Constant terms of near-Dyson polynomials

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Abstract

We formulate and prove a formula for the constant term for a certain class of Laurent polynomials, which include the Dyson conjecture and its generalizations by Bressoud and Goulden. Our method is explicit Combinatorial Nullstellensatz.

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We recall the following form of Alon's Combinatorial Nullstellensatz (appeared recently in [13, 14, 15], but essentially going back to Jacobi [8], see [9] for a modern exposition). It proved to be very useful [5, 11, 12, 13] for finding coefficients of polynomials.

Theorem 1. *Let $f(x_1, \ldots, x_n)$ be a polynomial over a field $\mathbb{F}$ of degree $\leqslant d_1 + \cdots + d_n$.*

Consider the grid $G = G_1 \times \cdots \times G_n$, $G_i \subset \mathbb{F}$, $\#G_i = d_i + 1$. The coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ in f is

$$\sum_{(c_1,\ldots,c_n)\in G} \frac{f(c_1, \ldots, c_n)}{\prod_{i=1}^{n} \prod_{z\in G_i\setminus\{c_i\}}(c_i - z)}.$$

Let $x_1, \ldots, x_n, q$ be commuting indeterminates. Consider a Laurent polynomial $f(x_1, \ldots, x_n, q)$ over a field $\mathbb{F}$. Denote its constant term over the function field $\mathbb{F}(q)$ as $CT[f]$.

Let $a_1, \ldots, a_n$ be non-negative integers, $a = a_1 + \cdots + a_n$. In a seminal 1962 work [4] Dyson formulated the following conjecture:

$$CT\left[\prod_{i\neq j}(1 - x_i/x_j)^{a_i}\right] = \frac{a!}{\prod_{i=1}^{n} a_i!}$$

This was proved by Gunson [unpublished] and Wilson [16] in the same year. The elegant proof, based on Lagrange interpolation, was given by Good [7]. In [13] another proof based on the above-stated form of Combinatorial Nullstellensatz is given. It was generalized to a q-version (proved for the first time in [17] by a different method) in [11].

Constant term identities with Laurent polynomials (such as this one) often arise in quantum electrodynamics. They are also closely related to Selberg-type integrals, which play an important role in random matrix theory, statistical mechanics and special function theory (see the exposition in [6]).

There are versions of (a particular case of) Dyson's conjecture for arbitrary root systems, in which Dyson's original case corresponds to A_n. These are famous Macdonald's conjectures proved by Cherednik [3] with the help of the so-called double affine Hecke algebras.

Understanding for which Laurent polynomials such identities do exist is an important question. The application of Combinatorial Nullstellensatz allowed to make substantial progress in this area, and our results continue this development.

We start with recalling the proof of q-version of the Dyson conjecture. Define $[l, r] = \{l, l+1, \dots, r\}$. Let $\chi(\dots)$ be equal to 1 if the expression in the parentheses is true, and to 0 otherwise. Also, denote $(x)_n = \prod_{t=0}^{n-1}(1 - q^t x)$.

Theorem 2. *Let $a_1, \dots, a_n$ be non-negative integers, $a = a_1 + \dots + a_n$. Consider the Laurent polynomial*

$$f(x_1, \dots, x_n, q) = \prod_{1 \leqslant i < j \leqslant n} (x_i/x_j)_{a_i} (qx_j/x_i)_{a_j}$$

over a field $\mathbb{F}$. Then

$$CT[f] = \frac{(q)_a}{(q)_{a_1}(q)_{a_2} \cdots (q)_{a_n}}.$$

Proof. We can assume that all $a_i > 0$ (if $a_i = 0$, then each factor of f contains x_i only in non-negative degree. Since we are interested in the constant term of f, we can assume that f does not depend on x_i).

