3$); [37] for K cyclic and Σ the Heawood graph ($p=7$ and $r=3$); [30] for K cyclic and $\Sigma=\mathsf{K}_{p,p}$ the complete bipartite graph with $p>3$; and see [42] for K a dihedral group and $r=3$.

The terminology and notation used in this paper are standard. For example, for a positive integer n, we denote by $\mathbb{Z}_n$, D_n with n even, A_n and S_n the cyclic group and the dihedral group of order n, the alternating and the symmetric group of degree n, respectively. For two groups N and H, we denote by $N\times H$ the direct product of N and H, by $N.H$ an extension of N by H, and if such an extension is split, then we write $N:H$ instead of $N.H$. Also, we denote by F_n, $\mathrm{F}_n\mathrm{A}$, $\mathrm{F}_n\mathrm{B}$ etc. with n a positive integer the corresponding cubic graphs of order n in the Foster census, see [1] or [4].

The main result of this paper is the following theorem. For convenience, the graphs appearing in the theorem which are not explained above are introduced in Section 2.

Theorem 1. *Let Γ be an X-arc-transitive K-cover of a graph Σ, where $X\leqslant \mathsf{Aut}\,\Gamma$, Σ is a non-complete graph of odd prime valency r and order $2p$ with p a prime, and $K\cong\mathbb{Z}_m:\mathbb{Z}_q$ is a metacyclic but not cyclic group with $q<r$ a prime. Then one of the following statements holds.*

(1) $\Sigma\cong\mathsf{K}_{3,3}$, $K\cong\mathbb{Z}_{2n}\times\mathbb{Z}_2$ with n odd, and Γ is characterized in [5, Theorem 5.1];

(2) $\Sigma\cong\mathsf{CD}(2p,3)$ with $p\equiv 1 \pmod 3$, and either

(i) $K\cong\mathbb{Z}_2^2$ and $\Gamma\cong\mathsf{CGD}(8p,3)$ is 1-arc-regular. or

(ii) $K\cong\mathbb{Z}_m\times\mathbb{Z}_2$, and $\Gamma\cong\mathsf{CGD}(2,\frac{mp}{2},\lambda)$, or $\mathsf{CGD}(2p,\frac{m}{2p},\lambda)$ with $p\mid m$, where $m=2\cdot 3^s p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}\geqslant 6$, $s\leqslant 1$, $0\leqslant t$ and $p_1,p_2,\ldots,p_t$ are distinct primes such that $3\mid(p_i-1)$ for $i=1,2,\ldots,t$.

(3) $\Sigma\cong\mathsf{O}_2$, and the tuple (Γ,K) is listed in the following table.

Row	Γ	K	s − arc − regular
1	F40	D_4	3 − arc − regular
2	F60	D_6	2 − arc − regular
3	F120B	D_{12}	2 − arc − regular

The structure of this paper as follows. We give some preliminary results and introduce examples that appear in Theorems 1 in Section 2, and prove some technical lemmas in Section 3. Then, we classify the mentioned covers of the complete bipartite graph $\mathsf{K}_{p,p}$ and graphs $\mathsf{CD}(2p,r)$ in Sections 3 and 4 respectively, and complete the proof of Theorem 1 in Sections 5.

2 Preliminaries and examples

2.1 Preliminary results

For a group G and its subgroup H, let $C_G(H)$ and $N_G(H)$ denote the centralizer and normalizer of H in G, respectively.

Lemma 2. ([22, Ch. I, Lemma 4.5]) *Let G be a group and H a subgroup of G. Then $N_G(H)/C_G(H) \leqslant \mathsf{Aut}(H)$.*

A graph Γ is called a *Cayley graph* of a group G if there is a subset $S \subseteq G \setminus \{1\}$, with $S = S^{-1} := \{g^{-1} \mid g \in S\}$, such that $V\Gamma = G$ and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(G, S)$. It is well known that a graph Σ is isomorphic to a Cayley graph of a group R if and only if $\mathsf{Aut}\,\Sigma$ contains a subgroup which is isomorphic to R and acts regularly on $V\Sigma$, see [2, Proposition 16.3]. If this regular subgroup is normal in X with $X \leqslant \mathsf{Aut}\,\Sigma$, then Σ is called an *X-normal Cayley graph*. In particular, if this regular subgroup is normal in $\mathsf{Aut}\,\Sigma$, then Σ is called a *normal Cayley graph*.

Let $\Gamma = \mathsf{Cay}(G, S)$. Let

$$\hat{G} = \{\hat{g} \mid \hat{g} : \; x \mapsto xg, \text{ for all } g, x \in G\},$$

$$\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}.$$

Then both $\hat{G}$ and $\mathsf{Aut}(G, S)$ are subgroups of $\mathsf{Aut}\,\Gamma$. Further, the following nice property holds.

Lemma 3. ([19, Lemma 2.1]) *Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph. Then the normalizer $N_{\mathsf{Aut}\,\Gamma}(\hat{G}) = \hat{G} : \mathsf{Aut}(G, S)$.*

We remark that $\hat{G}$ is the right regular representation of G, and so isomorphic to G. For convenience, we will often write $\hat{G}$ as G.

Let H be a group acting transitively on a set Ω. Then H is called *primitive* on Ω if H preserves no nontrivial partition of Ω. A graph Γ is called *X-locally-primitive* with $X \leqslant \mathsf{Aut}\,\Gamma$, if the vertex stabilizer X_α: $= \{x \in X \mid \alpha^x = \alpha\}$ acts primitively on the neighbor set $\Gamma(\alpha)$ for each vertex α. Clearly, an edge-transitive graph with odd prime valency is locally-primitive. By Lemma 3, one easily has the following assertion.

Lemma 4. *Let $\Gamma = \mathsf{Cay}(G, S)$ be an X-normal locally-primitive Cayley graph of a group G, where $G \lhd X \leqslant \mathsf{Aut}\,\Gamma$. Then $X_{\mathbf{1}} \leqslant \mathsf{Aut}(G, S)$ and elements in S are involutions, where $\mathbf{1}$ denotes the vertex of Γ corresponding to the identity element of G. In particular, if G is abelian, then G is an elementary abelian 2-group.*

The following theorem provides a basic method for studying vertex-transitive locally-primitive graphs (see [18] for the vertex-intransitive case), where parts (1–3) were first proved by Praeger [33, Theorem 4.1] for 2-arc-transitive graphs and slightly generalized to locally-primitive graphs by [24, Lemma 2.5]; part (4) follows easily by part (1).

Theorem 5. *Let Γ be an X-vertex-transitive locally-primitive graph, and let $N \lhd X$ have at least three orbits on $V\Gamma$. Then the following statements hold.*

(1) N is semiregular on $V\Gamma$, $X/N \leqslant \mathsf{Aut}\,\Gamma_N$, Γ_N is X/N-locally-primitive, and Γ is an X-locally-primitive N-cover of Γ_N.

(2) Γ is (X,s)-arc-transitive if and only if Γ_N is $(X/N,s)$-arc-transitive, where $1 \leqslant s \leqslant 5$ or $s = 7$.

(3) $X_\alpha \cong (X/N)_\delta$, where $\alpha \in V\Gamma$ and $\delta \in V\Gamma_N$.

(4) If X has a normal subgroup M which is contained in N, then Γ_M is an X/M-locally-primitive N/M-cover of Γ_N.

Given a group G and two subgroups L, R such that $L \cap R$ is *core-free* in G (that is, $L \cap R$ contains no nontrivial normal subgroup of G). Define a bipartite graph $\Gamma = \mathsf{Cos}(G,L,R)$, called *bi-coset graph*, with vertex set $[G:L] \cup [G:R]$ and

$$\{Lx, Ry\} \text{ is an edge in } \Gamma \iff yx^{-1} \in RL.$$

The following lemma gives some basic properties of the bi-coset graphs, refer to [18, Lemmas 3.5, 3.7].