$CT[f]$ equals to the coefficient of $\prod_{i=1}^{n} x_i^{a-a_i}$ in the polynomial h, where

$$h(x_1, \dots, x_n, q) = \prod_{1 \leqslant i < j \leqslant n} (x_i/x_j)_{a_i} (qx_j/x_i)_{a_j} \times x_j^{a_i} x_i^{a_j}.$$

We will calculate this coefficient of h using Theorem 1.

Consider the grid

$$G = \{(q^{y_1}, \dots, q^{y_n}) \mid y_i \in R_i\},$$

where

$$R_i = [0, a - a_i].$$

Let us assume that $x = (x_1, \dots, x_n) = (q^{y_1}, \dots, q^{y_n}) = q^y \in G$ is not a zero of h. Then, for each $i < j$,

$$y_j - y_i \geqslant a_i \text{ or } y_i - y_j \geqslant a_j + 1,$$

otherwise one of the factors in $(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j} \times x_j^{a_i} x_i^{a_j}$ equals to zero.

In particular, it means that all y_i are pairwise distinct. Let $\pi \in S_n$ be such a permutation that

$$y_{\pi_1} < y_{\pi_2} < \dots < y_{\pi_n}.$$

We know that

$$y_{\pi_{i+1}} - y_{\pi_i} \geqslant a_{\pi_i} + \chi(\pi_{i+1} < \pi_i).$$

Adding up these inequalities and taking into account that $y_{\pi_1} \geqslant 0$, we get

$$y_{\pi_n} - y_{\pi_1} \geqslant a - a_{\pi_n} + \sum_{i=1}^{n-1} \chi(\pi_{i+1} < \pi_i).$$

But $y_{\pi_n} \leqslant a - a_{\pi_n}$, so $\pi_i < \pi_{i+1}$ for all i, which means that $\pi = id$.

Let us note that all intermediate inequalities have to become equalities, so the only point on the grid which is not a zero of h is q^y, where $y_i = a_1 + a_2 + \cdots + a_{i-1}$.

Define for convenience $y_{n+1} = a$. By Theorem 1,

$$\begin{aligned} CT[f] &= \left(\prod_{1 \leqslant i < j \leqslant n} (q^{y_i - y_j})_{a_i} (q^{y_j + 1 - y_i})_{a_j} \times q^{y_j a_i} q^{y_i a_j} \right) \Big/ \left(\prod_{i=1}^{n} \prod_{z \in [0, a - a_i] \setminus \{y_i\}} (q^{y_i} - q^z) \right) \\ &= \frac{\left(\prod_{1 \leqslant i < j \leqslant n} \left(\prod_{k=0}^{a_i - 1} (q^{y_j} - q^{y_i + k}) \times \prod_{k=0}^{a_j - 1} (q^{y_i} - q^{y_j + 1 + k}) \right) \right)}{\left(\prod_{i=1}^{n} (-1)^{y_i} \left(\prod_{t=0}^{y_i - 1} q^t \right) (q)_{y_i} q^{y_i (a - a_i - y_i)} (q)_{a - a_i - y_i} \right)} \\ &= \frac{\left(\prod_{1 \leqslant i < j \leqslant n} \left((-1)^{a_i} \left(\prod_{t=y_i}^{y_{i+1} - 1} q^t \right) \frac{(q)_{y_j - y_i}}{(q)_{y_j - y_{i+1}}} \times q^{y_i a_j} \frac{(q)_{y_{j+1} - y_i}}{(q)_{y_j - y_i}} \right) \right)}{\left(\prod_{i=1}^{n} (-1)^{y_i} \left(\prod_{t=0}^{y_i - 1} q^t \right) (q)_{y_i - y_1} q^{y_i (a - a_i - y_i)} (q)_{y_{n+1} - y_{i+1}} \right)} \\ &= (q)_{y_{n+1} - y_1} \Big/ \left(\prod_{i=1}^{n} (q)_{y_{i+1} - y_i} \right) = \frac{(q)_a}{(q)_{a_1} \cdots (q)_{a_n}}. \end{aligned}$$

□

Next we give simple proofs of the master theorem and its transitive analogue from [2] using a similar technique. A similar proof in a different context can be found in [10] (see Theorems 1.2 and 3.5), though our proof is more direct.

A tournament T on n vertices is a set of ordered pairs (i, j) such that $1 \leqslant i \neq j \leqslant n$ and $(i, j) \in T$ if and only if $(j, i) \notin T$. One way of interpreting a tournament is as a relation on a set $[1, n]$: $i \to j$ if and only if $(i, j) \in T$.

A tournament T is transitive if the relation $\to$ is transitive. For a transitive tournament T, a winner permutation $\sigma \in S_n$ is such a permutation that $\sigma_i \to \sigma_j$ if and only if $i < j$. Note that every transitive tournament T has a unique winner permutation $\sigma = \sigma(T)$.

Theorem 3. *Let T be a tournament on n vertices. Let $a_1, \ldots, a_n$ be positive integers, $a = a_1 + \cdots + a_n$. Consider the Laurent polynomial*

$$f(x_1, \ldots, x_n, q) = \prod_{(i,j) \in T} (x_i / x_j)_{a_i} (q x_j / x_i)_{a_j - 1}$$

over a field $\mathbb{F}$. If T is nontransitive, then $CT[f] = 0$. If T is transitive with a winner permutation σ, then

$$CT[f] = \frac{(q)_a}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_n}} \times \prod_{i=1}^{n} \frac{1 - q^{a_{\sigma_i}}}{1 - q^{a_{\sigma_1} + \cdots + a_{\sigma_i}}}.$$

Proof. Let $deg(i) = \#\{j \mid (i,j) \in T\}$. Consider a permutation $\delta \in S_n$ such that for each $1 \leqslant i < j \leqslant n$ $deg(\delta_i) \geqslant deg(\delta_j)$, and $deg(\delta_i) = deg(\delta_j)$ only when $\delta_i < \delta_j$. Note that $\sigma = \delta$ in the case when T is transitive.

$CT[f]$ equals to the coefficient of $\prod_{i=1}^n x_i^{a-a_i-deg(i)}$ in the polynomial h, where

$$h(x_1, \ldots, x_n, q) = \prod_{(i,j)\in T} (x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-1} \times x_j^{a_i} x_i^{a_j-1}.$$

Once again, we will calculate this coefficient using Theorem 1.

Consider the grid

$$G = \{(q^{y_1}, \ldots, q^{y_n}) \mid y_i \in R_i\},$$

where

$$R_i = [0, a - a_i] \setminus S_i,$$

$$S_{\delta_i} = \{a - a_{\delta_i} - \sum_{v=j}^{n} a_{\delta_v} \mid n + 1 - deg(\delta_i) < j \leqslant n + 1\}.$$

Assume that $x = (x_1, \ldots, x_n) = (q^{y_1}, \ldots, q^{y_n}) = q^y \in G$ is not a zero of h. For each $(i,j) \in T$,

$$y_j - y_i \geqslant a_i \text{ or } y_i - y_j \geqslant a_j,$$

otherwise one of the factors in $(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-1} \times x_j^{a_i} x_i^{a_j-1}$ equals to zero.

It follows that all y_i are pairwise distinct. Let $\pi \in S_n$ be such a permutation that

$$y_{\pi_1} < y_{\pi_2} < \cdots < y_{\pi_n}.$$

We know that

$$y_{\pi_{i+1}} - y_{\pi_i} \geqslant a_{\pi_i}.$$

Adding up these inequalities and taking into account that $y_{\pi_1} \geqslant 0$, we get

$$y_{\pi_n} - y_{\pi_1} \geqslant a - a_{\pi_n}.$$

But $y_{\pi_n} \leqslant a - a_{\pi_n}$, so all intermediate inequalities have to become equalities, and

$$y_{\pi_i} = a - \sum_{j=i}^{n} a_{\pi_j}.$$

Since $y_{\pi_n} \notin S_{\pi_n}$, from definition of S_{π_n} it follows that $deg(\pi_n) = 0$. But T is a tournament, so $deg(i) = 0$ for at most one i. Since q^y is not a zero of h, such i exists (and equals to δ_n), so $deg(\delta_n) = 0$ and $\pi_n = \delta_n$.