Lemma 6. *With notation above, Γ satisfies the following properties:*

(1) Γ is connected if and only if $\langle L, R\rangle = G$;

(2) $G \leqslant \mathsf{Aut}\,\Gamma$, and Γ is G-edge-transitive and G-vertex-intransitive;

(3) $\Gamma(L) = \{Rx \mid x \in L\}$ and $\Gamma(R) = \{Lx \mid x \in R\}$;

(4) $|\Gamma(\alpha)| = |L : L \cap R|$ and $|\Gamma(\beta)| = |R : L \cap R|$, where $\alpha \in [G:L]$ and $\beta \in [G:R]$.

Conversely, each G-vertex-intransitive and G-edge-transitive graph is isomorphic to $\mathsf{Cos}(G, G_\alpha, G_\beta)$, where α and β are adjacent vertices.

A group G is called perfect if $G = G'$, where G' is the commutator subgroup of G; and an extension $G = N.H$ is called a *central extension* if N is contained in the centre $Z(G)$ of G. If a group G is perfect and $G/Z(G)$ is isomorphic to a simple group T, then G is called a *covering group* of T. Schur [35] showed that a simple (and more generally, perfect) group T possesses a 'universal' covering group G with the property that every covering group of T is a homomorphic image of G; in this case, the center $Z(G)$ is called the Schur multiplier of T, refer to [20, P.43]. The Schur multipliers of all simple groups are known ([20, P. 302–303]). The following lemma is known.

Lemma 7. *Let $G = N.T$, where N is cyclic or isomorphic to $\mathbb{Z}_p^2$ with p a prime, and T is a nonabelian simple group. Then $G = N.T$ is a central extension, $G = NG'$, and $G' = M.T$ is a perfect group, where M is a subgroup of both N and the Schur Multiplier of T.*

The next lemma gives an observation regarding the vertex stabilizer of permutation groups, that will be used later.

Lemma 8. ([32, Lemma 3.4]) *Suppose that $G = NH$ is a permutation group on a set Ω, where $N \lhd G$ and $H \leqslant G$. Then $G_\alpha \cong N_\alpha.o$ for $\alpha \in \Omega$, where $o \leqslant H/(H \cap N)$. In particular, if N is transitive on Ω, then $o \cong H/(H \cap N)$.*

2.2 Examples

We now introduce examples appearing in Theorem 1.

The first family of examples arises from Cayley graphs of dihedral groups, stated in Example 9, where the first two letters CD of the notation $\mathsf{CD}(2p, r)$ stand for 'Cayley graph of a dihedral group'.

Example 9. Let $G = \langle a, b \mid a^p = b^2 = 1, a^b = a^{-1}\rangle \cong \mathrm{D}_{2p}$ be a dihedral group, with p an odd prime. Let r be an odd prime and k a solution of the congruence equation

$$x^{r-1} + x^{r-2} + \cdots + x + 1 \equiv 0 \ (\mathsf{mod}\ \ p).$$

Define

$$\mathsf{CD}(2p, r) = \mathsf{Cay}(G, \{b, ab, a^{k+1}b, \ldots, a^{k^{r-2}+k^{r-3}+\cdots+1}b\}).$$

We give some remarks here.

(a) The congruence equation above has a solution if and only if $p = r$ or $r \mid (p-1)$, see [17, Lemma 3.3].

(b) The graph $\mathsf{CD}(2p, r)$ is a bipartite graph, and up to isomorphism, it is independent of the choice of k by [17, Corollary 3.2]. This is the reason why the notation used does not involve k.

(c) The graph $\mathsf{CD}(2p, p) \cong \mathsf{K}_{p,p}$, $\mathsf{CD}(14, 3)$ is the Heawood graph, and $\mathsf{CD}(22, 5)$ is the incidence graph of valency 5 (the another one is of valency 6) of the Hadamard design on 22 points.

A group G is called a *generalized dihedral group*, if $G = H : \langle a\rangle$ for some abelian subgroup H and an involution a such that $h^a = h^{-1}$ for each $h \in H$. This group is denoted by $\mathrm{Dih}(H)$. Obviously, $\mathrm{Dih}(\mathbb{Z}_n) \cong \mathrm{D}_{2n}$. The next family of examples arises from Cayley graphs of generalized dihedral groups, which was first obtained in [42].

Example 10. Let m and k be positive integers. Let

$$\mathrm{Dih}(\mathbb{Z}_{mk} \times \mathbb{Z}_m) = \langle a, b, c \mid a^2 = b^{mk} = c^m = 1, b^a = b^{-1}, c^a = c^{-1}, bc = cb\rangle$$

be a generalized dihedral group. Assume that $\lambda = 0$ for $k = 1$, and $\lambda^2 + \lambda + 1 \equiv 0$ $(\mathsf{mod}\ k)$ for $k > 1$. Define

$$\mathsf{CGD}(m, k, \lambda) = \mathsf{Cay}(\mathrm{Dih}(\mathbb{Z}_{mk} \times \mathbb{Z}_m), \{a, ab, ab^\lambda c\}).$$

The first three letters CGD of the graph $\mathsf{CGD}(m,k,\lambda)$ stand for 'Cayley graph of a generalized dihedral group'. By the notice before [42, Theorem 3.1], up to isomorphism, the cubic graph $\mathsf{CGD}(2,p,\lambda)$ with order $8p$ is independent of the choice of λ, we thus denote it simply by $\mathsf{CGD}(8p,3)$.

The next lemma is quoted from [3], which classifies arc-transitive prime-valent graphs of order twice a prime.

Lemma 11. *Suppose that $\varSigma$ is a non-complete arc-transitive graph of valency r and order $2p$, where r and p are odd primes. Then $r \leqslant p$ and one of the following holds.*

(1) $p=5$, $r=3$, $\varSigma \cong \mathsf{O}_2$ *and* $\mathsf{Aut}\,\varSigma \cong \mathrm{S}_5$.

(2) $\varSigma \cong \mathsf{K}_{p,p}$ *and* $\mathsf{Aut}\,\varSigma \cong \mathrm{S}_p \wr \mathrm{S}_2$.

(3) $\varSigma \cong \mathsf{CD}(2p,r)$ *with* $r \mid (p-1)$, *and one of the following is true.*

  (i) $(p,r)=(7,3)$ *and* $\mathsf{Aut}\,\varSigma \cong \mathrm{PGL}(2,7)$;

  (ii) $(p,r)=(11,5)$ *and* $\mathsf{Aut}\,\varSigma \cong \mathrm{PGL}(2,11)$;

  (iii) $(p,r)\neq(7,3)$ *and* $(11,5)$, *and* $\mathsf{Aut}\,\varSigma \cong \mathrm{D}_{2p}:\mathbb{Z}_r$.

A classification of arc-transitive cyclic covers of prime-valent graphs with order twice a prime is obtained by [32]. By [32, Theorem 1.1] (together with [30, Theorem 1.1] for a detailed description in one case), we have the following result.

Theorem 12. *Let $\varGamma$ be an arc-transitive $\mathbb{Z}_n$-cover of a graph $\varSigma$, with odd prime valency r and twice a prime order $2p$. Then one of the following statements holds.*

(1) $\varSigma \cong \mathsf{K}_4$, $n=2$ *and* $\varGamma \cong \mathsf{K}_{4,4}-4\mathsf{K}_2$, *or* $n=4$ *and* $\varGamma \cong \mathsf{P}(8,3)$ *is the generalized Petersen graph.*

(2) $\varSigma \cong \mathsf{O}_2$, $n=2$, *and $\varGamma$ is the Dodecahedron graph or the standard double cover of* O_2.

(3) $\varSigma \cong \mathsf{K}_{2p}$ *with* $p \geqslant 3$ *and* $2p-1=r$, $n=2$ *and* $\varGamma \cong \mathsf{K}_{2p,2p}-2p\mathsf{K}_2$.