Assume that we have already showed that $\pi_k = \delta_k$ and $deg(\delta_k) = n - k$ for $j < k \leqslant n$. Note that these conditions on deg imply that $(\delta_i, \delta_k) \in T$ for all $1 \leqslant i < k$, $j < k \leqslant n$. Then $deg(\delta_i) \geqslant n - j$ for all $1 \leqslant i \leqslant j$, and, since T is a tournament, $deg(\delta_i) > n - j$ for all $1 \leqslant i < j$.

$y_{\pi_j} \notin S_{\pi_j}$, so $deg(\pi_j) \leqslant n - j$. The only case in which it is possible is when $\pi_j = \delta_j$ and $deg(\delta_j) = n - j$.

Finally, either all elements of G are zeros of h and $CT[f] = 0$ or $\pi = \delta$ and $deg(\delta_i) = n - i$ for all i. If the latter is the case, obviously T is transitive and $\pi = \delta = \sigma$.

The only thing left is to calculate the coefficient in the case of transitive T. We will omit the calculations here since they are given in a more general case in the next theorem. $\square$

The main result is the following theorem.

Theorem 4. *Let k, $\{l_i\}_{i=1}^k$, $\{m_i\}_{i=1}^k$, $\{r_i\}_{i=1}^k$ be integers such that $1 \leqslant l_1 \leqslant m_1 \leqslant r_1 < l_2 \leqslant \ldots \leqslant r_{k-1} < l_k \leqslant m_k \leqslant r_k \leqslant n$. Let*

$$C_i = \bigcup_{j=m_i+1}^{r_i} [l_i, j-2] \times \{j\}, B_i \subseteq C_i,$$

$$U_i = \left(B_i \cup ([l_i, m_i - 1] \times \{m_i\}) \cup \bigcup_{j=m_i}^{r_i - 1} (j, j+1) \right),$$

$$U = \bigcup_{i=1}^k U_i.$$

Let $a_1, \ldots, a_n$ be positive integers, $a = a_1 + \cdots + a_n$. Consider the Laurent polynomial

$$f(x_1, \ldots, x_n, q) = \prod_{1 \leqslant i < j \leqslant n} (x_i/x_j)_{a_i} (q x_j / x_i)_{a_j - \chi((i,j) \in U)}$$

over a field $\mathbb{F}$. Then

$$CT[f] = \frac{(q)_a}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_n}} \times \prod_{i=1}^k \prod_{j=m_i}^{r_i} \frac{1 - q^{a_j}}{1 - q^{a_{l_i} + \cdots + a_j}}.$$

Remark 5. The statement of the theorem is long and cumbersome, therefore we provide an illustration that can help to understand the idea behind the formal definitions.

Consider the correspondence between a Laurent polynomial

$$f(x_1, \ldots, x_n, q) = \prod_{1 \leqslant i < j \leqslant n} (x_i/x_j)_{a_{i,j}} (q x_j / x_i)_{a_{j,i}}$$

and a square matrix of non-negative integers with zeroes on the main diagonal $A = \{a_{i,j}\}_{1 \leqslant i,j \leqslant n}$.

The polynomial from Theorem 2 corresponds to the matrix

$$\begin{pmatrix} 0 & a_1 & a_1 & \ldots & a_1 \\ a_2 & 0 & a_2 & \ldots & a_2 \\ a_3 & a_3 & 0 & \ldots & a_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_n & \ldots & a_n & 0 \end{pmatrix}.$$

The polynomial from transitive part of Theorem 3 (for winner permutation $\sigma = id$) corresponds to the matrix

$$\begin{pmatrix} 0 & a_1 & a_1 & \dots & a_1 \\ a_2-1 & 0 & a_2 & \dots & a_2 \\ a_3-1 & a_3-1 & 0 & \dots & a_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n-1 & a_n-1 & \dots & a_n-1 & 0 \end{pmatrix}.$$

The polynomial from Theorem 4 in the case $n = 9$, $k = 1$, $l_1 = 2$, $m_1 = 5$, $r_1 = 8$, $B_1 = \{(2,6),(4,6),(3,7),(4,7),(2,8),(4,8)\}$ corresponds to the matrix

$$\begin{pmatrix} 0 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & 0 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & 0 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & 0 & a_4 & a_4 & a_4 & a_4 & a_4 \\ a_5 & a_5-1 & a_5-1 & a_5-1 & 0 & a_5 & a_5 & a_5 & a_5 \\ a_6 & a_6-1 & a_6 & a_6-1 & a_6-1 & 0 & a_6 & a_6 & a_6 \\ a_7 & a_7 & a_7-1 & a_7-1 & a_7 & a_7-1 & 0 & a_7 & a_7 \\ a_8 & a_8-1 & a_8 & a_8-1 & a_8 & a_8 & a_8-1 & 0 & a_8 \\ a_9 & a_9 & a_9 & a_9 & a_9 & a_9 & a_9 & a_9 & 0 \end{pmatrix}.$$

As we can see, this matrix is the deformed version of the matrix from Theorem 2, with some coefficients decreased. Decreased coefficients are grouped into k blocks. Each block consists of a segment of some row ($[l_i, m_i - 1] \times \{m_i\}$), a segment of the diagonal (the one under the main diagonal, $\bigcup_{j=m_i}^{r_i-1} (j, j+1)$) and an arbitrary subset of elements "under" them (B_i is an arbitrary subset of C_i).

Remark 6. This theorem gives Theorem 2 when $k = 0$. It also gives transitive part of Theorem 3 (for winner permutation $\sigma = id$) when $k = 1$, $l_1 = m_1 = 1$, $r_1 = n$, $B_1 = C_1$.

Remark 7. This is a generalization of Theorem 2.5 from [2]. Specifically, it gives the said theorem when $B_i = C_i$ for all i.

Proof. $CT[f]$ equals to the coefficient of

$$\prod_{i=1}^{n} x_i^{a-a_i-\sum_{j=1}^{n}\chi((i,j)\in U)}$$

in the polynomial h, where

$$h(x_1,\dots,x_n,q) = \prod_{1\leqslant i<j\leqslant n} (x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-\chi((i,j)\in U)} \times x_j^{a_i} x_i^{a_j-\chi((i,j)\in U)}.$$

We will use Theorem 1 again. Consider the grid

$$G = \{(q^{y_1},\dots,q^{y_n}) \mid y_i \in R_i\},$$

where

$$R_i = [0, a - a_i] \setminus S_i,$$

$$S_i = \{a - a_i - \sum_{v=r_t+1}^{n} a_v\} \cup \{a - a_i - \sum_{v=j}^{n} a_v \mid (i,j) \in B_t\},$$

if there is a t such that $i \in [l_t, r_t - 1]$, $S_i = \varnothing$ otherwise.

Denote $A \rtimes B = \{(i,j) \in A \times B \mid i < j\}$.

Consider

$$N = [1,n] \rtimes [1,n], V_i = [l_i, r_i] \rtimes [m_i, r_i], V = \bigcup_{i=1}^{k} V_i.$$

Note that $U_i \subseteq V_i$, and thus $U \subseteq V$. Also note that $C_i \setminus B_i = V_i \setminus U_i$.