(4) $\varSigma \cong \mathsf{K}_{p,p}$ *and* $n=p^s p_1^{r_1} p_2^{r_2}\cdots p_t^{r_t}$, *where* $s\leqslant 1$, $0\leqslant t$ *and* $p_1,p_2,\ldots,p_t$ *are distinct primes such that* $p \mid (p_i-1)$ *for* $i=1,2,\ldots,t$.

(5) $\varSigma \cong \mathsf{CD}(2p,r)$ *with* $r \mid (p-1)$, *and* $n=r^s p_1^{r_1} p_2^{r_2}\cdots p_t^{r_t}$, *where* $s\leqslant 1$, $0\leqslant t$ *and* $p_1,p_2,\ldots,p_t$ *are distinct primes such that* $r \mid (p_i-1)$ *for* $i=1,2,\ldots,t$.

3 Technical lemmas

In this section, we prove some technical lemmas for the subsequent discussion.

For a group G, the socle of G, denoted by $\mathsf{soc}(G)$, is the product of all minimal normal subgroups of G. Obviously, $\mathsf{soc}(G)$ is a characteristic subgroup of G.

Lemma 13. *Suppose $G \cong (\mathbb{Z}_{p^s} \times \mathbb{Z}_p) : \mathbb{Z}_3$, where $p \equiv 1 \pmod 3$ is a prime and $s \geqslant 1$. Then G has a normal cyclic subgroup of order p^s.*

Proof. By assumption, we may write $G = (\langle a\rangle \times \langle b\rangle) : \langle c\rangle$, where $o(a) = p^s$, $o(b) = p$ and $o(c) = 3$.

If $s = 1$, then $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_3$. Noting that $\mathbb{Z}_p^2$ has exactly $p+1$ subgroups isomorphic to $\mathbb{Z}_p$. Since $3 \mid (p-1)$, $p+1 \equiv 2 \pmod 3$, hence the action of $\langle c\rangle \cong \mathbb{Z}_3$ on these $p+1$ subgroups is not fixed-point free. So at least one such subgroup must be invariant under this action, namely $\langle c\rangle$ normalizes a cyclic subgroup $\mathbb{Z}_p$, which is clearly normal in G, the lemma is true in this case.

Assume $s \geqslant 2$. Let $H = \langle a\rangle \times \langle b\rangle$. Then $\mathsf{soc}(H) = \langle a^{p^{s-1}}\rangle \times \langle b\rangle \cong \mathbb{Z}_p^2$ is a characteristic subgroup of H, and so is normal in G. Hence $\langle c\rangle$ normalizes $\mathsf{soc}(H)$. As $\langle c\rangle \cong \mathbb{Z}_3$ acts reducibly on $\mathsf{soc}(H)$ and $(3, p) = 1$, by Maschke's theorem (see [38, Theorem 1.4]), $\langle c\rangle$ acts completely reducible on $\mathsf{soc}(H)$, that is, we may write $\mathsf{soc}(H) = \langle b_1, b_2\rangle$ such that c normalizes both $\langle b_1\rangle$ and $\langle b_2\rangle$. Noting that at least one of b_1 and b_2 is not in $\langle a\rangle$, say b_1, then $H = \langle a\rangle \times \langle b_1\rangle$. Hence, replacing b by b_1 if necessary, we may assume $b^c = b^k$ for some k. As $o(b^k) = o(b) = p$, $(k, p) = 1$.

If $a^c \in \langle a\rangle$, the group $\langle a\rangle$ is a required normal cyclic subgroup. Suppose $a^c = a^i b^j$ with $p \nmid j$. If $p \mid (k-i)$, then $b^i = b^k$ and $(i, p) = 1$, by a simple computation, we have $a^{c^2} = (a^i b^j)^c = (a^i b^j)^i b^{kj} = a^{i^2} b^{2ij}$, and $a^{c^3} = (a^i b^j)^{i^2} b^{2ijk} = a^{i^3} b^{3i^2 j} \notin \langle a\rangle$ as $p \nmid 3i^2 j$, which is a contradiction because $o(c) = 3$. So $p \nmid (k-i)$, and hence the congruence equation

$$(k-i)x \equiv -j \pmod p$$

has a solution, say $x = l$. Now, as $o(ab^l) = p^s$ and $(ab^l)^c = a^i b^j b^{kl} = a^i b^{il} = (ab^l)^i$, $\langle ab^l\rangle \cong \mathbb{Z}_{p^s}$ is normal in G. □

The next lemma presents some properties of the automorphism groups of metacyclic groups isomorphic to $\mathbb{Z}_m : \mathbb{Z}_2$.

Lemma 14. *Let $K = \langle a\rangle : \langle b\rangle \cong \mathbb{Z}_m : \mathbb{Z}_2$ be a metacyclic group, where $m = 2^d n > 1$ with $d \geqslant 0$ and n odd. Then the following statements hold.*

(1) If m is odd, then $K \cong \mathbb{Z}_s \times \mathrm{D}_{2t}$, where $st = m$, and $(s, t) = 1$.

(2) If $d = 1$, then either K is abelian, or the center $Z(K) \cong \mathbb{Z}_{2n_1}$ and $K/Z(K) \cong \mathrm{D}_{2n_2}$, where $n_1 n_2 = n$, $n_2 > 1$ is odd and $(n_1, n_2) = 1$.

(3) If $d \geqslant 1$, then $\langle a^2\rangle$ is a characteristic subgroup of K with index 4.

(4) Suppose $d \geqslant 2$ and K is normal in an overgroup X. Then X has a normal subgroup which is contained in K with index 2.

Proof. (1) Suppose that $\langle a^e\rangle \cong \mathbb{Z}_{p^l}$ is a Sylow p-subgroup of $\langle a\rangle$, where p is an odd prime dividing m and l is a positive integer. Then $\langle a^e\rangle \lhd K$, so $(a^e)^b = (a^e)^k$ for some integer k. Since $b^2 = 1$, $k^2 \equiv 1 \pmod{p^l}$, and so $p^l \mid (k-1)(k+1)$. As $(k-1, k+1)$ divides 2 and $p^l \geqslant 3$, we have $k \equiv \pm 1 \pmod{p^l}$. Hence either b centralizes $\langle a^e \rangle$, or $\langle a^e, b\rangle \cong \mathrm{D}_{2p^l}$. Now, as $\langle a\rangle \cong \mathbb{Z}_{p_1^{s_1}} \times \cdots \times \mathbb{Z}_{p_k^{s_k}}$ for some distinct odd primes $p_1, \ldots, p_k$, and $\langle b\rangle \cong \mathbb{Z}_2$ acts trivially or dihedrally on each Sylow p-subgroup of $\langle a\rangle$, collecting factors on which $\langle b\rangle$ acts trivially gives the center $\mathbb{Z}_s$ of K while collecting the remaining factors together with $\langle b\rangle$ gives the dihedral group D_{2t}. Hence $K \cong \mathbb{Z}_s \times \mathrm{D}_{2t}$.

(2) Since $d = 1$, $\langle a^n\rangle \in Z(K)$ and $K = \langle a^n\rangle \times (\langle a^2\rangle : \langle b\rangle)$. By part (1), $\langle a^2\rangle : \langle b\rangle \cong \mathbb{Z}_{n_1} \times \mathrm{D}_{2n_2}$, where $n_1n_2 = n$ and $(n_1, n_2) = 1$. If $n_2 = 1$, K is abelian; if $n_2 > 1$, then $Z(K) \cong \mathbb{Z}_{2n_1}$ and $K/Z(K) \cong \mathrm{D}_{2n_2}$.

(3) Suppose $a^b = a^q$ with q an integer. Then $q^2 \equiv 1 \pmod{m}$, so q is odd as m is even. Let $K^{(2)} = \langle g^2 \mid g \in K\rangle$. Clearly, $K^{(2)} \supseteq \langle a^2\rangle$ is a characteristic subgroup of K. Since each element of K has a form a^i or a^ib for some integer i, $(a^i)^2 \in \langle a^2\rangle$ and $(a^ib)^2 = a^{(1+q)i} \in \langle a^2\rangle$, we conclude that $\langle a^2\rangle = K^{(2)}$ is a characteristic subgroup of K with index 4.

(4) By part (3), $\langle a^2\rangle$ is a characteristic subgroup of K, so is $\langle a^4\rangle$, hence $\langle a^4\rangle \lhd X$ and $K/\langle a^4\rangle \lhd X/\langle a^4\rangle$. Since $K/\langle a^4\rangle \cong \mathbb{Z}_4 : \mathbb{Z}_2$ is a split extension, we have $K/\langle a^4\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or D_8. For the former case, $K/\langle a^4\rangle$ has a characteristic subgroup, say $M/\langle a^4\rangle$, such that $M/\langle a^4\rangle \cong \mathbb{Z}_2^2$, so $M/\langle a^4\rangle \lhd X/\langle a^4\rangle$, it follows that M is normal in X and is of index 2 in K. For the latter case, $K/\langle a^4\rangle$ has a characteristic subgroup, say $N/\langle a^4\rangle$, such that $N/\langle a^4\rangle \cong \mathbb{Z}_4$, it follows that N is normal in X and is of index 2 in K. $\square$

The following lemma is regarding normal arc-transitive Cayley graphs.

Lemma 15. *Let $G \cong \mathbb{Z}_2^2 \times \mathrm{D}_{2p}$ with $p \geqslant 5$ a prime. Then there is no normal arc-transitive Cayley graph of G with prime valency $r \geqslant 5$.*

Proof. Suppose that, on the contrary, $\Gamma = \mathsf{Cay}(G, S)$ is an X-normal arc-transitive Cayley graph of G, where $G \lhd X \leqslant \mathsf{Aut}\,\Gamma$ and $S \subseteq G \setminus \{1\}$. Set $G = (\langle a\rangle \times \langle b\rangle) \times (\langle c\rangle : \langle d\rangle)$, where $\langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_2^2$ and $\langle c\rangle : \langle d\rangle \cong \mathrm{D}_{2p}$. By Lemma 4, $S = s^{\langle\sigma\rangle}$, where $s \in G$ is an involution and $\sigma \in \mathsf{Aut}(G)$ is of order r. Noting that $\langle a, b\rangle \cong \mathbb{Z}_2^2$ is a characteristic subgroup of G, the restriction $\sigma|_{\langle a,b\rangle}$ of σ on $\langle a, b\rangle$ is an automorphism of $\langle a, b\rangle$. Clearly, the order of $\sigma|_{\langle a,b\rangle}$ divides $o(\sigma) = r$, and so is 1 or r. As $r \geqslant 5$ and $\mathsf{Aut}(\mathbb{Z}_2^2) \cong \mathrm{S}_3$ has no element with order bigger than 3, we further conclude that $\sigma|_{\langle a,b\rangle}$ is the identity automorphism of $\langle a, b\rangle$. If $s \in \langle a, b\rangle$, then $\langle S\rangle = \langle s^{\langle\sigma\rangle}\rangle \subseteq \langle a, b\rangle < G$, Γ is disconnected, which is a contradiction. Thus $s \in G \setminus \langle a, b\rangle$, and so $s = a^ib^jc^kd$ for some integers i, j, k. Noting that there is an automorphism τ which fixes a, b, c and maps s to d, and $\mathsf{Cay}(G, S) \cong \mathsf{Cay}(G, S^\tau)$, we may, up to isomorphism, assume $s = d$. Now, since $\langle c\rangle \cong \mathbb{Z}_p$ is a characteristic subgroup of G, σ has a form:

$$\sigma : a \to a,\ b \to b,\ c \to c^k,\ d \to a^ub^vc^wd.$$

By computation, $d = d^{\sigma^r} = a^{ru}b^{rv}c^{w(1+k+\cdots+k^{r-1})}d$. So $a^{ru} = b^{rv} = 1$, implying $a^u = b^v = 1$ as r is odd and a, b are involutions. It follows that $\langle s^{\langle\sigma\rangle}\rangle = \langle d^{\langle\sigma\rangle}\rangle \subseteq \langle c, d\rangle < G$, which contradicts the connectivity of Γ. $\square$

4 Metacyclic covers of $K_{p,p}$

For convenience, we give the following Hypothesis 4.1 and Notations 4.2, which will be used throughout the rest of the paper.

Hypothesis 4.1. Let $\varGamma$ be an X-arc-transitive K-cover of a graph $\varSigma$, where $X \leqslant \mathsf{Aut}\,\varGamma$, $\varSigma$ is a non-complete graph of odd prime valency r and order $2p$ with p a prime, and $K := \langle a\rangle : \langle b\rangle \cong \mathbb{Z}_m : \mathbb{Z}_q$ is a metacyclic but not cyclic group with $q < r$ a prime. Then $K \lhd X$ is semiregular on $V\varGamma$, and $Y := X/K \leqslant \mathsf{Aut}\,\varSigma$ acts transitively on the arc set of $\varSigma$.

Notation 4.2. If $\varSigma$ is a bipartite graph, let $\varDelta_1$ and $\varDelta_2$ be the biparts of $\varSigma$, let $Y^+ = Y_{\varDelta_1} = Y_{\varDelta_2}$ be the stabilizer of Y on the biparts, and let $X^+ = K.Y^+$ (the full preimages of Y^+ under the natural homomorphism of Y to Y/K). Then $|X : X^+| = |Y : Y^+| = 2$ and $Y = \langle Y^+, y\rangle$ for some $y \in Y \setminus Y^+$.

By Hypothesis 4.1, the tuple $(\varSigma, \mathsf{Aut}\,\varSigma)$ is listed in Lemma 11. We will prove Theorem 1 by analysing all the candidates $\varSigma$ in Lemma 11. Recall that, for a group G, a subgroup H is called the *Frattini subgroup* of G if H is the intersection of all maximal subgroups of G; and for a prime $t \mid |G|$, H is called a *Hall t'-subgroup* if $(|H|, t) = 1$ and $|G : H|$ is a t-power. It is well known that the Frattini subgroup and normal Hall t'-subgroups are characteristic subgroups. As usual, we use G_t and $G_{t'}$ to denote a Sylow t-subgroup and a Hall t'-subgroup of G, respectively.

First, we prove an induction lemma.

Lemma 16. *With notation above, the following statements hold.*

(1) If $q \nmid m$, then $q = 2$ and $\varSigma \cong \mathsf{O}_2$.

(2) If $q \mid m$, then there exists a graph $\varOmega$ being an arc-transitive $\mathbb{Z}_q^2$-cover of $\varSigma$.

Proof. If $q \nmid m$, then $\langle a\rangle$ is a characteristic subgroup of K, so is normal in X. Since $K/\langle a\rangle \cong \mathbb{Z}_q$, by Theorem 5(4), the normal quotient graph $\varGamma_{\langle a\rangle}$ is an $X/\langle a\rangle$-arc-transitive $\mathbb{Z}_q$-cover of $\varSigma$. Then, as $q < r = \mathsf{val}(\varSigma)$ and $\varGamma$ is not a complete graph, by Theorem 12, we have $q = 2$ and $\varSigma \cong \mathsf{O}_2$, as in part (1) of the lemma.