We replace the linear factors of h

$$(x_i - q^{a_j} x_j), \text{ where } (i,j) \in V \setminus U$$

by

$$(x_i - q^{a_j} x_j - (q^{a - a_i - \sum_{v=j}^{n} a_v} - q^{a - \sum_{v=j}^{n} a_v})),$$

and call the modified polynomial h'. The coefficient of h we are interested in coincides with the corresponding coefficient of h' because it has the maximal sum of degrees of x_i, and the polynomials differ only by constants in linear factors.

Assume that h' does not vanish at $x = q^y \in G$. Let

$$\chi_1(i,j) = \chi\left((i,j) \in V \setminus U \text{ and } y_i = a - a_i - \sum_{v=j}^{n} a_v \text{ and } y_j = a - \sum_{v=j}^{n} a_v\right).$$

For each $i < j$ either $y_j - y_i \geqslant a_i + \chi_1(i,j)$ or $y_i - y_j \geqslant a_j + \chi((i,j) \in N \setminus V)$ (otherwise one of the linear factors of h' is zero).

It follows that all y_i are pairwise distinct. Let $\pi \in S_n$ be such a permutation that

$$y_{\pi_1} < y_{\pi_2} < \cdots < y_{\pi_n}.$$

Since $U \subseteq V$, we know that

$$y_{\pi_{i+1}} - y_{\pi_i} \geqslant a_{\pi_i} + \chi((\pi_{i+1}, \pi_i) \in N \setminus V) + \chi_1(\pi_i, \pi_{i+1}).$$

Adding up these inequalities and taking into account that $y_{\pi_1} \geqslant 0$, we get

$$y_{\pi_n} \geqslant a - a_{\pi_n} + \sum_{i=1}^{n-1} \Big(\chi((\pi_{i+1}, \pi_i) \in N \setminus V) + \chi_1(\pi_i, \pi_{i+1})\Big).$$

But $y_{\pi_n} \leqslant a - a_{\pi_n}$, so $(\pi_{i+1}, \pi_i) \notin N \setminus V$ and $\chi_1(\pi_i, \pi_{i+1}) = 0$ for all i.

Note that all intermediate inequalities have to become equalities, so

$$y_{\pi_i} = a - \sum_{j=i}^{n} a_{\pi_j}.$$

Let us denote the event $\pi_{i+1} < \pi_i$ as the descent. The descent is possible only when $(\pi_{i+1}, \pi_i) \in V$. From the definition of V it follows that descents happen only if $\pi_i, \pi_{i+1} \in [l_t, r_t]$ for some t. Then for each t all elements of π in the range $[l_t, r_t]$ should go in a row, all elements less than them should go before them, and all elements bigger should go after.

We will show that elements from $[l_t, r_t]$ go not just in a row, but in the ascending order. Therefore the only possible choice for π is id, and there is only one point on the grid at which h' does not vanish.

If $\pi_{r_t} \in [l_t, r_t - 1]$, then

$$y_{\pi_{r_t}} = a - a_{\pi_{r_t}} - \sum_{j=r_t+1}^{n} a_{\pi_j} = a - a_{\pi_{r_t}} - \sum_{j=r_t+1}^{n} a_j,$$

which contradicts the definition of $R_{\pi_{r_t}}$. So, $\pi_{r_t} = r_t$.

Consider $s \geqslant m_t$, and we have already showed that $\pi_{s+1} = s+1, \ldots, \pi_{r_t} = r_t$. Let us assume that $\pi_s < s$.

$\chi_1(\pi_s, \pi_{s+1}) = \chi_1(\pi_s, s+1) = 0$. Additionally,

$$y_{\pi_s} = a - a_{\pi_s} - \sum_{j=s+1}^{n} a_j, y_{s+1} = a - \sum_{j=s+1}^{n} a_j$$

and $(\pi_s, s+1) \in V$, then from the definition of χ_1 it follows that $(\pi_s, s+1) \in U$. $\pi_s < s$, so $(\pi_s, s+1) \in B_t$. But

$$y_{\pi_s} = a - a_{\pi_s} - \sum_{j=s+1}^{n} a_j,$$

which contradicts the definition of R_{π_s}. So, $\pi_s = s$.