Assume now $q \mid m$. Then $m = q^d n$ with $d \geqslant 1$ and $(q, n) = 1$. Clearly, $\langle a^{q^d}\rangle \cong \mathbb{Z}_n$ is a normal Hall q'-subgroup of K, so is characteristic in K and normal in X. Let $\varGamma_1$ denote the normal quotient graph $\varGamma_{\langle a^{q^d}\rangle}$. By Theorem 5, $\varGamma_1$ is an $X/\langle a^{q^d}\rangle$-arc-transitive $K/\langle a^{q^d}\rangle$-cover of $\varSigma$. Let Φ be the Frattini subgroup of $K/\langle a^{q^d}\rangle$. Then Φ is characteristic in $K/\langle a^{q^d}\rangle$ and normal in $X/\langle a^{q^d}\rangle$. Since $K/\langle a^{q^d}\rangle \cong \mathbb{Z}_{q^d} : \mathbb{Z}_q$, by [34, 5.3.2], $(K/\langle a^{q^d}\rangle)/\Phi \cong \mathbb{Z}_q^2$. It then follows from Theorem 5 that $\varOmega := (\varGamma_1)_\Phi$ is an arc-transitive $\mathbb{Z}_q^2$-cover of $\varSigma$. $\square$

In this section, we will determine all arc-transitive K-covers of the complete bipartite graph $\mathsf{K}_{p,p}$.

The following lemma excludes the case where $p \geqslant 5$.

Lemma 17. *Suppose $\varSigma \cong \mathsf{K}_{p,p}$ with $p \geqslant 5$. Then no graph $\varGamma$ exists.*

Proof. By Lemma 16, it is sufficient to show that Lemma 17 is true for the case where $K \cong \mathbb{Z}_q^2$. Recall that $q < r = \mathsf{val}(\varSigma) = p$.

Let K_1 and K_2 be the kernels of Y^+ acting on $\varDelta_1$ and $\varDelta_2$, respectively. For any $\alpha \in \varDelta_1$, since $y \notin Y^+$, we have $\alpha^{y^{-1}} \in \varDelta_2$ and $\alpha^{y^{-1}K_2y} = (\alpha^{y^{-1}})^{K_2y} = (\alpha^{y^{-1}})^y = \alpha$, so $K_2^y = y^{-1}K_2y \subseteq K_1$. Similarly, $K_1^y \subseteq K_2$. Thus $K_1 = K_2^y$ and $K_2 = K_1^y$. Since Y is arc-transitive on $\varSigma$, Y_α is transitive on the neighbor set $\varSigma(\alpha) = \varDelta_2$. If Y^+ acts faithfully on $\varDelta_1$, then Y^+ can be viewed as a transitive permutation group of degree p on $\varDelta_1$, so $Y^+ \leqslant \mathrm{S}_p$. It follows that $Y_\alpha^+ \leqslant \mathrm{S}_{p-1}$, which contradicts that $Y_\alpha^+ = Y_\alpha$ is transitive on $\varDelta_2$ of size p.

Hence, Y^+ acts unfaithfully on both $\varDelta_1$ and $\varDelta_2$, that is, $K_1 \cong K_2 \neq 1$. Clearly, $K_1 \cap K_2 = 1$, so $K_1K_2 = K_1 \times K_2 \lhd Y^+$ and $Y^+ \cong (K_1 \times K_2).P$, where $P \cong Y^+/(K_1 \times K_2)$. It follows that $K_i.P \cong Y^+/K_{3-i} = (Y^+)^{\varDelta_{3-i}}$ is a transitive permutation group of degree p, where $i = 1, 2$. By the classification of transitive permutation groups of prime degree (see [8, P. 99]), we have that either $K_i.P \leqslant \mathbb{Z}_p : \mathbb{Z}_{p-1}$ is an affine group or $K_i.P$ is an almost simple 2-transitive group. Let $T_i = \mathsf{soc}(K_i.P)$ and let $G = K.(T_1 \times T_2)$.

If T_i is nonabelian, then T_i is 2-transitive on $\varDelta_i$, so $T_1 \times T_2$ is transitive on the set of all 4-cycles of $\varSigma \cong \mathsf{K}_{p,p}$. Noting that $G \cong \mathbb{Z}_q^2.(T_1 \times T_2)$ is a central extension by Lemma 7, then with quite similar discussion as in [10, Theorem 4.2] (or [30, Lemma 5.1]), we have that $\varGamma$ is disconnected, which is a contradiction.

Assume now that T_i is abelian. Then $T_1 \times T_2 \cong \mathbb{Z}_p^2$ is edge-transitive and vertex-intransitive on $\varSigma$, and so $G = K.(T_1 \times T_2) \cong \mathbb{Z}_q^2 : \mathbb{Z}_p^2$ is edge-transitive and vertex-intransitive on $\varGamma$. By Lemma 6, $\varGamma \cong \mathsf{Cos}(G, G_\alpha, G_\beta)$ for some adjacent vertices α and β of $\varGamma$. Clearly, $G_\alpha \cong G_\beta \cong \mathbb{Z}_p$. If $p \geqslant q+2$, then $|\mathsf{Aut}(\mathbb{Z}_q^2)| = |\mathrm{GL}(2,q)| = q(q-1)^2(q+1)$ is not a multiple of p, by Lemma 2, the centralizer $C_G(K) = G$, and so $G \cong \mathbb{Z}_q^2 \times \mathbb{Z}_p^2$ is abelian. It follows that $\langle G_\alpha, G_\beta \rangle \leqslant G_p < G$, so $\varGamma$ is disconnected by Lemma 6, which is a contradiction. Thus $p \leqslant q+1$ and hence $q = p-1$ as $q < p$. Now, both p and $p-1$ are primes, the only possibility is $p = 3$ and $q = 2$, which is a contradicts to the assumption $p \geqslant 5$. $\square$

We next consider the case $\varSigma \cong \mathsf{K}_{3,3}$.

Lemma 18. *Suppose $\varSigma \cong \mathsf{K}_{3,3}$. Then $K \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$ with n odd, and $\varGamma$ is characterized in [5, Theorem 5.1].*

Proof. By assumption, $r = \mathsf{val}(\varSigma) = 3$, so $q = 2$ and $K = \langle a \rangle : \langle b \rangle \cong \mathbb{Z}_m : \mathbb{Z}_2$. Then Lemma 16(1) implies that m is even. Set $m = 2^d n$ with $d \geqslant 1$ and n odd.

If $d \geqslant 2$, since $K \lhd X$, by Lemma 14(4), X has a normal subgroup, say M, such that M is contained in K with index 2. Then, by Theorem 5, $\varGamma_M$ is an arc-transitive $\mathbb{Z}_2$-cover of $\varSigma$, so $|V\varGamma_M| = 12$; however, there is no cubic arc-transitive graph of order 12 by [4], which is a contradiction.

Therefore, $d = 1$. Assume K is nonabelian. Then $\langle a^n \rangle \cong \mathbb{Z}_2$ is in the center of K, $\langle a^2 \rangle$ is a characteristic subgroup of K and as K is nonabelian, b is not in the center of K, we conclude that $\langle a \rangle = \langle a^2 \rangle \times \langle a^n \rangle$ is normal in X. By Theorem 5, $\varGamma_{\langle a \rangle}$ is an arc-transitive $\mathbb{Z}_2$-cover of $\varSigma$, in particular, $|V\varGamma| = 12$, which is a contradiction by [4]. Thus K is abelian, $K \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$ with n odd, and $\varGamma$ is characterized in [5, Theorem 5.1]. $\square$

5 Metacyclic covers of $\mathsf{CD}(2p, r)$

Let $\mathit{\Gamma}, X, \mathit{\Sigma}, K$ and Y be as in Hypothesis 4.1, and as $\mathsf{CD}(2p, r)$ is a bipartite graph, we use Notation 4.2.

In this section, we will determine all arc-transitive K-covers of the bipartite graphs $\mathsf{CD}(2p, r)$ with $r \mid (p-1)$. Noting that $K = \langle a \rangle : \langle b \rangle \cong \mathbb{Z}_m : \mathbb{Z}_q$ and $q < r$ are primes.

Lemma 19. *Assume $\mathit{\Sigma} \cong \mathsf{CD}(2p, r)$ with $r \mid (p-1)$ and $Y \cong \mathrm{D}_{2p} : \mathbb{Z}_r$. Then $q = 2$ and $r = 3$.*

Proof. Suppose that, on the contrary, $q \geqslant 3$. Since $Y \cong \mathrm{D}_{2p} : \mathbb{Z}_r$, $\mathit{\Sigma}$ is a Y-normal arc-transitive Cayley graph of the dihedral group D_{2p}, so $\mathit{\Gamma}$ is an X-normal arc-transitive Cayley graph of $R := K.\mathrm{D}_{2p}$. Also, by Lemma 16(1), we have $q \mid m$.

We first consider the case where $m = q$. Then $R \cong \mathbb{Z}_q^2 : \mathrm{D}_{2p}$ and $X = K.Y \cong R : \mathbb{Z}_r$. Since K is abelian, the centralizer $C := C_R(K) \supseteq K$. If $C = K$, Lemma 2 implies $\mathrm{D}_{2p} \cong R/K = R/C \leqslant \mathsf{Aut}(K) \cong \mathsf{Aut}(\mathbb{Z}_q^2) \cong \mathrm{GL}(2, q)$, as $3 \leqslant q < r < p$, $|\mathrm{GL}(2, q)| = q(q-1)^2(q+1)$ is not a multiple of p, which is a contradiction. On the other hand, if $C = R$, then $R = K \times R_{q'}$, so $R_{q'} \cong \mathrm{D}_{2p}$ is normal in X and has q^2 orbits on $V\mathit{\Gamma}$, by Theorem 5, $\mathit{\Gamma}_{R_{q'}}$ is arc-transitive of odd valency r and odd order q^2, which is also a contradiction. Hence $K < C < R$, and $C/K \neq 1$ is a proper normal subgroup of $R/K \cong \mathrm{D}_{2p}$, we further conclude that $C = R_{2'} \cong \mathbb{Z}_q^2 \times \mathbb{Z}_p$ and $\mathbb{Z}_p \cong R_p \lhd X$. Now, Theorem 5 implies that $\mathit{\Gamma}_{R_p}$ is an X/R_p-normal arc-transitive Cayley graph of the group R/R_p. Clearly, $R/R_p \cong R_{p'} = K : \mathbb{Z}_2 \cong \mathbb{Z}_q^2 : \mathbb{Z}_2$. As $q \geqslant 3$, $R_{p'}$ is nonabelian by Lemma 4. Suppose that $c \in R_{p'}$ is an involution. If $x^c = y \notin \langle x \rangle$ for some $x \in K$, as $o(c) = 2$, $y^c = x$, so $(xy)^c = yx = xy$ and $(xy^{-1})^c = yx^{-1} = (xy^{-1})^{-1}$, hence $R_{p'} = \langle xy \rangle \times \langle xy^{-1}, c \rangle \cong \mathbb{Z}_q \times \mathrm{D}_{2q}$. Since $\mathit{\Gamma}_{R_p}$ is an X/R_p-normal Cayley graph with valency r of R/R_p, we may assume $\mathit{\Gamma}_{R_p} = \mathsf{Cay}(R/R_q, S)$, where $S \subseteq R/R_p \setminus \{1\}$. By Lemma 4, elements in S are involutions, and as $R/R_p \cong R_{p'} \cong \mathbb{Z}_q \times \mathrm{D}_{2q}$, we have $\langle S \rangle \leqslant \mathrm{D}_{2q} < R/R_p$, $\mathit{\Gamma}$ is disconnected, which is a contradiction. Therefore, $x^c \in \langle x \rangle$ for each $x \in K$, and $\langle x, c \rangle \cong \mathbb{Z}_{2q}$ or D_{2q} by Lemma 14(1). Since $R_{p'}$ is nonabelian, and as proved above, $R_{p'} \not\cong \mathbb{Z}_q \times \mathrm{D}_{2q}$, we further have that $R_{p'} \cong \mathrm{Dih}(\mathbb{Z}_q^2)$ is a generalized dihedral group. Now, as $X/R_p \cong (R/R_p) : \mathbb{Z}_r \cong \mathbb{Z}_q^2 : \mathbb{Z}_r : \mathbb{Z}_2$, and $(X/R_p)_u \cong \mathbb{Z}_r$ with $u \in V(\mathit{\Gamma}_{R_p})$ is not normal in X/R_p, by Lemma 2, $\mathbb{Z}_q^2$ in self-centralized in X/R_p and $\mathbb{Z}_r : \mathbb{Z}_2 \leqslant \mathsf{Aut}(\mathbb{Z}_q^2) \cong \mathrm{GL}(2, q)$. Hence r divides $|\mathrm{GL}(2, q)| = q(q-1)^2(q+1)$. However, as $q < r$ are primes, the only possibility is $q = 2$ and $r = 3$, which contradicts the assumption that $q \geqslant 3$.

For the general case, by Lemma 16, there is a graph $\mathit{\Omega}$ which is an arc-transitive $\mathbb{Z}_q^2$-cover of $\mathit{\Sigma}$, then above discussion also leads to a contradiction.

Therefore, $q = 2$ and $R \cong \mathbb{Z}_2^2.\mathrm{D}_{2p}$. We now prove $r = 3$. Since $r \geqslant 3$ divides $p - 1$, $p \geqslant 7$, and as $|\mathsf{Aut}(\mathbb{Z}_2^2)| = |\mathrm{S}_3| = 6$, Lemma 2 implies $R = (\mathbb{Z}_2^2 \times \mathbb{Z}_p).\mathbb{Z}_2$ and so $R_p \lhd X$. By Theorem 5, $\mathit{\Gamma}_{R_p}$ is an X/R_p-normal arc-transitive Cayley graph of the group $R/R_p \cong R_2 \cong \mathbb{Z}_2^2.\mathbb{Z}_2$. If R_2 is nonabelian, then $R_2 \cong Q_8$ is a quaternion group or isomorphic to D_8. The former case is not possible by Lemma 4 as Q_8 has unique involution, and the latter case is not possible by [29, Lemma 3.1]. Thus R_2 is abelian.

By Lemma 4, $R_2 \cong \mathbb{Z}_2^3$ and $R \cong \mathbb{Z}_2^2 \times \mathrm{D}_{2p}$. Recall that Γ is an X-normal arc-transitive Cayley graph of R, by Lemma 15, we have $r < 5$, hence $r = 3$, as required. $\square$

The next lemma excludes the case where $\Sigma \cong \mathsf{CD}(22, 5)$.

Lemma 20. *Assume $\Sigma \cong \mathsf{CD}(22, 5)$. Then no graph Γ exists.*

Proof. With Lemma 16, we only need to prove the lemma for the case $K \cong \mathbb{Z}_q^2$.

Since $\Sigma \cong \mathsf{CD}(22, 5)$, $\mathsf{Aut}\,\Sigma \cong \mathrm{PGL}(2, 11)$ by Lemma 11. Since $Y \leqslant \mathsf{Aut}\,\Sigma$ acts arc-transitively on Σ, 110 divides $|Y|$, by Atlas [7], $Y \cong \mathrm{D}_{22} : \mathbb{Z}_5, \mathrm{PSL}(2, 11)$ or $\mathrm{PGL}(2, 11)$. Noting that $\mathsf{CD}(22, 5)$ is a bipartite graph, Y has a normal subgroup Y^+ with index 2, so $Y \ncong \mathrm{PSL}(2, 11)$. If $Y \cong \mathrm{D}_{22} : \mathbb{Z}_5$, by Lemma 19, we have $\mathsf{val}(\Sigma) = 3$, yielding a contradiction.