We proved that $\pi_{m_t} = m_t, \ldots, \pi_{r_t} = r_t$. By the definition of V, no descents are possible when $\pi_i, \pi_{i+1} \in [l_t, m_t - 1]$, so all elements of $[l_t, m_t - 1]$ also go in the ascending order.

So, $\pi = id$, the only element of G which is not a zero of h' is $x = q^y$, where $y = (0, a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{n-1})$.

Let us see what changes happen to the calculation of the coefficient compared to Theorem 2. For convenience denote $y_{n+1} = a$.

Fix $1 \leqslant t \leqslant k$.

Firstly, elements of S_i $(l_t \leqslant i < r_t)$ disappear from R_i, so the coefficient increases by a factor of

$$\left(\prod_{i=l_t}^{r_t-1} \left(q^{y_i} - q^{a - a_i - \sum_{v=r_t+1}^{n} a_v}\right)\right) \times \prod_{(i,j) \in B_t} \left(q^{y_i} - q^{a - a_i - \sum_{v=j}^{n} a_v}\right).$$

Secondly, we add linear factor

$$(x_i - q^{a_j} x_j - (q^{a - a_i - \sum_{v=j}^{n} a_v} - q^{a - \sum_{s=j}^{n} a_v}))$$

for all $(i,j) \in V_t \setminus U_t = C_t \setminus B_t$, so the coefficient increases by a factor of

$$\prod_{(i,j)\in C_t\setminus B_t} \left(q^{y_i} - q^{y_{j+1}} - (q^{a-a_i-\sum_{v=j}^n a_v} - q^{a-\sum_{v=j}^n a_v})\right) = \prod_{(i,j)\in C_t\setminus B_t} \left(q^{y_i} - q^{a-a_i-\sum_{v=j}^n a_v}\right).$$

Thirdly, we remove linear factor $(x_i - q^{a_j}x_j)$ for all $(i,j) \in V_t$, so the coefficient decreases by a factor of

$$\prod_{(i,j)\in V_t} (q^{y_i} - q^{y_{j+1}}).$$

In total, the coefficient increases by a factor of

$$\begin{aligned}
&\prod_{i=l_t}^{r_t-1} \left(q^{y_i} - q^{a-a_i-\sum_{v=r_t+1}^n a_v}\right) \times \prod_{(i,j)\in C_t} \left(q^{y_i} - q^{a-a_i-\sum_{v=j}^n a_v}\right) / \left(\prod_{(i,j)\in V_t} (q^{y_i} - q^{y_{j+1}})\right)\\
&= \prod_{j=m_t}^{r_t}\prod_{i=l_t}^{j-1} \left(q^{y_i} - q^{a-a_i-\sum_{v=j+1}^n a_v}\right) / \left(\prod_{j=m_t}^{r_t}\prod_{i=l_t}^{j-1} (q^{y_i} - q^{y_{j+1}})\right)\\
&= \prod_{j=m_t}^{r_t}\prod_{i=l_t}^{j-1} (1 - q^{a_{i+1}+\cdots+a_j}) / \left(\prod_{j=m_t}^{r_t}\prod_{i=l_t}^{j-1} (1 - q^{a_i+\cdots+a_j})\right)\\
&= \prod_{j=m_t}^{r_t}\prod_{i=l_t+1}^{j} (1 - q^{a_i+\cdots+a_j}) / \left(\prod_{j=m_t}^{r_t}\prod_{i=l_t}^{j-1} (1 - q^{a_i+\cdots+a_j})\right)\\
&= \prod_{j=m_t}^{r_t} \frac{1-q^{a_j}}{1-q^{a_{l_t}+\cdots+a_j}}.
\end{aligned}$$

All that remains is to multiply the results for $1 \leqslant t \leqslant k$. □

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