Assume $Y \cong \mathrm{PGL}(2, 11)$. Then $Y^+ \cong \mathrm{PSL}(2, 11)$ and $Y_\delta = Y_\delta^+ \cong \mathrm{A}_5$ as $|V\Sigma| = 22$, where $\delta \in V\Sigma$. By Lemma 7, $X^+ = K.Y^+ \cong \mathbb{Z}_q^2.\mathrm{PSL}(2.11)$ is a central extension, $X^+ = K(X^+)'$, and as the Schur multiplier of $\mathrm{PSL}(2, 11)$ is $\mathbb{Z}_2$ (see [20, P. 302]), $(X^+)' \cong \mathrm{PSL}(2, 11)$ or $\mathrm{SL}(2, 11)$. For $\alpha \in V\Gamma$, by Theorem 5(3), $(X^+)'_\alpha \lhd X_\alpha^+ = X_\alpha \cong Y_\delta \cong \mathrm{A}_5$, so $(X^+)'_\alpha = 1$ or A_5. If $(X^+)'_\alpha = 1$, as $X^+ = K(X^+)'$, Lemma 8 implies $\mathrm{A}_5 \cong X_\alpha^+ \leqslant K/(K \cap (X^+)')$, which is a contradiction as K is abelian. Thus $(X^+)'_\alpha \cong \mathrm{A}_5$, and hence $(X^+)' \ncong \mathrm{SL}(2, 11)$ since $\mathrm{SL}(2, s)$ with s an odd prime power has a unique involution by [23, Lemma 7.4]. Therefore, $(X^+)' \cong \mathrm{PSL}(2, 11)$, $X^+ = K \times (X^+)'$ and each orbit of $(X^+)'$ in $V\Gamma$ has length $|\mathrm{PSL}(2, 11) : \mathrm{A}_5| = 11$. Since $|V\Gamma| = 22q^2$, $(X^+)' \lhd X$ has $2q^2 \geqslant 8$ orbits on $V\Gamma$. By Theorem 5, $\Gamma_{(X^+)'}$ is an $X^+/(X^+)'$-arc-transitive graph, which is also a contradiction as $X^+/(X^+)' \cong \mathbb{Z}_q^2$ is abelian. $\square$

With Lemmas 19 and 20, we may now determine all arc-transitive K-covers of $\mathsf{CD}(2p, r)$ with $r \mid (p - 1)$.

Lemma 21. *Assume $\Sigma \cong \mathsf{CD}(2p, r)$ with $r \mid (p - 1)$. Then $r = 3$ and Γ is as in part* (2) *of Theorem* 1.

Proof. By Lemma 20, $(p, r) \neq (11, 5)$. If $(p, r) = (7, 3)$, the $q = 2$ as $q < r$; if $(p, r) \neq (11, 5)$ and $(7, 3)$, by Lemma 11, $\mathsf{Aut}\,\Sigma \cong \mathrm{D}_{2p} : \mathbb{Z}_r$ is arc-regular on Σ, so $Y = \mathsf{Aut}\,\Sigma$ as Y is arc-transitive on Σ, by Lemma 19, we also have $q = 2$ and $r = 3$. Now, by Lemma 16, m is even. Suppose $m = 2^d n$ with $(2, n) = 1$ and $d \geqslant 1$.

Assume $d \geqslant 2$. By Lemma 14(4), X has a normal subgroup, say M, such that M is contained in K with index 2. It then follows from Theorem 5 that Γ_M is an arc-transitive $\mathbb{Z}_2$-cover of $\mathsf{CD}(2p, 3)$, which is impossible by Theorem 12(5).

Hence, $d = 1$ and $K = \langle a \rangle : \langle b \rangle \cong \mathbb{Z}_{2n} : \mathbb{Z}_2$. If $n = 1$, by [42, Theorem 3.1], $\Gamma \cong \mathsf{CGD}(8p, 3)$ (as CD_{8p} there), part (2)(i) of Theorem 1 holds. If K is nonabelian, by Lemma 14(2), the center $Z(K) \cong \mathbb{Z}_{2n_1}$ and $K/Z(K) \cong \mathrm{D}_{2n_2}$, where $n_1 n_2 = n$, $n_2 > 1$ and $(n_1, n_2) = 1$. Then, by Theorem 5, $\Gamma_{Z(K)}$ is an arc-transitive D_{2n_2}-cover of Σ, as n_2 is odd, which is not possible by [42, Theorem 3.1]. Hence $K \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$ is abelian. Since $\langle a^2 \rangle \lhd X$ and $K/\langle a^2 \rangle \cong \mathbb{Z}_2^2$, by Theorem 5, $\Gamma_{\langle a^2 \rangle}$ is an $X/\langle a^2 \rangle$-arc-transitive $\mathbb{Z}_2^2$-cover of $\Sigma \cong \mathsf{CD}(2p, 3)$, then [42, Theorem 3.1] implies $\Gamma_{\langle a^2 \rangle} \cong \mathsf{CGD}(8p, 3)$ is a 1-arc-regular normal

Cayley graph of the generalized dihedral group $\mathrm{Dih}(\mathbb{Z}_{2p}\times\mathbb{Z}_2)$, so $X/\langle a^2\rangle \cong \mathrm{Dih}(\mathbb{Z}_{2p}\times\mathbb{Z}_2):\mathbb{Z}_3$. Noting that $\langle a^2\rangle.\mathbb{Z}_2^2 \cong K \lhd X$, we further conclude $X \cong (\mathbb{Z}_{2n}\times\mathbb{Z}_2).\mathrm{D}_{2p}.\mathbb{Z}_3$. Further, as $K_2 \cong \mathbb{Z}_2^2$ is normal in X and $K/K_2 \cong \mathbb{Z}_n$, by Theorem 5, $\mathit{\Gamma}_{K_2}$ is an arc-transitive $\mathbb{Z}_n$-cover of $\mathsf{CD}(2p,3)$. It then follows from Theorem 12(5) that $n = 3^s p_1^{r_1} p_2^{r_2}\cdots p_t^{r_t}$, where $s \leqslant 1$, $0 \leqslant t$, and $p_1, p_2, \ldots, p_t$ are distinct primes such that $3 \mid p_i - 1$ for $i = 1, 2, \ldots, t$.

Recall that $X \cong (\mathbb{Z}_{2n}\times\mathbb{Z}_2).\mathrm{D}_{2p}.\mathbb{Z}_3$. Let $R \lhd X$ such that $R \cong K.\mathrm{D}_{2p}$, and let $C = C_R(K_{2'})$. Then $K \subseteq C \lhd R$. If $C = K$, Lemma 2 implies $\mathrm{D}_{2p} \cong Y/K = Y/C \leqslant \mathsf{Aut}(K_{2'}) \cong \mathbb{Z}_n^*$, yielding a contradiction. Thus $C > K$ and $1 \neq C/K \lhd Y/K \cong \mathrm{D}_{2p}$, implying $C \supseteq K.\mathbb{Z}_p$. Let $L \subseteq C$ such that $L/K \cong \mathbb{Z}_p$. Then L is abelian and $L \cong \mathbb{Z}_{2np}\times\mathbb{Z}_2$, or $\mathbb{Z}_{2n}\times\mathbb{Z}_{2p}$ with $p \mid n$. Clearly, $L \subseteq K.\mathrm{D}_{2p}$ is semiregular and has exactly two orbits on $V\mathit{\Gamma}$, by [42, Lemma 3.1], $\mathit{\Gamma}$ is a Cayley graph of the generalized dihedral group $\mathrm{Dih}(L)$. Then, by [42, Proposition 2.6], we have $\mathit{\Gamma} \cong \mathsf{CGD}(2, np, \lambda)$, or $\mathsf{CGD}(2p, n/p, \lambda)$ with $p \mid n$, as in part (2)(ii) of Theorem 1. $\square$

6 Proof of Theorem 1

To complete the proof of Theorem 1, we first prove that the graphs $\mathit{\Gamma}$ in part (2)(ii) of Theorem 1 are really examples.

Lemma 22. *Suppose that $p \equiv 1 \pmod 3$ is a prime and $m = 2\cdot 3^s p_1^{r_1} p_2^{r_2}\cdots p_t^{r_t}$, where $s \leqslant 1$, $0 \leqslant t$, and $p_1, p_2, \ldots, p_t$ are distinct primes such that $3 \mid (p_i - 1)$ for $i = 1, 2, \ldots, t$. Then the graphs $\mathsf{CGD}(2, \frac{mp}{2}, \lambda)$ and $\mathsf{CGD}(2p, \frac{m}{2p}, \lambda)$ with $p \mid m$ are arc-transitive $(\mathbb{Z}_m\times\mathbb{Z}_2)$-covers of $\mathsf{CD}(2p, 3)$.*

Proof. (i) Let $\Omega = \mathsf{CGD}(2, \frac{mp}{2}, \lambda)$. By Example 10, $\Omega = \mathsf{Cay}(G, \{a, b, ab^{-\lambda}c\})$, where

$$G = \langle a, b, c \mid a^2 = b^{mp} = c^2 = 1, b^a = b^{-1}, c^a = c^{-1}, bc = cb\rangle,$$

and $\lambda^2 + \lambda + 1 \equiv 0 \pmod{\frac{mp}{2}}$. By [42, Proposition 2.6(4)], Ω is an arc-regular normal Cayley graph of G, hence $\mathsf{Aut}(\Omega) \cong G : \mathbb{Z}_3$.

Clearly, the center $Z(G) = \langle b^{\frac{mp}{2}}\rangle \times \langle c\rangle \cong \mathbb{Z}_2^2$ and $G^{(2p)} := \langle g^{2p} \mid g \in G\rangle = \langle b^{2p}\rangle \cong \mathbb{Z}_{\frac{m}{2}}$ are characteristic subgroups of G. Set $M = Z(G)G^{(2p)}$. Since $(\frac{mp}{2}, 2p) = (3^s p_1^{r_1} p_2^{r_2}\cdots p_t^{r_t} p, 2p) = p$, we have $M = \langle b^{\frac{mp}{2}}, b^{2p}\rangle \times \langle c\rangle = \langle b^p\rangle \times \langle c\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_2$ is a characteristic subgroup of G, and hence $M \lhd \mathsf{Aut}(\Omega)$. By Theorem 5, Ω is an arc-transitive $(\mathbb{Z}_m\times\mathbb{Z}_2)$-cover of $\mathit{\Gamma}_M$, and as $|V(\Omega_M)| = 2p$ with $p \equiv 1 \pmod 3$ and $\mathsf{val}(\Omega_M) = 3$, by Lemma 11, $\Omega_M \cong \mathsf{CD}(2p, 3)$.

(ii) Set $\Omega' = \mathsf{CGD}(2p, \frac{m}{2p}, \lambda)$ with $p \mid m$. Then $\Omega' = \mathsf{Cay}(G, \{a, b, ab^{-\lambda}c\})$ by Example 10, where

$$G = \langle a, b, c \mid a^2 = b^m = c^{2p} = 1, b^a = b^{-1}, c^a = c^{-1}, bc = cb\rangle,$$

and $\lambda = 0$ if $m = 2p$, and $\lambda^2+\lambda+1 \equiv 0 \pmod{\frac{m}{2p}}$ if $\frac{m}{2p} > 1$. By [42, Proposition 2.6], Ω' is a normal Cayley graph of G, and Ω' is 2-arc-regular if $\frac{m}{2p} = 1$ of 3, or is 1-arc-regular if $\frac{m}{2p} > 3$. Hence $\mathsf{Aut}(\Omega') \cong \mathrm{G} : \mathbb{Z}_3$ or $G : \mathrm{S}_3$, it follows that there exists an automorphism group $X \leqslant$

$\mathsf{Aut}(\Omega')$ such that $X \cong G : \mathbb{Z}_3$ acts arc-regularly on Ω'. Let $\langle f \rangle = X_\alpha \cong \mathbb{Z}_3$ with $\alpha \in V\Omega'$. Noting that $Z(G) = \langle b^{\frac{m}{2}} \rangle \times \langle c^p \rangle \cong \mathbb{Z}_2^2$ and $G_{2'} = \langle b^2 \rangle \times \langle c^2 \rangle \cong \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_p$ are characteristic subgroups of G, we have $\langle b, c \rangle = \langle b^{\frac{m}{2}}, b^2 \rangle \times \langle c^p, c^2 \rangle = Z(G)G_{2'}$ is characteristic in G and so normal in X. Let $H = \langle b, c, f \rangle$. Then $H = \langle b, c \rangle : \langle f \rangle \cong (\mathbb{Z}_m \times \mathbb{Z}_{2p}) : \mathbb{Z}_3$ and $X = \langle H \rangle : \langle a \rangle$.

Claim. X has a normal subgroup $N \subseteq G$ such that $N \cong \mathbb{Z}_m \times \mathbb{Z}_2$.

Since $p \equiv 1 \pmod 3$ and $p \mid m$, we may suppose $p = p_1$ and $s_1 \geqslant 1$. Now, $\langle b, c \rangle_{p'} \cong \mathbb{Z}_{\frac{m}{p^{s_1}}} \times \mathbb{Z}_2$ is normal in H, with order $\frac{2m}{p^{s_1}}$. Consider the group $R := (\langle b^{\frac{m}{p^{s_1}}} \rangle \times \langle c^2 \rangle) : \langle f \rangle$. Then $R \cong (\mathbb{Z}_{p^{s_1}} \times \mathbb{Z}_p) : \mathbb{Z}_3$. As $3 \mid (p-1)$, by Lemma 13, R has a normal cyclic subgroup $M \subseteq \langle b, c \rangle$ with order p^{s_1}. Noting that M is normal in H, then $N := \langle b, c \rangle_{p'} M \cong \mathbb{Z}_m \times \mathbb{Z}_2$ is contained in $\langle b, c \rangle$ and normal in H, and as $x^a = x^{-1}$ for each $x \in \langle b, c \rangle$, N is normal in $\langle H, a \rangle = X$, the claim is true.

Now, by Theorem 5, Ω' is an arc-transitive $(\mathbb{Z}_m \times \mathbb{Z}_2)$-cover of Ω'_N. Since $p \equiv 1 \pmod 3$, and Ω'_N is of valency 3 and order $2p$, by Lemma 11, $\Omega'_M \cong \mathsf{CD}(2p, 3)$. □

Now, we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. By assumption, Σ is X/K-arc-transitive of order $2p$, and so Σ is listed in Lemma 11. To be precise, $\Sigma \cong \mathsf{O}_2, \mathsf{K}_{p,p}$ or $\mathsf{CD}(2p, r)$ with $r \mid (p-1)$.

Assume $\Gamma \cong \mathsf{O}_2$. All edge-transitive metacyclic covers of O_2 are determined in [31], which are arc-transitive and exactly consist of seven graphs. Noting that K is not a cyclic group and a quaternion group, by [31, Theorem 1.1], the tuple (Γ, K) satisfies part (3) of Theorem 1.

Assume $\Sigma \cong \mathsf{K}_{p,p}$. By Lemma 17, we have $p < 5$, then by Lemma 18, we have $K \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$ with n odd, and Γ is characterized in [5, Theorem 5.1], as in part (1) of Theorem 1.

Assume $\Sigma \cong \mathsf{CD}(2p, r)$ with $r \mid (p-1)$. By Lemma 21, we have $r = 3$ and part (2) of Theorem 1 holds.

Finally, by Lemma 22, the graphs Γ in part (2)(ii) of Theorem 1 are arc-transitive $\mathbb{Z}_m \times \mathbb{Z}_2$-covers of $\mathsf{CD}(2p, r)$, and the graph $\mathsf{CGD}(8p, 3)$ in part (2)(i) is arc-transitive D_4-cover of $\mathsf{CD}(2p, 3)$ by [42, Theorem 3.1], thus the graphs Γ in Theorem 1 are really examples.

This completes the proof of Theorem 1. □

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